11 PARAMETRIC EQUATIONS, POLAR COORDINATES, AND CONIC SECTIONS

11.1 Parametric Equations

Preliminary Questions

1. Describe the shape of the curve $x = 3\cos t$, $y = 3\sin t$.

SOLUTION For all t,

$$x^2 + y^2 = (3\cos t)^2 + (3\sin t)^2 = 9(\cos^2 t + \sin^2 t) = 9 \cdot 1 = 9,$$

therefore the curve is on the circle $x^2 + y^2 = 9$. Also, each point on the circle $x^2 + y^2 = 9$ can be represented in the form $(3\cos t, 3\sin t)$ for some value of t. We conclude that the curve $x = 3\cos t$, $y = 3\sin t$ is the circle of radius 3 centered at the origin.

2. How does $x = 4 + 3\cos t$, $y = 5 + 3\sin t$ differ from the curve in the previous question?

SOLUTION In this case we have

$$(x-4)^2 + (y-5)^2 = (3\cos t)^2 + (3\sin t)^2 = 9(\cos^2 t + \sin^2 t) = 9 \cdot 1 = 9$$

Therefore, the given equations parametrize the circle of radius 3 centered at the point $(4, 5)$.

3. What is the maximum height of a particle whose path has parametric equations $x = t^9$, $y = 4 - t^2$?

SOLUTION The particle's height is $y = 4 - t^2$. To find the maximum height we set the derivative equal to zero and solve:

$$\frac{dy}{dt} = \frac{d}{dt}(4 - t^2) = -2t = 0 \quad \text{or} \quad t = 0$$

The maximum height is $y(0) = 4 - 0^2 = 4$.

4. Can the parametric curve $(t, \sin t)$ be represented as a graph $y = f(x)$? What about $(\sin t, t)$?

SOLUTION In the parametric curve $(t, \sin t)$ we have $x = t$ and $y = \sin t$, therefore, $y = \sin x$. That is, the curve can be represented as a graph of a function. In the parametric curve $(\sin t, t)$ we have $x = \sin t$, $y = t$, therefore $x = \sin y$. This equation does not define y as a function of x, therefore the parametric curve $(\sin t, t)$ cannot be represented as a graph of a function $y = f(x)$.

5. Match the derivatives with a verbal description:

(a) $\dfrac{dx}{dt}$ **(b)** $\dfrac{dy}{dt}$ **(c)** $\dfrac{dy}{dx}$

(i) Slope of the tangent line to the curve

(ii) Vertical rate of change with respect to time

(iii) Horizontal rate of change with respect to time

SOLUTION

(a) The derivative $\dfrac{dx}{dt}$ is the horizontal rate of change with respect to time.

(b) The derivative $\dfrac{dy}{dt}$ is the vertical rate of change with respect to time.

(c) The derivative $\dfrac{dy}{dx}$ is the slope of the tangent line to the curve.

Hence, (a) $\leftrightarrow$ (iii), (b) $\leftrightarrow$ (ii), (c) $\leftrightarrow$ (i)

Exercises

1. Find the coordinates at times $t = 0, 2, 4$ of a particle following the path $x = 1 + t^3$, $y = 9 - 3t^2$.

SOLUTION Substituting $t = 0$, $t = 2$, and $t = 4$ into $x = 1 + t^3$, $y = 9 - 3t^2$ gives the coordinates of the particle at these times respectively. That is,

$$\begin{aligned}
&(t = 0) \quad x = 1 + 0^3 = 1,\ y = 9 - 3\cdot 0^2 = 9 && \Rightarrow (1, 9)\\
&(t = 2) \quad x = 1 + 2^3 = 9,\ y = 9 - 3\cdot 2^2 = -3 && \Rightarrow (9, -3)\\
&(t = 4) \quad x = 1 + 4^3 = 65,\ y = 9 - 3\cdot 4^2 = -39 && \Rightarrow (65, -39).
\end{aligned}$$

2. Find the coordinates at $t = 0, \frac{\pi}{4}, \pi$ of a particle moving along the path $c(t) = (\cos 2t, \sin^2 t)$.

SOLUTION Setting $t = 0, t = \frac{\pi}{4}$, and $t = \pi$ in $c(t) = (\cos 2t, \sin^2 t)$ we obtain the following coordinates of the particle:

$$\begin{aligned}
t = 0&: \quad (\cos 2\cdot 0, \sin^2 0) = (1, 0)\\
t = \tfrac{\pi}{4}&: \quad (\cos \tfrac{2\pi}{4}, \sin^2 \tfrac{\pi}{4}) = (0, \tfrac{1}{2})\\
t = \pi&: \quad (\cos 2\pi, \sin^2 \pi) = (1, 0)
\end{aligned}$$

3. Show that the path traced by the bullet in Example 3 is a parabola by eliminating the parameter.

SOLUTION The path traced by the bullet is given by the following parametric equations:

$$x = 200t, \quad y = 400t - 16t^2$$

We eliminate the parameter. Since $x = 200t$, we have $t = \dfrac{x}{200}$. Substituting into the equation for y we obtain:

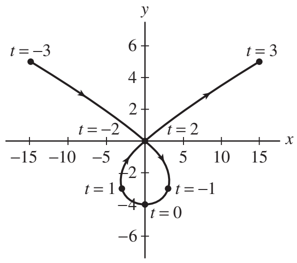
$$y = 400t - 16t^2 = 400\cdot\frac{x}{200} - 16\left(\frac{x}{200}\right)^2 = 2x - \frac{x^2}{2500}$$

The equation $y = -\dfrac{x^2}{2500} + 2x$ is the equation of a parabola.

4. Use the table of values to sketch the parametric curve $(x(t), y(t))$, indicating the direction of motion.

t	-3	-2	-1	0	1	2	3
x	-15	0	3	0	-3	0	15
y	5	0	-3	-4	-3	0	5

SOLUTION We mark the given points on the xy-plane and connect the points corresponding to successive values of t in the direction of increasing t. We get the following trajectory (there are other correct answers):



5. Graph the parametric curves. Include arrows indicating the direction of motion.

(a) (t, t), $\quad -\infty < t < \infty$

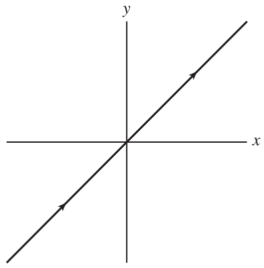
(b) $(\sin t, \sin t)$, $\quad 0 \le t \le 2\pi$

(c) (e^t, e^t), $\quad -\infty < t < \infty$

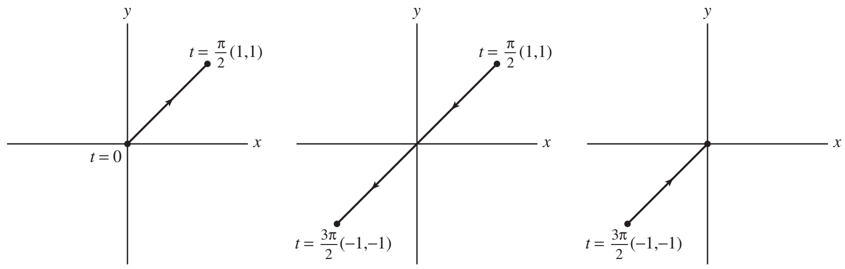
(d) (t^3, t^3), $\quad -1 \le t \le 1$

SOLUTION

(a) For the trajectory $c(t) = (t, t)$, $-\infty < t < \infty$ we have $y = x$. Also the two coordinates tend to ∞ and $-\infty$ as $t \to \infty$ and $t \to -\infty$ respectively. The graph is shown next:

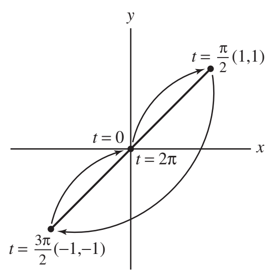


(b) For the curve $c(t) = (\sin t, \sin t)$, $0 \le t \le 2\pi$, we have $y = x$. $\sin t$ is increasing for $0 \le t \le \frac{\pi}{2}$, decreasing for $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$ and increasing again for $\frac{3\pi}{2} \le t \le 2\pi$. Hence the particle moves from $c(0) = (0, 0)$ to $c(\frac{\pi}{2}) = (1, 1)$, then moves back to $c(\frac{3\pi}{2}) = (-1, -1)$ and then returns to $c(2\pi) = (0, 0)$. We obtain the following trajectory:

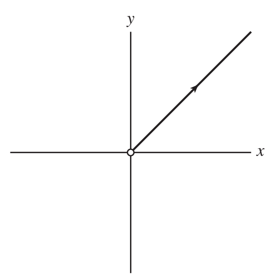


$0 < t \le \frac{\pi}{2}$ $\qquad \frac{\pi}{2} \le t \le \frac{3\pi}{2}$ $\qquad \frac{3\pi}{2} \le t < 2\pi$

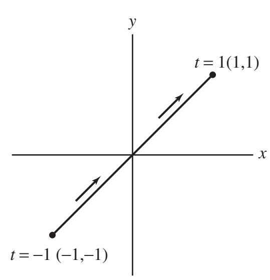
These three parts of the trajectory are shown together in the next figure:



(c) For the trajectory $c(t) = (e^t, e^t)$, $-\infty < t < \infty$, we have $y = x$. However since $\lim_{t \to -\infty} e^t = 0$ and $\lim_{t \to \infty} e^t = \infty$, the trajectory is the part of the line $y = x$, $0 < x$.



(d) For the trajectory $c(t) = (t^3, t^3)$, $-1 \le t \le 1$, we have again $y = x$. Since the function t^3 is increasing the particle moves in one direction starting at $((-1)^3, (-1)^3) = (-1, -1)$ and ending at $(1^3, 1^3) = (1, 1)$. The trajectory is shown next:



6. Give two different parametrizations of the line through $(4, 1)$ with slope 2.

SOLUTION The equation of the line through $(4, 1)$ with slope 2 is $y - 1 = 2(x - 4)$ or $y = 2x - 7$. One parametrization is obtained by choosing the x coordinate as the parameter. That is, $x = t$. Hence $y = 2t - 7$ and we get $x = t$, $y = 2t - 7$, $-\infty < t < \infty$. Another parametrization is given by $x = \frac{t}{2}$, $y = t - 7$, $-\infty < t < \infty$.

In Exercises 7–14, express in the form $y = f(x)$ by eliminating the parameter.

7. $x = t + 3, \quad y = 4t$

SOLUTION We eliminate the parameter. Since $x = t + 3$, we have $t = x - 3$. Substituting into $y = 4t$ we obtain

$$y = 4t = 4(x - 3) \Rightarrow y = 4x - 12$$

8. $x = t^{-1}, \quad y = t^{-2}$

SOLUTION From $x = t^{-1}$, we have $t = x^{-1}$. Substituting in $y = t^{-2}$ we obtain

$$y = t^{-2} = (x^{-1})^{-2} = x^2 \Rightarrow y = x^2, \quad x \neq 0.$$

9. $x = t, \quad y = \tan^{-1}(t^3 + e^t)$

SOLUTION Replacing t by x in the equation for y we obtain $y = \tan^{-1}(x^3 + e^x)$.

10. $x = t^2, \quad y = t^3 + 1$

SOLUTION From $x = t^2$ we get $t = \pm\sqrt{x}$. Substituting into $y = t^3 + 1$ we obtain

$$y = t^3 + 1 = (\pm\sqrt{x})^3 + 1 = \pm\sqrt{x^3} + 1, \quad x \geq 0.$$

Since we must have y a function of x, we should probably choose either the positive or negative root.

11. $x = e^{-2t}, \quad y = 6e^{4t}$

SOLUTION We eliminate the parameter. Since $x = e^{-2t}$, we have $-2t = \ln x$ or $t = -\frac{1}{2}\ln x$. Substituting in $y = 6e^{4t}$ we get

$$y = 6e^{4t} = 6e^{4\cdot(-\frac{1}{2}\ln x)} = 6e^{-2\ln x} = 6e^{\ln x^{-2}} = 6x^{-2} \Rightarrow y = \frac{6}{x^2}, \quad x > 0.$$

12. $x = 1 + t^{-1}, \quad y = t^2$

SOLUTION From $x = 1 + t^{-1}$, we get $t^{-1} = x - 1$ or $t = \frac{1}{x-1}$. We now substitute $t = \frac{1}{x-1}$ in $y = t^2$ to obtain

$$y = t^2 = \left(\frac{1}{x - 1}\right)^2 \Rightarrow y = \frac{1}{(x - 1)^2}, \quad x \neq 1.$$

13. $x = \ln t, \quad y = 2 - t$

SOLUTION Since $x = \ln t$ we have $t = e^x$. Substituting in $y = 2 - t$ we obtain $y = 2 - e^x$.

14. $x = \cos t, \quad y = \tan t$

SOLUTION We use the trigonometric identity $\sin t = \pm\sqrt{1 - \cos^2 t}$ to write

$$y = \tan t = \frac{\sin t}{\cos t} = \pm\frac{\sqrt{1 - \cos^2 t}}{\cos t}.$$

We now express y in terms of x:

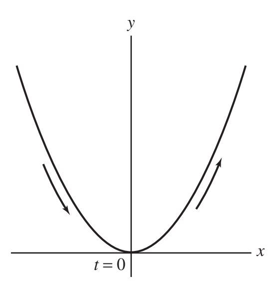
$$y = \tan t = \pm\frac{\sqrt{1 - x^2}}{x} \Rightarrow y = \pm\frac{\sqrt{1 - x^2}}{x}, \quad x \neq 0.$$

Since we must have y a function of x, we should probably choose either the positive or negative root.

In Exercises 15–18, graph the curve and draw an arrow specifying the direction corresponding to motion.

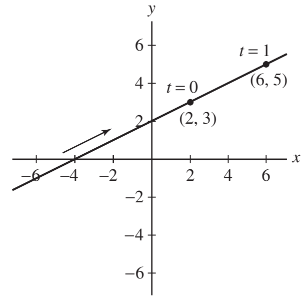
15. $x = \frac{1}{2}t, \quad y = 2t^2$

SOLUTION Let $c(t) = (x(t), y(t)) = (\frac{1}{2}t, 2t^2)$. Then $c(-t) = (-x(t), y(t))$ so the curve is symmetric with respect to the y-axis. Also, the function $\frac{1}{2}t$ is increasing. Hence there is only one direction of motion on the curve. The corresponding function is the parabola $y = 2 \cdot (2x)^2 = 8x^2$. We obtain the following trajectory:



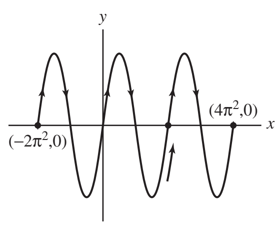
16. $x = 2 + 4t, \quad y = 3 + 2t$

SOLUTION We find the function by eliminating the parameter. Since $x = 2 + 4t$ we have $t = \frac{x-2}{4}$, hence $y = 3 + 2(\frac{x-2}{4})$ or $y = \frac{x}{2} + 2$. Also, since $2 + 4t$ and $3 + 2t$ are increasing functions, the direction of motion is the direction of increasing t. We obtain the following curve:



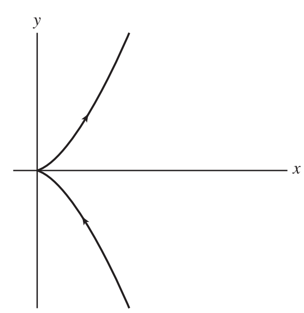
17. $x = \pi t, \quad y = \sin t$

SOLUTION We find the function by eliminating t. Since $x = \pi t$, we have $t = \frac{x}{\pi}$. Substituting $t = \frac{x}{\pi}$ into $y = \sin t$ we get $y = \sin \frac{x}{\pi}$. We obtain the following curve:



18. $x = t^2, \quad y = t^3$

SOLUTION From $x = t^2$ we have $t = \pm x^{1/2}$. Hence, $y = \pm x^{3/2}$. Since the functions t^2 and t^3 are increasing, there is only one direction of motion, which is the direction of increasing t. Notice that for $c(t) = (t^2, t^3)$ we have $c(-t) = (t^2, -t^3) = (x(t), -y(t))$. Hence the curve is symmetric with respect to the x axis. We obtain the following curve:



19. Match the parametrizations (a)–(d) below with their plots in Figure 14, and draw an arrow indicating the direction of motion.

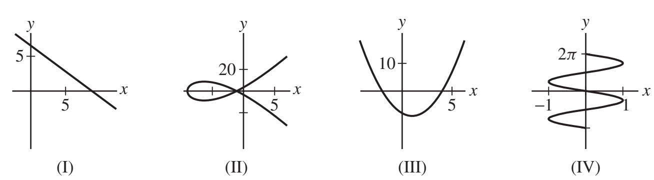


FIGURE 14

(a) $c(t) = (\sin t, -t)$

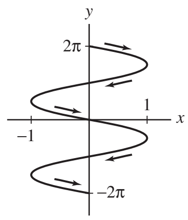
(b) $c(t) = (t^2 - 9, 8t - t^3)$

(c) $c(t) = (1 - t, t^2 - 9)$

(d) $c(t) = (4t + 2, 5 - 3t)$

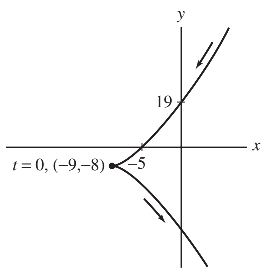
SOLUTION

(a) In the curve $c(t) = (\sin t, -t)$ the x-coordinate is varying between -1 and 1 so this curve corresponds to plot IV. As t increases, the y-coordinate $y = -t$ is decreasing so the direction of motion is downward.



(IV) $c(t) = (\sin t, -t)$

(b) The curve $c(t) = (t^2 - 9, -t^3 - 8)$ intersects the x-axis where $y = -t^3 - 8 = 0$, or $t = -2$. The x-intercept is $(-5, 0)$. The y-intercepts are obtained where $x = t^2 - 9 = 0$, or $t = \pm 3$. The y-intercepts are $(0, -35)$ and $(0, 19)$. As t increases from $-\infty$ to 0, x and y decrease, and as t increases from 0 to ∞, x increases and y decreases. We obtain the following trajectory:

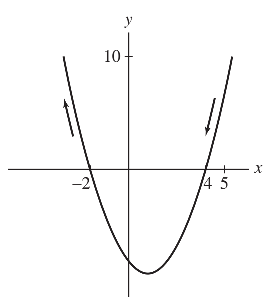


(II)

(c) The curve $c(t) = (1 - t, t^2 - 9)$ intersects the y-axis where $x = 1 - t = 0$, or $t = 1$. The y-intercept is $(0, -8)$. The x-intercepts are obtained where $t^2 - 9 = 0$ or $t = \pm 3$. These are the points $(-2, 0)$ and $(4, 0)$. Setting $t = 1 - x$ we get

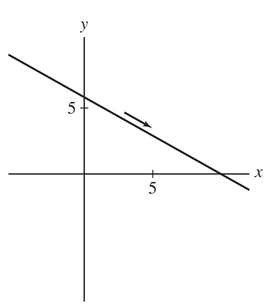
$$y = t^2 - 9 = (1 - x)^2 - 9 = x^2 - 2x - 8.$$

As t increases the x coordinate decreases and we obtain the following trajectory:



(III)

(d) The curve $c(t) = (4t + 2, 5 - 3t)$ is a straight line, since eliminating t in $x = 4t + 2$ and substituting in $y = 5 - 3t$ gives $y = 5 - 3 \cdot \frac{x-2}{4} = -\frac{3}{4}x + \frac{13}{2}$ which is the equation of a line. As t increases, the x coordinate $x = 4t + 2$ increases and the y-coordinate $y = 5 - 3t$ decreases. We obtain the following trajectory:



(I)

20. A particle follows the trajectory

$$x(t) = \frac{1}{4}t^3 + 2t, \qquad y(t) = 20t - t^2$$

with t in seconds and distance in centimeters.
(a) What is the particle's maximum height?
(b) When does the particle hit the ground and how far from the origin does it land?

SOLUTION
(a) To find the maximum height $y(t)$, we set the derivative of $y(t)$ equal to zero and solve:

$$\frac{dy}{dt} = \frac{d}{dt}(20t - t^2) = 20 - 2t = 0 \Rightarrow t = 10.$$

The maximum height is $y(10) = 20 \cdot 10 - 10^2 = 100$ cm.
(b) The object hits the ground when its height is zero. That is, when $y(t) = 0$. Solving for t we get

$$20t - t^2 = t(20 - t) = 0 \Rightarrow t = 0, t = 20.$$

$t = 0$ is the initial time, so the solution is $t = 20$. At that time, the object's x coordinate is $x(20) = \frac{1}{4} \cdot 20^3 + 2 \cdot 20 = 2040$. Thus, when it hits the ground, the object is 2040 cm away from the origin.

21. Find an interval of t-values such that $c(t) = (\cos t, \sin t)$ traces the lower half of the unit circle.

SOLUTION For $t = \pi$, we have $c(\pi) = (-1, 0)$. As t increases from π to 2π, the x-coordinate of $c(t)$ increases from -1 to 1, and the y-coordinate decreases from 0 to -1 (at $t = 3\pi/2$) and then returns to 0. Thus, for t in $[\pi, 2\pi]$, the equation traces the lower part of the circle.

22. Find an interval of t-values such that $c(t) = (2t + 1, 4t - 5)$ parametrizes the segment from $(0, -7)$ to $(7, 7)$.

SOLUTION Note that $2t + 1 = 0$ at $t = -1/2$, and $2t + 1 = 7$ at $t = 3$. Also, $4t - 5$ takes on the values of -7 and 7 at $t = -1/2$ and $t = 3$. Thus, the interval is $[-1/2, 3]$.

In Exercises 23–38, find parametric equations for the given curve.

23. $y = 9 - 4x$

SOLUTION This is a line through $P = (0, 9)$ with slope $m = -4$. Using the parametric representation of a line, as given in Example 3, we obtain $c(t) = (t, 9 - 4t)$.

24. $y = 8x^2 - 3x$

SOLUTION Letting $t = x$ yields the parametric representation $c(t) = (t, 8t^2 - 3t)$.

25. $4x - y^2 = 5$

SOLUTION We define the parameter $t = y$. Then, $x = \dfrac{5 + y^2}{4} = \dfrac{5 + t^2}{4}$, giving us the parametrization $c(t) = \left(\dfrac{5 + t^2}{4}, t\right)$.

26. $x^2 + y^2 = 49$

SOLUTION The curve $x^2 + y^2 = 49$ is a circle of radius 7 centered at the origin. We use the parametric representation of a circle to obtain the representation $c(t) = (7\cos t, 7\sin t)$.

27. $(x + 9)^2 + (y - 4)^2 = 49$

SOLUTION This is a circle of radius 7 centered at $(-9, 4)$. Using the parametric representation of a circle we get $c(t) = (-9 + 7\cos t, 4 + 7\sin t)$.

28. $\left(\dfrac{x}{5}\right)^2 + \left(\dfrac{y}{12}\right)^2 = 1$

SOLUTION This is an ellipse centered at the origin with $a = 5$ and $b = 12$. Using the parametric representation of an ellipse we get $c(t) = (5\cos t, 12\sin t)$ for $-\pi \le t \le \pi$.

29. Line of slope 8 through $(-4, 9)$

SOLUTION Using the parametric representation of a line given in Example 3, we get the parametrization $c(t) = (-4 + t, 9 + 8t)$.

30. Line through $(2, 5)$ perpendicular to $y = 3x$

SOLUTION The line perpendicular to $y = 3x$ has slope $m = -\frac{1}{3}$. We use the parametric representation of a line given in Example 3 to obtain the parametrization $c(t) = (2 + t, 5 - \frac{1}{3}t)$.

31. Line through (3, 1) and (−5, 4)

SOLUTION We use the two-point parametrization of a line with $P = (a, b) = (3, 1)$ and $Q = (c, d) = (-5, 4)$. Then $c(t) = (3 - 8t, 1 + 3t)$ for $-\infty < t < \infty$.

32. Line through $\left(\frac{1}{3}, \frac{1}{6}\right)$ and $\left(-\frac{7}{6}, \frac{5}{3}\right)$

SOLUTION We use the two-point parametrization of a line with $P = (a, b) = \left(\frac{1}{3}, \frac{1}{6}\right)$ and $Q = (c, d) = \left(-\frac{7}{6}, \frac{5}{3}\right)$. Then

$$c(t) = \left(\frac{1}{3} - \frac{3}{2}t, \frac{1}{6} + \frac{3}{2}t\right)$$

for $-\infty < t < \infty$.

33. Segment joining (1, 1) and (2, 3)

SOLUTION We use the two-point parametrization of a line with $P = (a, b) = (1, 1)$ and $Q = (c, d) = (2, 3)$. Then $c(t) = (1 + t, 1 + 2t)$; since we want only the segment joining the two points, we want $0 \le t \le 1$.

34. Segment joining (−3, 0) and (0, 4)

SOLUTION We use the two-point parametrization of a line with $P = (a, b) = (-3, 0)$ and $Q = (c, d) = (0, 4)$. Then $c(t) = (-3 + 3t, 4t)$; since we want only the segment joining the two points, we want $0 \le t \le 1$.

35. Circle of radius 4 with center (3, 9)

SOLUTION Substituting $(a, b) = (3, 9)$ and $R = 4$ in the parametric equation of the circle we get $c(t) = (3 + 4\cos t, 9 + 4\sin t)$.

36. Ellipse of Exercise 28, with its center translated to (7, 4)

SOLUTION Since the center is translated by (7, 4), so is every point. Thus the original parametrization becomes $c(t) = (7 + 5\cos t, 4 + 12\sin t)$ for $-\pi \le t \le \pi$.

37. $y = x^2$, translated so that the minimum occurs at (−4, −8)

SOLUTION We may parametrize $y = x^2$ by (t, t^2) for $-\infty < t < \infty$. The minimum of $y = x^2$ occurs at (0, 0), so the desired curve is translated by (−4, −8) from $y = x^2$. Thus a parametrization of the desired curve is $c(t) = (-4 + t, -8 + t^2)$.

38. $y = \cos x$ translated so that a maximum occurs at (3, 5)

SOLUTION A maximum value 1 of $y = \cos x$ occurs at $x = 0$. Hence, the curve $y - 4 = \cos(x - 3)$, or $y = 4 + \cos(x - 3)$ has a maximum at the point (3, 5). We let $t = x - 3$, then $x = t + 3$ and $y = 4 + \cos t$. We obtain the representation $c(t) = (t + 3, 4 + \cos t)$.

In Exercises 39–42, find a parametrization $c(t)$ of the curve satisfying the given condition.

39. $y = 3x - 4$, $c(0) = (2, 2)$

SOLUTION Let $x(t) = t + a$ and $y(t) = 3x - 4 = 3(t + a) - 4$. We want $x(0) = 2$, thus we must use $a = 2$. Our line is $c(t) = (x(t), y(t)) = (t + 2, 3(t + 2) - 4) = (t + 2, 3t + 2)$.

40. $y = 3x - 4$, $c(3) = (2, 2)$

SOLUTION Let $x(t) = t + a$; since $x(3) = 2$ we have $2 = 3 + a$ so that $a = -1$. Then $y = 3x - 4 = 3(t - 1) - 4 = 3t - 7$, so that the line is $c(t) = (t - 1, 3t - 7)$ for $-\infty < t < \infty$.

41. $y = x^2$, $c(0) = (3, 9)$

SOLUTION Let $x(t) = t + a$ and $y(t) = x^2 = (t + a)^2$. We want $x(0) = 3$, thus we must use $a = 3$. Our curve is $c(t) = (x(t), y(t)) = (t + 3, (t + 3)^2) = (t + 3, t^2 + 6t + 9)$.

42. $x^2 + y^2 = 4$, $c(0) = (1, \sqrt{3})$

SOLUTION This is a circle of radius 2 centered at the origin, so we are looking for a parametrization of that circle that starts at a different point. Thus instead of the standard parametrization $(2\cos\theta, 2\sin\theta)$, $\theta = 0$ must correspond to some other angle ω. We choose the parametrization $(2\cos(\theta + \omega), 2\sin(\theta + \omega))$ and must determine the value of ω. Now,

$$x(0) = 1, \quad \text{so} \quad 1 = 2\cos(0 + \omega) = 2\cos\omega \quad \text{and} \quad \omega = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

Since

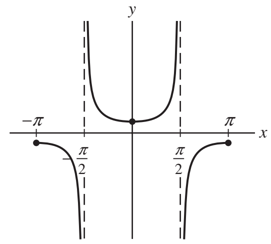
$$y(0) = \sqrt{3}, \quad \text{we have} \quad \sqrt{3} = 2\sin(0 + \omega) = 2\sin\omega \quad \text{and} \quad \omega = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

Comparing these results we see that we must have $\omega = \frac{\pi}{3}$ so that the parametrization is

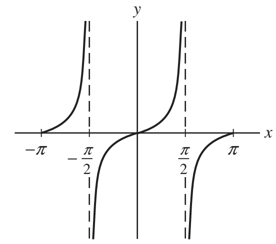
$$c(t) = \left(2\cos\left(\theta + \frac{\pi}{3}\right), 2\sin\left(\theta + \frac{\pi}{3}\right)\right)$$

43. Describe $c(t) = (\sec t, \tan t)$ for $0 \le t < \frac{\pi}{2}$ in the form $y = f(x)$. Specify the domain of x.

SOLUTION The function $x = \sec t$ has period 2π and $y = \tan t$ has period π. The graphs of these functions in the interval $-\pi \le t \le \pi$, are shown below:



$x = \sec t$

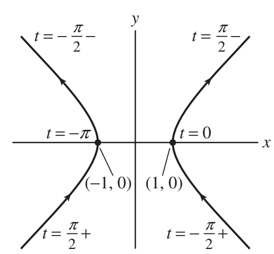


$y = \tan t$

$$x = \sec t \Rightarrow x^2 = \sec^2 t$$

$$y = \tan t \Rightarrow y^2 = \tan^2 t = \frac{\sin^2 t}{\cos^2 t} = \frac{1 - \cos^2 t}{\cos^2 t} = \sec^2 t - 1 = x^2 - 1$$

Hence the graph of the curve is the hyperbola $x^2 - y^2 = 1$. The function $x = \sec t$ is an even function while $y = \tan t$ is odd. Also x has period 2π and y has period π. It follows that the intervals $-\pi \le t < -\frac{\pi}{2}$, $\frac{-\pi}{2} < t < \frac{\pi}{2}$ and $\frac{\pi}{2} < t < \pi$ trace the curve exactly once. The corresponding curve is shown next:



$c(t) = (\sec t, \tan t)$

44. Find a parametrization of the right branch ($x > 0$) of the hyperbola

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

using the functions $\cosh t$ and $\sinh t$. How can you parametrize the branch $x < 0$?

SOLUTION We show first that $x = \cosh t$, $y = \sinh t$ parametrizes the hyperbola when $a = b = 1$: then

$$x^2 - y^2 = (\cosh t)^2 - (\sinh t)^2 = 1.$$

using the identity $\cosh^2 - \sinh^2 = 1$. Generalize this parametrization to get a parametrization for the general hyperbola $(\frac{x}{a})^2 - (\frac{y}{b})^2 = 1$:

$$x = a \cosh t, \ \ y = b \sinh t.$$

We must of course check that this parametrization indeed parametrizes the curve, i.e. that $x = a \cosh t$ and $y = b \sin t$ satisfy the equation $(\frac{x}{a})^2 - (\frac{y}{b})^2 = 1$:

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \left(\frac{a \cosh t}{a}\right)^2 - \left(\frac{b \sinh t}{b}\right)^2 = (\cosh t)^2 - (\sinh t)^2 = 1.$$

The left branch of the hyperbola is the reflection of the right branch around the line $x = 0$, so it clearly has the parametrization

$$x = -a \cosh t, \ \ y = b \sinh t.$$

45. The graphs of $x(t)$ and $y(t)$ as functions of t are shown in Figure 15(A). Which of (I)–(III) is the plot of $c(t) = (x(t), y(t))$? Explain.

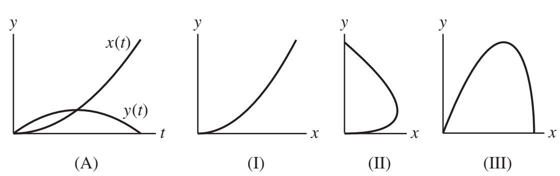


FIGURE 15

SOLUTION As seen in Figure 15(A), the x-coordinate is an increasing function of t, while $y(t)$ is first increasing and then decreasing. In Figure I, x and y are both increasing or both decreasing (depending on the direction on the curve). In Figure II, x does not maintain one tendency, rather, it is decreasing and increasing for certain values of t. The plot $c(t) = (x(t), y(t))$ is plot III.

46. Which graph, (I) or (II), is the graph of $x(t)$ and which is the graph of $y(t)$ for the parametric curve in Figure 16(A)?

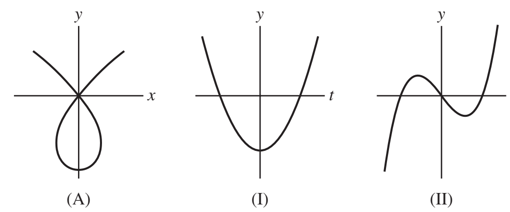
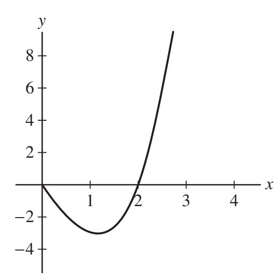


FIGURE 16

SOLUTION As indicated by Figure 16(A), the y-coordinate is decreasing and then increasing, so plot I is the graph of y. Figure 16(A) also shows that the x-coordinate is increasing, decreasing and then increasing, so plot II is the graph for x.

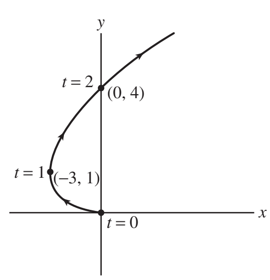
47. Sketch $c(t) = (t^3 - 4t, t^2)$ following the steps in Example 7.

SOLUTION We note that $x(t) = t^3 - 4t$ is odd and $y(t) = t^2$ is even, hence $c(-t) = (x(-t), y(-t)) = (-x(t), y(t))$. It follows that $c(-t)$ is the reflection of $c(t)$ across y-axis. That is, $c(-t)$ and $c(t)$ are symmetric with respect to the y-axis; thus, it suffices to graph the curve for $t \geq 0$. For $t = 0$, we have $c(0) = (0, 0)$ and the y-coordinate $y(t) = t^2$ tends to ∞ as $t \to \infty$. To analyze the x-coordinate, we graph $x(t) = t^3 - 4t$ for $t \geq 0$:

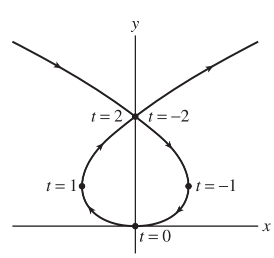


$x = t^3 - 4t$

We see that $x(t) < 0$ and decreasing for $0 < t < 2/\sqrt{3}$, $x(t) < 0$ and increasing for $2/\sqrt{3} < t < 2$ and $x(t) > 0$ and increasing for $t > 2$. Also $x(t)$ tends to ∞ as $t \to \infty$. Therefore, starting at the origin, the curve first directs to the left of the y-axis, then at $t = 2/\sqrt{3}$ it turns to the right, always keeping an upward direction. The part of the path for $t \leq 0$ is obtained by reflecting across the y-axis. We also use the points $c(0) = (0, 0)$, $c(1) = (-3, 1)$, $c(2) = (0, 4)$ to obtain the following graph for $c(t)$:



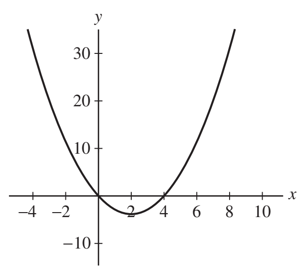
Graph of $c(t)$ for $t \geq 0$.



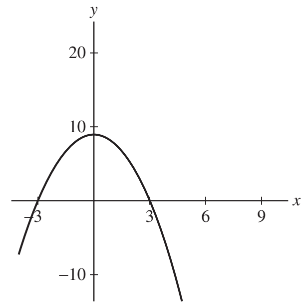
Graph of $c(t)$ for all t.

48. Sketch $c(t) = (t^2 - 4t, 9 - t^2)$ for $-4 \leq t \leq 10$.

SOLUTION The graphs of $x(t) = t^2 - 4t$ and $y(t) = 9 - t^2$ for $-4 \leq t \leq 10$ are shown in the following figures:



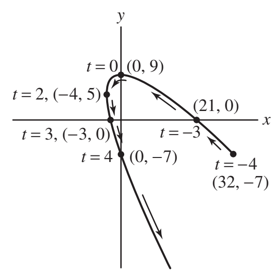
$x(t) = t^2 - 4t$



$y(t) = 9 - t^2$

The curve starts at $c(-4) = (32, -7)$. For $-4 < t < 0$, $x(t)$ is decreasing and $y(t)$ is increasing, so the graph turns to the left and upwards to $c(0) = (0, 9)$. Then for $0 < t < 2$, $x(t)$ is decreasing and so is $y(t)$, hence the graph turns to the left and downwards towards $c(2) = (-4, 5)$.

For $2 < t < 10$, $x(t)$ is increasing and $y(t)$ is decreasing, hence the graph turns to the right and downwards, ending at $c(10) = (60, -91)$. The intercept are the points where $t^2 - 4t = t(t-4) = 0$ or $9 - t^2 = 0$, that is $t = 0, 4, \pm 3$. These are the points $c(0) = (0, 9)$, $c(4) = (0, -7)$, $c(3) = (-3, 0)$, $c(-3) = (21, 0)$. These properties lead to the following path:



In Exercises 49–52, use Eq. (7) to find dy/dx at the given point.

49. $(t^3, t^2 - 1), \quad t = -4$

SOLUTION By Eq. (7) we have

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(t^2-1)'}{(t^3)'} = \frac{2t}{3t^2} = \frac{2}{3t}$$

Substituting $t = -4$ we get

$$\frac{dy}{dx} = \left.\frac{2}{3t}\right|_{t=-4} = \frac{2}{3 \cdot (-4)} = -\frac{1}{6}.$$

50. $(2t + 9, 7t - 9), \quad t = 1$

SOLUTION We find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{(7t-9)'}{(2t+9)'} = \frac{7}{2} \Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = \frac{7}{2}.$$

51. $(s^{-1} - 3s, s^3), \quad s = -1$

SOLUTION Using Eq. (7) we get

$$\frac{dy}{dx} = \frac{y'(s)}{x'(s)} = \frac{(s^3)'}{(s^{-1} - 3s)'} = \frac{3s^2}{-s^{-2} - 3} = \frac{3s^4}{-1 - 3s^2}$$

Substituting $s = -1$ we obtain

$$\frac{dy}{dx} = \left.\frac{3s^4}{-1 - 3s^2}\right|_{s=-1} = \frac{3 \cdot (-1)^4}{-1 - 3 \cdot (-1)^2} = -\frac{3}{4}.$$

52. $(\sin 2\theta, \cos 3\theta), \quad \theta = \frac{\pi}{6}$

SOLUTION Using Eq. (7) we get

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{-3\sin 3\theta}{2\cos 2\theta}$$

Substituting $\theta = \dfrac{\pi}{6}$ we get

$$\frac{dy}{dx} = \frac{-3\sin 3\theta}{2\cos 2\theta}\bigg|_{\theta=\pi/6} = \frac{-3\sin\frac{\pi}{2}}{2\cos\frac{\pi}{3}} = \frac{-3}{2\cdot\frac{1}{2}} = -3$$

In Exercises 53–56, find an equation $y = f(x)$ for the parametric curve and compute dy/dx in two ways: using Eq. (7) and by differentiating $f(x)$.

53. $c(t) = (2t+1, 1-9t)$

SOLUTION Since $x = 2t+1$, we have $t = \dfrac{x-1}{2}$. Substituting in $y = 1-9t$ we have

$$y = 1 - 9\left(\frac{x-1}{2}\right) = -\frac{9}{2}x + \frac{11}{2}$$

Differentiating $y = -\dfrac{9}{2}x + \dfrac{11}{2}$ gives $\dfrac{dy}{dx} = -\dfrac{9}{2}$. We now find $\dfrac{dy}{dx}$ using Eq. (7):

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(1-9t)'}{(2t+1)'} = -\frac{9}{2}$$

54. $c(t) = \left(\frac{1}{2}t, \frac{1}{4}t^2 - t\right)$

SOLUTION Since $x = \frac{1}{2}t$ we have $t = 2x$. Substituting in $y = \frac{1}{4}t^2 - t$ yields

$$y = \frac{1}{4}(2x)^2 - 2x = x^2 - 2x.$$

We differentiate $y = x^2 - 2x$:

$$\frac{dy}{dx} = 2x - 2$$

Now, we find $\frac{dy}{dx}$ using Eq. (7). Thus,

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\left(\frac{1}{4}t^2 - t\right)'}{\left(\frac{1}{2}t\right)'} = \frac{\frac{1}{2}t - 1}{\frac{1}{2}} = t - 2.$$

Since $t = 2x$, then this $t - 2$ is the same as $2x - 2$.

55. $x = s^3, \quad y = s^6 + s^{-3}$

SOLUTION We find y as a function of x:

$$y = s^6 + s^{-3} = \left(s^3\right)^2 + \left(s^3\right)^{-1} = x^2 + x^{-1}.$$

We now differentiate $y = x^2 + x^{-1}$. This gives

$$\frac{dy}{dx} = 2x - x^{-2}.$$

Alternatively, we can use Eq. (7) to obtain the following derivative:

$$\frac{dy}{dx} = \frac{y'(s)}{x'(s)} = \frac{\left(s^6 + s^{-3}\right)'}{\left(s^3\right)'} = \frac{6s^5 - 3s^{-4}}{3s^2} = 2s^3 - s^{-6}.$$

Hence, since $x = s^3$,

$$\frac{dy}{dx} = 2x - x^{-2}.$$

56. $x = \cos\theta, \quad y = \cos\theta + \sin^2\theta$

SOLUTION To find y as a function of x, we first use the trigonometric identity $\sin^2\theta = 1 - \cos^2\theta$ to write

$$y = \cos\theta + 1 - \cos^2\theta.$$

We substitute $x = \cos\theta$ to obtain $y = x + 1 - x^2$. Differentiating this function yields

$$\frac{dy}{dx} = 1 - 2x.$$

Alternatively, we can compute $\frac{dy}{dx}$ using Eq. (7). That is,

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{\left(\cos\theta + \sin^2\theta\right)'}{(\cos\theta)'} = \frac{-\sin\theta + 2\sin\theta\cos\theta}{-\sin\theta} = 1 - 2\cos\theta.$$

Hence, since $x = \cos\theta$,

$$\frac{dy}{dx} = 1 - 2x.$$

57. Find the points on the curve $c(t) = (3t^2 - 2t, t^3 - 6t)$ where the tangent line has slope 3.

SOLUTION We solve

$$\frac{dy}{dx} = \frac{3t^2 - 6}{6t - 2} = 3$$

or $3t^2 - 6 = 18t - 6$, or $t^2 - 6t = 0$, so the slope is 3 at $t = 0, 6$ and the points are $(0, 0)$ and $(96, 180)$

58. Find the equation of the tangent line to the cycloid generated by a circle of radius 4 at $t = \frac{\pi}{2}$.

SOLUTION The cycloid generated by a circle of radius 4 can be parametrized by

$$c(t) = (4t - 4\sin t, 4 - 4\cos t)$$

Then we compute

$$\left.\frac{dy}{dx}\right|_{t=\pi/2} = \left.\frac{4\sin t}{4 - 4\cos t}\right|_{t=\pi/2} = \frac{4}{4} = 1$$

so that the slope of the tangent line is 1 and the equation of the tangent line is

$$y - \left(4 - 4\cos\frac{\pi}{2}\right) = 1 \cdot \left(x - \left(4 \cdot \frac{\pi}{2} - 4\sin\frac{\pi}{2}\right)\right) \quad \text{or} \quad y = x + 8 - 2\pi$$

In Exercises 59–62, let $c(t) = (t^2 - 9, t^2 - 8t)$ (see Figure 17).

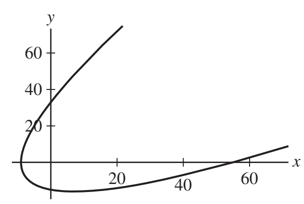
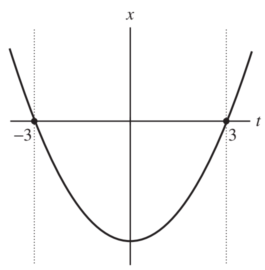


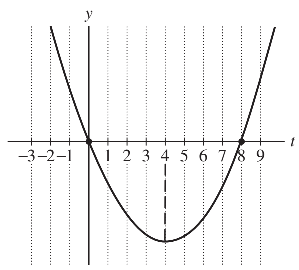
FIGURE 17 Plot of $c(t) = (t^2 - 9, t^2 - 8t)$.

59. Draw an arrow indicating the direction of motion, and determine the interval of t-values corresponding to the portion of the curve in each of the four quadrants.

SOLUTION We plot the functions $x(t) = t^2 - 9$ and $y(t) = t^2 - 8t$:

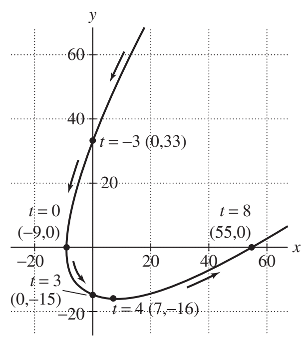


$x = t^2 - 9$



$y = t^2 - 8t$

We note carefully where each of these graphs are positive or negative, increasing or decreasing. In particular, $x(t)$ is decreasing for $t < 0$, increasing for $t > 0$, positive for $|t| > 3$, and negative for $|t| < 3$. Likewise, $y(t)$ is decreasing for $t < 4$, increasing for $t > 4$, positive for $t > 8$ or $t < 0$, and negative for $0 < t < 8$. We now draw arrows on the path following the decreasing/increasing behavior of the coordinates as indicated above. We obtain:



This plot also shows that:

- The graph is in the first quadrant for $t < -3$ or $t > 8$.
- The graph is in the second quadrant for $-3 < t < 0$.
- The graph is in the third quadrant for $0 < t < 3$.
- The graph is in the fourth quadrant for $3 < t < 8$.

60. Find the equation of the tangent line at $t = 4$.

SOLUTION Using the formula for the slope m of the tangent line we have:

$$m = \frac{dy}{dx}\bigg|_{t=4} = \frac{\left(t^2 - 8t\right)'}{\left(t^2 - 9\right)'}\bigg|_{t=4} = \frac{2t - 8}{2t}|_{t=4} = 1 - \frac{4}{t}|_{t=4} = 0.$$

Since the slope is zero, the tangent line is horizontal. The y-coordinate corresponding to $t = 4$ is $y = 4^2 - 8 \cdot 4 = -16$. Hence the equation of the tangent line is $y = -16$.

61. Find the points where the tangent has slope $\frac{1}{2}$.

SOLUTION The slope of the tangent at t is

$$\frac{dy}{dx} = \frac{\left(t^2 - 8t\right)'}{\left(t^2 - 9\right)'} = \frac{2t - 8}{2t} = 1 - \frac{4}{t}$$

The point where the tangent has slope $\frac{1}{2}$ corresponds to the value of t that satisfies

$$\frac{dy}{dx} = 1 - \frac{4}{t} = \frac{1}{2} \Rightarrow \frac{4}{t} = \frac{1}{2} \Rightarrow t = 8.$$

We substitute $t = 8$ in $x(t) = t^2 - 9$ and $y(t) = t^2 - 8t$ to obtain the following point:

$$\begin{aligned} x(8) &= 8^2 - 9 = 55 \\ y(8) &= 8^2 - 8 \cdot 8 = 0 \end{aligned} \quad \Rightarrow \quad (55, 0)$$

62. Find the points where the tangent is horizontal or vertical.

SOLUTION In Exercise 61 we found that the slope of the tangent at t is

$$\frac{dy}{dx} = 1 - \frac{4}{t} = \frac{t - 4}{t}$$

The tangent is horizontal where its slope is zero. We set the slope equal to zero and solve for t. This gives

$$\frac{t - 4}{t} = 0 \Rightarrow t = 4.$$

The corresponding point is

$$(x(4), y(4)) = (4^2 - 9, 4^2 - 8 \cdot 4) = (7, -16).$$

The tangent is vertical where it has infinite slope; that is, at $t = 0$. The corresponding point is

$$(x(0), y(0)) = (0^2 - 9, 0^2 - 8 \cdot 0) = (-9, 0).$$

63. Let A and B be the points where the ray of angle θ intersects the two concentric circles of radii $r < R$ centered at the origin (Figure 18). Let P be the point of intersection of the horizontal line through A and the vertical line through B. Express the coordinates of P as a function of θ and describe the curve traced by P for $0 \le \theta \le 2\pi$.

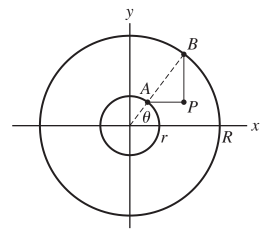


FIGURE 18

SOLUTION We use the parametric representation of a circle to determine the coordinates of the points A and B. That is,

$$A = (r\cos\theta, r\sin\theta), \quad B = (R\cos\theta, R\sin\theta)$$

The coordinates of P are therefore

$$P = (R\cos\theta, r\sin\theta)$$

In order to identify the curve traced by P, we notice that the x and y coordinates of P satisfy $\frac{x}{R} = \cos\theta$ and $\frac{y}{r} = \sin\theta$. Hence

$$\left(\frac{x}{R}\right)^2 + \left(\frac{y}{r}\right)^2 = \cos^2\theta + \sin^2\theta = 1.$$

The equation

$$\left(\frac{x}{R}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

is the equation of ellipse. Hence, the coordinates of P, $(R\cos\theta, r\sin\theta)$ describe an ellipse for $0 \le \theta \le 2\pi$.

64. A 10-ft ladder slides down a wall as its bottom B is pulled away from the wall (Figure 19). Using the angle θ as parameter, find the parametric equations for the path followed by (a) the top of the ladder A, (b) the bottom of the ladder B, and (c) the point P located 4 ft from the top of the ladder. Show that P describes an ellipse.

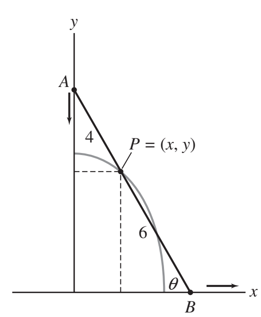
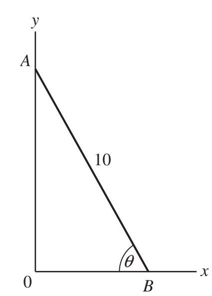


FIGURE 19

SOLUTION

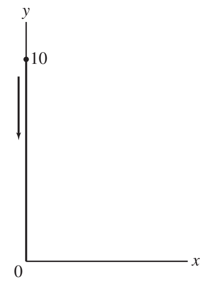
(a) We define the xy-coordinate system as shown in the figure:



As the ladder slides down the wall, the x-coordinate of A is always zero and the y-coordinate is $y = 10\sin\theta$. The parametric equations for the path followed by A are thus

$$x = 0,\ \ y = 10\sin\theta, \quad \theta \text{ is between } \tfrac{\pi}{2} \text{ and } 0.$$

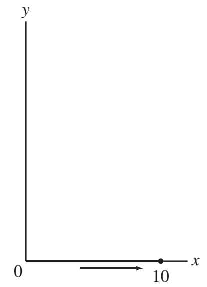
The path described by A is the segment [0, 10] on the y-axis.



(b) As the ladder slides down the wall, the y-coordinate of B is always zero and the x-coordinate is $x = 10\cos\theta$. The parametric equations for the path followed by B are therefore

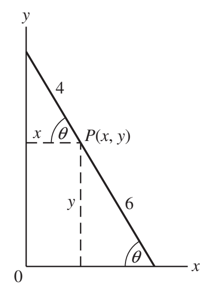
$$x = 10\cos\theta,\ y = 0, \quad \theta \text{ is between } \tfrac{\pi}{2} \text{ and } 0.$$

The path is the segment [0, 10] on the x-axis.

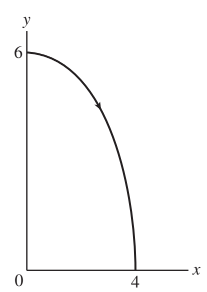


(c) The x and y coordinates of P are $x = 4\cos\theta$, $y = 6\sin\theta$. The path followed by P has the following parametrization:

$$c(\theta) = (4\cos\theta, 6\sin\theta), \quad \theta \text{ is between } \tfrac{\pi}{2} \text{ and } 0.$$



As shown in Example 4, the corresponding path is a part of an ellipse. Since θ is varying between $\frac{\pi}{2}$ and 0, we obtain the part of the ellipse in the first quadrant.



In Exercises 65–68, refer to the Bézier curve defined by Eqs. (8) and (9).

65. Show that the Bézier curve with control points

$$P_0 = (1, 4), \quad P_1 = (3, 12), \quad P_2 = (6, 15), \quad P_3 = (7, 4)$$

has parametrization

$$c(t) = (1 + 6t + 3t^2 - 3t^3, 4 + 24t - 15t^2 - 9t^3)$$

Verify that the slope at $t = 0$ is equal to the slope of the segment $\overline{P_0P_1}$.

SOLUTION For the given Bézier curve we have $a_0 = 1, a_1 = 3, a_2 = 6, a_3 = 7$, and $b_0 = 4, b_1 = 12, b_2 = 15, b_3 = 4$. Substituting these values in Eq. (8)–(9) and simplifying gives

$$\begin{aligned} x(t) &= (1-t)^3 + 9t(1-t)^2 + 18t^2(1-t) + 7t^3 \\ &= 1 - 3t + 3t^2 - t^3 + 9t(1 - 2t + t^2) + 18t^2 - 18t^3 + 7t^3 \\ &= 1 - 3t + 3t^2 - t^3 + 9t - 18t^2 + 9t^3 + 18t^2 - 18t^3 + 7t^3 \\ &= -3t^3 + 3t^2 + 6t + 1 \\ y(t) &= 4(1-t)^3 + 36t(1-t)^2 + 45t^2(1-t) + 4t^3 \\ &= 4(1 - 3t + 3t^2 - t^3) + 36t(1 - 2t + t^2) + 45t^2 - 45t^3 + 4t^3 \\ &= 4 - 12t + 12t^2 - 4t^3 + 36t - 72t^2 + 36t^3 + 45t^2 - 45t^3 + 4t^3 \\ &= 4 + 24t - 15t^2 - 9t^3 \end{aligned}$$

Then

$$c(t) = (1 + 6t + 3t^2 - 3t^3, 4 + 24t - 15t^2 - 9t^3), \quad 0 \le t \le 1.$$

We find the slope at $t = 0$. Using the formula for slope of the tangent line we get

$$\frac{dy}{dx} = \frac{(4 + 24t - 15t^2 - 9t^3)'}{(1 + 6t + 3t^2 - 3t^3)'} = \frac{24 - 30t - 27t^2}{6 + 6t - 9t^2} \Rightarrow \left.\frac{dy}{dx}\right|_{t=0} = \frac{24}{6} = 4.$$

The slope of the segment $\overline{P_0P_1}$ is the slope of the line determined by the points $P_0 = (1, 4)$ and $P_1 = (3, 12)$. That is, $\frac{12-4}{3-1} = \frac{8}{2} = 4$. We see that the slope of the tangent line at $t = 0$ is equal to the slope of the segment $\overline{P_0P_1}$, as expected.

66. Find an equation of the tangent line to the Bézier curve in Exercise 65 at $t = \frac{1}{3}$.

SOLUTION We have

$$\frac{dy}{dx} = \frac{y(t)'}{x(t)'} = \frac{24 - 30t - 27t^2}{6_6t - 9t^2}$$

so that at $t = \frac{1}{3}$,

$$\left.\frac{dy}{dx}\right|_{t=1/3} = \left.\frac{24 - 30t - 27t^2}{6 + 6t - 9t^2}\right|_{t=1/3} = \frac{11}{7}$$

and

$$x\left(\frac{1}{3}\right) = \frac{29}{9}, \qquad y\left(\frac{1}{3}\right) = 10$$

Thus the tangent line is

$$y - 10 = \frac{11}{7}\left(x - \frac{29}{9}\right) \qquad \text{or} \qquad y = \frac{11}{7}x + \frac{311}{63}$$

67. CAS Find and plot the Bézier curve $c(t)$ passing through the control points

$$P_0 = (3, 2), \quad P_1 = (0, 2), \quad P_2 = (5, 4), \quad P_3 = (2, 4)$$

SOLUTION Setting $a_0 = 3, a_1 = 0, a_2 = 5, a_3 = 2$, and $b_0 = 2, b_1 = 2, b_2 = 4, b_3 = 4$ into Eq. (8)–(9) and simplifying gives

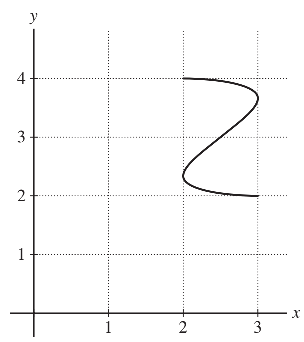
$$\begin{aligned} x(t) &= 3(1-t)^3 + 0 + 15t^2(1-t) + 2t^3 \\ &= 3(1 - 3t + 3t^2 - t^3) + 15t^2 - 15t^3 + 2t^3 = 3 - 9t + 24t^2 - 16t^3 \end{aligned}$$

$$\begin{aligned} y(t) &= 2(1-t)^3 + 6t(1-t)^2 + 12t^2(1-t) + 4t^3 \\ &= 2(1-3t+3t^2-t^3) + 6t(1-2t+t^2) + 12t^2 - 12t^3 + 4t^3 \\ &= 2 - 6t + 6t^2 - 2t^3 + 6t - 12t^2 + 6t^3 + 12t^2 - 12t^3 + 4t^3 = 2 + 6t^2 - 4t^3 \end{aligned}$$

We obtain the following equation

$$c(t) = (3 - 9t + 24t^2 - 16t^3, 2 + 6t^2 - 4t^3), \quad 0 \le t \le 1.$$

The graph of the Bézier curve is shown in the following figure:



68. Show that a cubic Bézier curve is tangent to the segment $\overline{P_2P_3}$ at P_3.

SOLUTION The equations of the cubic Bézier curve are

$$\begin{aligned} x(t) &= a_0(1-t)^3 + 3a_1t(1-t)^2 + 3a_2t^2(1-t) + a_3t^3 \\ y(t) &= b_0(1-t)^3 + 3b_1t(1-t)^2 + 3b_2t^2(1-t) + b_3t^3 \end{aligned}$$

We use the formula for the slope of the tangent line to find the slope of the tangent line at P_3. We obtain

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-3b_0(1-t)^2 + 3b_1((1-t)^2 - 2t(1-t)) + 3b_2(2t(1-t) - t^2) + 3b_3t^2}{-3a_0(1-t)^2 + 3a_1((1-t)^2 - 2t(1-t)) + 3a_2(2t(1-t) - t^2) + 3a_3t^2} \tag{1}$$

The slope of the tangent line at P_3 is obtained by setting $t = 1$ in (1). That is,

$$m_1 = \frac{0 + 0 - 3b_2 + 3b_3}{0 + 0 - 3a_2 + 3a_3} = \frac{b_3 - b_2}{a_3 - a_2} \tag{2}$$

We compute the slope of the segment $\overline{P_2P_3}$ for $P_2 = (a_2, b_2)$ and $P_3 = (a_3, b_3)$. We get

$$m_2 = \frac{b_3 - b_2}{a_3 - a_2}$$

Since the two slopes are equal, we conclude that the tangent line to the curve at the point P_3 is the segment $\overline{P_2P_3}$.

69. A bullet fired from a gun follows the trajectory

$$x = at, \qquad y = bt - 16t^2 \quad (a, b > 0)$$

Show that the bullet leaves the gun at an angle $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ and lands at a distance $ab/16$ from the origin.

SOLUTION The height of the bullet equals the value of the y-coordinate. When the bullet leaves the gun, $y(t) = t(b - 16t) = 0$. The solutions to this equation are $t = 0$ and $t = \frac{b}{16}$, with $t = 0$ corresponding to the moment the bullet leaves the gun. We find the slope m of the tangent line at $t = 0$:

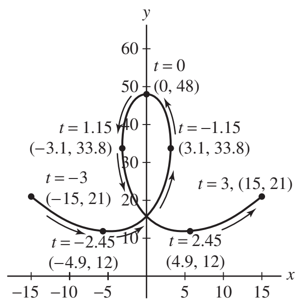
$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{b - 32t}{a} \Rightarrow m = \left.\frac{b - 32t}{a}\right|_{t=0} = \frac{b}{a}$$

It follows that $\tan\theta = \frac{b}{a}$ or $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. The bullet lands at $t = \frac{b}{16}$. We find the distance of the bullet from the origin at this time, by substituting $t = \frac{b}{16}$ in $x(t) = at$. This gives

$$x\left(\frac{b}{16}\right) = \frac{ab}{16}$$

70. CAS Plot $c(t) = (t^3 - 4t, t^4 - 12t^2 + 48)$ for $-3 \le t \le 3$. Find the points where the tangent line is horizontal or vertical.

SOLUTION The graph of $c(t) = (t^3 - 4t, t^4 - 12t^2 + 48)$, $-3 \le t \le 3$ is shown in the following figure:



We find the slope of the tangent line at t:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(x)} = \frac{(t^4 - 12t^2 + 48)'}{(t^3 - 4t)'} = \frac{4t^3 - 24t}{3t^2 - 4} \tag{1}$$

The tangent line is horizontal where $\frac{dy}{dx} = 0$. That is,

$$\frac{4t^3 - 24t}{3t^2 - 4} = 0 \Rightarrow 4t(t^2 - 6) = 0 \Rightarrow t = 0,\ t = -\sqrt{6},\ t = \sqrt{6}.$$

We find the corresponding points by substituting these values of t in $c(t)$. We obtain:

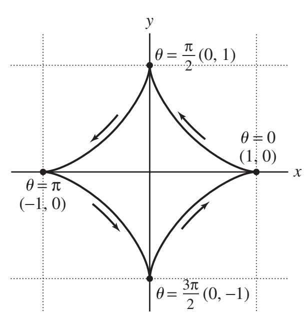
$$c(0) = (0, 48),\ c(-\sqrt{6}) \approx (-4.9, 12),\ c(\sqrt{6}) \approx (4.9, 12).$$

The tangent line is vertical where the slope in (1) is infinite, that is, where $3t^2 - 4 = 0$ or $t = \pm\frac{2}{\sqrt{3}} \approx \pm 1.15$. We find the points by setting $t = \pm\frac{2}{\sqrt{3}}$ in $c(t)$. We get

$$c\left(\frac{2}{\sqrt{3}}\right) \approx (-3.1, 33.8), \quad c\left(-\frac{2}{\sqrt{3}}\right) \approx (3.1, 33.8).$$

71. CAS Plot the astroid $x = \cos^3\theta$, $y = \sin^3\theta$ and find the equation of the tangent line at $\theta = \frac{\pi}{3}$.

SOLUTION The graph of the astroid $x = \cos^3\theta$, $y = \sin^3\theta$ is shown in the following figure:



The slope of the tangent line at $\theta = \frac{\pi}{3}$ is

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \left.\frac{(\sin^3\theta)'}{(\cos^3\theta)'}\right|_{\theta=\pi/3} = \left.\frac{3\sin^2\theta\cos\theta}{3\cos^2\theta(-\sin\theta)}\right|_{\theta=\pi/3} = -\left.\frac{\sin\theta}{\cos\theta}\right|_{\theta=\pi/3} = -\tan\theta\Big|_{\pi/3} = -\sqrt{3}$$

We find the point of tangency:

$$\left(x\left(\frac{\pi}{3}\right), y\left(\frac{\pi}{3}\right)\right) = \left(\cos^3\frac{\pi}{3}, \sin^3\frac{\pi}{3}\right) = \left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$$

The equation of the tangent line at $\theta = \frac{\pi}{3}$ is, thus,

$$y - \frac{3\sqrt{3}}{8} = -\sqrt{3}\left(x - \frac{1}{8}\right) \Rightarrow y = -\sqrt{3}x + \frac{\sqrt{3}}{2}$$

72. Find the equation of the tangent line at $t = \frac{\pi}{4}$ to the cycloid generated by the unit circle with parametric equation (5).

SOLUTION We find the equation of the tangent line at $t = \frac{\pi}{4}$ to the cycloid $x = t - \sin t$, $y = 1 - \cos t$. We first find the derivative $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(1-\cos t)'}{(t-\sin t)'} = \frac{\sin t}{1-\cos t}$$

The slope of the tangent line at $t = \frac{\pi}{4}$ is therefore:

$$m = \left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{\sin\frac{\pi}{4}}{1-\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{2}-1}$$

We find the point of tangency:

$$\left(x\left(\frac{\pi}{4}\right), y\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4} - \sin\frac{\pi}{4}, 1-\cos\frac{\pi}{4}\right) = \left(\frac{\pi}{4} - \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\right)$$

The equation of the tangent line is, thus,

$$y - \left(1 - \frac{\sqrt{2}}{2}\right) = \frac{1}{\sqrt{2}-1}\left(x - \left(\frac{\pi}{4} - \frac{\sqrt{2}}{2}\right)\right) \Rightarrow y = \frac{1}{\sqrt{2}-1}x + \left(2 - \frac{\frac{\pi}{4}}{\sqrt{2}-1}\right)$$

73. Find the points with horizontal tangent line on the cycloid with parametric equation (5).

SOLUTION The parametric equations of the cycloid are

$$x = t - \sin t, \quad y = 1 - \cos t$$

We find the slope of the tangent line at t:

$$\frac{dy}{dx} = \frac{(1-\cos t)'}{(t-\sin t)'} = \frac{\sin t}{1-\cos t}$$

The tangent line is horizontal where it has slope zero. That is,

$$\frac{dy}{dx} = \frac{\sin t}{1-\cos t} = 0 \quad\Rightarrow\quad \begin{array}{c}\sin t = 0\\ \cos t \neq 1\end{array} \quad\Rightarrow\quad t = (2k-1)\pi, \quad k = 0, \pm 1, \pm 2, \dots$$

We find the coordinates of the points with horizontal tangent line, by substituting $t = (2k-1)\pi$ in $x(t)$ and $y(t)$. This gives

$$x = (2k-1)\pi - \sin(2k-1)\pi = (2k-1)\pi$$
$$y = 1 - \cos((2k-1)\pi) = 1 - (-1) = 2$$

The required points are

$$((2k-1)\pi, 2), \quad k = 0, \pm 1, \pm 2, \dots$$

74. Property of the Cycloid Prove that the tangent line at a point P on the cycloid always passes through the top point on the rolling circle as indicated in Figure 20. Assume the generating circle of the cycloid has radius 1.

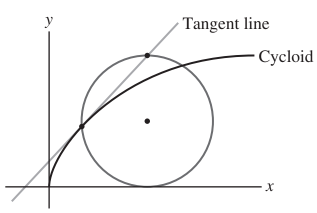


FIGURE 20

SOLUTION The definition of the cycloid is such that at time t, the top of the circle has coordinates $Q = (t, 2)$ (since at time $t = 2\pi$ the circle has rotated exactly once, and its circumference is 2π). Let L be the line through P and Q. To show that L is tangent to the cycloid at P it suffices to show that the slope of L equals the slope of the tangent at P. Recall that the cycloid is parametrized by $c(t) = (t - \sin t, 1 - \cos t)$. Then the slope of L is

$$\frac{2-(1-\cos t)}{t-(t-\sin t)} = \frac{1+\cos t}{\sin t}$$

and the slope of the tangent line is

$$\frac{y'(t)}{x'(t)} = \frac{(1-\cos t)'}{(t-\sin t)'} = \frac{\sin t}{1-\cos t} = \frac{\sin t(1+\cos t)}{1-\cos^2 t} = \frac{\sin t(1+\cos t)}{\sin^2 t} = \frac{1+\cos t}{\sin t}$$

and the two are equal.

75. A *curtate cycloid* (Figure 21) is the curve traced by a point at a distance h from the center of a circle of radius R rolling along the x-axis where $h < R$. Show that this curve has parametric equations $x = Rt - h\sin t$, $y = R - h\cos t$.

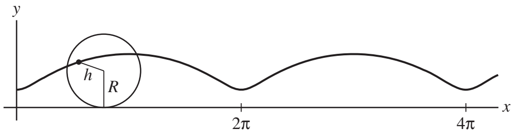
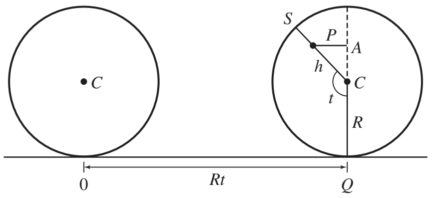


FIGURE 21 Curtate cycloid.

SOLUTION Let P be a point at a distance h from the center C of the circle. Assume that at $t = 0$, the line of CP is passing through the origin. When the circle rolls a distance Rt along the x-axis, the length of the arc $\widehat{SQ}$ (see figure) is also Rt and the angle $\angle SCQ$ has radian measure t. We compute the coordinates x and y of P.



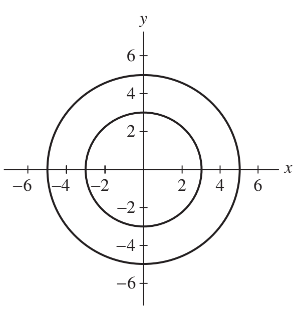
$$x = Rt - \overline{PA} = Rt - h\sin(\pi - t) = Rt - h\sin t$$
$$y = R + \overline{AC} = R + h\cos(\pi - t) = R - h\cos t$$

We obtain the following parametrization:

$$x = Rt - h\sin t, \ \ y = R - h\cos t.$$

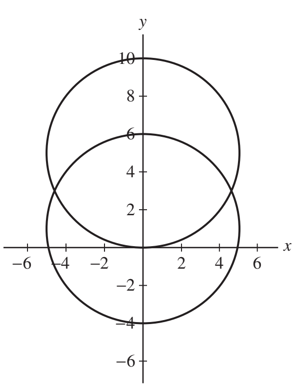
76. CAS Use a computer algebra system to explore what happens when $h > R$ in the parametric equations of Exercise 75. Describe the result.

SOLUTION Look first at the parametric equations $x = -h\sin t$, $y = -h\cos t$. These describe a circle of radius h. See for instance the graphs below obtained for $h = 3$ and $h = 5$.



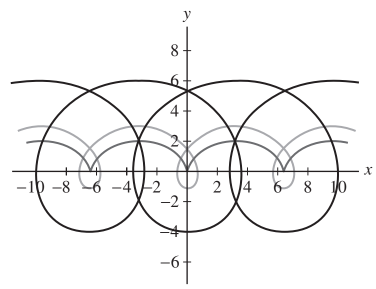
$c(t) = (-h*\sin(t), -h*\cos(t))$ $h = 3, 5$

Adding R to the y coordinate to obtain the parametric equations $x = -h\sin t$, $y = R - h\cos t$, yields a circle with its center moved up by R units:



$c(t) = (-h*\sin(t), R-h*\cos(t))$ $R = 1, 5$ $h = 5$

Now, we add Rt to the x coordinate to obtain the given parametric equation; the curve becomes a spring. The figure below shows the graphs obtained for $R = 1$ and various values of h. We see the inner loop formed for $h > R$.



77. Show that the line of slope t through $(-1, 0)$ intersects the unit circle in the point with coordinates

$$x = \frac{1-t^2}{t^2+1}, \qquad y = \frac{2t}{t^2+1}$$ **10**

Conclude that these equations parametrize the unit circle with the point $(-1, 0)$ excluded (Figure 22). Show further that $t = y/(x+1)$.

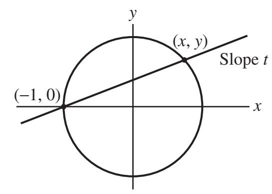


FIGURE 22 Unit circle.

SOLUTION The equation of the line of slope t through $(-1, 0)$ is $y = t(x+1)$. The equation of the unit circle is $x^2 + y^2 = 1$. Hence, the line intersects the unit circle at the points (x, y) that satisfy the equations:

$$y = t(x+1) \tag{1}$$
$$x^2 + y^2 = 1 \tag{2}$$

Substituting y from equation (1) into equation (2) and solving for x we obtain

$$\begin{aligned} x^2 + t^2(x+1)^2 &= 1 \\ x^2 + t^2x^2 + 2t^2x + t^2 &= 1 \\ (1+t^2)x^2 + 2t^2x + (t^2-1) &= 0 \end{aligned}$$

This gives

$$x_{1,2} = \frac{-2t^2 \pm \sqrt{4t^4 - 4(t^2+1)(t^2-1)}}{2(1+t^2)} = \frac{-2t^2 \pm 2}{2(1+t^2)} = \frac{\pm 1 - t^2}{1+t^2}$$

So $x_1 = -1$ and $x_2 = \frac{1-t^2}{t^2+1}$. The solution $x = -1$ corresponds to the point $(-1, 0)$. We are interested in the second point of intersection that is varying as t varies. Hence the appropriate solution is

$$x = \frac{1-t^2}{t^2+1}$$

We find the y-coordinate by substituting x in equation (1). This gives

$$y = t(x+1) = t\left(\frac{1-t^2}{t^2+1} + 1\right) = t \cdot \frac{1-t^2+t^2+1}{t^2+1} = \frac{2t}{t^2+1}$$

We conclude that the line and the unit circle intersect, besides at $(-1, 0)$, at the point with the following coordinates:

$$x = \frac{1-t^2}{t^2+1}, \quad y = \frac{2t}{t^2+1} \tag{3}$$

Since these points determine all the points on the unit circle except for $(-1, 0)$ and no other points, the equations in (3) parametrize the unit circle with the point $(-1, 0)$ excluded.

We show that $t = \frac{y}{x+1}$. Using (3) we have

$$\frac{y}{x+1} = \frac{\frac{2t}{t^2+1}}{\frac{1-t^2}{t^2+1}+1} = \frac{\frac{2t}{t^2+1}}{\frac{1-t^2+t^2+1}{t^2+1}} = \frac{\frac{2t}{t^2+1}}{\frac{2}{t^2+1}} = \frac{2t}{2} = t.$$

78. The **folium of Descartes** is the curve with equation $x^3 + y^3 = 3axy$, where $a \neq 0$ is a constant (Figure 23).

(a) Show that the line $y = tx$ intersects the folium at the origin and at one other point P for all $t \neq -1, 0$. Express the coordinates of P in terms of t to obtain a parametrization of the folium. Indicate the direction of the parametrization on the graph.

(b) Describe the interval of t-values parametrizing the parts of the curve in quadrants I, II, and IV. Note that $t = -1$ is a point of discontinuity of the parametrization.

(c) Calculate dy/dx as a function of t and find the points with horizontal or vertical tangent.

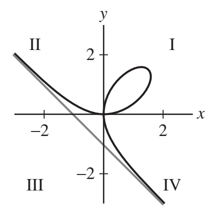


FIGURE 23 Folium $x^3 + y^3 = 3axy$.

SOLUTION

(a) We find the points where the line $y = tx$ $(t \neq -1, 0)$ and the folium intersect, by solving the following equations:

$$y = tx \tag{1}$$

$$x^3 + y^3 = 3axy \tag{2}$$

Substituting y from (1) in (2) and solving for x we get

$$x^3 + t^3x^3 = 3axtx$$

$$(1 + t^3)x^3 - 3atx^2 = 0$$

$$x^2(x(1 + t^3) - 3at) = 0 \Rightarrow x_1 = 0,\ x_2 = \frac{3at}{1+t^3}$$

Substituting in (1) we find the corresponding y-coordinates. That is,

$$y_1 = t \cdot 0 = 0,\ y_2 = t \cdot \frac{3at}{1+t^3} = \frac{3at^2}{1+t^3}$$

We conclude that the line $y = \text{tx}$, $t \neq 0, -1$ intersects the folium in a unique point P besides the origin. The coordinates of P are:

$$x = \frac{3at}{1+t^3},\ y = \frac{3at^2}{1+t^3}, \quad t \neq 0, -1$$

The coordinates of P determine a parametrization for the folium. We add the origin so $t = 0$ must be included in the interval of t. We get

$$c(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right), \quad t \neq -1$$

To indicate the direction on the curve (for $a > 0$), we first consider the following limits:

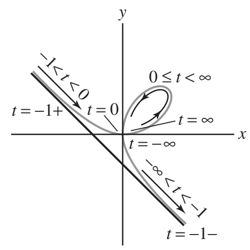
$$\lim_{t\to -1-} x(t) = \infty \qquad \lim_{t\to -1-} y(t) = -\infty$$

$$\lim_{t\to -\infty} x(t) = \lim_{t\to\infty} x(t) = 0 \qquad \lim_{t\to -\infty} y(t) = \lim_{t\to\infty} y(t) = 0$$

$$\lim_{t\to -1+} x(t) = -\infty \qquad \lim_{t\to -1+} y(t) = \infty$$

$$\lim_{t\to 0} x(t) = 0 \qquad \lim_{t\to 0} y(t) = 0$$

These limits determine the directions of the two parts of the folium in the second and fourth quadrant. The loop in the first quadrant, corresponds to the values $0 \le t < \infty$, and it is directed from $c(1) = \left(\frac{3a}{2}, \frac{3a}{2}\right)$ to $c(2) = \left(\frac{2a}{3}, \frac{4a}{3}\right)$ where $t = 1$ and $t = 2$ are two chosen values in the interval $0 \le t < \infty$. The following graph shows the directed folium:



(b) The limits computed in part (a) indicate that the parts of the curve in the second and fourth quadrants correspond to the values $-1 < t < 0$ and $-\infty < t < -1$ respectively. The loop in the first quadrant corresponds to the remaining interval $0 \le t < \infty$.

(c) We find the derivative $\frac{dy}{dx}$, using the Formula for the Slope of the Tangent Line. We get

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\left(\frac{3at^2}{1+t^3}\right)'}{\left(\frac{3at}{1+t^3}\right)'} = \frac{\frac{6at(1+t^3)-3at^2\cdot 3t^2}{(1+t^3)^2}}{\frac{3a(1+t^3)-3at\cdot 3t^2}{(1+t^3)^2}} = \frac{6at - 3at^4}{3a - 6at^3} = \frac{t(2-t^3)}{1-2t^3}$$

Horizontal tangent occurs when $\frac{dy}{dx} = 0$. That is,

$$\frac{t(2-t^3)}{1-2t^3} = 0 \Rightarrow t(2-t^3) = 0,\ 1 - 2t^3 \ne 0 \Rightarrow t = 0,\ t = \sqrt[3]{2}.$$

The corresponding points are:

$$c(0) = (x(0), y(0)) = (0, 0)$$

$$c\left(\sqrt[3]{2}\right) = \left(x\left(\sqrt[3]{2}\right), y\left(\sqrt[3]{2}\right)\right) = \left(\frac{3a\sqrt[3]{2}}{1+2}, \frac{3a\sqrt[3]{4}}{1+2}\right) = \left(a\sqrt[3]{2}, a\sqrt[3]{4}\right)$$

Vertical tangent line occurs when $\frac{dy}{dx}$ is infinite. That is,

$$1 - 2t^3 = 0 \Rightarrow t = \frac{1}{\sqrt[3]{2}}$$

The corresponding point is

$$c\left(\frac{1}{\sqrt[3]{2}}\right) = \left(x\left(\frac{1}{\sqrt[3]{2}}\right), y\left(\frac{1}{\sqrt[3]{2}}\right)\right) = \left(\frac{\frac{3a}{\sqrt[3]{2}}}{1+\frac{1}{2}}, \frac{\frac{3a}{\sqrt[3]{4}}}{1+\frac{1}{2}}\right) = \left(\sqrt[3]{4}a, \sqrt[3]{2}a\right).$$

79. Use the results of Exercise 78 to show that the asymptote of the folium is the line $x + y = -a$. *Hint:* Show that $\lim_{t\to -1}(x + y) = -a$.

SOLUTION We must show that as $x \to \infty$ or $x \to -\infty$ the graph of the folium is getting arbitrarily close to the line $x + y = -a$, and the derivative $\frac{dy}{dx}$ is approaching the slope -1 of the line.

In Exercise 78 we showed that $x \to \infty$ when $t \to (-1^-)$ and $x \to -\infty$ when $t \to (-1^+)$. We first show that the graph is approaching the line $x + y = -a$ as $x \to \infty$ or $x \to -\infty$, by showing that $\lim_{t\to -1-} x + y = \lim_{t\to -1+} x + y = -a$.

For $x(t) = \dfrac{3at}{1+t^3}$, $y(t) = \dfrac{3at^2}{1+t^3}$, $a > 0$, calculated in Exercise 78, we obtain using L'Hôpital's Rule:

$$\lim_{t\to -1-}(x+y) = \lim_{t\to -1-}\frac{3at + 3at^2}{1+t^3} = \lim_{t\to -1-}\frac{3a+6at}{3t^2} = \frac{3a-6a}{3} = -a$$

$$\lim_{t\to -1+}(x+y) = \lim_{t\to -1+}\frac{3at + 3at^2}{1+t^3} = \lim_{t\to -1+}\frac{3a+6at}{3t^2} = \frac{3a-6a}{3} = -a$$

We now show that $\dfrac{dy}{dx}$ is approaching -1 as $t \to -1-$ and as $t \to -1+$. We use $\dfrac{dy}{dx} = \dfrac{6at - 3at^4}{3a - 6at^3}$ computed in Exercise 78 to obtain

$$\lim_{t\to -1-}\frac{dy}{dx} = \lim_{t\to -1-}\frac{6at - 3at^4}{3a - 6at^3} = \frac{-9a}{9a} = -1$$

$$\lim_{t\to -1+}\frac{dy}{dx}=\lim_{t\to -1+}\frac{6at-3at^4}{3a-6at^3}=\frac{-9a}{9a}=-1$$

We conclude that the line $x+y=-a$ is an asymptote of the folium as $x\to\infty$ and as $x\to-\infty$.

80. Find a parametrization of $x^{2n+1}+y^{2n+1}=ax^ny^n$, where a and n are constants.

SOLUTION Following the method in Exercise 78, we first find the coordinates of the point P where the curve and the line $y=tx$ intersect. We solve the following equations:

$$\begin{aligned}y&=tx\\ x^{2n+1}+y^{2n+1}&=ax^ny^n\end{aligned}$$

Substituting $y=tx$ in the second equation and solving for x yields

$$\begin{aligned}x^{2n+1}+t^{2n+1}x^{2n+1}&=ax^n\cdot t^nx^n\\ (1+t^{2n+1})x^{2n+1}-at^nx^{2n}&=0\\ x^{2n}((1+t^{2n+1})x-at^n)&=0\Rightarrow x=0,\ x=\frac{at^n}{1+t^{2n+1}}\end{aligned}$$

We assume that $t\neq-1$ (so $1+t^{2n+1}\neq 0$) and obtain one solution besides the origin. The corresponding y coordinates are

$$y=tx=t\cdot\frac{at^n}{1+t^{2n+1}}=\frac{at^{n+1}}{1+t^{2n+1}}$$

Hence, the points $x=\dfrac{at^n}{1+t^{2n+1}}$, $y=\dfrac{at^{n+1}}{1+t^{2n+1}}$, $t\neq-1$, are exactly the points on the curve. We obtain the following parametrization:

$$x=\frac{at^n}{1+t^{2n+1}},\ y=\frac{at^{n+1}}{1+t^{2n+1}},\quad t\neq-1.$$

81. Second Derivative for a Parametrized Curve Given a parametrized curve $c(t)=(x(t),y(t))$, show that

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{x'(t)y''(t)-y'(t)x''(t)}{x'(t)^2}$$

Use this to prove the formula

$$\boxed{\frac{d^2y}{dx^2}=\frac{x'(t)y''(t)-y'(t)x''(t)}{x'(t)^3}} \qquad \boxed{11}$$

SOLUTION By the formula for the slope of the tangent line we have

$$\frac{dy}{dx}=\frac{y'(t)}{x'(t)}$$

Differentiating with respect to t, using the Quotient Rule, gives

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)=\frac{x'(t)y''(t)-y'(t)x''(t)}{x'(t)^2}$$

By the Chain Rule we have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

Substituting into the above equation $\left(\text{and using } \dfrac{dt}{dx}=\dfrac{1}{dx/dt}=\dfrac{1}{x'(t)}\right)$ gives

$$\frac{d^2y}{dx^2}=\frac{x'(t)y''(t)-y'(t)x''(t)}{x'(t)^2}\cdot\frac{1}{x'(t)}=\frac{x'(t)y''(t)-y'(t)x''(t)}{x'(t)^3}$$

82. The second derivative of $y = x^2$ is $dy^2/d^2x = 2$. Verify that Eq. (11) applied to $c(t) = (t, t^2)$ yields $dy^2/d^2x = 2$. In fact, any parametrization may be used. Check that $c(t) = (t^3, t^6)$ and $c(t) = (\tan t, \tan^2 t)$ also yield $dy^2/d^2x = 2$.

SOLUTION For the parametrization $c(t) = (t, t^2)$, we have

$$x'(t) = 1, \quad x''(t) = 0, \quad y'(t) = 2t, \quad y''(t) = 2$$

so that indeed

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} = \frac{1 \cdot 2 - 2t \cdot 0}{1^3} = 2$$

For $c(t) = (t^3, t^6)$, we have

$$x'(t) = 3t^2, \quad x''(t) = 6t, \quad y'(t) = 6t^5, \quad y''(t) = 30t^4$$

so that again

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} = \frac{3t^2 \cdot 30t^4 - 6t^5 \cdot 6t}{(3t^2)^3} = \frac{54t^6}{27t^6} = 2$$

Finally, for $c(t) = (\tan t, \tan^2 t)$,

$$x'(t) = \sec^2 t, \quad x''(t) = 2\tan t \sec^2 t, \quad y'(t) = 2\tan t \sec^2 t, \quad y''(t) = 6\sec^4 t - 4\sec^2 t$$

and

$$\begin{aligned}\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} &= \frac{\sec^2 t(6\sec^4 t - 4\sec^2 t) - 2\tan t \sec^2 t(2\tan t \sec^2 t)}{\sec^6 t} \\ &= \frac{6\sec^6 t - 4\sec^4 t - 4\sec^4 t\tan^2 t}{\sec^6 t} = \frac{6\sec^6 t - 4\sec^4 t(1 - (1 + \sec^2 t)))}{\sec^6 t} \\ &= \frac{2\sec^6 t}{\sec^6 t} = 2\end{aligned}$$

In Exercises 83–86, use Eq. (11) to find d^2y/dx^2.

83. $x = t^3 + t^2, \quad y = 7t^2 - 4, \quad t = 2$

SOLUTION We find the first and second derivatives of $x(t)$ and $y(t)$:

$$\begin{aligned} x'(t) &= 3t^2 + 2t \Rightarrow x'(2) = 3 \cdot 2^2 + 2 \cdot 2 = 16 \\ x''(t) &= 6t + 2 \quad \Rightarrow x''(2) = 6 \cdot 2 + 2 = 14 \\ y'(t) &= 14t \quad\quad \Rightarrow y'(2) = 14 \cdot 2 = 28 \\ y''(t) &= 14 \quad\quad\; \Rightarrow y''(2) = 14 \end{aligned}$$

Using Eq. (11) we get

$$\left.\frac{d^2y}{dx^2}\right|_{t=2} = \left.\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3}\right|_{t=2} = \frac{16 \cdot 14 - 28 \cdot 14}{16^3} = \frac{-21}{512}$$

84. $x = s^{-1} + s, \quad y = 4 - s^{-2}, \quad s = 1$

SOLUTION Since $x'(s) = -s^{-2} + 1 = 1 - \frac{1}{s^2}$, we have $x'(1) = 0$. Hence, Eq. (11) cannot be used to compute $\frac{d^2y}{dx^2}$ at $s = 1$.

85. $x = 8t + 9, \quad y = 1 - 4t, \quad t = -3$

SOLUTION We compute the first and second derivatives of $x(t)$ and $y(t)$:

$$\begin{aligned} x'(t) &= 8 \quad\; \Rightarrow x'(-3) = 8 \\ x''(t) &= 0 \quad\; \Rightarrow x''(-3) = 0 \\ y'(t) &= -4 \Rightarrow y'(-3) = -4 \\ y''(t) &= 0 \quad\; \Rightarrow y''(-3) = 0 \end{aligned}$$

Using Eq. (11) we get

$$\left.\frac{d^2y}{dx^2}\right|_{t=-3} = \frac{x'(-3)y''(-3) - y'(-3)x''(-3)}{x'(-3)^3} = \frac{8 \cdot 0 - (-4) \cdot 0}{8^3} = 0$$

86. $x = \cos\theta, \quad y = \sin\theta, \quad \theta = \frac{\pi}{4}$

SOLUTION We find the first and second derivatives of $x(\theta)$ and $y(\theta)$:

$$x'(\theta) = -\sin\theta \Rightarrow x'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$
$$x''(\theta) = -\cos\theta \Rightarrow x''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$
$$y'(\theta) = \cos\theta \Rightarrow y'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$y''(\theta) = -\sin\theta \Rightarrow y''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

Using Eq. (11) we get

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{4}} = \frac{x'\left(\frac{\pi}{4}\right)y''\left(\frac{\pi}{4}\right) - y'\left(\frac{\pi}{4}\right)x''\left(\frac{\pi}{4}\right)}{\left(x'\left(\frac{\pi}{4}\right)\right)^3} = \frac{\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}\cdot\left(-\frac{\sqrt{2}}{2}\right)}{\left(-\frac{\sqrt{2}}{2}\right)^3} = -2\sqrt{2}$$

87. Use Eq. (11) to find the t-intervals on which $c(t) = (t^2, t^3 - 4t)$ is concave up.

SOLUTION The curve is concave up where $\frac{d^2y}{dx^2} > 0$. Thus,

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} > 0 \qquad (1)$$

We compute the first and second derivatives:

$$x'(t) = 2t, \qquad x''(t) = 2$$
$$y'(t) = 3t^2 - 4, \quad y''(t) = 6t$$

Substituting in (1) and solving for t gives

$$\frac{12t^2 - (6t^2 - 8)}{8t^3} = \frac{6t^2 + 8}{8t^3}$$

Since $6t^2 + 8 > 0$ for all t, the quotient is positive if $8t^3 > 0$. We conclude that the curve is concave up for $t > 0$.

88. Use Eq. (11) to find the t-intervals on which $c(t) = (t^2, t^4 - 4t)$ is concave up.

SOLUTION The curve is concave up where $\frac{d^2y}{dx^2} > 0$. That is,

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} > 0 \qquad (1)$$

We compute the first and second derivatives:

$$x'(t) = 2t, \qquad x''(t) = 2$$
$$y'(t) = 4t^3 - 4, \quad y''(t) = 12t^2$$

Substituting in (1) and solving for t gives

$$\frac{24t^3 - (8t^3 - 8)}{8t^3} = \frac{16t^3 + 8}{8t^3} = 1 + \frac{1}{2t^3}$$

This is clearly positive for $t > 0$. For $t < 0$, we want $1 + \frac{1}{2t^3} > 0$, which means $\frac{1}{2t^3} > -1$, so $2t^3 < -1$ (by taking the reciprocal of both sides), so $t < -\frac{1}{\sqrt[3]{2}}$. Thus, we see that our curve is concave up for $t < -\frac{1}{\sqrt[3]{2}}$ and for $t > 0$.

89. Area Under a Parametrized Curve Let $c(t) = (x(t), y(t))$, where $y(t) > 0$ and $x'(t) > 0$ (Figure 24). Show that the area A under $c(t)$ for $t_0 \le t \le t_1$ is

$$A = \int_{t_0}^{t_1} y(t)x'(t)\,dt \qquad \boxed{12}$$

Hint: Because it is increasing, the function $x(t)$ has an inverse $t = g(x)$ and $c(t)$ is the graph of $y = y(g(x))$. Apply the change-of-variables formula to $A = \int_{x(t_0)}^{x(t_1)} y(g(x))\,dx$.

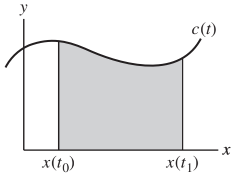


FIGURE 24

SOLUTION Let $x_0 = x(t_0)$ and $x_1 = x(t_1)$. We are given that $x'(t) > 0$, hence $x = x(t)$ is an increasing function of t, so it has an inverse function $t = g(x)$. The area A is given by $\int_{x_0}^{x_1} y(g(x))\,dx$. Recall that y is a function of t and $t = g(x)$, so the height y at any point x is given by $y = y(g(x))$. We find the new limits of integration. Since $x_0 = x(t_0)$ and $x_1 = x(t_1)$, the limits for t are t_0 and t_1, respectively. Also since $x'(t) = \frac{dx}{dt}$, we have $dx = x'(t)dt$. Performing this substitution gives

$$A = \int_{x_0}^{x_1} y(g(x))\,dx = \int_{t_0}^{t_1} y(g(x))x'(t)\,dt.$$

Since $g(x) = t$, we have $A = \displaystyle\int_{t_0}^{t_1} y(t)x'(t)\,dt$.

90. Calculate the area under $y = x^2$ over $[0, 1]$ using Eq. (12) with the parametrizations (t^3, t^6) and (t^2, t^4).

SOLUTION The area A under $y = x^2$ on $[0, 1]$ is given by the integral

$$A = \int_{t_0}^{t_1} y(t)x'(t)\,dt$$

where $x(t_0) = 0$ and $x(t_1) = 1$. We first use the parametrization (t^3, t^6). We have $x(t) = t^3$, $y(t) = t^6$. Hence,

$$0 = x(t_0) = t_0^3 \Rightarrow t_0 = 0$$
$$1 = x(t_1) = t_1^3 \Rightarrow t_1 = 1$$

Also $x'(t) = 3t^2$. Substituting these values in Eq. (12) we obtain

$$A = \int_0^1 t^6 \cdot 3t^2\,dt = \int_0^1 3t^8\,dt = \frac{3}{9}t^9\Big|_0^1 = \frac{3}{9} = \frac{1}{3}$$

Using the parametrization $x(t) = t^2$, $y(t) = t^4$, we have $x'(t) = 2t$. We find t_0 and t_1:

$$0 = x(t_0) = t_0^2 \Rightarrow t_0 = 0$$
$$1 = x(t_1) = t_1^2 \Rightarrow t_1 = 1 \quad \text{or} \quad t_1 = -1.$$

Equation (12) is valid if $x'(t) > 0$, that is if $t > 0$. Hence we choose the positive value, $t_1 = 1$. We now use Eq. (12) to obtain

$$A = \int_0^1 t^4 \cdot 2t\,dt = \int_0^1 2t^5\,dt = \frac{2}{6}t^6\Big|_0^1 = \frac{2}{6} = \frac{1}{3}$$

Both answers agree, as expected.

91. What does Eq. (12) say if $c(t) = (t, f(t))$?

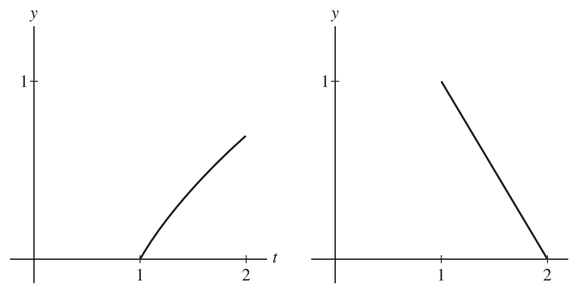
SOLUTION In the parametrization $x(t) = t$, $y(t) = f(t)$ we have $x'(t) = 1$, $t_0 = x(t_0)$, $t_1 = x(t_1)$. Hence Eq. (12) becomes

$$A = \int_{t_0}^{t_1} y(t)x'(t)\,dt = \int_{x(t_0)}^{x(t_1)} f(t)\,dt$$

We see that in this parametrization Eq. (12) is the familiar formula for the area under the graph of a positive function.

92. Sketch the graph of $c(t) = (\ln t, 2 - t)$ for $1 \le t \le 2$ and compute the area under the graph using Eq. (12).

SOLUTION We use the following graphs of $x(t) = \ln t$ and $y(t) = 2 - t$ for $1 \le t \le 2$:



$x(t) = \ln t$ $\qquad$ $y(t) = 2 - t$

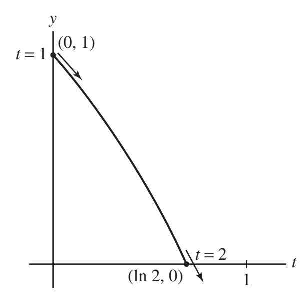
We see that for $1 < t < 2$, $x(t)$ is positive and increasing and $y(t)$ is positive and decreasing. Also $c(1) = (\ln 1, 2 - 1) = (0, 1)$ and $c(2) = (\ln 2, 2 - 2) = (\ln 2, 0)$. Additional information is obtained from the derivative

$$\frac{dy}{dx} = \frac{(2-t)'}{(\ln t)'} = -\frac{1}{1/t} = -t,$$

yielding

$$\left.\frac{dy}{dx}\right|_{t=1} - 1 \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{t=2} - 2.$$

We obtain the following graph:



We now use Eq. (12) to compute the area A under the graph. We have $x(t) = \ln t$, $x'(t) = \frac{1}{t}$, $y(t) = 2 - t$, $t_0 = 1$, $t_1 = 2$. Hence,

$$\begin{aligned} A &= \int_{t_0}^{t_1} y(t)x'(t)\,dt = \int_1^2 (2-t)\cdot\frac{1}{t}\,dt = \int_1^2 \left(\frac{2}{t} - 1\right)dt \\ &= 2\ln t - t\Big|_1^2 = (2\ln 2 - 2) - (2\ln 1 - 1) = 2\ln 2 - 1 \approx 0.386 \end{aligned}$$

93. Galileo tried unsuccessfully to find the area under a cycloid. Around 1630, Gilles de Roberval proved that the area under one arch of the cycloid $c(t) = (Rt - R\sin t, R - R\cos t)$ generated by a circle of radius R is equal to three times the area of the circle (Figure 25). Verify Roberval's result using Eq. (12).

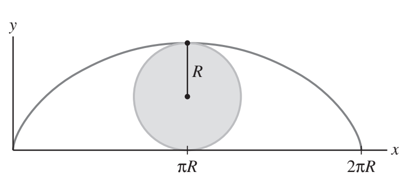


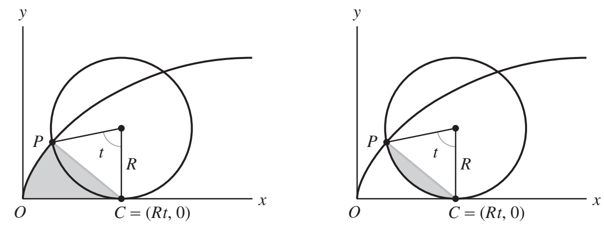
FIGURE 25 The area of one arch of the cycloid equals three times the area of the generating circle.

SOLUTION This reduces to

$$\int_0^{2\pi} (R - R\cos t)(Rt - R\sin t)'\,dt = \int_0^{2\pi} R^2(1-\cos t)^2\,dt = 3\pi R^2.$$

Further Insights and Challenges

94. Prove the following generalization of Exercise 93: For all $t > 0$, the area of the cycloidal sector OPC is equal to three times the area of the circular segment cut by the chord PC in Figure 26.



(A) Cycloidal sector OPC

(B) Circular segment cut by the chord PC

FIGURE 26

SOLUTION Drop a perpendicular from point P to the x-axis and label the point of intersection T, and denote by D the center of the circle. Then the area of the cycloidal sector is equal to the area of OPT plus the area of PTC. The latter is a triangle with height $y(t) = R - R\cos t$ and base $Rt - (Rt - R\sin t) = R\sin t$, so its area is $\frac{1}{2}R^2 \sin t(1 - \cos t)$. The area of OPT, using Eq. (12), is

$$\int_0^t y(u)x'(u)\,du = \int_0^t (R - R\cos u)(Ru - R\sin u)'\,du = R^2\int_0^t (1-\cos u)^2\,du$$
$$= R^2\left(\frac{3}{2}t - 2\sin t + \frac{1}{2}\sin t\cos t\right)$$

so that the total area of the cycloidal sector is

$$R^2\left(\frac{3}{2}t - 2\sin t + \frac{1}{2}\sin t\cos t\right) + R^2\frac{1}{2}\sin t(1-\cos t) = 3\left(\frac{1}{2}R^2 t - \frac{1}{2}R^2\sin t\right) = 3\cdot\frac{1}{2}R^2(t - \sin t)$$

The area of the circular segment is the area of the circular sector DPC subtended by the angle t less the area of the triangle DPC. The triangle DPC has height $R\cos\frac{t}{2}$ and base $2R\sin\frac{t}{2}$ so that its area is $R^2\cos\frac{t}{2}\sin\frac{t}{2} = \frac{1}{2}R^2\sin t$, and the area of the circular sector is $\pi R^2\cdot\frac{t}{2\pi} = \frac{1}{2}R^2 t$. Thus the area of the circular segment is

$$\frac{1}{2}R^2(t - \sin t)$$

which is one third the area of the cycloidal sector.

95. Derive the formula for the slope of the tangent line to a parametric curve $c(t) = (x(t), y(t))$ using a method different from that presented in the text. Assume that $x'(t_0)$ and $y'(t_0)$ exist and that $x'(t_0) \neq 0$. Show that

$$\lim_{h\to 0}\frac{y(t_0+h) - y(t_0)}{x(t_0+h) - x(t_0)} = \frac{y'(t_0)}{x'(t_0)}$$

Then explain why this limit is equal to the slope dy/dx. Draw a diagram showing that the ratio in the limit is the slope of a secant line.

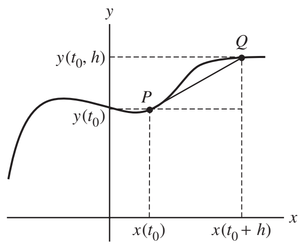
SOLUTION Since $y'(t_0)$ and $x'(t_0)$ exist, we have the following limits:

$$\lim_{h\to 0}\frac{y(t_0+h) - y(t_0)}{h} = y'(t_0),\quad \lim_{h\to 0}\frac{x(t_0+h) - x(t_0)}{h} = x'(t_0) \tag{1}$$

We use Basic Limit Laws, the limits in (1) and the given data $x'(t_0) \neq 0$, to write

$$\lim_{h\to 0}\frac{y(t_0+h) - y(t_0)}{x(t_0+h) - x(t_0)} = \lim_{h\to 0}\frac{\frac{y(t_0+h)-y(t_0)}{h}}{\frac{x(t_0+h)-x(t_0)}{h}} = \frac{\lim_{h\to 0}\frac{y(t_0+h)-y(t_0)}{h}}{\lim_{h\to 0}\frac{x(t_0+h)-x(t_0)}{h}} = \frac{y'(t_0)}{x'(t_0)}$$

Notice that the quotient $\frac{y(t_0+h) - y(t_0)}{x(t_0+h) - x(t_0)}$ is the slope of the secant line determined by the points $P = (x(t_0), y(t_0))$ and $Q = (x(t_0+h), y(t_0+h))$. Hence, the limit of the quotient as $h \to 0$ is the slope of the tangent line at P, that is the derivative $\frac{dy}{dx}$.



96. Verify that the **tractrix** curve ($\ell > 0$)

$$c(t) = \left(t - \ell \tanh \frac{t}{\ell}, \ell \operatorname{sech} \frac{t}{\ell}\right)$$

has the following property: For all t, the segment from $c(t)$ to $(t, 0)$ is tangent to the curve and has length ℓ (Figure 27).

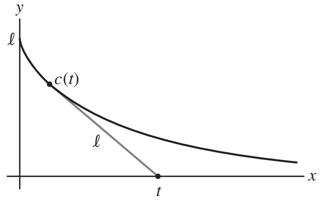
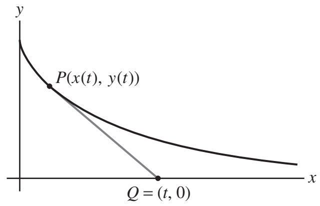


FIGURE 27 The tractrix $c(t) = \left(t - \ell \tanh \frac{t}{\ell}, \ell \operatorname{sech} \frac{t}{\ell}\right)$.

SOLUTION Let $P = c(t)$ and $Q = (t, 0)$.



The slope of the segment $\overline{PQ}$ is

$$m_1 = \frac{y(t) - 0}{x(t) - t} = \frac{\ell \operatorname{sech}\left(\frac{t}{\ell}\right)}{-\ell \tanh\left(\frac{t}{\ell}\right)} = -\frac{1}{\sinh\left(\frac{t}{\ell}\right)}$$

We compute the slope of the tangent line at P:

$$m_2 = \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\left(\ell \operatorname{sech}\left(\frac{t}{\ell}\right)\right)'}{\left(t - \ell \tanh\left(\frac{t}{\ell}\right)\right)'} = \frac{\ell \cdot \frac{1}{\ell}\left(-\operatorname{sech}\left(\frac{t}{\ell}\right)\tanh\left(\frac{t}{\ell}\right)\right)}{1 - \ell \cdot \frac{1}{\ell}\operatorname{sech}^2\left(\frac{t}{\ell}\right)}$$

$$= -\frac{-\operatorname{sech}\left(\frac{t}{\ell}\right)\tanh\left(\frac{t}{\ell}\right)}{1 - \operatorname{sech}^2\left(\frac{t}{\ell}\right)} = \frac{-\operatorname{sech}\left(\frac{t}{\ell}\right)\tanh\left(\frac{t}{\ell}\right)}{\tanh^2\left(\frac{t}{\ell}\right)} = \frac{-\operatorname{sech}\left(\frac{t}{\ell}\right)}{\tanh\left(\frac{t}{\ell}\right)} = -\frac{1}{\sinh\left(\frac{t}{\ell}\right)}$$

Since $m_1 = m_2$, we conclude that the segment from $c(t)$ to $(t, 0)$ is tangent to the curve.

We now show that $|\overline{PQ}| = \ell$:

$$|\overline{PQ}| = \sqrt{(x(t) - t)^2 + (y(t) - 0)^2} = \sqrt{\left(-\ell \tanh \frac{t}{\ell}\right)^2 + \left(\ell \operatorname{sech}\left(\frac{t}{\ell}\right)\right)^2}$$

$$= \sqrt{\ell^2\left(\tanh^2\left(\frac{t}{\ell}\right) + \operatorname{sech}^2\left(\frac{t}{\ell}\right)\right)} = \ell\sqrt{\operatorname{sech}^2\left(\frac{t}{\ell}\right)\sinh^2\left(\frac{t}{\ell}\right) + \operatorname{sech}^2\left(\frac{t}{\ell}\right)}$$

$$= \ell \operatorname{sech}\left(\frac{t}{\ell}\right)\sqrt{\sinh^2\left(\frac{t}{\ell}\right) + 1} = \ell \operatorname{sech}\left(\frac{t}{\ell}\right)\cosh\left(\frac{t}{\ell}\right) = \ell \cdot 1 = \ell$$

97. In Exercise 54 of Section 10.1 (ET Exercise 54 of Section 9.1), we described the tractrix by the differential equation

$$\frac{dy}{dx} = -\frac{y}{\sqrt{\ell^2 - y^2}}$$

Show that the curve $c(t)$ identified as the tractrix in Exercise 96 satisfies this differential equation. Note that the derivative on the left is taken with respect to x, not t.

SOLUTION Note that $dx/dt = 1 - \operatorname{sech}^2(t/\ell) = \tanh^2(t/\ell)$ and $dy/dt = -\operatorname{sech}(t/\ell)\tanh(t/\ell)$. Thus,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\operatorname{sech}(t/\ell)}{\tanh(t/\ell)} = \frac{-y/\ell}{\sqrt{1 - y^2/\ell^2}}$$

Multiplying top and bottom by ℓ/ℓ gives

$$\frac{dy}{dx} = \frac{-y}{\sqrt{\ell^2 - y^2}}$$

In Exercises 98 and 99, refer to Figure 28.

98. In the parametrization $c(t) = (a\cos t, b\sin t)$ of an ellipse, t is *not* an angular parameter unless $a = b$ (in which case the ellipse is a circle). However, t can be interpreted in terms of area: Show that if $c(t) = (x, y)$, then $t = (2/ab)A$, where A is the area of the shaded region in Figure 28. *Hint:* Use Eq. (12).

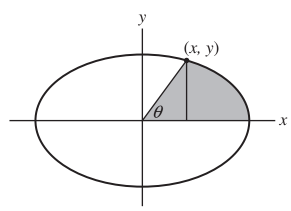
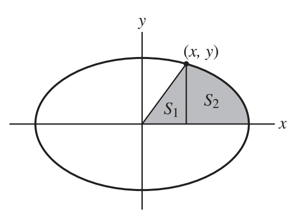


FIGURE 28 The parameter θ on the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

SOLUTION We compute the area A of the shaded region as the sum of the area S_1 of the triangle and the area S_2 of the region under the curve. The area of the triangle is

$$S_1 = \frac{xy}{2} = \frac{(a\cos t)(b\sin t)}{2} = \frac{ab\sin 2t}{4} \tag{1}$$



The area S_2 under the curve can be computed using Eq. (12). The lower limit of the integration is $t_0 = 0$ (corresponds to $(a, 0)$) and the upper limit is t (corresponds to $(x(t), y(t))$). Also $y(t) = b\sin t$ and $x'(t) = -a\sin t$. Since $x'(t) < 0$ on the interval $0 < t < \frac{\pi}{2}$ (which represents the ellipse on the first quadrant), we use the positive value $a\sin t$ to obtain a positive value for the area. This gives

$$\begin{aligned} S_2 &= \int_0^t b\sin u \cdot a\sin u\, du = ab\int_0^t \sin^2 u\, du \\ &= ab\int_0^t \left(\frac{1}{2} - \frac{1}{2}\cos 2u\right) du = ab\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]\Bigg|_0^t \\ &= ab\left[\frac{t}{2} - \frac{\sin 2t}{4} - 0\right] = \frac{abt}{2} - \frac{ab\sin 2t}{4} \end{aligned} \tag{2}$$

Combining (1) and (2) we obtain

$$A = S_1 + S_2 = \frac{ab\sin 2t}{4} + \frac{abt}{2} - \frac{ab\sin 2t}{4} = \frac{abt}{2}$$

Hence, $t = \frac{2A}{ab}$.

99. Show that the parametrization of the ellipse by the angle θ is

$$x = \frac{ab\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$
$$y = \frac{ab\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

SOLUTION We consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

For the angle θ we have $\tan\theta = \frac{y}{x}$, hence,

$$y = x\tan\theta \tag{1}$$

Substituting in the equation of the ellipse and solving for x we obtain

$$\frac{x^2}{a^2} + \frac{x^2\tan^2\theta}{b^2} = 1$$
$$b^2x^2 + a^2x^2\tan^2\theta = a^2b^2$$
$$(a^2\tan^2\theta + b^2)x^2 = a^2b^2$$
$$x^2 = \frac{a^2b^2}{a^2\tan^2\theta + b^2} = \frac{a^2b^2\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

We now take the square root. Since the sign of the x-coordinate is the same as the sign of $\cos\theta$, we take the positive root, obtaining

$$x = \frac{ab\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \tag{2}$$

Hence by (1), the y-coordinate is

$$y = x\tan\theta = \frac{ab\cos\theta\tan\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} = \frac{ab\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \tag{3}$$

Equalities (2) and (3) give the following parametrization for the ellipse:

$$c_1(\theta) = \left(\frac{ab\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}, \frac{ab\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right)$$

11.2 Arc Length and Speed

Preliminary Questions

1. What is the definition of arc length?

SOLUTION A curve can be approximated by a polygonal path obtained by connecting points

$$p_0 = c(t_0),\ p_1 = c(t_1),\ \dots,\ p_N = c(t_N)$$

on the path with segments. One gets an approximation by summing the lengths of the segments. The definition of arc length is the limit of that approximation when increasing the number of points so that the lengths of the segments approach zero. In doing so, we obtain the following theorem for the arc length:

$$S = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt,$$

which is the length of the curve $c(t) = (x(t), y(t))$ for $a \le t \le b$.

2. What is the interpretation of $\sqrt{x'(t)^2 + y'(t)^2}$ for a particle following the trajectory $(x(t), y(t))$?

SOLUTION The expression $\sqrt{x'(t)^2 + y'(t)^2}$ denotes the speed at time t of a particle following the trajectory $(x(t), y(t))$.

3. A particle travels along a path from (0, 0) to (3, 4). What is the displacement? Can the distance traveled be determined from the information given?

SOLUTION The net displacement is the distance between the initial point (0, 0) and the endpoint (3, 4). That is

$$\sqrt{(3-0)^2+(4-0)^2}=\sqrt{25}=5.$$

The distance traveled can be determined only if the trajectory $c(t)=(x(t),\,y(t))$ of the particle is known.

4. A particle traverses the parabola $y=x^2$ with constant speed 3 cm/s. What is the distance traveled during the first minute? *Hint:* No computation is necessary.

SOLUTION Since the speed is constant, the distance traveled is the following product: $L=st=3\cdot 60=180$ cm.

Exercises

In Exercises 1–10, use Eq. (3) to find the length of the path over the given interval.

1. $(3t+1, 9-4t), \quad 0\le t\le 2$

SOLUTION Since $x=3t+1$ and $y=9-4t$ we have $x'=3$ and $y'=-4$. Hence, the length of the path is

$$S=\int_0^2\sqrt{3^2+(-4)^2}\,dt=5\int_0^2 dt=10.$$

2. $(1+2t, 2+4t), \quad 1\le t\le 4$

SOLUTION We have $x=1+2t$ and $y=2+4t$, hence $x'=2$ and $y'=4$. Using the formula for arc length we obtain

$$S=\int_1^4\sqrt{2^2+4^2}\,dt=\int_1^4\sqrt{20}\,dt=\sqrt{20}(4-1)=6\sqrt{5}$$

3. $(2t^2, 3t^2-1), \quad 0\le t\le 4$

SOLUTION Since $x=2t^2$ and $y=3t^2-1$, we have $x'=4t$ and $y'=6t$. By the formula for the arc length we get

$$S=\int_0^4\sqrt{x'(t)^2+y'(t)^2}\,dt=\int_0^4\sqrt{16t^2+36t^2}\,dt=\sqrt{52}\int_0^4 t\,dt=\sqrt{52}\cdot\frac{t^2}{2}\bigg|_0^4=16\sqrt{13}$$

4. $(3t, 4t^{3/2}), \quad 0\le t\le 1$

SOLUTION We have $x=3t$ and $y=4t^{3/2}$, hence $x'=3$ and $y'=6t^{1/2}$. Using the formula for the arc length we obtain

$$S=\int_0^1\sqrt{x'(t)^2+y'(t)^2}\,dt=\int_0^1\sqrt{3^2+\left(6t^{1/2}\right)^2}\,dt=\int_0^1\sqrt{9+36t}\,dt=3\int_0^1\sqrt{1+4t}\,dt$$

Setting $u=1+4t$ we get

$$S=\frac{3}{4}\int_1^5\sqrt{u}\,du=\frac{3}{4}\cdot\frac{2}{3}u^{3/2}\bigg|_1^5=\frac{1}{2}(5^{3/2}-1)\approx 5.09$$

5. $(3t^2, 4t^3), \quad 1\le t\le 4$

SOLUTION We have $x=3t^2$ and $y=4t^3$. Hence $x'=6t$ and $y'=12t^2$. By the formula for the arc length we get

$$S=\int_1^4\sqrt{x'(t)^2+y'(t)^2}\,dt=\int_1^4\sqrt{36t^2+144t^4}\,dt=6\int_1^4\sqrt{1+4t^2}\,t\,dt.$$

Using the substitution $u=1+4t^2$, $du=8t\,dt$ we obtain

$$S=\frac{6}{8}\int_5^{65}\sqrt{u}\,du=\frac{3}{4}\cdot\frac{2}{3}u^{3/2}\bigg|_5^{65}=\frac{1}{2}(65^{3/2}-5^{3/2})\approx 256.43$$

6. $(t^3+1, t^2-3), \quad 0 \le t \le 1$

SOLUTION We have $x = t^3+1$, $y = t^2-3$, hence, $x' = 3t^2$ and $y' = 2t$. By the formula for the arc length we get

$$S = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^1 \sqrt{9t^4 + 4t^2}\,dt = \int_0^1 t\sqrt{9t^2+4}\,dt$$

We compute the integral using the substitution $u = 4 + 9t^2$. This gives

$$S = \frac{1}{18}\int_4^{13} \sqrt{u}\,du = \frac{1}{18}\cdot\frac{2}{3}u^{3/2}\Big|_4^{13} = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{1}{27}(13^{3/2} - 8) \approx 1.44.$$

7. $(\sin 3t, \cos 3t), \quad 0 \le t \le \pi$

SOLUTION We have $x = \sin 3t$, $y = \cos 3t$, hence $x' = 3\cos 3t$ and $y' = -3\sin 3t$. By the formula for the arc length we obtain:

$$S = \int_0^\pi \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^\pi \sqrt{9\cos^2 3t + 9\sin^2 3t}\,dt = \int_0^\pi \sqrt{9}\,dt = 3\pi$$

8. $(\sin\theta - \theta\cos\theta, \cos\theta + \theta\sin\theta), \quad 0 \le \theta \le 2$

SOLUTION We have $x = \sin\theta - \theta\cos\theta$ and $y = \cos\theta + \theta\sin\theta$. Hence, $x' = \cos\theta - (\cos\theta - \theta\sin\theta) = \theta\sin\theta$ and $y' = -\sin\theta + \sin\theta + \theta\cos\theta = \theta\cos\theta$. Using the formula for the arc length we obtain:

$$\begin{aligned} S &= \int_0^2 \sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta = \int_0^2 \sqrt{(\theta\sin\theta)^2 + (\theta\cos\theta)^2}\,d\theta \\ &= \int_0^2 \sqrt{\theta^2(\sin^2\theta + \cos^2\theta)}\,d\theta = \int_0^2 \theta\,d\theta = \frac{\theta^2}{2}\Big|_0^2 = 2 \end{aligned}$$

In Exercises 9 and 10, use the identity

$$\frac{1-\cos t}{2} = \sin^2\frac{t}{2}$$

9. $(2\cos t - \cos 2t, 2\sin t - \sin 2t), \quad 0 \le t \le \frac{\pi}{2}$

SOLUTION We have $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$. Thus, $x' = -2\sin t + 2\sin 2t$ and $y' = 2\cos t - 2\cos 2t$. We get

$$\begin{aligned} x'(t)^2 + y'(t)^2 &= (-2\sin t + 2\sin 2t)^2 + (2\cos t - 2\cos 2t)^2 \\ &= 4\sin^2 t - 8\sin t\sin 2t + 4\sin^2 2t + 4\cos^2 t - 8\cos t\cos 2t + 4\cos^2 2t \\ &= 4(\sin^2 t + \cos^2 t) + 4(\sin^2 2t + \cos^2 2t) - 8(\sin t\sin 2t + \cos t\cos 2t) \\ &= 4 + 4 - 8\cos(2t - t) = 8 - 8\cos t = 8(1 - \cos t) \end{aligned}$$

We now use the formula for the arc length to obtain

$$\begin{aligned} S &= \int_0^{\pi/2} \sqrt{x'(t)^2 + y'(t)^2} = \int_0^{\pi/2} \sqrt{8(1-\cos t)}\,dt = \int_0^{\pi/2} \sqrt{16\sin^2\frac{t}{2}}\,dt = 4\int_0^{\pi/2} \sin\frac{t}{2}\,dt \\ &= -8\cos\frac{t}{2}\Big|_0^{\pi/2} = -8\left(\cos\frac{\pi}{4} - \cos 0\right) = -8\left(\frac{\sqrt{2}}{2} - 1\right) \approx 2.34 \end{aligned}$$

10. $(5(\theta - \sin\theta), 5(1-\cos\theta)), \quad 0 \le \theta \le 2\pi$

SOLUTION Since $x = 5(\theta - \sin\theta)$ and $y = 5(1-\cos\theta)$, we have $x' = 5(1-\cos\theta)$ and $y' = 5\sin\theta$. Using the formula for the arc length we obtain:

$$\begin{aligned} S &= \int_0^{2\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{25(1-\cos\theta)^2 + 25\sin^2\theta}\,d\theta \\ &= 5\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\,d\theta = 5\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\,d\theta \\ &= 5\int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\,d\theta = 10\int_0^{2\pi} \sin\frac{\theta}{2}\,d\theta = 20\int_0^{\pi} \sin u\,du \\ &= 20(-\cos u)\Big|_0^{\pi} = -20(-1-1) = 40. \end{aligned}$$

11. Show that one arch of a cycloid generated by a circle of radius R has length $8R$.

SOLUTION Recall from earlier that the cycloid generated by a circle of radius R has parametric equations $x = Rt - R\sin t$, $y = R - R\cos t$. Hence, $x' = R - R\cos t$, $y' = R\sin t$. Using the identity $\sin^2\frac{t}{2} = \frac{1-\cos t}{2}$, we get

$$\begin{aligned} x'(t)^2 + y'(t)^2 &= R^2(1-\cos t)^2 + R^2\sin^2 t = R^2(1 - 2\cos t + \cos^2 t + \sin^2 t) \\ &= R^2(1 - 2\cos t + 1) = 2R^2(1-\cos t) = 4R^2\sin^2\frac{t}{2} \end{aligned}$$

One arch of the cycloid is traced as t varies from 0 to 2π. Hence, using the formula for the arc length we obtain:

$$\begin{aligned} S &= \int_0^{2\pi}\sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{2\pi}\sqrt{4R^2\sin^2\frac{t}{2}}\,dt = 2R\int_0^{2\pi}\sin\frac{t}{2}\,dt = 4R\int_0^{\pi}\sin u\,du \\ &= -4R\cos u\Big|_0^{\pi} = -4R(\cos\pi - \cos 0) = 8R \end{aligned}$$

12. Find the length of the spiral $c(t) = (t\cos t, t\sin t)$ for $0 \le t \le 2\pi$ to three decimal places (Figure 7). *Hint:* Use the formula

$$\int\sqrt{1+t^2}\,dt = \frac{1}{2}t\sqrt{1+t^2} + \frac{1}{2}\ln\left(t + \sqrt{1+t^2}\right)$$

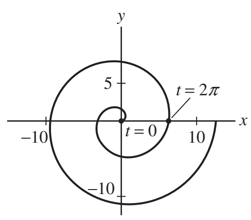


FIGURE 7 The spiral $c(t) = (t\cos t, t\sin t)$.

SOLUTION We use the formula for the arc length:

$$S = \int_0^{2\pi}\sqrt{x'(t)^2 + y'(t)^2}\,dt \tag{1}$$

Differentiating $x = t\cos t$ and $y = t\sin t$ yields

$$\begin{aligned} x'(t) &= \frac{d}{dt}(t\cos t) = \cos t - t\sin t \\ y'(t) &= \frac{d}{dt}(t\sin t) = \sin t + t\cos t \end{aligned}$$

Thus,

$$\begin{aligned} \sqrt{x'(t)^2 + y'(t)^2} &= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} \\ &= \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t} \\ &= \sqrt{(\cos^2 t + \sin^2 t)(1+t^2)} = \sqrt{1+t^2} \end{aligned}$$

We substitute into (1) and use the integral given in the hint to obtain the following arc length:

$$\begin{aligned} S &= \int_0^{2\pi}\sqrt{1+t^2}\,dt = \frac{1}{2}t\sqrt{1+t^2} + \frac{1}{2}\ln\left(t + \sqrt{1+t^2}\right)\Big|_0^{2\pi} \\ &= \frac{1}{2}\cdot 2\pi\sqrt{1+(2\pi)^2} + \frac{1}{2}\ln\left(2\pi + \sqrt{1+(2\pi)^2}\right) - \left(0 + \frac{1}{2}\ln 1\right) \\ &= \pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln\left(2\pi + \sqrt{1+4\pi^2}\right) \approx 21.256 \end{aligned}$$

13. Find the length of the tractrix (see Figure 6)

$$c(t) = (t - \tanh(t), \operatorname{sech}(t)), \qquad 0 \le t \le A$$

SOLUTION Since $x = t - \tanh(t)$ and $y = \operatorname{sech}(t)$ we have $x' = 1 - \operatorname{sech}^2(t)$ and $y' = -\operatorname{sech}(t)\tanh(t)$. Hence,

$$\begin{aligned} x'(t)^2 + y'(t)^2 &= (1 - \operatorname{sech}^2(t))^2 + \operatorname{sech}^2(t)\tanh^2(t) \\ &= 1 - 2\operatorname{sech}^2(t) + \operatorname{sech}^4(t) + \operatorname{sech}^2(t)\tanh^2(t) \\ &= 1 - 2\operatorname{sech}^2(t) + \operatorname{sech}^2(t)(\operatorname{sech}^2(t) + \tanh^2(t)) \\ &= 1 - 2\operatorname{sech}^2(t) + \operatorname{sech}^2(t) = 1 - \operatorname{sech}^2(t) = \tanh^2(t) \end{aligned}$$

Hence, using the formula for the arc length we get:

$$\begin{aligned} S &= \int_0^A \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^A \sqrt{\tanh^2(t)}\,dt = \int_0^A \tanh(t)\,dt = \ln(\cosh(t))\Big|_0^A \\ &= \ln(\cosh(A)) - \ln(\cosh(0)) = \ln(\cosh(A)) - \ln 1 = \ln(\cosh(A)) \end{aligned}$$

14. CAS Find a numerical approximation to the length of $c(t) = (\cos 5t, \sin 3t)$ for $0 \le t \le 2\pi$ (Figure 8).

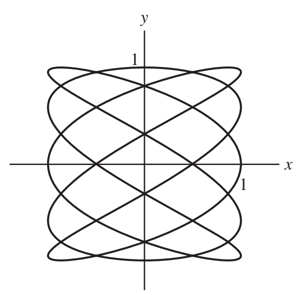


FIGURE 8

SOLUTION Since $x = \cos 5t$ and $y = \sin 3t$, we have

$$x'(t) = -5\sin 5t, \qquad y'(t) = 3\cos 3t$$

so that

$$x'(t)^2 + y'(t)^2 = 25\sin^2 5t + 9\cos^2 3t$$

Then the arc length is

$$\int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{2\pi} \sqrt{25\sin^2 5t + 9\cos^2 3t}\,dt \approx 24.60296$$

In Exercises 15–18, determine the speed s at time t (assume units of meters and seconds).

15. $(t^3, t^2), \quad t = 2$

SOLUTION We have $x(t) = t^3$, $y(t) = t^2$ hence $x'(t) = 3t^2$, $y'(t) = 2t$. The speed of the particle at time t is thus, $\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{9t^4 + 4t^2} = t\sqrt{9t^2 + 4}$. At time $t = 2$ the speed is

$$\frac{ds}{dt}\Big|_{t=2} = 2\sqrt{9 \cdot 2^2 + 4} = 2\sqrt{40} = 4\sqrt{10} \approx 12.65 \text{ m/s}.$$

16. $(3\sin 5t, 8\cos 5t), \quad t = \frac{\pi}{4}$

SOLUTION We have $x = 3\sin 5t$, $y = 8\cos 5t$, hence $x' = 15\cos 5t$, $y' = -40\sin 5t$. Thus, the speed of the particle at time t is

$$\begin{aligned} \frac{ds}{dt} &= \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{225\cos^2 5t + 1600\sin^2 5t} \\ &= \sqrt{225(\cos^2 5t + \sin^2 5t) + 1375\sin^2 5t} = 5\sqrt{9 + 55\sin^2 5t} \end{aligned}$$

Thus,

$$\frac{ds}{dt} = 5\sqrt{9 + 55\sin^2 5t}.$$

The speed at time $t = \frac{\pi}{4}$ is thus

$$\left.\frac{ds}{dt}\right|_{t=\pi/4} = 5\sqrt{9 + 55\sin^2\left(5 \cdot \frac{\pi}{4}\right)} \cong 30.21 \text{ m/s}$$

17. $(5t + 1, 4t - 3), \quad t = 9$

SOLUTION Since $x = 5t + 1$, $y = 4t - 3$, we have $x' = 5$ and $y' = 4$. The speed of the particle at time t is

$$\frac{ds}{dt} = \sqrt{x'(t) + y'(t)} = \sqrt{5^2 + 4^2} = \sqrt{41} \approx 6.4 \text{ m/s}.$$

We conclude that the particle has constant speed of 6.4 m/s.

18. $(\ln(t^2 + 1), t^3), \quad t = 1$

SOLUTION We have $x = \ln(t^2 + 1)$, $y = t^3$, so $x' = \dfrac{2t}{t^2 + 1}$ and $y' = 3t^2$. The speed of the particle at time t is thus

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\frac{4t^2}{(t^2+1)^2} + 9t^4} = t\sqrt{\frac{4}{(t^2+1)^2} + 9t^2}.$$

The speed at time $t = 1$ is

$$\left.\frac{ds}{dt}\right|_{t=1} = \sqrt{\frac{4}{2^2} + 9} = \sqrt{10} \approx 3.16 \text{ m/s}.$$

19. Find the minimum speed of a particle with trajectory $c(t) = (t^3 - 4t, t^2 + 1)$ for $t \geq 0$. *Hint:* It is easier to find the minimum of the square of the speed.

SOLUTION We first find the speed of the particle. We have $x(t) = t^3 - 4t$, $y(t) = t^2 + 1$, hence $x'(t) = 3t^2 - 4$ and $y'(t) = 2t$. The speed is thus

$$\frac{ds}{dt} = \sqrt{(3t^2 - 4)^2 + (2t)^2} = \sqrt{9t^4 - 24t^2 + 16 + 4t^2} = \sqrt{9t^4 - 20t^2 + 16}.$$

The square root function is an increasing function, hence the minimum speed occurs at the value of t where the function $f(t) = 9t^4 - 20t^2 + 16$ has minimum value. Since $\lim_{t\to\infty} f(t) = \infty$, f has a minimum value on the interval $0 \leq t < \infty$, and it occurs at a critical point or at the endpoint $t = 0$. We find the critical point of f on $t \geq 0$:

$$f'(t) = 36t^3 - 40t = 4t(9t^2 - 10) = 0 \Rightarrow t = 0, t = \sqrt{\frac{10}{9}}.$$

We compute the values of f at these points:

$$f(0) = 9 \cdot 0^4 - 20 \cdot 0^2 + 16 = 16$$

$$f\left(\sqrt{\frac{10}{9}}\right) = 9\left(\sqrt{\frac{10}{9}}\right)^4 - 20\left(\sqrt{\frac{10}{9}}\right)^2 + 16 = \frac{44}{9} \approx 4.89$$

We conclude that the minimum value of f on $t \geq 0$ is 4.89. The minimum speed is therefore

$$\left(\frac{ds}{dt}\right)_{\min} \approx \sqrt{4.89} \approx 2.21.$$

20. Find the minimum speed of a particle with trajectory $c(t) = (t^3, t^{-2})$ for $t \geq 0.5$.

SOLUTION We first compute the speed of the particle. Since $x(t) = t^3$ and $y(t) = t^{-2}$, we have $x'(t) = 3t^2$ and $y'(t) = -2t^{-3}$. The speed is

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{9t^4 + 4t^{-6}}.$$

The square root function is an increasing function, hence the minimum value of $\frac{ds}{dt}$ occurs at the point where the function $f(t) = 9t^4 + 4t^{-6}$ attains its minimum value. We find the critical points of f on the interval $t \geq 0.5$:

$$f'(t) = 36t^3 - 24t^{-7} = 0$$

$$3t^{10} - 2 = 0 \Rightarrow t = \sqrt[10]{\frac{2}{3}} \approx 0.96$$

Since $\lim_{t \to \infty} f(t) = \infty$, the minimum value on $0.5 \le t < \infty$ exists, and it occurs at the critical point $t = 0.96$ or at the endpoint $t = 0.5$. We compute the values of f at these points:

$$f(0.96) = 9 \cdot (0.96)^4 + 4 \cdot (0.96)^{-6} = 12.75$$
$$f(0.5) = 9(0.5)^4 + 4(0.5)^{-6} = 256.56$$

We conclude that the minimum value of f on the interval $t \ge 0.5$ is 12.75. The minimum speed for $t \ge 0.5$ is therefore

$$\left(\frac{ds}{dt}\right)_{\min} = \sqrt{12.75} \approx 3.57$$

21. Find the speed of the cycloid $c(t) = (4t - 4\sin t, 4 - 4\cos t)$ at points where the tangent line is horizontal.

SOLUTION We first find the points where the tangent line is horizontal. The slope of the tangent line is the following quotient:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\sin t}{4 - 4\cos t} = \frac{\sin t}{1 - \cos t}.$$

To find the points where the tangent line is horizontal we solve the following equation for $t \ge 0$:

$$\frac{dy}{dx} = 0, \quad \frac{\sin t}{1 - \cos t} = 0 \Rightarrow \sin t = 0 \quad \text{and} \quad \cos t \ne 1.$$

Now, $\sin t = 0$ and $t \ge 0$ at the points $t = \pi k$, $k = 0, 1, 2, \ldots$. Since $\cos \pi k = (-1)^k$, the points where $\cos t \ne 1$ are $t = \pi k$ for k odd. The points where the tangent line is horizontal are, therefore:

$$t = \pi(2k - 1), \quad k = 1, 2, 3, \ldots$$

The speed at time t is given by the following expression:

$$\begin{aligned}\frac{ds}{dt} &= \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(4 - 4\cos t)^2 + (4\sin t)^2} \\ &= \sqrt{16 - 32\cos t + 16\cos^2 t + 16\sin^2 t} = \sqrt{16 - 32\cos t + 16} \\ &= \sqrt{32(1 - \cos t)} = \sqrt{32 \cdot 2\sin^2 \frac{t}{2}} = 8\left|\sin \frac{t}{2}\right|\end{aligned}$$

That is, the speed of the cycloid at time t is

$$\frac{ds}{dt} = 8\left|\sin \frac{t}{2}\right|.$$

We now substitute

$$t = \pi(2k - 1), \quad k = 1, 2, 3, \ldots$$

to obtain

$$\frac{ds}{dt} = 8\left|\sin \frac{\pi(2k - 1)}{2}\right| = 8|(-1)^{k+1}| = 8$$

22. Calculate the arc length integral $s(t)$ for the *logarithmic spiral* $c(t) = (e^t \cos t, e^t \sin t)$.

SOLUTION We have $x'(t) = e^t(\cos t - \sin t)$, $y'(t) = e^t(\cos t + \sin t)$ so that

$$x'(t)^2 + y'(t)^2 = e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t) = 2e^{2t}(\cos^2 t + \sin^2 t) = 2e^{2t}$$

so that the arc length integral is

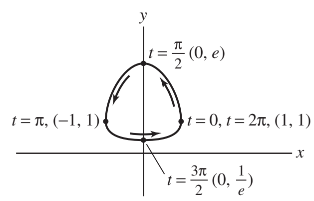
$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt = \sqrt{2}\int_a^b e^t\, dt$$

If neither a nor b is $\pm\infty$, then this equals $\sqrt{2}(e^b - e^a)$. Note that the origin corresponds to $t = -\infty$.

CAS *In Exercises 23–26, plot the curve and use the Midpoint Rule with $N = 10, 20, 30$, and 50 to approximate its length.*

23. $c(t) = (\cos t, e^{\sin t})$ for $0 \le t \le 2\pi$

SOLUTION The curve of $c(t) = (\cos t, e^{\sin t})$ for $0 \le t \le 2\pi$ is shown in the figure below:



$c(t) = (\cos t, e^{\sin t}), 0 \le t \le 2\pi.$

The length of the curve is given by the following integral:

$$S = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + \left(\cos t\, e^{\sin t}\right)^2}\, dt.$$

That is, $S = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t\, e^{2\sin t}}\, dt$. We approximate the integral using the Mid-Point Rule with $N = 10, 20, 30, 50$. For $f(t) = \sqrt{\sin^2 t + \cos^2 t\, e^{2\sin t}}$ we obtain

$$(N = 10):\quad \Delta x = \frac{2\pi}{10} = \frac{\pi}{5}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{5}$$

$$M_{10} = \frac{\pi}{5} \sum_{i=1}^{10} f(c_i) = 6.903734$$

$$(N = 20):\quad \Delta x = \frac{2\pi}{20} = \frac{\pi}{10}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{10}$$

$$M_{20} = \frac{\pi}{10} \sum_{i=1}^{20} f(c_i) = 6.915035$$

$$(N = 30):\quad \Delta x = \frac{2\pi}{30} = \frac{\pi}{15}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{15}$$

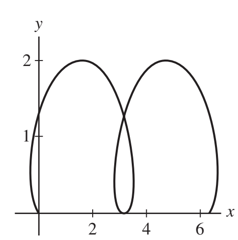
$$M_{30} = \frac{\pi}{15} \sum_{i=1}^{30} f(c_i) = 6.914949$$

$$(N = 50):\quad \Delta x = \frac{2\pi}{50} = \frac{\pi}{25}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{25}$$

$$M_{50} = \frac{\pi}{25} \sum_{i=1}^{50} f(c_i) = 6.914951$$

24. $c(t) = (t - \sin 2t, 1 - \cos 2t)$ for $0 \le t \le 2\pi$

SOLUTION The curve is shown in the figure below:



$c(t) = (t - \sin 2t, 1 - \cos 2t), 0 \le t \le 2\pi.$

The length of the curve is given by the following integral:

$$S = \int_0^{2\pi} \sqrt{(1 - 2\cos 2t)^2 + (2\sin 2t)^2}\, dt = \int_0^{2\pi} \sqrt{1 - 4\cos 2t + 4\cos^2 2t + 4\sin^2 2t}\, dt = \int_0^{2\pi} \sqrt{5 - 4\cos 2t}\, dt.$$

That is,

$$S = \int_0^{2\pi} \sqrt{5 - 4\cos 2t}\, dt.$$

Approximating the length using the Mid-Point Rule with $N = 10, 20, 30, 50$ for $f(t) = \sqrt{5 - 4\cos 2t}$ we obtain

$$(N = 10): \quad \Delta x = \frac{2\pi}{10} = \frac{\pi}{5}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{5}$$

$$M_{10} = \frac{\pi}{5} \sum_{i=1}^{10} f(c_i) = 13.384047$$

$$(N = 20): \quad \Delta x = \frac{2\pi}{20} = \frac{\pi}{10}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{10}$$

$$M_{20} = \frac{\pi}{10} \sum_{i=1}^{20} f(c_i) = 13.365095$$

$$(N = 30): \quad \Delta x = \frac{2\pi}{30} = \frac{\pi}{15}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{15}$$

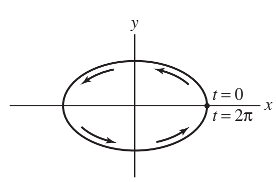
$$M_{30} = \frac{\pi}{15} \sum_{i=1}^{30} f(c_i) = 13.364897$$

$$(N = 50): \quad \Delta x = \frac{2\pi}{50} = \frac{\pi}{25}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{25}$$

$$M_{50} = \frac{\pi}{25} \sum_{i=1}^{50} f(c_i) = 13.364893$$

25. The ellipse $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

SOLUTION We use the parametrization given in Example 4, section 12.1, that is, $c(t) = (5\cos t, 3\sin t)$, $0 \le t \le 2\pi$. The curve is shown in the figure below:



$c(t) = (5\cos t, 3\sin t)$, $0 \le t \le 2\pi$.

The length of the curve is given by the following integral:

$$S = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} \sqrt{(-5\sin t)^2 + (3\cos t)^2}\, dt$$
$$= \int_0^{2\pi} \sqrt{25\sin^2 t + 9\cos^2 t}\, dt = \int_0^{2\pi} \sqrt{9(\sin^2 t + \cos^2 t) + 16\sin^2 t}\, dt = \int_0^{2\pi} \sqrt{9 + 16\sin^2 t}\, dt.$$

That is,

$$S = \int_0^{2\pi} \sqrt{9 + 16\sin^2 t}\, dt.$$

We approximate the integral using the Mid-Point Rule with $N = 10, 20, 30, 50$, for $f(t) = \sqrt{9 + 16\sin^2 t}$. We obtain

$$(N = 10): \quad \Delta x = \frac{2\pi}{10} = \frac{\pi}{5}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{5}$$

$$M_{10} = \frac{\pi}{5} \sum_{i=1}^{10} f(c_i) = 25.528309$$

$$(N = 20): \quad \Delta x = \frac{2\pi}{20} = \frac{\pi}{10},\ c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{10}$$

$$M_{20} = \frac{\pi}{10}\sum_{i=1}^{20} f(c_i) = 25.526999$$

$$(N = 30): \quad \Delta x = \frac{2\pi}{30} = \frac{\pi}{15},\ c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{15}$$

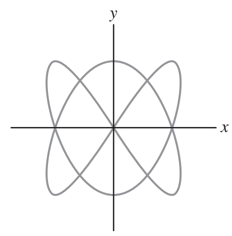
$$M_{30} = \frac{\pi}{15}\sum_{i=1}^{30} f(c_i) = 25.526999$$

$$(N = 50): \quad \Delta x = \frac{2\pi}{50} = \frac{\pi}{25},\ c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{25}$$

$$M_{50} = \frac{\pi}{25}\sum_{i=1}^{50} f(c_i) = 25.526999$$

26. $x = \sin 2t, \quad y = \sin 3t \quad \text{for } 0 \le t \le 2\pi$

SOLUTION The curve is shown in the figure below:



$c(t) = (\sin 2t, \sin 3t),\ 0 \le t \le 2\pi.$

The length of the curve is given by the following integral:

$$S = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} \sqrt{(2\cos 2t)^2 + (3\cos 3t)^2}\, dt = \int_0^{2\pi} \sqrt{4\cos^2 2t + 9\cos^2 3t}\, dt.$$

We approximate the length using the Mid-Point Rule with $N = 10, 20, 30, 50$ for $f(t) = \sqrt{4\cos^2 2t + 9\cos^2 3t}$. We obtain

$$(N = 10): \quad \Delta x = \frac{2\pi}{10} = \frac{\pi}{5},\ c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{5}$$

$$M_{10} = \frac{\pi}{5}\sum_{i=1}^{10} f(c_i) = 15.865169$$

$$(N = 20): \quad \Delta x = \frac{2\pi}{20} = \frac{\pi}{10},\ c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{10}$$

$$M_{20} = \frac{\pi}{10}\sum_{i=1}^{20} f(c_i) = 15.324697$$

$$(N = 30): \quad \Delta x = \frac{2\pi}{30} = \frac{\pi}{15},\ c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{15}$$

$$M_{30} = \frac{\pi}{15}\sum_{i=1}^{30} f(c_i) = 15.279322$$

$$(N = 50): \quad \Delta x = \frac{2\pi}{50} = \frac{\pi}{25},\ c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{25}$$

$$M_{50} = \frac{\pi}{25}\sum_{i=1}^{50} f(c_i) = 15.287976$$

27. If you unwind thread from a stationary circular spool, keeping the thread taut at all times, then the endpoint traces a curve $\mathcal{C}$ called the **involute** of the circle (Figure 9). Observe that $\overline{PQ}$ has length $R\theta$. Show that $\mathcal{C}$ is parametrized by

$$c(\theta) = \big(R(\cos\theta + \theta\sin\theta),\, R(\sin\theta - \theta\cos\theta)\big)$$

Then find the length of the involute for $0 \le \theta \le 2\pi$.

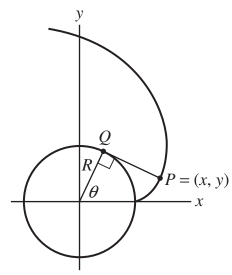


FIGURE 9 Involute of a circle.

SOLUTION Suppose that the arc $\widehat{QT}$ corresponding to the angle θ is unwound. Then the length of the segment $\overline{QP}$ equals the length of this arc. That is, $\overline{QP} = R\theta$. With the help of the figure we can see that

$$x = \overline{OA} + \overline{AB} = \overline{OA} + \overline{EP} = R\cos\theta + \overline{QP}\sin\theta = R\cos\theta + R\theta\sin\theta = R(\cos\theta + \theta\sin\theta).$$

Furthermore,

$$y = \overline{QA} - \overline{QE} = R\sin\theta - \overline{QP}\cos\theta = R\sin\theta - R\theta\cos\theta = R(\sin\theta - \theta\cos\theta)$$

The coordinates of P with respect to the parameter θ form the following parametrization of the curve:

$$c(\theta) = (R(\cos\theta + \theta\sin\theta), R(\sin\theta - \theta\cos\theta)), \qquad 0 \le \theta \le 2\pi.$$

We find the length of the involute for $0 \le \theta \le 2\pi$, using the formula for the arc length:

$$S = \int_0^{2\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta.$$

We compute the integrand:

$$x'(\theta) = \frac{d}{d\theta}(R(\cos\theta + \theta\sin\theta)) = R(-\sin\theta + \sin\theta + \theta\cos\theta) = R\theta\cos\theta$$

$$y'(\theta) = \frac{d}{d\theta}(R(\sin\theta - \theta\cos\theta)) = R(\cos\theta - (\cos\theta - \theta\sin\theta)) = R\theta\sin\theta$$

$$\sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{(R\theta\cos\theta)^2 + (R\theta\sin\theta)^2} = \sqrt{R^2\theta^2(\cos^2\theta + \sin^2\theta)} = \sqrt{R^2\theta^2} = R\theta$$

We now compute the arc length:

$$S = \int_0^{2\pi} R\theta\, d\theta = \frac{R\theta^2}{2}\bigg|_0^{2\pi} = \frac{R\cdot(2\pi)^2}{2} = 2\pi^2 R.$$

28. Let $a > b$ and set

$$k = \sqrt{1 - \frac{b^2}{a^2}}$$

Use a parametric representation to show that the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ has length $L = 4aG\left(\frac{\pi}{2}, k\right)$, where

$$G(\theta, k) = \int_0^{\theta} \sqrt{1 - k^2\sin^2 t}\, dt$$

is the *elliptic integral of the second kind*.

SOLUTION Since the ellipse is symmetric with respect to the x and y axis, its length L is four times the length of the part of the ellipse which is in the first quadrant. This part is represented by the following parametrization: $x(t) = a\sin t$, $y(t) = b\cos t$, $0 \le t \le \frac{\pi}{2}$. Using the formula for the arc length we get:

$$L = 4\int_0^{\pi/2} \sqrt{x'(t)^2 + y'(t)^2}\, dt = 4\int_0^{\pi/2} \sqrt{(a\cos t)^2 + (-b\sin t)^2}\, dt$$

$$= 4\int_0^{\pi/2} \sqrt{a^2\cos^2 t + b^2\sin^2 t}\, dt$$

We rewrite the integrand as follows:

$$\begin{aligned}
L &= 4\int_0^{\pi/2}\sqrt{a^2\cos^2 t + a^2\sin^2 t + (b^2-a^2)\sin^2 t}\,dt \\
&= 4\int_0^{\pi/2}\sqrt{a^2(\cos^2 t+\sin^2 t) + (b^2-a^2)\sin^2 t}\,dt \\
&= 4\int_0^{\pi/2}\sqrt{a^2 + (b^2-a^2)\sin^2 t}\,dt = 4a\int_0^{\pi/2}\sqrt{\frac{a^2}{a^2} + \frac{b^2-a^2}{a^2}\sin^2 t}\,dt \\
&= 4a\int_0^{\pi/2}\sqrt{1-\left(1-\frac{b^2}{a^2}\right)\sin^2 t}\,dt = 4a\int_0^{\pi/2}\sqrt{1-k^2\sin^2 t}\,dt = 4aG\left(\frac{\pi}{2},k\right)
\end{aligned}$$

where $k = \sqrt{1-\frac{b^2}{a^2}}$.

In Exercises 29–32, use Eq. (4) to compute the surface area of the given surface.

29. The cone generated by revolving $c(t) = (t, mt)$ about the x-axis for $0 \le t \le A$

SOLUTION Substituting $y(t) = mt$, $y'(t) = m$, $x'(t) = 1$, $a = 0$, and $b = 0$ in the formula for the surface area, we get

$$S = 2\pi\int_0^A mt\sqrt{1+m^2}\,dt = 2\pi\sqrt{1+m^2}\,m\int_0^A t\,dt = 2\pi m\sqrt{1+m^2}\cdot\left.\frac{t^2}{2}\right|_0^A = m\sqrt{1+m^2}\pi A^2$$

30. A sphere of radius R

SOLUTION The sphere of radius R is generated by revolving the half circle $c(t) = (R\cos t, R\sin t)$, $0 \le t \le \pi$ about the x-axis. We have $x(t) = R\cos t$, $x'(t) = -R\sin t$, $y(t) = R\sin t$, $y'(t) = R\cos t$. Using the formula for the surface area, we get

$$\begin{aligned}
S &= 2\pi\int_0^{\pi} y(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt = 2\pi\int_0^{\pi} R\sin t\sqrt{R^2\sin^2 t + R^2\cos^2 t}\,dt \\
&= 2\pi R^2\int_0^{\pi}\sin t\,dt = -2\pi R^2\cos t\Big|_0^{\pi} = -2\pi R^2(-1-1) = 4\pi R^2
\end{aligned}$$

31. The surface generated by revolving one arch of the cycloid $c(t) = (t-\sin t, 1-\cos t)$ about the x-axis

SOLUTION One arch of the cycloid is traced as t varies from 0 to 2π. Since $x(t) = t - \sin t$ and $y(t) = 1 - \cos t$, we have $x'(t) = 1 - \cos t$ and $y'(t) = \sin t$. Hence, using the identity $1 - \cos t = 2\sin^2\frac{t}{2}$, we get

$$x'(t)^2 + y'(t)^2 = (1-\cos t)^2 + \sin^2 t = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2 - 2\cos t = 4\sin^2\frac{t}{2}$$

By the formula for the surface area we obtain:

$$\begin{aligned}
S &= 2\pi\int_0^{2\pi} y(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt = 2\pi\int_0^{2\pi}(1-\cos t)\cdot 2\sin\frac{t}{2}\,dt \\
&= 2\pi\int_0^{2\pi} 2\sin^2\frac{t}{2}\cdot 2\sin\frac{t}{2}\,dt = 8\pi\int_0^{2\pi}\sin^3\frac{t}{2}\,dt = 16\pi\int_0^{\pi}\sin^3 u\,du
\end{aligned}$$

We use a reduction formula to compute this integral, obtaining

$$S = 16\pi\left[\frac{1}{3}\cos^3 u - \cos u\right]\Bigg|_0^{\pi} = 16\pi\left[\frac{4}{3}\right] = \frac{64\pi}{3}$$

32. The surface generated by revolving the astroid $c(t) = (\cos^3 t, \sin^3 t)$ about the x-axis for $0 \le t \le \frac{\pi}{2}$

SOLUTION We have $x(t) = \cos^3 t$, $y(t) = \sin^3 t$, $x'(t) = -3\cos^2 t\sin t$, $y'(t) = 3\sin^2 t\cos t$. Hence,

$$x'(t)^2 + y'(t)^2 = 9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t = 9\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t) = 9\cos^2 t\sin^2 t$$

Using the formula for the surface area we get

$$S = 2\pi\int_0^{\pi/2} y(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt = 2\pi\int_0^{\pi/2}\sin^3 t\cdot 3\cos t\sin t\,dt = 6\pi\int_0^{\pi/2}\sin^4 t\cos t\,dt$$

We compute the integral using the substitution $u = \sin t\ du = \cos t\,dt$. We obtain

$$S = 6\pi\int_0^1 u^4\,du = 6\pi\left.\frac{u^5}{5}\right|_0^1 = \frac{6\pi}{5}.$$

Further Insights and Challenges

33. CAS Let $b(t)$ be the "Butterfly Curve":

$$x(t) = \sin t \left(e^{\cos t} - 2\cos 4t - \sin\left(\frac{t}{12}\right)^5 \right)$$

$$y(t) = \cos t \left(e^{\cos t} - 2\cos 4t - \sin\left(\frac{t}{12}\right)^5 \right)$$

(a) Use a computer algebra system to plot $b(t)$ and the speed $s'(t)$ for $0 \le t \le 12\pi$.
(b) Approximate the length $b(t)$ for $0 \le t \le 10\pi$.

SOLUTION

(a) Let $f(t) = e^{\cos t} - 2\cos 4t - \sin\left(\frac{t}{12}\right)^5$, then

$$x(t) = \sin t f(t)$$
$$y(t) = \cos t f(t)$$

and so

$$(x'(t))^2 + (y'(t))^2 = [\sin t f'(t) + \cos t f(t)]^2 + [\cos t f'(t) - \sin t f(t)]^2$$

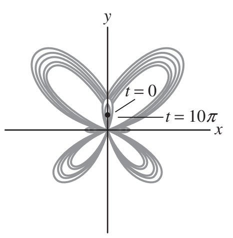
Using the identity $\sin^2 t + \cos^2 t = 1$, we get

$$(x'(t))^2 + (y'(t))^2 = (f'(t))^2 + (f(t))^2.$$

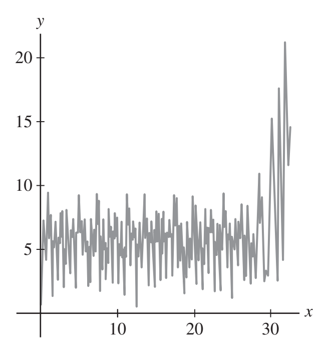
Thus, $s'(t)$ is the following:

$$\sqrt{\left[e^{\cos t} - 2\cos 4t - \sin\left(\frac{t}{12}\right)^5\right]^2 + \left[-\sin t e^{\cos t} + 8\sin 4t - \frac{5}{12}\left(\frac{t}{12}\right)^4 \cos\left(\frac{t}{12}\right)^5\right]^2}.$$

The following figures show the curves of $b(t)$ and the speed $s'(t)$ for $0 \le t \le 10\pi$:



The "Butterfly Curve" $b(t)$, $0 \le t \le 10\pi$



$s'(t)$, $0 \le t \le 10\pi$

Looking at the graph, we see it would be difficult to compute the length using numeric integration; due to the high frequency oscillations, very small steps would be needed.

(b) The length of $b(t)$ for $0 \le t \le 10\pi$ is given by the integral: $L = \int_0^{10\pi} s'(t)\,dt$ where $s'(t)$ is given in part (a). We approximate the length using the Midpoint Rule with $N = 30$. The numerical methods in Mathematica approximate the answer by 211.952. Using the Midpoint Rule with $N = 50$, we get 204.48; with $N = 500$, we get 211.6; and with $N = 5000$, we get 212.09.

34. CAS Let $a \ge b > 0$ and set $k = \dfrac{2\sqrt{ab}}{a-b}$. Show that the **trochoid**

$$x = at - b\sin t, \qquad y = a - b\cos t, \quad 0 \le t \le T$$

has length $2(a-b)G\left(\frac{T}{2}, k\right)$ with $G(\theta, k)$ as in Exercise 28.

SOLUTION We have $x'(t) = a - b\cos t$, $y'(t) = b\sin t$. Hence,

$$x'(t)^2 + y'(t)^2 = (a - b\cos t)^2 + (b\sin t)^2 = a^2 - 2ab\cos t + b^2\cos^2 t + b^2\sin^2 t$$
$$= a^2 + b^2 - 2ab\cos t$$

The length of the trochoid for $0 \le t \le T$ is

$$L = \int_0^T \sqrt{a^2 + b^2 - 2ab\cos t}\, dt$$

We rewrite the integrand as follows to bring it to the required form. We use the identity $1 - \cos t = 2\sin^2 \frac{t}{2}$ to obtain

$$\begin{aligned} L &= \int_0^T \sqrt{(a-b)^2 + 2ab - 2ab\cos t}\, dt = \int_0^T \sqrt{(a-b)^2 + 2ab(1-\cos t)}\, dt \\ &= \int_0^T \sqrt{(a-b)^2 + 4ab\sin^2 \frac{t}{2}}\, dt = \int_0^T \sqrt{(a-b)^2\left(1 + \frac{4ab}{(a-b)^2}\sin^2\frac{t}{2}\right)}\, dt \\ &= (a-b)\int_0^T \sqrt{1 + k^2\sin^2\frac{t}{2}}\, dt \end{aligned}$$

(where $k = \frac{2\sqrt{ab}}{a-b}$).

Substituting $u = \frac{t}{2}$, $du = \frac{1}{2}\,dt$, we get

$$L = 2(a-b)\int_0^{T/2} \sqrt{1 + k^2\sin^2 u}\, du = 2(a-b)E(T/2, k)$$

35. A satellite orbiting at a distance R from the center of the earth follows the circular path $x = R\cos\omega t$, $y = R\sin\omega t$.

(a) Show that the period T (the time of one revolution) is $T = 2\pi/\omega$.

(b) According to Newton's laws of motion and gravity,

$$x''(t) = -Gm_e\frac{x}{R^3}, \qquad y''(t) = -Gm_e\frac{y}{R^3}$$

where G is the universal gravitational constant and m_e is the mass of the earth. Prove that $R^3/T^2 = Gm_e/4\pi^2$. Thus, R^3/T^2 has the same value for all orbits (a special case of Kepler's Third Law).

SOLUTION

(a) As shown in Example 4, the circular path has constant speed of $\frac{ds}{dt} = \omega R$. Since the length of one revolution is $2\pi R$, the period T is

$$T = \frac{2\pi R}{\omega R} = \frac{2\pi}{\omega}.$$

(b) Differentiating $x = R\cos\omega t$ twice with respect to t gives

$$x'(t) = -R\omega\sin\omega t$$
$$x''(t) = -R\omega^2\cos\omega t$$

Substituting $x(t)$ and $x''(t)$ in the equation $x''(t) = -Gm_e\frac{x}{R^3}$ and simplifying, we obtain

$$-R\omega^2\cos\omega t = -Gm_e\cdot\frac{R\cos\omega t}{R^3}$$
$$-R\omega^2 = -\frac{Gm_e}{R^2} \Rightarrow R^3 = \frac{Gm_e}{\omega^2}$$

By part (a), $T = \dfrac{2\pi}{\omega}$. Hence, $\omega = \dfrac{2\pi}{T}$. Substituting yields

$$R^3 = \frac{Gm_e}{\frac{4\pi^2}{T^2}} = \frac{T^2 Gm_e}{4\pi^2} \Rightarrow \frac{R^3}{T^2} = \frac{Gm_e}{4\pi^2}$$

36. The acceleration due to gravity on the surface of the earth is

$$g = \frac{Gm_e}{R_e^2} = 9.8\ \text{m/s}^2, \quad \text{where } R_e = 6378\ \text{km}$$

Use Exercise 35(b) to show that a satellite orbiting at the earth's surface would have period $T_e = 2\pi\sqrt{R_e/g} \approx 84.5$ min. Then estimate the distance R_m from the moon to the center of the earth. Assume that the period of the moon (sidereal month) is $T_m \approx 27.43$ days.

SOLUTION By part (b) of Exercise 35, it follows that

$$\frac{R_e^3}{T_e^2} = \frac{Gm_e}{4\pi^2} \Rightarrow T_e^2 = \frac{4\pi^2 R_e^3}{Gm_e} = \frac{4\pi^2 R_e}{\frac{Gm_e}{R_e^2}} = \frac{4\pi^2 R_e}{g}$$

Hence,

$$T_e = 2\pi\sqrt{\frac{R_e}{g}} = 2\pi\sqrt{\frac{6378 \cdot 10^3}{9.8}} \approx 5068.8 \text{ s} \approx 84.5 \text{ min.}$$

In part (b) of Exercise 35 we showed that $\frac{R^3}{T^2}$ is the same for all orbits. It follows that this quotient is the same for the satellite orbiting at the earth's surface and for the moon orbiting around the earth. Thus,

$$\frac{R_m^3}{T_m^2} = \frac{R_e^3}{T_e^2} \Rightarrow R_m = R_e\left(\frac{T_m}{T_e}\right)^{2/3}.$$

Setting $T_m = 27.43 \cdot 1440 = 39{,}499.2$ minutes, $T_e = 84.5$ minutes, and $R_e = 6378$ km we get

$$R_m = 6378\left(\frac{39{,}499.2}{84.5}\right)^{2/3} \approx 384{,}154 \text{ km.}$$

11.3 Polar Coordinates

Preliminary Questions

1. Points P and Q with the same radial coordinate (choose the correct answer):

(a) Lie on the same circle with the center at the origin.

(b) Lie on the same ray based at the origin.

SOLUTION Two points with the same radial coordinate are equidistant from the origin, therefore they lie on the same circle centered at the origin. The angular coordinate defines a ray based at the origin. Therefore, if the two points have the same angular coordinate, they lie on the same ray based at the origin.

2. Give two polar representations for the point $(x, y) = (0, 1)$, one with negative r and one with positive r.

SOLUTION The point $(0, 1)$ is on the y-axis, distant one unit from the origin, hence the polar representation with positive r is $(r, \theta) = \left(1, \frac{\pi}{2}\right)$. The point $(r, \theta) = \left(-1, \frac{\pi}{2}\right)$ is the reflection of $(r, \theta) = \left(1, \frac{\pi}{2}\right)$ through the origin, hence we must add π to return to the original point.

We obtain the following polar representation of $(0, 1)$ with negative r:

$$(r, \theta) = \left(-1, \frac{\pi}{2} + \pi\right) = \left(-1, \frac{3\pi}{2}\right).$$

3. Describe each of the following curves:

(a) $r = 2$ **(b)** $r^2 = 2$ **(c)** $r\cos\theta = 2$

SOLUTION

(a) Converting to rectangular coordinates we get

$$\sqrt{x^2 + y^2} = 2 \quad \text{or} \quad x^2 + y^2 = 2^2.$$

This is the equation of the circle of radius 2 centered at the origin.

(b) We convert to rectangular coordinates, obtaining $x^2 + y^2 = 2$. This is the equation of the circle of radius $\sqrt{2}$, centered at the origin.

(c) We convert to rectangular coordinates. Since $x = r\cos\theta$ we obtain the following equation: $x = 2$. This is the equation of the vertical line through the point $(2, 0)$.

4. If $f(-\theta) = f(\theta)$, then the curve $r = f(\theta)$ is symmetric with respect to the (choose the correct answer):

(a) x-axis **(b)** y-axis **(c)** origin

SOLUTION The equality $f(-\theta) = f(\theta)$ for all θ implies that whenever a point (r, θ) is on the curve, also the point $(r, -\theta)$ is on the curve. Since the point $(r, -\theta)$ is the reflection of (r, θ) with respect to the x-axis, we conclude that the curve is symmetric with respect to the x-axis.

Exercises

1. Find polar coordinates for each of the seven points plotted in Figure 16.

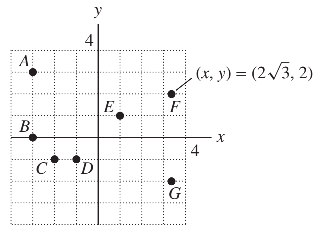
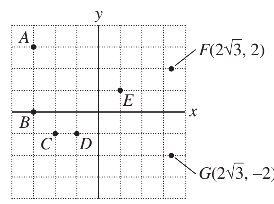


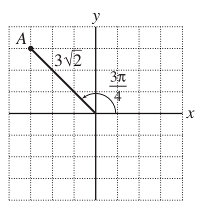
FIGURE 16

SOLUTION We mark the points as shown in the figure.

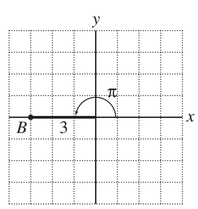


Using the data given in the figure for the x and y coordinates and the quadrants in which the point are located, we obtain:

(A), with rectangular coordinates $(-3, 4)$: $\begin{array}{l} r = \sqrt{(-3)^2 + 3^2} = \sqrt{18} \\ \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \end{array} \Rightarrow (r, \theta) = \left(3\sqrt{2}, \frac{3\pi}{4}\right)$

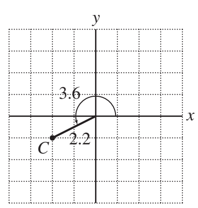


(B), with rectangular coordinates $(-3, 0)$: $\begin{array}{l} r = 3 \\ \theta = \pi \end{array} \Rightarrow (r, \theta) = (3, \pi)$

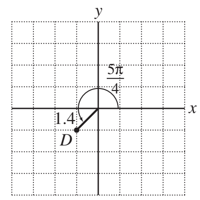


(C), with rectangular coordinates $(-2, -1)$:

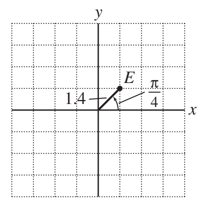
$$\begin{array}{l} r = \sqrt{2^2 + 1^2} = \sqrt{5} \approx 2.2 \\ \theta = \tan^{-1}\left(\frac{-1}{-2}\right) = \tan^{-1}\left(\frac{1}{2}\right) = \pi + 0.46 \approx 3.6 \end{array} \Rightarrow (r, \theta) \approx \left(\sqrt{5}, 3.6\right)$$



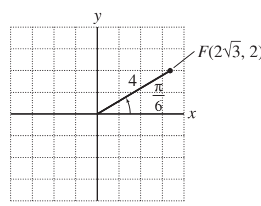
(D), with rectangular coordinates $(-1,-1)$: $\begin{array}{l} r=\sqrt{1^2+1^2}=\sqrt{2}\approx 1.4 \\ \theta=\pi+\frac{\pi}{4}=\frac{5\pi}{4} \end{array} \Rightarrow (r,\theta)\approx\left(\sqrt{2},\frac{5\pi}{4}\right)$



(E), with rectangular coordinates $(1,1)$: $\begin{array}{l} r=\sqrt{1^2+1^2}=\sqrt{2}\approx 1.4 \\ \theta=\tan^{-1}\left(\frac{1}{1}\right)=\frac{\pi}{4} \end{array} \Rightarrow (r,\theta)\approx\left(\sqrt{2},\frac{\pi}{4}\right)$



(F), with rectangular coordinates $(2\sqrt{3},2)$: $\begin{array}{l} r=\sqrt{\left(2\sqrt{3}\right)^2+2^2}=\sqrt{16}=4 \\ \theta=\tan^{-1}\left(\frac{2}{2\sqrt{3}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6} \end{array} \Rightarrow (r,\theta)=\left(4,\frac{\pi}{6}\right)$

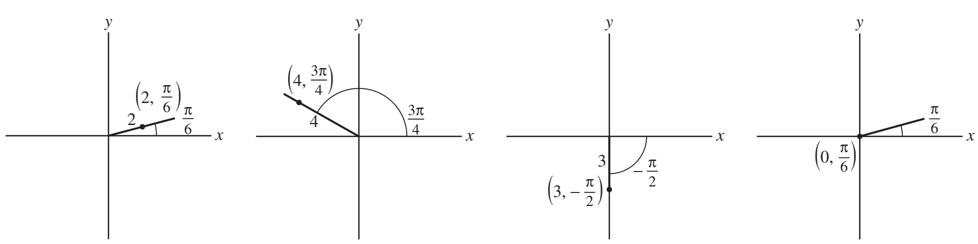


(G), with rectangular coordinates $(2\sqrt{3},-2)$: G is the reflection of F about the x axis, hence the two points have equal radial coordinates, and the angular coordinate of G is obtained from the angular coordinate of F: $\theta=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$. Hence, the polar coordinates of G are $\left(4,\frac{11\pi}{6}\right)$.

2. Plot the points with polar coordinates:

(a) $\left(2,\frac{\pi}{6}\right)$ **(b)** $\left(4,\frac{3\pi}{4}\right)$ **(c)** $\left(3,-\frac{\pi}{2}\right)$ **(d)** $\left(0,\frac{\pi}{6}\right)$

SOLUTION We first plot the ray $\theta=\theta_0$ for the given angle θ_0, and then mark the point on this line distanced $r=r_0$ from the origin. We obtain the following points:



(a) (b) (c) (d)

$R=0$ is the point $(0,0)$ in rect. coords.

3. Convert from rectangular to polar coordinates.

(a) $(1,0)$ **(b)** $(3,\sqrt{3})$ **(c)** $(-2,2)$ **(d)** $(-1,\sqrt{3})$

SOLUTION

(a) The point $(1,0)$ is on the positive x axis distanced one unit from the origin. Hence, $r=1$ and $\theta=0$. Thus, $(r,\theta)=(1,0)$.

(b) The point $\left(3,\sqrt{3}\right)$ is in the first quadrant so $\theta=\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)=\frac{\pi}{6}$. Also, $r=\sqrt{3^2+\left(\sqrt{3}\right)^2}=\sqrt{12}$. Hence, $(r,\theta)=\left(\sqrt{12},\frac{\pi}{6}\right)$.

(c) The point $(-2, 2)$ is in the second quadrant. Hence,

$$\theta = \tan^{-1}\left(\frac{2}{-2}\right) = \tan^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Also, $r = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$. Hence, $(r, \theta) = \left(\sqrt{8}, \frac{3\pi}{4}\right)$.

(d) The point $\left(-1, \sqrt{3}\right)$ is in the second quadrant, hence,

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \tan^{-1}\left(-\sqrt{3}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Also, $r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$. Hence, $(r, \theta) = \left(2, \frac{2\pi}{3}\right)$.

4. Convert from rectangular to polar coordinates using a calculator (make sure your choice of θ gives the correct quadrant).

(a) $(2, 3)$ **(b)** $(4, -7)$ **(c)** $(-3, -8)$ **(d)** $(-5, 2)$

SOLUTION

(a) The point $(2, 3)$ is in the first quadrant, with $x = 2$ and $y = 3$. Hence

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{3}{2}\right) \approx 0.98 \\ r &= \sqrt{2^2 + 3^2} = \sqrt{13} \approx 3.6 \end{aligned} \quad \Rightarrow \quad (r, \theta) \approx (3.6, 0.98)\,.$$

(b) The point $(4, -7)$ is in the fourth quadrant with $x = 4$ and $y = -7$. We have

$$\tan^{-1}\left(\frac{-7}{4}\right) \approx -1.05$$

$$r = \sqrt{(-7)^2 + 4^2} = \sqrt{65} \approx 8.1$$

Note that $\tan^{-1}$ an angle less that zero in the fourth quadrant; since we want an angle between 0 and 2π, we add 2π to get $\theta \approx 2\pi - 1.05 \approx 5.232$. Thus $(r, \theta) \approx (8.1, 5.2)$.

(c) The point $(-3, -8)$ is in the third quadrant, with $x = -3$ and $y = -8$. We have

$$\tan^{-1}\left(\frac{-8}{-3}\right) = \tan^{-1}\left(\frac{8}{3}\right) \approx 1.212$$

$$r = \sqrt{(-3)^2 + (-8)^2} = \sqrt{73} \approx 8.54$$

Note that $\tan^{-1}$ produced an angle in the first quadrant; we want the third quadrant angle with the same tangent, so we add π to get $\theta \approx \pi + 1.212 \approx 4.35$. Thus $(r, \theta) \approx (8.54, 4.35)$

(d) The point $(-5, 2)$ is in the second quadrant, with $x = -5$ and $y = 2$. We have

$$\tan^{-1}\left(\frac{2}{-5}\right) \approx -0.38$$

$$r = \sqrt{2^2 + (-5)^2} = \sqrt{29} \approx 5.39$$

Note that the angle is in the fourth quadrant; to get the second quadrant angle with the same tangent and in the range $[0, 2\pi)$, we add π to get $\theta \approx \pi - 0.38 \approx 2.76$. Thus $(r, \theta) \approx (5.39, 2.76)$.

5. Convert from polar to rectangular coordinates:

(a) $\left(3, \frac{\pi}{6}\right)$ **(b)** $\left(6, \frac{3\pi}{4}\right)$ **(c)** $\left(0, \frac{\pi}{5}\right)$ **(d)** $\left(5, -\frac{\pi}{2}\right)$

SOLUTION

(a) Since $r = 3$ and $\theta = \frac{\pi}{6}$, we have:

$$\begin{aligned} x &= r\cos\theta = 3\cos\frac{\pi}{6} = 3\cdot\frac{\sqrt{3}}{2} \approx 2.6 \\ y &= r\sin\theta = 3\sin\frac{\pi}{6} = 3\cdot\frac{1}{2} = 1.5 \end{aligned} \quad \Rightarrow \quad (x, y) \approx (2.6, 1.5)\,.$$

(b) For $\left(6, \frac{3\pi}{4}\right)$ we have $r = 6$ and $\theta = \frac{3\pi}{4}$. Hence,

$$\begin{aligned} x &= r\cos\theta = 6\cos\frac{3\pi}{4} \approx -4.24 \\ y &= r\sin\theta = 6\sin\frac{3\pi}{4} \approx 4.24 \end{aligned} \quad\Rightarrow\quad (x, y) \approx (-4.24, 4.24).$$

(c) For $\left(0, \frac{\pi}{5}\right)$, we have $r = 0$, so that the rectangular coordinates are $(x, y) = (0, 0)$.

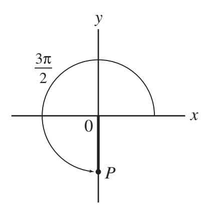
(d) Since $r = 5$ and $\theta = -\frac{\pi}{2}$ we have

$$\begin{aligned} x &= r\cos\theta = 5\cos\left(-\frac{\pi}{2}\right) = 5 \cdot 0 = 0 \\ y &= r\sin\theta = 5\sin\left(-\frac{\pi}{2}\right) = 5 \cdot (-1) = -5 \end{aligned} \quad\Rightarrow\quad (x, y) = (0, -5)$$

6. Which of the following are possible polar coordinates for the point P with rectangular coordinates $(0, -2)$?

(a) $\left(2, \frac{\pi}{2}\right)$
(b) $\left(2, \frac{7\pi}{2}\right)$
(c) $\left(-2, -\frac{3\pi}{2}\right)$
(d) $\left(-2, \frac{7\pi}{2}\right)$
(e) $\left(-2, -\frac{\pi}{2}\right)$
(f) $\left(2, -\frac{7\pi}{2}\right)$

SOLUTION The point P has distance 2 from the origin and the angle between $\overline{OP}$ and the positive x-axis in the positive direction is $\frac{3\pi}{2}$. Hence, $(r, \theta) = \left(2, \frac{3\pi}{2}\right)$ is one choice for the polar coordinates for P.



The polar coordinates $(2, \theta)$ are possible for P if $\theta - \frac{3\pi}{2}$ is a multiple of 2π. The polar coordinate $(-2, \theta)$ are possible for P if $\theta - \frac{3\pi}{2}$ is an odd multiple of π. These considerations lead to the following conclusions:

(a) $\left(2, \frac{\pi}{2}\right)$ $\frac{\pi}{2} - \frac{3\pi}{2} = -\pi \Rightarrow \left(2, \frac{\pi}{2}\right)$ does not represent P.

(b) $\left(2, \frac{7\pi}{2}\right)$ $\frac{7\pi}{2} - \frac{3\pi}{2} = 2\pi \Rightarrow \left(2, \frac{7\pi}{2}\right)$ represents P.

(c) $\left(-2, -\frac{3\pi}{2}\right)$ $-\frac{3\pi}{2} - \frac{3\pi}{2} = -3\pi \Rightarrow \left(-2, -\frac{3\pi}{2}\right)$ represents P.

(d) $\left(-2, \frac{7\pi}{2}\right)$ $\frac{7\pi}{2} - \frac{3\pi}{2} = 2\pi \Rightarrow \left(-2, \frac{7\pi}{2}\right)$ does not represent P.

(e) $\left(-2, -\frac{\pi}{2}\right)$ $-\frac{\pi}{2} - \frac{3\pi}{2} = -2\pi \Rightarrow \left(-2, -\frac{\pi}{2}\right)$ does not represent P.

(f) $\left(2, -\frac{7\pi}{2}\right)$ $-\frac{7\pi}{2} - \frac{3\pi}{2} = -5\pi \Rightarrow \left(2, -\frac{7\pi}{2}\right)$ does not represent P.

7. Describe each shaded sector in Figure 17 by inequalities in r and θ.

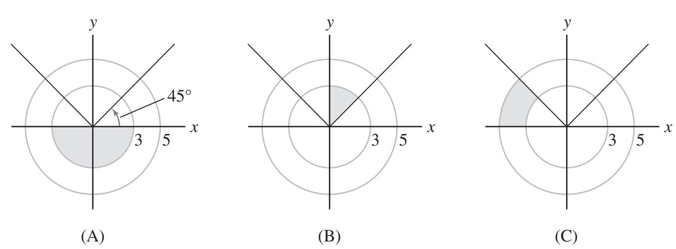


FIGURE 17

SOLUTION

(a) In the sector shown below r is varying between 0 and 3 and θ is varying between π and 2π. Hence the following inequalities describe the sector:

$$0 \le r \le 3$$
$$\pi \le \theta \le 2\pi$$

(b) In the sector shown below r is varying between 0 and 3 and θ is varying between $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Hence, the inequalities for the sector are:

$$0 \le r \le 3$$
$$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$$

(c) In the sector shown below r is varying between 3 and 5 and θ is varying between $\frac{3\pi}{4}$ and π. Hence, the inequalities are:

$$3 \le r \le 5$$
$$\frac{3\pi}{4} \le \theta \le \pi$$

8. Find the equation in polar coordinates of the line through the origin with slope $\frac{1}{2}$.

SOLUTION A line of slope $m = \frac{1}{2}$ makes an angle $\theta_0 = \tan^{-1}\frac{1}{2} \approx 0.46$ with the positive x-axis. The equation of the line is $\theta \approx 0.46$, while r is arbitrary.

9. What is the slope of the line $\theta = \frac{3\pi}{5}$?

SOLUTION This line makes an angle $\theta_0 = \frac{3\pi}{5}$ with the positive x-axis, hence the slope of the line is $m = \tan\frac{3\pi}{5} \approx -3.1$.

10. Which of $r = 2\sec\theta$ and $r = 2\csc\theta$ defines a horizontal line?

SOLUTION The equation $r = 2\csc\theta$ is the polar equation of a horizontal line, as it can be written as $r = 2/\sin\theta$, so $r\sin\theta = 2$, which becomes $y = 2$. On the other hand, the equation $r = 2\sec\theta$ is the polar equation of a vertical line, as it can be written as $r = 2/\cos\theta$, so $r\cos\theta = 2$, which becomes $x = 2$.

In Exercises 11–16, convert to an equation in rectangular coordinates.

11. $r = 7$

SOLUTION $r = 7$ describes the points having distance 7 from the origin, that is, the circle with radius 7 centered at the origin. The equation of the circle in rectangular coordinates is

$$x^2 + y^2 = 7^2 = 49.$$

12. $r = \sin\theta$

SOLUTION Multiplying by r and substituting $y = r\sin\theta$ and $r^2 = x^2 + y^2$ gives

$$r^2 = r\sin\theta$$
$$x^2 + y^2 = y$$

We move the y and then complete the square to obtain

$$x^2 + y^2 - y = 0$$
$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

Thus, $r = \sin\theta$ is the equation of a circle of radius $\frac{1}{2}$ and center $\left(0, \frac{1}{2}\right)$.

13. $r = 2\sin\theta$

SOLUTION We multiply the equation by r and substitute $r^2 = x^2 + y^2$, $r\sin\theta = y$. This gives

$$r^2 = 2r\sin\theta$$
$$x^2 + y^2 = 2y$$

Moving the $2y$ and completing the square yield: $x^2 + y^2 - 2y = 0$ and $x^2 + (y-1)^2 = 1$. Thus, $r = 2\sin\theta$ is the equation of a circle of radius 1 centered at $(0, 1)$.

14. $r = 2\csc\theta$

SOLUTION We multiply the equation by $\sin\theta$ and substitute $y = r\sin\theta$. We get

$$r\sin\theta = 2$$
$$y = 2$$

Thus, $r = 2\csc\theta$ is the equation of the line $y = 2$.

15. $r = \dfrac{1}{\cos\theta - \sin\theta}$

SOLUTION We multiply the equation by $\cos\theta - \sin\theta$ and substitute $y = r\sin\theta$, $x = r\cos\theta$. This gives

$$r(\cos\theta - \sin\theta) = 1$$
$$r\cos\theta - r\sin\theta = 1$$

$x - y = 1 \Rightarrow y = x - 1$. Thus,

$$r = \frac{1}{\cos\theta - \sin\theta}$$

is the equation of the line $y = x - 1$.

16. $r = \dfrac{1}{2 - \cos\theta}$

SOLUTION We multiply the equation by $2 - \cos\theta$. Then we substitute $x = r\cos\theta$ and $r = \sqrt{x^2 + y^2}$, to obtain

$$r(2 - \cos\theta) = 1$$
$$2r - r\cos\theta = 1$$
$$2\sqrt{x^2 + y^2} - x = 1$$

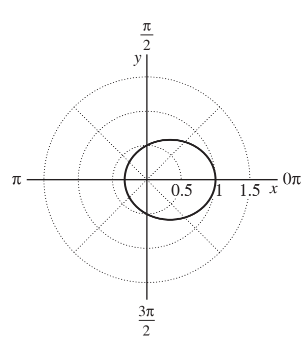
Moving the x, then squaring and simplifying, we obtain

$$2\sqrt{x^2 + y^2} = x + 1$$
$$4\left(x^2 + y^2\right) = x^2 + 2x + 1$$
$$3x^2 - 2x + 4y^2 = 1$$

We complete the square:

$$3\left(x^2 - \frac{2}{3}x\right) + 4y^2 = 1$$
$$3\left(x - \frac{1}{3}\right)^2 + 4y^2 = \frac{4}{3}$$
$$\frac{\left(x - \frac{1}{3}\right)^2}{\frac{4}{9}} + \frac{y^2}{\frac{1}{3}} = 1$$

This is the equation of the ellipse shown in the figure:



In Exercises 17–20, convert to an equation in polar coordinates.

17. $x^2 + y^2 = 5$

SOLUTION We make the substitution $x^2 + y^2 = r^2$ to obtain; $r^2 = 5$ or $r = \sqrt{5}$.

18. $x = 5$

SOLUTION Substituting $x = r\cos\theta$ gives the polar equation $r\cos\theta = 5$ or $r = 5\sec\theta$.

19. $y = x^2$

SOLUTION Substituting $y = r\sin\theta$ and $x = r\cos\theta$ yields

$$r\sin\theta = r^2\cos^2\theta.$$

Then, dividing by $r\cos^2\theta$ we obtain,

$$\frac{\sin\theta}{\cos^2\theta} = r \qquad \text{so} \qquad r = \tan\theta\sec\theta$$

20. $xy = 1$

SOLUTION We substitute $x = r\cos\theta$, $y = r\sin\theta$ to obtain

$$(r\cos\theta)(r\sin\theta) = 1$$
$$r^2\cos\theta\sin\theta = 1$$

Using the identity $\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$ yields

$$r^2 \cdot \frac{\sin 2\theta}{2} = 1 \Rightarrow r^2 = 2\csc 2\theta.$$

21. Match each equation with its description.

(a) $r = 2$	**(i)** Vertical line
(b) $\theta = 2$	**(ii)** Horizontal line
(c) $r = 2\sec\theta$	**(iii)** Circle
(d) $r = 2\csc\theta$	**(iv)** Line through origin

SOLUTION

(a) $r = 2$ describes the points 2 units from the origin. Hence, it is the equation of a circle.

(b) $\theta = 2$ describes the points P so that $\overline{OP}$ makes an angle of $\theta_0 = 2$ with the positive x-axis. Hence, it is the equation of a line through the origin.

(c) This is $r\cos\theta = 2$, which is $x = 2$, a vertical line.

(d) Converting to rectangular coordinates, we get $r = 2\csc\theta$, so $r\sin\theta = 2$ and $y = 2$. This is the equation of a horizontal line.

22. Find the values of θ in the plot of $r = 4\cos\theta$ corresponding to points A, B, C, D in Figure 18. Then indicate the portion of the graph traced out as θ varies in the following intervals:

(a) $0 \le \theta \le \frac{\pi}{2}$ **(b)** $\frac{\pi}{2} \le \theta \le \pi$ **(c)** $\pi \le \theta \le \frac{3\pi}{2}$

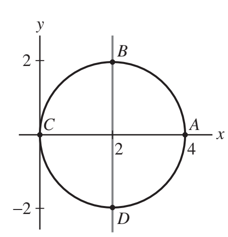


FIGURE 18 Plot of $r = 4\cos\theta$.

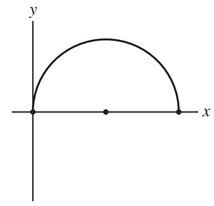
SOLUTION The point A is on the x-axis hence $\theta = 0$. The point B is in the first quadrant with $x = y = 2$ hence $\theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$. The point C is at the origin. Thus,

$$r = 0 \Rightarrow 4\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \ \frac{3\pi}{2}.$$

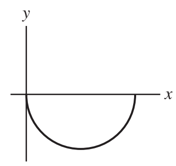
The point D is in the fourth quadrant with $x = 2$, $y = -2$, hence

$$\theta = \tan^{-1}\left(\frac{-2}{2}\right) = \tan^{-1}(-1) = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}.$$

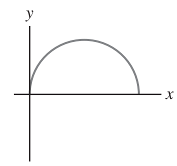
$0 \le \theta \le \frac{\pi}{2}$ represents the first quadrant, hence the points (r, θ) where $r = 4\cos\theta$ and $0 \le \theta \le \frac{\pi}{2}$ are the points on the circle which are in the first quadrant, as shown below:



If we insist that $r \geq 0$, then since $\frac{\pi}{2} \leq \theta \leq \pi$ represents the second quadrant and $\pi \leq \theta \leq \frac{3\pi}{2}$ represents the third quadrant, and since the circle $r = 4\cos\theta$ has no points in the left xy -plane, then there are no points for (b) and (c). However, if we allow $r < 0$ then (b) represents the semi-circle



and (c) like (a) represent

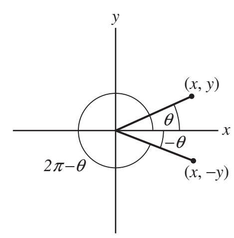


23. Suppose that $P = (x, y)$ has polar coordinates (r, θ). Find the polar coordinates for the points:

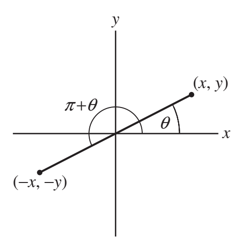
(a) $(x, -y)$ **(b)** $(-x, -y)$ **(c)** $(-x, y)$ **(d)** (y, x)

SOLUTION

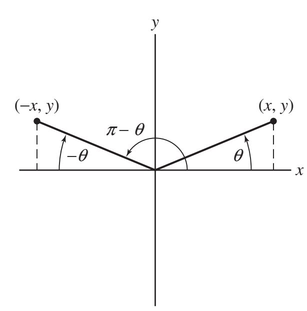
(a) $(x, -y)$ is the symmetric point of (x, y) with respect to the x-axis, hence the two points have the same radial coordinate, and the angular coordinate of $(x, -y)$ is $2\pi - \theta$. Hence, $(x, -y) = (r, 2\pi - \theta)$.



(b) $(-x, -y)$ is the symmetric point of (x, y) with respect to the origin. Hence, $(-x, -y) = (r, \theta + \pi)$.



(c) $(-x, y)$ is the symmetric point of (x, y) with respect to the y-axis. Hence the two points have the same radial coordinates and the angular coordinate of $(-x, y)$ is $\pi - \theta$. Hence, $(-x, y) = (r, \pi - \theta)$.

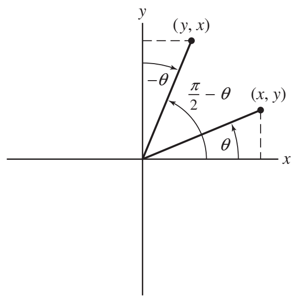


(d) Let (r_1, θ_1) denote the polar coordinates of (y, x). Hence,

$$r_1 = \sqrt{y^2 + x^2} = \sqrt{x^2 + y^2} = r$$

$$\tan\theta_1 = \frac{x}{y} = \frac{1}{y/x} = \frac{1}{\tan\theta} = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

Since the points (x, y) and (y, x) are in the same quadrant, the solution for θ_1 is $\theta_1 = \frac{\pi}{2} - \theta$. We obtain the following polar coordinates: $(y, x) = \left(r, \frac{\pi}{2} - \theta\right)$.



24. Match each equation in rectangular coordinates with its equation in polar coordinates.

(a) $x^2 + y^2 = 4$ **(i)** $r^2(1 - 2\sin^2\theta) = 4$
(b) $x^2 + (y - 1)^2 = 1$ **(ii)** $r(\cos\theta + \sin\theta) = 4$
(c) $x^2 - y^2 = 4$ **(iii)** $r = 2\sin\theta$
(d) $x + y = 4$ **(iv)** $r = 2$

SOLUTION

(a) Since $x^2 + y^2 = r^2$, we have $r^2 = 4$ or $r = 2$.
(b) Using Example 7, the equation of the circle $x^2 + (y - 1)^2 = 1$ has polar equation $r = 2\sin\theta$.
(c) Setting $x = r\cos\theta$, $y = r\sin\theta$ in $x^2 - y^2 = 4$ gives

$$x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2\left(\cos^2\theta - \sin^2\theta\right) = 4.$$

We now use the identity $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$ to obtain the following equation:

$$r^2\left(1 - 2\sin^2\theta\right) = 4.$$

(d) Setting $x = r\cos\theta$ and $y = r\sin\theta$ in $x + y = 4$ we get:

$$x + y = 4$$
$$r\cos\theta + r\sin\theta = 4$$

so

$$r\,(\cos\theta + \sin\theta) = 4$$

25. What are the polar equations of the lines parallel to the line $r\cos\left(\theta - \frac{\pi}{3}\right) = 1$?

SOLUTION The line $r\cos\left(\theta - \frac{\pi}{3}\right) = 1$, or $r = \sec\left(\theta - \frac{\pi}{3}\right)$, is perpendicular to the ray $\theta = \frac{\pi}{3}$ and at distance $d = 1$ from the origin. Hence, the lines parallel to this line are also perpendicular to the ray $\theta = \frac{\pi}{3}$, so the polar equations of these lines are $r = d\sec\left(\theta - \frac{\pi}{3}\right)$ or $r\cos\left(\theta - \frac{\pi}{3}\right) = d$.

26. Show that the circle with center at $\left(\frac{1}{2}, \frac{1}{2}\right)$ in Figure 19 has polar equation $r = \sin\theta + \cos\theta$ and find the values of θ between 0 and π corresponding to points A, B, C, and D.

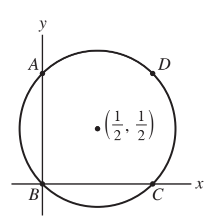


FIGURE 19 Plot of $r = \sin\theta + \cos\theta$.

SOLUTION We show that the rectangular equation of $r = \sin\theta + \cos\theta$ is

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}.$$

We multiply the polar equation by r and substitute $r^2 = x^2 + y^2$, $r\sin\theta = y$, $r\cos\theta = x$. This gives

$$r = \sin\theta + \cos\theta$$
$$r^2 = r\sin\theta + r\cos\theta$$
$$x^2 + y^2 = y + x$$

Transferring sides and completing the square yields

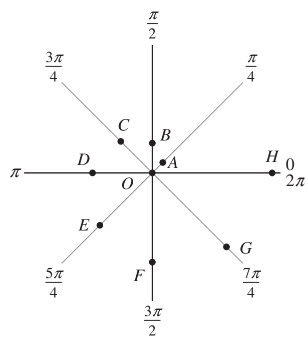
$$x^2 - x + y^2 - y = 0$$
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Clearly point C corresponds to $\theta = 0$ since $\cos 0 + \sin 0 = 1$. The circle is traced out counterclockwise as θ increases to π, so A corresponds to $\theta = \frac{\pi}{2}$ since again $\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0$. Next, D clearly corresponds to $\theta = \frac{\pi}{4}$, and indeed $\cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \sqrt{2}$, which is the diameter of the circle. Finally, point A corresponds to $\theta = \frac{3\pi}{4}$, since there $\cos\theta = -\sin\theta$.

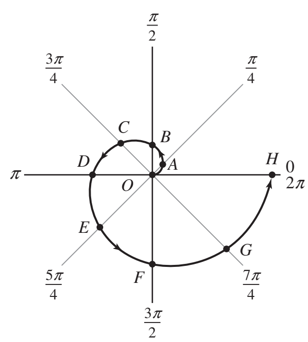
27. Sketch the curve $r = \frac{1}{2}\theta$ (the spiral of Archimedes) for θ between 0 and 2π by plotting the points for $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \dots, 2\pi$.

SOLUTION We first plot the following points (r, θ) on the spiral:

$$O = (0,0),\ A = \left(\frac{\pi}{8}, \frac{\pi}{4}\right),\ B = \left(\frac{\pi}{4}, \frac{\pi}{2}\right),\ C = \left(\frac{3\pi}{8}, \frac{3\pi}{4}\right),\ D = \left(\frac{\pi}{2}, \pi\right),$$
$$E = \left(\frac{5\pi}{8}, \frac{5\pi}{4}\right),\ F = \left(\frac{3\pi}{4}, \frac{3\pi}{2}\right),\ G = \left(\frac{7\pi}{8}, \frac{7\pi}{4}\right),\ H = (\pi, 2\pi).$$



Since $r(0) = \frac{0}{2} = 0$, the graph begins at the origin and moves toward the points A, B, C, D, E, F, G and H as θ varies from $\theta = 0$ to the other values stated above. Connecting the points in this direction we obtain the following graph for $0 \le \theta \le 2\pi$:



28. Sketch $r = 3\cos\theta - 1$ (see Example 8).

SOLUTION We first choose some values of θ between 0 and π and mark the corresponding points on the graph. Then we use symmetry (due to $\cos(2\pi - \theta) = \cos\theta$) to plot the other half of the graph by reflecting the first half through the x-axis. Since $r = 3\cos\theta - 1$ is periodic, the entire curve is obtained as θ varies from 0 to 2π. We start with the values $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$, and compute the corresponding values of r:

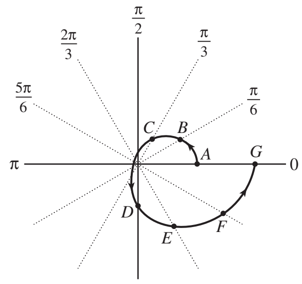
$$r = 3\cos 0 - 1 = 3 - 1 = 2 \Rightarrow A = (2, 0)$$
$$r = 3\cos\frac{\pi}{6} - 1 = \frac{3\sqrt{3}}{2} - 1 \approx 1.6 \Rightarrow B = \left(1.6, \frac{\pi}{6}\right)$$
$$r = 3\cos\frac{\pi}{3} - 1 = \frac{3}{2} - 1 = 0.5 \Rightarrow C = \left(0.5, \frac{\pi}{3}\right)$$
$$r = 3\cos\frac{\pi}{2} - 1 = 3 \cdot 0 - 1 = -1 \Rightarrow D = \left(-1, \frac{\pi}{2}\right)$$

$$r = 3\cos\frac{2\pi}{3} - 1 = -2.5 \Rightarrow E = \left(-2.5, \frac{2\pi}{3}\right)$$

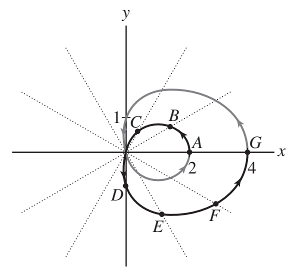
$$r = 3\cos\frac{5\pi}{6} - 1 = -3.6 \Rightarrow F = \left(-3.6, \frac{5\pi}{6}\right)$$

$$r = 3\cos\pi - 1 = -4 \Rightarrow G = (-4, \pi)$$

The graph begins at the point $(r, \theta) = (2, 0)$ and moves toward the other points in this order, as θ varies from 0 to π. Since r is negative for $\frac{\pi}{2} \le \theta \le \pi$, the curve continues into the fourth quadrant, rather than into the second quadrant. We obtain the following graph:



Now we have half the curve and we use symmetry to plot the rest. Reflecting the first half through the x axis we obtain the whole curve:

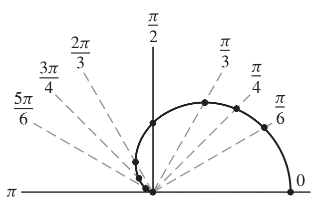


29. Sketch the cardioid curve $r = 1 + \cos\theta$.

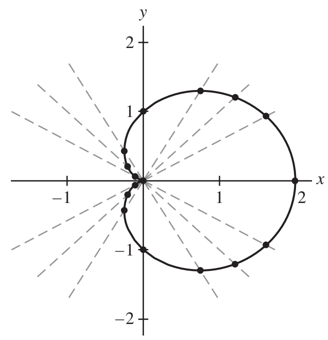
SOLUTION Since $\cos\theta$ is period with period 2π, the entire curve will be traced out as θ varies from 0 to 2π. Additionally, since $\cos(2\pi - \theta) = \cos(\theta)$, we can sketch the curve for θ between 0 and π and reflect the result through the x axis to obtain the whole curve. Use the values $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$, and π:

θ	r	point
0	$1 + \cos 0 = 2$	$(2, 0)$
$\frac{\pi}{6}$	$1 + \cos\frac{\pi}{6} = \frac{2+\sqrt{3}}{2}$	$\left(\frac{2+\sqrt{3}}{2}, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	$1 + \cos\frac{\pi}{4} = \frac{2+\sqrt{2}}{2}$	$\left(\frac{2+\sqrt{2}}{2}, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	$1 + \cos\frac{\pi}{3} = \frac{3}{2}$	$\left(\frac{3}{2}, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	$1 + \cos\frac{\pi}{2} = 1$	$\left(1, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$1 + \cos\frac{2\pi}{3} = \frac{1}{2}$	$\left(\frac{1}{2}, \frac{2\pi}{3}\right)$
$\frac{3\pi}{4}$	$1 + \cos\frac{3\pi}{4} = \frac{2-\sqrt{2}}{2}$	$\left(\frac{2-\sqrt{2}}{2}, \frac{3\pi}{4}\right)$
$\frac{5\pi}{6}$	$1 + \cos\frac{5\pi}{6} = \frac{2-\sqrt{3}}{2}$	$\left(\frac{2-\sqrt{3}}{2}, \frac{5\pi}{6}\right)$

$\theta = 0$ corresponds to the point $(2, 0)$, and the graph moves clockwise as θ increases from 0 to π. Thus the graph is



Reflecting through the x axis gives the other half of the curve:



30. Show that the cardioid of Exercise 29 has equation

$$(x^2 + y^2 - x)^2 = x^2 + y^2$$

in rectangular coordinates.

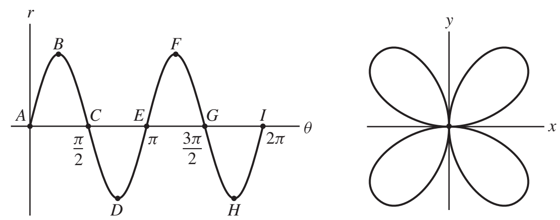
SOLUTION Multiply through by r and substitute for r, r^2, and $r\cos\theta$ to get

$$r = 1 + \cos\theta$$
$$r^2 = r + r\cos\theta$$
$$x^2 + y^2 = \sqrt{x^2 + y^2} + x$$
$$x^2 + y^2 - x = \sqrt{x^2 + y^2}$$
$$(x^2 + y^2 - x)^2 = x^2 + y^2$$

31. Figure 20 displays the graphs of $r = \sin 2\theta$ in rectangular coordinates and in polar coordinates, where it is a "rose with four petals." Identify:

(a) The points in (B) corresponding to points A–I in (A).

(b) The parts of the curve in (B) corresponding to the angle intervals $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$, $\left[\pi, \frac{3\pi}{2}\right]$, and $\left[\frac{3\pi}{2}, 2\pi\right]$.



(A) Graph of r as a function of θ, where $r = \sin 2\theta$

(B) Graph of $r = \sin 2\theta$ in polar coordinates

FIGURE 20

SOLUTION

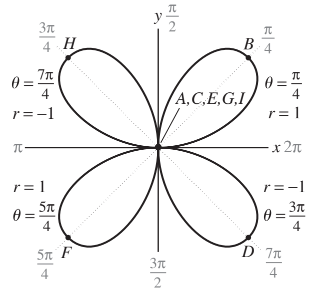
(a) The graph (A) gives the following polar coordinates of the labeled points:

$$A: \quad \theta = 0, \quad r = 0$$
$$B: \quad \theta = \frac{\pi}{4}, \quad r = \sin\frac{2\pi}{4} = 1$$
$$C: \quad \theta = \frac{\pi}{2}, \quad r = 0$$
$$D: \quad \theta = \frac{3\pi}{4}, \quad r = \sin\frac{2\cdot 3\pi}{4} = -1$$
$$E: \quad \theta = \pi, \quad r = 0$$
$$F: \quad \theta = \frac{5\pi}{4}, \quad r = 1$$
$$G: \quad \theta = \frac{3\pi}{2}, \quad r = 0$$

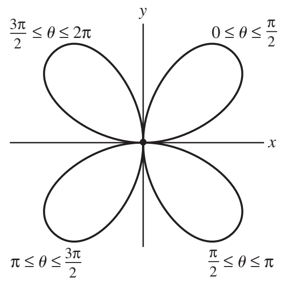
$$H: \quad \theta = \frac{7\pi}{4}, \quad r = -1$$

$$I: \quad \theta = 2\pi, \quad r = 0.$$

Since the maximal value of $|r|$ is 1, the points with $r = 1$ or $r = -1$ are the furthest points from the origin. The corresponding quadrant is determined by the value of θ and the sign of r. If $r_0 < 0$, the point (r_0, θ_0) is on the ray $\theta = -\theta_0$. These considerations lead to the following identification of the points in the xy plane. Notice that A, C, G, E, and I are the same point.



(b) We use the graph (A) to find the sign of $r = \sin 2\theta$: $0 \le \theta \le \frac{\pi}{2} \Rightarrow r \ge 0 \Rightarrow (r, \theta)$ is in the first quadrant. $\frac{\pi}{2} \le \theta \le \pi \Rightarrow r \le 0 \Rightarrow (r, \theta)$ is in the fourth quadrant. $\pi \le \theta \le \frac{3\pi}{2} \Rightarrow r \ge 0 \Rightarrow (r, \theta)$ is in the third quadrant. $\frac{3\pi}{2} \le \theta \le 2\pi \Rightarrow r \le 0 \Rightarrow (r, \theta)$ is in the second quadrant. That is,



32. Sketch the curve $r = \sin 3\theta$. First fill in the table of r-values below and plot the corresponding points of the curve. Notice that the three petals of the curve correspond to the angle intervals $\left[0, \frac{\pi}{3}\right]$, $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$, and $\left[\frac{\pi}{3}, \pi\right]$. Then plot $r = \sin 3\theta$ in rectangular coordinates and label the points on this graph corresponding to (r, θ) in the table.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\cdots$	$\frac{11\pi}{12}$	π
r									

SOLUTION We compute the values of r corresponding to the given values of θ:

$$\theta = 0, \qquad r = \sin 0 = 0 \qquad (A)$$

$$\theta = \frac{\pi}{12}, \qquad r = \sin \frac{3\pi}{12} \approx 0.71 \qquad (B)$$

$$\theta = \frac{\pi}{6}, \qquad r = \sin \frac{3\pi}{6} = 1 \qquad (C)$$

$$\theta = \frac{\pi}{4}, \qquad r = \sin \frac{3\pi}{4} \approx 0.71 \qquad (D)$$

$$\theta = \frac{\pi}{3}, \qquad r = \sin \frac{3\pi}{3} = 0 \qquad (E)$$

$$\theta = \frac{5\pi}{12}, \qquad r = \sin \frac{15\pi}{12} \approx -0.71 \qquad (F)$$

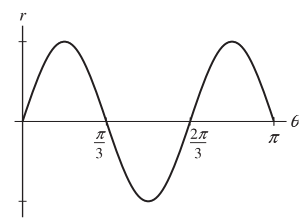
$$\theta = \frac{\pi}{2}, \qquad r = \sin \frac{3\pi}{2} = -1 \qquad (G)$$

$$\theta = \frac{7\pi}{12}, \qquad r = \sin \frac{21\pi}{12} \approx -0.71 \qquad (H)$$

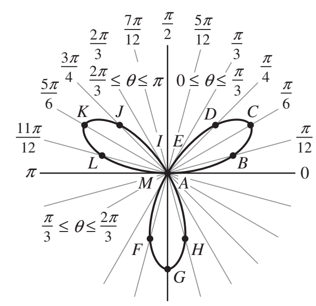
$$\theta = \frac{3\pi}{2}, \qquad r = \sin \frac{9\pi}{2} = 0 \qquad (I)$$

$$\theta = \frac{3\pi}{4}, \quad r = \sin\frac{9\pi}{4} \approx 0.71 \qquad (J)$$
$$\theta = \frac{5\pi}{6}, \quad r = \sin\frac{15\pi}{6} = 1 \qquad (K)$$
$$\theta = \frac{11\pi}{12}, \quad r = \sin\frac{33\pi}{12} \approx 0.71 \qquad (L)$$
$$\theta = \pi, \quad r = \sin 3\pi = 0 \qquad (M)$$

We plot the points on the xy -plane and join them to obtain the following curve:



Using the graph of $r = \sin 3\theta$ we find the sign of r and determine the parts of the graph corresponding to the angle intervals. We get

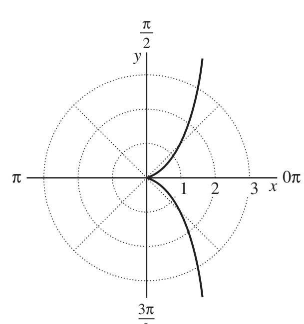


$$r = \sin 3\theta \quad \begin{cases} 0 \le \theta \le \frac{\pi}{3} \Rightarrow r \ge 0 \Rightarrow (r,\theta) \text{ in the first quadrant.} \\ \frac{\pi}{3} \le \theta \le \frac{2\pi}{3} \Rightarrow r \le 0 \Rightarrow (r,\theta) \text{ in the third and fourth quadrant.} \\ \frac{2\pi}{3} \le \theta \le \pi \Rightarrow r \ge 0 \Rightarrow (r,\theta) \text{ in the second quadrant.} \end{cases}$$

33. CAS Plot the **cissoid** $r = 2\sin\theta\tan\theta$ and show that its equation in rectangular coordinates is

$$y^2 = \frac{x^3}{2 - x}$$

SOLUTION Using a CAS we obtain the following curve of the cissoid:



We substitute $\sin\theta = \frac{y}{r}$ and $\tan\theta = \frac{y}{x}$ in $r = 2\sin\theta\tan\theta$ to obtain

$$r = 2\frac{y}{r} \cdot \frac{y}{x}.$$

Multiplying by rx, setting $r^2 = x^2 + y^2$ and simplifying, yields

$$r^2 x = 2y^2$$
$$(x^2 + y^2)x = 2y^2$$

$$x^3 + y^2x = 2y^2$$
$$y^2(2 - x) = x^3$$

so

$$y^2 = \frac{x^3}{2 - x}$$

34. Prove that $r = 2a\cos\theta$ is the equation of the circle in Figure 21 using only the fact that a triangle inscribed in a circle with one side a diameter is a right triangle.

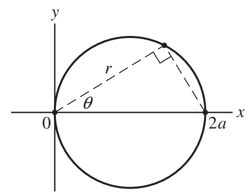


FIGURE 21

SOLUTION Since the triangle inscribed in the circle has a diameter as one of its sides, it is a right triangle, so we may use the definition of cosine for angles in right triangles to write

$$\cos\theta = \frac{r}{2a} \Rightarrow r = 2a\cos\theta.$$

35. Show that

$$r = a\cos\theta + b\sin\theta$$

is the equation of a circle passing through the origin. Express the radius and center (in rectangular coordinates) in terms of a and b.

SOLUTION We multiply the equation by r and then make the substitution $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$. This gives

$$r^2 = ar\cos\theta + br\sin\theta$$
$$x^2 + y^2 = ax + by$$

Transferring sides and completing the square yields

$$x^2 - ax + y^2 - by = 0$$
$$\left(x^2 - 2\cdot\frac{a}{2}x + \left(\frac{a}{2}\right)^2\right) + \left(y^2 - 2\cdot\frac{b}{2}y + \left(\frac{b}{2}\right)^2\right) = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2$$
$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}$$

This is the equation of the circle with radius $\frac{\sqrt{a^2+b^2}}{2}$ centered at the point $\left(\frac{a}{2}, \frac{b}{2}\right)$. By plugging in $x = 0$ and $y = 0$ it is clear that the circle passes through the origin.

36. Use the previous exercise to write the equation of the circle of radius 5 and center $(3, 4)$ in the form $r = a\cos\theta + b\sin\theta$.

SOLUTION In the previous exercise we showed that $r = a\cos\theta + b\sin\theta$ is the equation of the circle with radius $\frac{\sqrt{a^2+b^2}}{2}$ centered at $\left(\frac{a}{2}, \frac{b}{2}\right)$. Thus, we must have

$$\left(\frac{a}{2}, \frac{b}{2}\right) = (3, 4) \Rightarrow \frac{a}{2} = 3, \frac{b}{2} = 4 \Rightarrow a = 6, b = 8.$$

The radius of the circle is $\frac{\sqrt{a^2+b^2}}{2} = \frac{\sqrt{6^2+8^2}}{2} = 5$. Thus, the corresponding equation is $r = 6\cos\theta + 8\sin\theta$.

37. Use the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ to find a polar equation of the hyperbola $x^2 - y^2 = 1$.

SOLUTION We substitute $x = r\cos\theta$, $y = r\sin\theta$ in $x^2 - y^2 = 1$ to obtain

$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$
$$r^2(\cos^2\theta - \sin^2\theta) = 1$$

Using the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ we obtain the following equation of the hyperbola:

$$r^2\cos 2\theta = 1 \quad \text{or} \quad r^2 = \sec 2\theta.$$

38. Find an equation in rectangular coordinates for the curve $r^2 = \cos 2\theta$.

SOLUTION We first use the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ to rewrite the equation of the curve as follows: $r^2 = \cos^2\theta - \sin^2\theta$. Multiplying by r^2 and substituting $r^2 = x^2 + y^2$, $r\cos\theta = x$ and $r\sin\theta = y$, we get

$$r^4 = (r\cos\theta)^2 - (r\sin\theta)^2(x^2 + y^2)^2 = x^2 - y^2.$$

Thus, the curve has the equation $(x^2 + y^2)^2 = x^2 - y^2$ in rectangular coordinates.

39. Show that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and use this identity to find an equation in rectangular coordinates for the curve $r = \cos 3\theta$.

SOLUTION We use the identities $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$, and $\sin 2\alpha = 2\sin\alpha\cos\alpha$ to write

$$\begin{aligned}
\cos 3\theta = \cos(2\theta + \theta) &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\
&= (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta \\
&= \cos^3\theta - 3\sin^2\theta\cos\theta
\end{aligned}$$

Using this identity we may rewrite the equation $r = \cos 3\theta$ as follows:

$$r = \cos^3\theta - 3\sin^2\theta\cos\theta \tag{1}$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, we have $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$. Substituting into (1) gives:

$$r = \left(\frac{x}{r}\right)^3 - 3\left(\frac{y}{r}\right)^2\left(\frac{x}{r}\right)$$
$$r = \frac{x^3}{r^3} - \frac{3y^2x}{r^3}$$

We now multiply by r^3 and make the substitution $r^2 = x^2 + y^2$ to obtain the following equation for the curve:

$$r^4 = x^3 - 3y^2x$$
$$(x^2 + y^2)^2 = x^3 - 3y^2x$$

40. Use the addition formula for the cosine to show that the line $\mathcal{L}$ with polar equation $r\cos(\theta - \alpha) = d$ has the equation in rectangular coordinates $(\cos\alpha)x + (\sin\alpha)y = d$. Show that $\mathcal{L}$ has slope $m = -\cot\alpha$ and y-intercept $d/\sin\alpha$.

SOLUTION We use the identity $\cos(a - b) = \cos a\cos b + \sin a\sin b$ to rewrite the equation $r\cos(\theta - \alpha) = d$ as follows:

$$r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = d$$
$$r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = d$$

We now substitute $r\cos\theta = x$ and $r\sin\theta = y$ to obtain: $x\cos\alpha + y\sin\alpha = d$. Dividing by $\cos\alpha$, transferring sides and simplifying yields

$$\begin{aligned}
x + y\tan\alpha &= \frac{d}{\cos\alpha} \\
y\tan\alpha &= -x + \frac{d}{\cos\alpha} \\
y &= -\frac{x}{\tan\alpha} + \frac{d}{\tan\alpha\cos\alpha}
\end{aligned}$$

so

$$y = (-\cot\alpha)\, x + \frac{d}{\sin\alpha}$$

This equation of the line implies that $\mathcal{L}$ has slope $m = -\cot\alpha$ and y -intercept $\frac{d}{\sin\alpha}$.

In Exercises 41–44, find an equation in polar coordinates of the line $\mathcal{L}$ with the given description.

41. The point on $\mathcal{L}$ closest to the origin has polar coordinates $\left(2, \frac{\pi}{9}\right)$.

SOLUTION In Example 5, it is shown that the polar equation of the line where (r, α) is the point on the line closest to the origin is $r = d\sec(\theta - \alpha)$. Setting $(d, \alpha) = \left(2, \frac{\pi}{9}\right)$ we obtain the following equation of the line:

$$r = 2\sec\left(\theta - \frac{\pi}{9}\right).$$

42. The point on $\mathcal{L}$ closest to the origin has rectangular coordinates $(-2, 2)$.

SOLUTION We first convert the rectangular coordinates $(-2, 2)$ to polar coordinates (d, α). This point is in the second quadrant so $\frac{\pi}{2} < \alpha < \pi$. Hence,

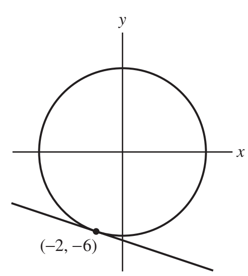
$$\begin{aligned} d &= \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2} \\ \alpha &= \tan^{-1}\left(\frac{2}{-2}\right) = \tan^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \end{aligned} \quad \Rightarrow \quad (d, \alpha) = \left(2\sqrt{2}, \frac{3\pi}{4}\right).$$

Substituting $d = 2\sqrt{2}$ and $\alpha = \frac{3\pi}{4}$ in the equation $r = d\sec(\theta - \alpha)$ gives us

$$r = 2\sqrt{2}\sec\left(\theta - \frac{3\pi}{4}\right).$$

43. $\mathcal{L}$ is tangent to the circle $r = 2\sqrt{10}$ at the point with rectangular coordinates $(-2, -6)$.

SOLUTION



Since $\mathcal{L}$ is tangent to the circle at the point $(-2, -6)$, this is the point on $\mathcal{L}$ closest to the center of the circle which is at the origin. Therefore, we may use the polar coordinates (d, α) of this point in the equation of the line:

$$r = d\sec(\theta - \alpha) \tag{1}$$

We thus must convert the coordinates $(-2, -6)$ to polar coordinates. This point is in the third quadrant so $\pi < \alpha < \frac{3\pi}{2}$. We get

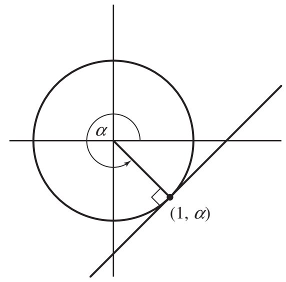
$$\begin{aligned} d &= \sqrt{(-2)^2 + (-6)^2} = \sqrt{40} = 2\sqrt{10} \\ \alpha &= \tan^{-1}\left(\frac{-6}{-2}\right) = \tan^{-1} 3 \approx \pi + 1.25 \approx 4.39 \end{aligned}$$

Substituting in (1) yields the following equation of the line:

$$r = 2\sqrt{10}\sec(\theta - 4.39).$$

44. $\mathcal{L}$ has slope 3 and is tangent to the unit circle in the fourth quadrant.

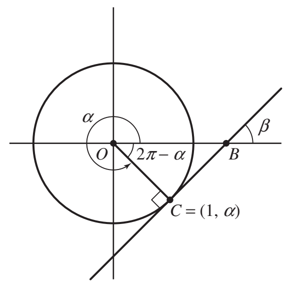
SOLUTION We denote the point of tangency by $P_0 = (1, \alpha)$ (in polar coordinates).



Since $\mathcal{L}$ is the tangent line to the circle at P_0, P_0 is the point on $\mathcal{L}$ closest to the center of the circle at the origin. Thus, the polar equation of $\mathcal{L}$ is

$$r = \sec(\theta - \alpha) \tag{1}$$

We now must find α. Let β be the given angle shown in the figure.



By the given information, $\tan\beta = 3$. Also, since the point of tangency is in the fourth quadrant, β must be an acute angle. Hence

$$\tan\beta = 3,\ 0 < \beta < \frac{\pi}{2} \Rightarrow \beta = 1.25 \text{ rad.}$$

Now, since $\frac{3\pi}{2} < \alpha < 2\pi$, we have for the triangle OBC

$$(2\pi - \alpha) + \frac{\pi}{2} + 1.25 = \pi \Rightarrow \alpha = \frac{3\pi}{2} + 1.25 = 5.96 \text{ rad.}$$

Substituting into (1) we obtain the following polar equation of the tangent line:

$$r = \sec(\theta - 5.96).$$

45. Show that every line that does not pass through the origin has a polar equation of the form

$$r = \frac{b}{\sin\theta - a\cos\theta}$$

where $b \neq 0$.

SOLUTION Write the equation of the line in rectangular coordinates as $y = ax + b$. Since the line does not pass through the origin, we have $b \neq 0$. Substitute for y and x to convert to polar coordinates, and simplify:

$$\begin{aligned} y &= ax + b \\ r\sin\theta &= ar\cos\theta + b \\ r(\sin\theta - a\cos\theta) &= b \\ r &= \frac{b}{\sin\theta - a\cos\theta} \end{aligned}$$

46. By the Law of Cosines, the distance d between two points (Figure 22) with polar coordinates (r, θ) and (r_0, θ_0) is

$$d^2 = r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)$$

Use this distance formula to show that

$$r^2 - 10r\cos\left(\theta - \frac{\pi}{4}\right) = 56$$

is the equation of the circle of radius 9 whose center has polar coordinates $\left(5, \frac{\pi}{4}\right)$.

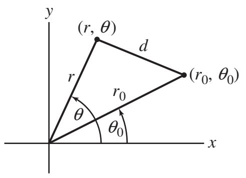


FIGURE 22

SOLUTION The distance d between a point (r, θ) on the circle and the center $(r_0, \theta_0) = \left(5, \frac{\pi}{4}\right)$ is the radius 9. Setting $d = 9$, $r_0 = 5$ and $\theta_0 = \frac{\pi}{4}$ in the distance formula we get

$$d^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)$$
$$9^2 = r^2 + 5^2 - 2 \cdot r \cdot 5 \cos\left(\theta - \frac{\pi}{4}\right)$$

Transferring sides we get

$$r^2 - 10r\cos\left(\theta - \frac{\pi}{4}\right) = 56.$$

47. For $a > 0$, a **lemniscate curve** is the set of points P such that the product of the distances from P to $(a, 0)$ and $(-a, 0)$ is a^2. Show that the equation of the lemniscate is

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

Then find the equation in polar coordinates. To obtain the simplest form of the equation, use the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$. Plot the lemniscate for $a = 2$ if you have a computer algebra system.

SOLUTION We compute the distances d_1 and d_2 of $P(x, y)$ from the points $(a, 0)$ and $(-a, 0)$ respectively. We obtain:

$$d_1 = \sqrt{(x-a)^2 + (y-0)^2} = \sqrt{(x-a)^2 + y^2}$$
$$d_2 = \sqrt{(x+a)^2 + (y-0)^2} = \sqrt{(x+a)^2 + y^2}$$

For the points $P(x, y)$ on the lemniscate we have $d_1 d_2 = a^2$. That is,

$$\begin{aligned}
a^2 &= \sqrt{(x-a)^2 + y^2}\sqrt{(x+a)^2 + y^2} = \sqrt{\left[(x-a)^2 + y^2\right]\left[(x+a)^2 + y^2\right]} \\
&= \sqrt{(x-a)^2(x+a)^2 + y^2(x-a)^2 + y^2(x+a)^2 + y^4} \\
&= \sqrt{(x^2-a^2)^2 + y^2\left[(x-a)^2 + (x+a)^2\right] + y^4} \\
&= \sqrt{x^4 - 2a^2x^2 + a^4 + y^2\left(x^2 - 2xa + a^2 + x^2 + 2xa + a^2\right) + y^4} \\
&= \sqrt{x^4 - 2a^2x^2 + a^4 + 2y^2x^2 + 2y^2a^2 + y^4} \\
&= \sqrt{x^4 + 2x^2y^2 + y^4 + 2a^2(y^2 - x^2) + a^4} \\
&= \sqrt{\left(x^2 + y^2\right)^2 + 2a^2(y^2 - x^2) + a^4}.
\end{aligned}$$

Squaring both sides and simplifying yields

$$a^4 = (x^2 + y^2)^2 + 2a^2(y^2 - x^2) + a^4$$
$$0 = (x^2 + y^2)^2 + 2a^2(y^2 - x^2)$$

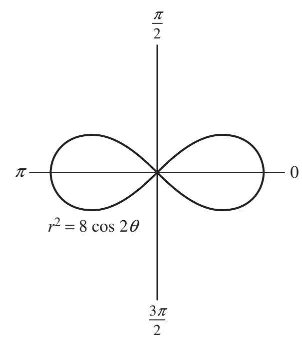
so

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

We now find the equation in polar coordinates. We substitute $x = r\cos\theta$, $y = r\sin\theta$ and $x^2 + y^2 = r^2$ into the equation of the lemniscate. This gives

$$\begin{aligned}
(r^2)^2 &= 2a^2(r^2\cos^2\theta - r^2\sin^2\theta) = 2a^2r^2(\cos^2\theta - \sin^2\theta) = 2a^2r^2\cos 2\theta \\
r^4 &= 2a^2r^2\cos 2\theta
\end{aligned}$$

$r = 0$ is a solution, hence the origin is on the curve. For $r \neq 0$ we divide the equation by r^2 to obtain $r^2 = 2a^2 \cos 2\theta$. This curve also includes the origin ($r = 0$ is obtained for $\theta = \frac{\pi}{4}$ for example), hence this is the polar equation of the lemniscate. Setting $a = 2$ we get $r^2 = 8 \cos 2\theta$.



48. Let c be a fixed constant. Explain the relationship between the graphs of:

(a) $y = f(x + c)$ and $y = f(x)$ (rectangular)

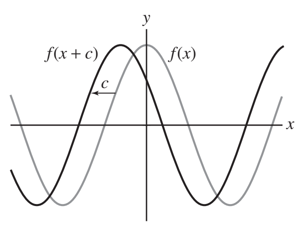
(b) $r = f(\theta + c)$ and $r = f(\theta)$ (polar)

(c) $y = f(x) + c$ and $y = f(x)$ (rectangular)

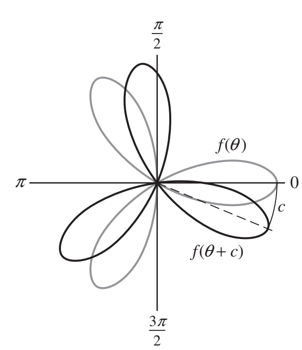
(d) $r = f(\theta) + c$ and $r = f(\theta)$ (polar)

SOLUTION

(a) For $c > 0$, $y = f(x + c)$ shifts the graph of $y = f(x)$ by c units to the left. If $c < 0$, the result is a shift to the right. It is a horizontal translation.

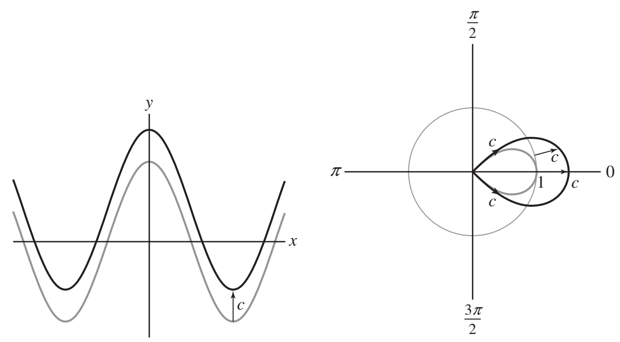


(b) As in part (a), the graph of $r = f(\theta + c)$ is a shift of the graph of $r = f(\theta)$ by c units in θ. Thus, the graph in polar coordinates is rotated by angle c as shown in the following figure:



(c) $y = f(x) + c$ shifts the graph vertically upward by c units if $c > 0$, and downward by $(-c)$ units if $c < 0$. It is a vertical translation.

(d) The graph of $r = f(\theta) + c$ is a shift of the graph of $r = f(\theta)$ by c units in r. In the corresponding graph, in polar coordinates, each point with $f(\theta) > 0$ moves on the ray connecting it to the origin c units away from the origin if $c > 0$ and $(-c)$ units toward the origin if $c < 0$, and vice-versa for $f(\theta) < 0$.



$c > 0$

49. The Derivative in Polar Coordinates Show that a polar curve $r = f(\theta)$ has parametric equations

$$x = f(\theta)\cos\theta, \qquad y = f(\theta)\sin\theta$$

Then apply Theorem 2 of Section 11.1 to prove

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} \qquad \boxed{2}$$

where $f'(\theta) = df/d\theta$.

SOLUTION Multiplying both sides of the given equation by $\cos\theta$ yields $r\cos\theta = f(\theta)\cos\theta$; multiplying both sides by $\sin\theta$ yields $r\sin\theta = f(\theta)\sin\theta$. The left-hand sides of these two equations are the x and y coordinates in rectangular coordinates, so for any θ we have $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$, showing that the parametric equations are as claimed. Now, by the formula for the derivative we have

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} \qquad (1)$$

We differentiate the functions $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$ using the Product Rule for differentiation. This gives

$$y'(\theta) = f'(\theta)\sin\theta + f(\theta)\cos\theta$$
$$x'(\theta) = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

Substituting in (1) gives

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

50. Use Eq. (2) to find the slope of the tangent line to $r = \sin\theta$ at $\theta = \frac{\pi}{3}$.

SOLUTION We have $f(\theta) = \sin\theta$, $f'(\theta) = \cos\theta$ and, by Eq. (2), the slope of the tangent line is

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} = \frac{\sin\theta\cos\theta + \cos\theta\sin\theta}{-\sin^2\theta + \cos^2\theta} = \frac{\sin 2\theta}{\cos 2\theta}$$

Evaluating at $\theta = \frac{\pi}{3}$ gives

$$\frac{dy}{dx} = \frac{\sin\frac{2\pi}{3}}{\cos\frac{2\pi}{3}} = \frac{\sqrt{3}2}{-1/2} = -\sqrt{3}$$

Thus the slope of the tangent line to $r = \sin\theta$ at $\theta = \frac{\pi}{3}$ is $-\sqrt{3}$.

51. Use Eq. (2) to find the slope of the tangent line to $r = \theta$ at $\theta = \frac{\pi}{2}$ and $\theta = \pi$.

SOLUTION In the given curve we have $r = f(\theta) = \theta$. Using Eq. (2) we obtain the following derivative, which is the slope of the tangent line at (r, θ).

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} = \frac{\theta\cos\theta + 1\cdot\sin\theta}{-\theta\sin\theta + 1\cdot\cos\theta} \qquad (1)$$

The slope, m, of the tangent line at $\theta = \frac{\pi}{2}$ and $\theta = \pi$ is obtained by substituting these values in (1). We get ($\theta = \frac{\pi}{2}$):

$$m = \frac{\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}}{-\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}} = \frac{\frac{\pi}{2}\cdot 0 + 1}{-\frac{\pi}{2}\cdot 1 + 0} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}.$$

$(\theta = \pi)$:

$$m = \frac{\pi\cos\pi + \sin\pi}{-\pi\sin\pi + \cos\pi} = \frac{-\pi}{-1} = \pi.$$

52. Find the equation in rectangular coordinates of the tangent line to $r = 4\cos 3\theta$ at $\theta = \frac{\pi}{6}$.

SOLUTION We have $f(\theta) = 4\cos 3\theta$. By Eq. (2),

$$m = \frac{4\cos 3\theta\cos\theta - 12\sin 3\theta\sin\theta}{-4\cos 3\theta\sin\theta - 12\sin 3\theta\cos\theta}.$$

Setting $\theta = \frac{\pi}{6}$ yields

$$m = \frac{4\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right) - 12\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right)}{-4\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right) - 12\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right)} = \frac{-12\sin\frac{\pi}{6}}{-12\cos\frac{\pi}{6}} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

We identify the point of tangency. For $\theta = \frac{\pi}{6}$ we have $r = 4\cos\frac{3\pi}{6} = 4\cos\frac{\pi}{2} = 0$. The point of tangency is the origin. The tangent line is the line through the origin with slope $\frac{1}{\sqrt{3}}$. This is the line $y = \frac{x}{\sqrt{3}}$.

53. Find the polar coordinates of the points on the lemniscate $r^2 = \cos 2t$ in Figure 23 where the tangent line is horizontal.

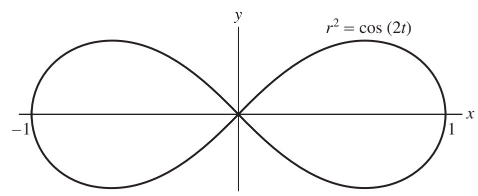


FIGURE 23

SOLUTION This curve is defined for $-\frac{\pi}{2} \le 2t \le \frac{\pi}{2}$ (where $\cos 2t \ge 0$), so for $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$. For each θ in that range, there are two values of r satisfying the equation ($\pm\sqrt{\cos 2t}$). By symmetry, we need only calculate the coordinates of the points corresponding to the positive square root (i.e. to the right of the y axis). Then the equation becomes $r = \sqrt{\cos 2t}$. Now, by Eq. (2), with $f(t) = \sqrt{\cos(2t)}$ and $f'(t) = -\sin(2t)(\cos(2t))^{-1/2}$, we have

$$\frac{dy}{dx} = \frac{f(t)\cos t + f'(t)\sin t}{-f(t)\sin t + f'(t)\cos t} = \frac{\cos t\sqrt{\cos(2t)} - \sin(2t)\sin t(\cos(2t))^{-1/2}}{-\sin t\sqrt{\cos(2t)} - \sin(2t)\cos t(\cos(2t))^{-1/2}}$$

The tangent line is horizontal when this derivative is zero, which occurs when the numerator of the fraction is zero and the denominator is not. Multiply top and bottom of the fraction by $\sqrt{\cos(2t)}$, and use the identities $\cos 2t = \cos^2 t - \sin^2 t$, $\sin 2t = 2\sin t\cos t$ to get

$$-\frac{\cos t\cos 2t - \sin t\sin 2t}{\sin t\cos 2t + \cos t\sin 2t} = -\frac{\cos t(\cos^2 t - 3\sin^2 t)}{\sin t\cos 2t + \cos t\sin 2t}$$

The numerator is zero when $\cos t = 0$, so when $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$, or when $\tan t = \pm\frac{1}{\sqrt{3}}$, so when $t = \pm\frac{\pi}{6}$ or $t = \pm\frac{5\pi}{6}$. Of these possibilities, only $t = \pm\frac{\pi}{6}$ lie in the range $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$. Note that the denominator is nonzero for $t = \pm\frac{\pi}{6}$, so these are the two values of t for which the tangent line is horizontal. The corresponding values of r are solutions to

$$r^2 = \cos\left(2\cdot\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$
$$r^2 = \cos\left(2\cdot\frac{-\pi}{6}\right) = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

Finally, the four points are $(r, t) =$

$$\left(\frac{1}{\sqrt{2}}, \frac{\pi}{6}\right), \qquad \left(-\frac{1}{\sqrt{2}}, \frac{\pi}{6}\right), \qquad \left(\frac{1}{\sqrt{2}}, -\frac{pi}{6}\right), \qquad \left(-\frac{1}{\sqrt{2}}, -\frac{\pi}{6}\right)$$

If desired, we can change the second and fourth points by adding π to the angle and making r positive, to get

$$\left(\frac{1}{\sqrt{2}}, \frac{\pi}{6}\right), \qquad \left(\frac{1}{\sqrt{2}}, \frac{7\pi}{6}\right), \qquad \left(\frac{1}{\sqrt{2}}, -\frac{pi}{6}\right), \qquad \left(\frac{1}{\sqrt{2}}, \frac{5\pi}{6}\right)$$

54. Find the polar coordinates of the points on the cardioid $r = 1 + \cos\theta$ where the tangent line is horizontal (see Figure 24).

SOLUTION Use Eq. (2) with $f(\theta) = 1 + \cos\theta$ and $f'(\theta) = -\sin\theta$. Then

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - \cos\theta\sin\theta - \sin\theta\cos\theta} = -\frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$$

The tangent line is horizontal when the numerator is zero but the denominator is not. The numerator is zero when $\cos\theta + \cos 2\theta = 0$. But

$$\cos\theta + \cos 2\theta = \cos\theta + 2\cos^2\theta - 1 = \left(\cos\theta - \frac{1}{2}\right)(\cos\theta + 1)$$

So for $0 \le \theta < 2\pi$, the numerator is zero when $\theta = \pi$ and when $\theta = \pm\frac{\pi}{3}$. For the latter two points, the denominator is nonzero, so the tangent is horizontal at the points

$$(r, \theta) = \left(\frac{3}{2}, \frac{\pi}{3}\right), \qquad \left(\frac{3}{2}, -\frac{\pi}{3}\right) = \left(\frac{3}{2}, \frac{5\pi}{3}\right)$$

When $\theta = \pi$, both numerator and denominator vanish. However, using L'Hôpital's Rule, we have

$$-\lim_{\theta\to\pi} \frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta} = -\lim_{\theta\to\pi} \frac{-\sin\theta - 2\sin 2\theta}{\cos\theta + 2\cos 2\theta} = 0$$

so that the tangent is defined at $\theta = \pi$, and it is horizontal. Thus the tangent is also horizontal at the point

$$(r, \theta) = (0, \pi)$$

55. Use Eq. (2) to show that for $r = \sin\theta + \cos\theta$,

$$\frac{dy}{dx} = \frac{\cos 2\theta + \sin 2\theta}{\cos 2\theta - \sin 2\theta}$$

Then calculate the slopes of the tangent lines at points A, B, C in Figure 19.

SOLUTION In Exercise 49 we proved that for a polar curve $r = f(\theta)$ the following formula holds:

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} \tag{1}$$

For the given circle we have $r = f(\theta) = \sin\theta + \cos\theta$, hence $f'(\theta) = \cos\theta - \sin\theta$. Substituting in (1) we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{(\sin\theta + \cos\theta)\cos\theta + (\cos\theta - \sin\theta)\sin\theta}{-(\sin\theta + \cos\theta)\sin\theta + (\cos\theta - \sin\theta)\cos\theta} = \frac{\sin\theta\cos\theta + \cos^2\theta + \cos\theta\sin\theta - \sin^2\theta}{-\sin^2\theta - \cos\theta\sin\theta + \cos^2\theta - \sin\theta\cos\theta} \\ &= \frac{\cos^2\theta - \sin^2\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta}\end{aligned}$$

We use the identities $\cos^2\theta - \sin^2\theta = \cos 2\theta$ and $2\sin\theta\cos\theta = \sin 2\theta$ to obtain

$$\frac{dy}{dx} = \frac{\cos 2\theta + \sin 2\theta}{\cos 2\theta - \sin 2\theta} \tag{2}$$

The derivative $\frac{dy}{dx}$ is the slope of the tangent line at (r, θ). The slopes of the tangent lines at the points with polar coordinates $A = \left(1, \frac{\pi}{2}\right)$ $B = \left(0, \frac{3\pi}{4}\right)$ $C = (1, 0)$ are computed by substituting the values of θ in (2). This gives

$$\begin{aligned}\left.\frac{dy}{dx}\right|_A &= \frac{\cos\left(2\cdot\frac{\pi}{2}\right) + \sin\left(2\cdot\frac{\pi}{2}\right)}{\cos\left(2\cdot\frac{\pi}{2}\right) - \sin\left(2\cdot\frac{\pi}{2}\right)} = \frac{\cos\pi + \sin\pi}{\cos\pi - \sin\pi} = \frac{-1+0}{-1-0} = 1 \\ \left.\frac{dy}{dx}\right|_B &= \frac{\cos\left(2\cdot\frac{3\pi}{4}\right) + \sin\left(2\cdot\frac{3\pi}{4}\right)}{\cos\left(2\cdot\frac{3\pi}{4}\right) - \sin\left(2\cdot\frac{3\pi}{4}\right)} = \frac{\cos\frac{3\pi}{2} + \sin\frac{3\pi}{2}}{\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2}} = \frac{0-1}{0+1} = -1 \\ \left.\frac{dy}{dx}\right|_C &= \frac{\cos(2\cdot 0) + \sin(2\cdot 0)}{\cos(2\cdot 0) - \sin(2\cdot 0)} = \frac{\cos 0 + \sin 0}{\cos 0 - \sin 0} = \frac{1+0}{1-0} = 1\end{aligned}$$

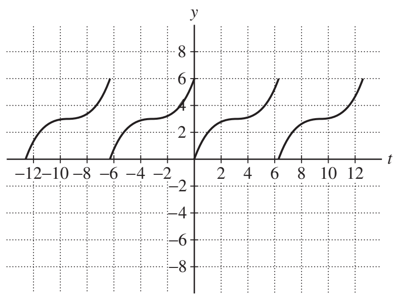
Further Insights and Challenges

56. Let $f(x)$ be a periodic function of period 2π—that is, $f(x) = f(x + 2\pi)$. Explain how this periodicity is reflected in the graph of:

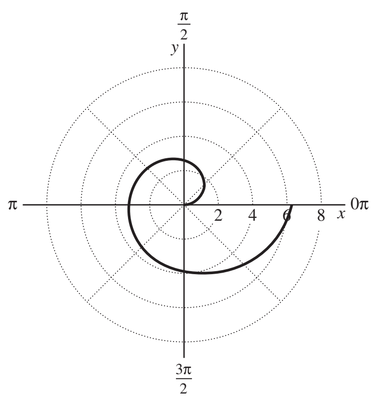
(a) $y = f(x)$ in rectangular coordinates
(b) $r = f(\theta)$ in polar coordinates

SOLUTION

(a) The graph of $y = f(x)$ on an interval of length 2π repeats itself on successive intervals of length 2π. For instance:

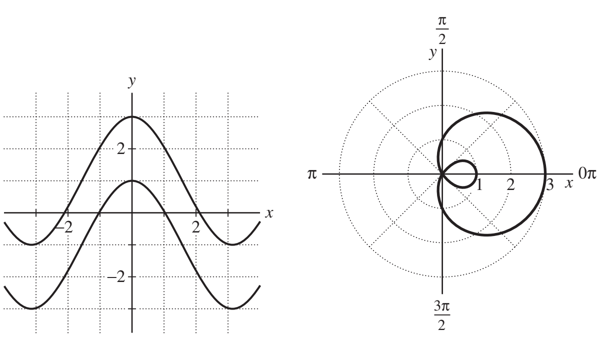


(b) Shown below is the graph of the function above, this time drawn in polar coordinates. The graphs of the various branches repeat themselves and are drawn one on the top of the other.



57. GU Use a graphing utility to convince yourself that the polar equations $r = f_1(\theta) = 2\cos\theta - 1$ and $r = f_2(\theta) = 2\cos\theta + 1$ have the same graph. Then explain why. *Hint:* Show that the points $(f_1(\theta + \pi), \theta + \pi)$ and $(f_2(\theta), \theta)$ coincide.

SOLUTION The graphs of $r = 2\cos\theta - 1$ and $r = 2\cos\theta + 1$ in the xy -plane coincide as shown in the graph obtained using a CAS.



Recall that (r, θ) and $(-r, \theta + \pi)$ represent the same point. Replacing θ by $\theta + \pi$ and r by $(-r)$ in $r = 2\cos\theta - 1$ we obtain

$$-r = 2\cos(\theta + \pi) - 1$$
$$-r = -2\cos\theta - 1$$
$$r = 2\cos\theta + 1$$

Thus, the two equations define the same graph. (One could also convert both equations to rectangular coordinates and note that they come out identical.)

58. CAS We investigate how the shape of the limaçon curve $r = b + \cos\theta$ depends on the constant b (see Figure 24).

(a) Argue as in Exercise 57 to show that the constants b and $-b$ yield the same curve.

(b) Plot the limaçon for $b = 0, 0.2, 0.5, 0.8, 1$ and describe how the curve changes.

(c) Plot the limaçon for $b = 1.2, 1.5, 1.8, 2, 2.4$ and describe how the curve changes.

(d) Use Eq. (2) to show that

$$\frac{dy}{dx} = -\left(\frac{b\cos\theta + \cos 2\theta}{b + 2\cos\theta}\right)\csc\theta$$

(e) Find the points where the tangent line is vertical. Note that there are three cases: $0 \le b < 2$, $b = 1$, and $b > 2$. Do the plots constructed in (b) and (c) reflect your results?

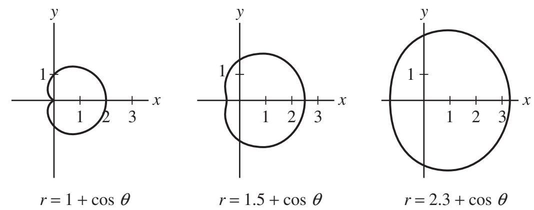


FIGURE 24

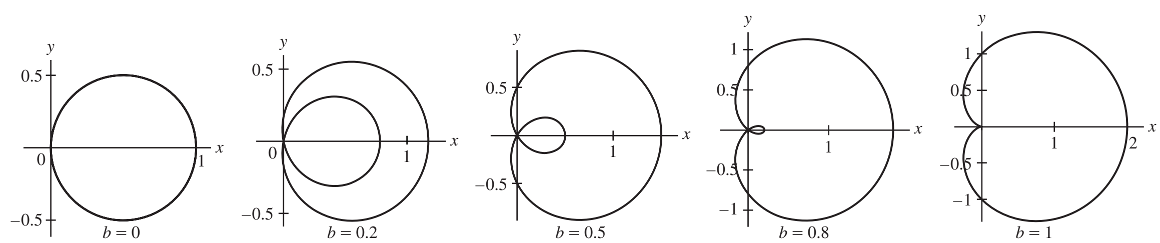
SOLUTION

(a) If (r, θ) is on the curve $r = -b + \cos\theta$, then so is $(-r, \theta + \pi)$ since they represent the same point. Thus

$$-r = -b + \cos(\theta + \pi)$$
$$-r = -b - \cos\theta$$
$$r = b + \cos\theta$$

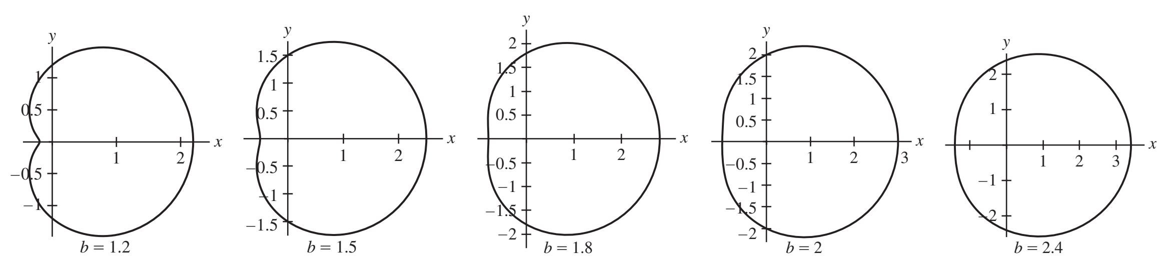
Thus the same set of points lie on the graph of both equations, so they define the same curve.

(b)



For $0 < b < 1$, there is a "loop" inside the curve. For $b = 0$, the curve is a circle, although actually for $0 \le \theta \le 2\pi$ the circle is traversed twice, so in fact the loop is as large as the circle and overlays it. When $b = 1$, the loop is pinched to a point.

(c)



For b between 1 and 2, the pinch at $b = 1$ smooths out into a concavity in the curve, which decreases in size. By $b = 2$ it appears to be gone; further increases in b push the left-hand section of the curve out, making it more convex.

(d) By Eq. (2), with $f(\theta) = b + \cos\theta$ and $f'(\theta) = -\sin\theta$, we have (using the double-angle identities for sin and cos)

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} = \frac{(b+\cos\theta)\cos\theta - \sin^2\theta}{-(b+\cos\theta)\sin\theta - \sin\theta\cos\theta} = \frac{b\cos\theta + \cos 2\theta}{-b\sin\theta - 2\sin\theta\cos\theta}$$

$$= -\frac{b\cos\theta + \cos 2\theta}{\sin\theta(b + 2\cos\theta)} = -\left(\frac{b\cos\theta + \cos 2\theta}{b + 2\cos\theta}\right)\csc\theta$$

(e) From part **(d)**, the tangent line is vertical when either $\csc\theta$ is undefined or when $b + 2\cos\theta = 0$ (as long as the numerator $b\cos\theta + \cos 2\theta \neq 0$). Consider first the case when $\csc\theta$ is undefined, so that $\theta = 0$ or $\theta = \pi$. If $\theta = 0$, the numerator of the fraction is $b + 1 \neq 0$ and the denominator is $b + 2 \neq 0$, so that the tangent is vertical here.

$$\text{For any } b, \text{ the limaçon has a vertical tangent at } (b + \cos 0, 0) = (b + 1, 0)$$

If $\theta = \pi$, the numerator of the fraction is $1 - b$ and the denominator is $b + 2 \neq 0$. As long as $b \neq 1$, the numerator does not vanish and we have found a point of vertical tangency. If $b = 1$, then by L'Hôpital's Rule,

$$-\lim_{\theta\to\pi}\left(\frac{b\cos\theta + \cos 2\theta}{b + 2\cos\theta}\right)\csc\theta = -\lim_{\theta\to\pi}\left(\frac{b\cos\theta + \cos 2\theta}{(b + 2\cos\theta)\sin\theta}\right) = \lim_{\theta\to\pi}\frac{\sin t + \sin 2t}{2\cos^2 t - 2\sin^2 t + \cos t} = 0$$

so that the tangent is not vertical here. Thus

$$\text{If } b \neq 1, \text{ the limaçon has a vertical tangent at } (b + \cos\pi, \pi) = (b - 1, \pi)$$

Next consider the possibility that $b + 2\cos\theta = 0$; this happens when $\cos\theta = -\frac{b}{2}$. First assume that $0 \le b < 2$. This equation holds for two values of θ: $\cos^{-1}\left(-\frac{b}{2}\right)$ and $-\cos^{-1}\left(-\frac{b}{2}\right)$. Neither of these angles is 0 or π, so that $\csc\theta$ is defined. Additionally, the numerator is

$$b\cos\theta + \cos 2\theta = b\cos\theta + 2\cos^2\theta - 1 = -\frac{b^2}{2} + 2\cdot\frac{b^2}{4} - 1 = -1$$

so that the numerator does not vanish. Thus

$$\text{For } 0 \le b < 2, \text{ the limaçon has a vertical tangent at } \left(\frac{b}{2}, \cos^{-1}\left(-\frac{b}{2}\right)\right) \text{ and } \left(\frac{b}{2}, -\cos^{-1}\left(-\frac{b}{2}\right)\right)$$

Next assume that $b = 2$; then $\cos\theta = -1$ holds for $\theta = \pi$; we have considered that case above. Finally assume that $b > 2$; then $\cos\theta = -\frac{b}{2}$ has no solutions. Thus, in summary, vertical tangents of the limaçon occur as follows:

$$\begin{aligned}
0 \le b < 2, b \neq 1: &\quad \left(\frac{b}{2}, \cos^{-1}\left(-\frac{b}{2}\right)\right),\ \left(\frac{b}{2}, -\cos^{-1}\left(-\frac{b}{2}\right)\right),\ (b - 1, \pi),\ (b + 1, 0)\\
b = 1: &\quad \left(\frac{b}{2}, \cos^{-1}\left(-\frac{b}{2}\right)\right),\ \left(\frac{b}{2}, -\cos^{-1}\left(-\frac{b}{2}\right)\right),\ (b + 1, 0)\\
b \ge 2: &\quad (b + 1, 0),\ (b - 1, \pi)
\end{aligned}$$

These do correspond to the figures in parts **(b)** and **(c)**.

11.4 Area and Arc Length in Polar Coordinates

Preliminary Questions

1. Polar coordinates are suited to finding the area (choose one):

(a) Under a curve between $x = a$ and $x = b$.

(b) Bounded by a curve and two rays through the origin.

SOLUTION Polar coordinates are best suited to finding the area bounded by a curve and two rays through the origin. The formula for the area in polar coordinates gives the area of this region.

2. Is the formula for area in polar coordinates valid if $f(\theta)$ takes negative values?

SOLUTION The formula for the area

$$\frac{1}{2}\int_\alpha^\beta f(\theta)^2\,d\theta$$

always gives the actual (positive) area, even if $f(\theta)$ takes on negative values.

3. The horizontal line $y = 1$ has polar equation $r = \csc\theta$. Which area is represented by the integral $\frac{1}{2}\int_{\pi/6}^{\pi/2} \csc^2\theta\, d\theta$ (Figure 12)?

(a) $\square ABCD$ **(b)** $\triangle ABC$ **(c)** $\triangle ACD$

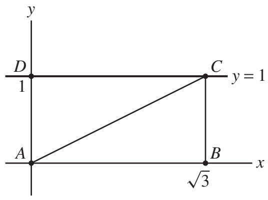


FIGURE 12

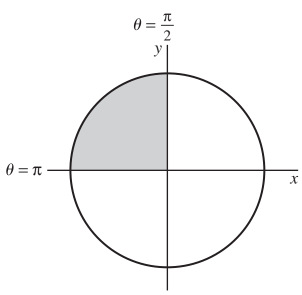
SOLUTION This integral represents an area taken from $\theta = \pi/6$ to $\theta = \pi/2$, which can only be the triangle $\triangle ACD$, as seen in part (c).

Exercises

1. Sketch the area bounded by the circle $r = 5$ and the rays $\theta = \frac{\pi}{2}$ and $\theta = \pi$, and compute its area as an integral in polar coordinates.

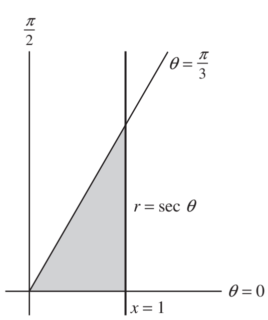
SOLUTION The region bounded by the circle $r = 5$ and the rays $\theta = \frac{\pi}{2}$ and $\theta = \pi$ is the shaded region in the figure. The area of the region is given by the following integral:

$$\frac{1}{2}\int_{\pi/2}^{\pi} r^2\, d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} 5^2\, d\theta = \frac{25}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{25\pi}{4}$$



2. Sketch the region bounded by the line $r = \sec\theta$ and the rays $\theta = 0$ and $\theta = \frac{\pi}{3}$. Compute its area in two ways: as an integral in polar coordinates and using geometry.

SOLUTION The region bounded by the line $r = \sec\theta$ and the rays $\theta = 0$ and $\theta = \frac{\pi}{3}$ is shown here:

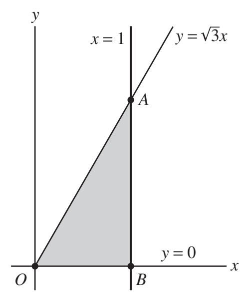


Using the area in polar coordinates, the area of the region is given by the following integral:

$$A = \frac{1}{2}\int_0^{\pi/3} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/3} \sec^2\theta\, d\theta = \frac{1}{2}\tan\theta\Big|_0^{\pi/3} = \frac{1}{2}\left(\tan\frac{\pi}{3} - \tan 0\right) = \frac{\sqrt{3}}{2}$$

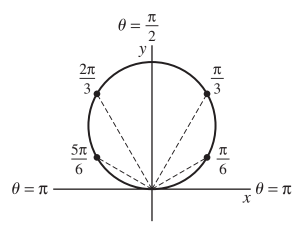
We now compute the area using the formula for the area of a triangle. The equations of the lines $\theta = \frac{\pi}{3}$, $\theta = 0$, and $r = \sec\theta$ in rectangular coordinates are $y = \sqrt{3}x$, $y = 0$ and $x = 1$ respectively (see Example 5 in Section 12.3 for the equation of the line $r = \sec\theta$). Denoting the vertices of the triangle by O, A, B (see figure) we have $O = (0, 0)$, $A = \left(1, \sqrt{3}\right)$ and $B = (1, 0)$. The area of the triangle is thus

$$A = \frac{OB \cdot AB}{2} = \frac{1 \cdot \sqrt{3}}{2} = \frac{\sqrt{3}}{2}.$$



3. Calculate the area of the circle $r = 4\sin\theta$ as an integral in polar coordinates (see Figure 4). Be careful to choose the correct limits of integration.

SOLUTION The equation $r = 4\sin\theta$ defines a circle of radius 2 tangent to the x-axis at the origin as shown in the figure:



The circle is traced as θ varies from 0 to π. We use the area in polar coordinates and the identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

to obtain the following area:

$$A = \frac{1}{2}\int_0^{\pi} r^2\,d\theta = \frac{1}{2}\int_0^{\pi}(4\sin\theta)^2\,d\theta = 8\int_0^{\pi}\sin^2\theta\,d\theta = 4\int_0^{\pi}(1 - \cos 2\theta)\,d\theta = 4\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$
$$= 4\left(\left(\pi - \frac{\sin 2\pi}{2}\right) - 0\right) = 4\pi.$$

4. Find the area of the shaded triangle in Figure 13 as an integral in polar coordinates. Then find the rectangular coordinates of P and Q and compute the area via geometry.

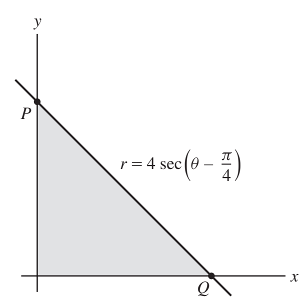


FIGURE 13

SOLUTION The boundary of the region is traced as θ varies from 0 to $\frac{\pi}{2}$, so the area is

$$\frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} 16\sec^2\left(\theta - \frac{\pi}{4}\right)d\theta = 8\tan\left(\theta - \frac{\pi}{4}\right)\Big|_0^{\pi/2} = 8(1+1) = 16$$

5. Find the area of the shaded region in Figure 14. Note that θ varies from 0 to $\frac{\pi}{2}$.

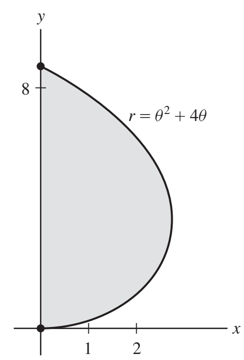


FIGURE 14

SOLUTION Since θ varies from 0 to $\frac{\pi}{2}$, the area is

$$\frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} (\theta^2+4\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} \theta^4+8\theta^3+16\theta^2\,d\theta$$
$$= \frac{1}{2}\left(\frac{1}{5}\theta^5 + 2\theta^4 + \frac{16}{3}\theta^3\right)\Big|_0^{\pi/2} = \frac{\pi^5}{320}+\frac{\pi^4}{16}+\frac{\pi^2}{3}$$

6. Which interval of θ-values corresponds to the the shaded region in Figure 15? Find the area of the region.

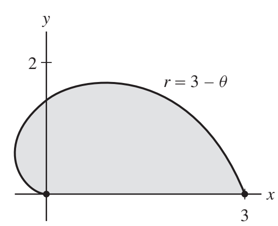
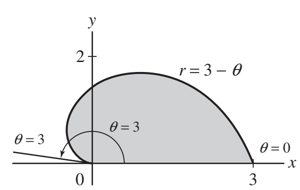


FIGURE 15

SOLUTION We first find the interval of θ. At the origin $r = 0$, so $\theta = 3$. At the endpoint on the x-axis, $\theta = 0$. Thus, θ varies from 0 to 3.



Using the area in polar coordinates we obtain

$$A = \frac{1}{2}\int_0^3 r^2\,d\theta = \frac{1}{2}\int_0^3 (3-\theta)^2\,d\theta = -\frac{(3-\theta)^3}{6}\Big|_0^3 = 4.5.$$

7. Find the total area enclosed by the cardioid in Figure 16.

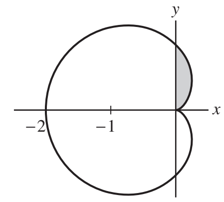
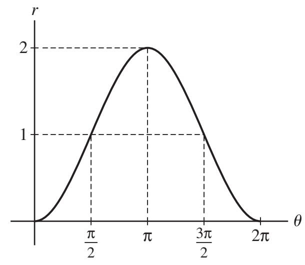


FIGURE 16 The cardioid $r = 1 - \cos\theta$.

SOLUTION We graph $r = 1 - \cos\theta$ in r and θ (cartesian, not polar, this time):



We see that as θ varies from 0 to π, the radius r increases from 0 to 2, so we get the upper half of the cardioid (the lower half is obtained as θ varies from π to 2π and consequently r decreases from 2 to 0). Since the cardioid is symmetric with respect to the x-axis we may compute the upper area and double the result. Using

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

we get

$$\begin{aligned} A &= 2\cdot\frac{1}{2}\int_0^{\pi} r^2\,d\theta = \int_0^{\pi}(1-\cos\theta)^2\,d\theta = \int_0^{\pi}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta \\ &= \int_0^{\pi}\left(1 - 2\cos\theta + \frac{\cos 2\theta + 1}{2}\right)d\theta = \int_0^{\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta \\ &= \frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\Big|_0^{\pi} = \frac{3\pi}{2} \end{aligned}$$

The total area enclosed by the cardioid is $A = \frac{3\pi}{2}$.

8. Find the area of the shaded region in Figure 16.

SOLUTION The shaded region is traced as θ varies from 0 to $\frac{\pi}{2}$. Using the formula for the area in polar coordinates we get:

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta \\ &= \frac{1}{2}\int_0^{\pi/2}\left(1 - 2\cos\theta + \frac{\cos 2\theta + 1}{2}\right)d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta \\ &= \frac{1}{2}\left(\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2} = \frac{1}{2}\left(\left(\frac{3}{2}\cdot\frac{\pi}{2} - 2\sin\frac{\pi}{2} + \frac{1}{4}\sin\pi\right) - 0\right) \\ &= \frac{1}{2}\left(\frac{3\pi}{4} - 2\right) = \frac{3\pi}{8} - 1 \approx 0.18 \end{aligned}$$

9. Find the area of one leaf of the "four-petaled rose" $r = \sin 2\theta$ (Figure 17). Then prove that the total area of the rose is equal to one-half the area of the circumscribed circle.

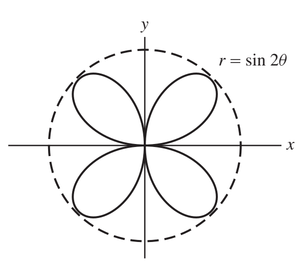
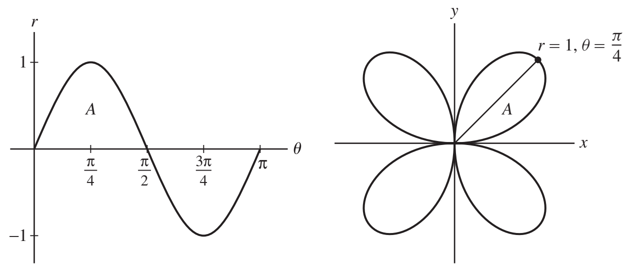


FIGURE 17 Four-petaled rose $r = \sin 2\theta$.

SOLUTION We consider the graph of $r = \sin 2\theta$ in cartesian and in polar coordinates:



We see that as θ varies from 0 to $\frac{\pi}{4}$ the radius r is increasing from 0 to 1, and when θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r is decreasing back to zero. Hence, the leaf in the first quadrant is traced as θ varies from 0 to $\frac{\pi}{2}$. The area of the leaf (the four leaves have equal areas) is thus

$$A = \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\,d\theta.$$

Using the identity

$$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$$

we get

$$A = \frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{2} - \frac{\cos 4\theta}{2}\right)d\theta = \frac{1}{2}\left(\frac{\theta}{2} - \frac{\sin 4\theta}{8}\right)\Bigg|_0^{\pi/2} = \frac{1}{2}\left(\left(\frac{\pi}{4} - \frac{\sin 2\pi}{8}\right) - 0\right) = \frac{\pi}{8}$$

The area of one leaf is $A = \frac{\pi}{8} \approx 0.39$. It follows that the area of the entire rose is $\frac{\pi}{2}$. Since the "radius" of the rose (the point where $\theta = \frac{\pi}{4}$) is 1, and the circumscribed circle is tangent there, the circumscribed circle has radius 1 and thus area π. Hence the area of the rose is half that of the circumscribed circle.

10. Find the area enclosed by one loop of the lemniscate with equation $r^2 = \cos 2\theta$ (Figure 18). Choose your limits of integration carefully.

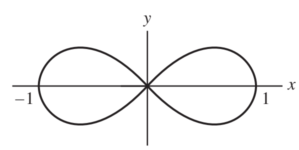
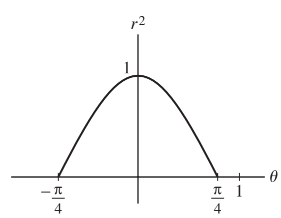
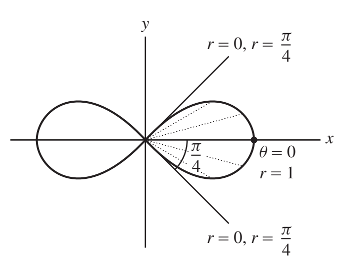


FIGURE 18 The lemniscate $r^2 = \cos 2\theta$.

SOLUTION We sketch the graph of $r^2 = \cos 2\theta$ in the $\left(r^2, \theta\right)$ plane; for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$:



We see that as θ varies from $-\frac{\pi}{4}$ to 0, r^2 increases from 0 to 1, hence r also increases from 0 to 1. Then, as θ varies from 0 to $\frac{\pi}{4}$, r^2, so r decreases from 1 to 0. This gives the right-hand loop of the lemniscate.

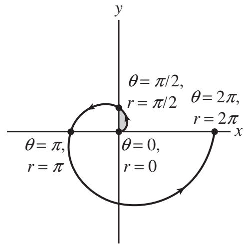


Therefore, the area enclosed by the right-hand loop is:

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2\,d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4}\cos 2\theta\,d\theta = \frac{1}{2}\frac{\sin 2\theta}{2}\bigg|_{-\pi/4}^{\pi/4} = \frac{1}{4}\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) = \frac{1}{2}$$

11. Sketch the spiral $r = \theta$ for $0 \le \theta \le 2\pi$ and find the area bounded by the curve and the first quadrant.

SOLUTION The spiral $r = \theta$ for $0 \le \theta \le 2\pi$ is shown in the following figure in the xy-plane:



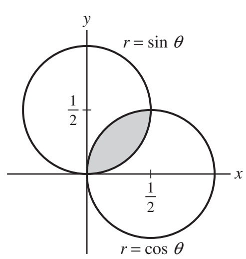
The spiral $r = \theta$

We must compute the area of the shaded region. This region is traced as θ varies from 0 to $\frac{\pi}{2}$. Using the formula for the area in polar coordinates we get

$$A = \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}\theta^2\,d\theta = \frac{1}{2}\frac{\theta^3}{3}\bigg|_0^{\pi/2} = \frac{1}{6}\left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{48}$$

12. Find the area of the intersection of the circles $r = \sin\theta$ and $r = \cos\theta$.

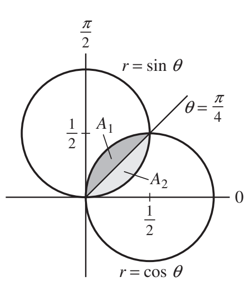
SOLUTION The region of intersection between the two circles is shown in the following figure:



We first find the value of θ at the point of intersection (besides the origin) of the two circles, by solving the following equation for $0 \le a \le \frac{\pi}{2}$:

$$\sin\theta = \cos\theta$$
$$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

We now compute the area as the sum of the two areas A_1 and A_2, shown in the figure:



Using the formula for the area in polar coordinates we get

$$A_1 = \frac{1}{2}\int_{\pi/4}^{\pi/2}\cos^2\theta\,d\theta = \frac{1}{2}\int_{\pi/2}^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{4}\int_{\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta$$
$$= \frac{1}{4}\left(\theta + \frac{\sin 2\theta}{2}\right)\bigg|_{\pi/2}^{\pi/2} = \frac{1}{4}\left(\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right) - \left(\frac{\pi}{4}+\frac{\sin\frac{\pi}{2}}{2}\right)\right) = \frac{1}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi}{16}-\frac{1}{8}$$

$$A_2 = \frac{1}{2}\int_0^{\pi/4} \sin^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/4}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta = \frac{1}{4}\int_0^{\pi/4} (1 - \cos 2\theta)\ d\theta$$
$$= \frac{1}{4}\left(\theta - \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\pi/4} = \frac{1}{4}\left(\left(\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right) - 0\right) = \frac{\pi}{16} - \frac{1}{8}$$

Notice that $A_2 = A_1$ as shown in the figure due to symmetry. The total area enclosed by the two circles is the sum

$$A = A_1 + A_2 = \left(\frac{\pi}{16} - \frac{1}{8}\right) + \left(\frac{\pi}{16} - \frac{1}{8}\right) = \frac{\pi}{8} - \frac{1}{4} \approx 0.14.$$

13. Find the area of region A in Figure 19.

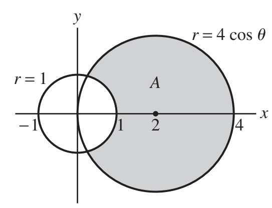
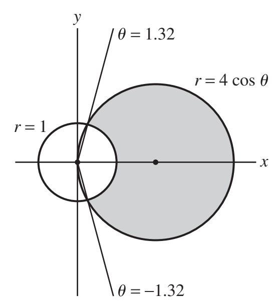


FIGURE 19

SOLUTION We first find the values of θ at the points of intersection of the two circles, by solving the following equation for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$:

$$4\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{4} \Rightarrow \theta_1 = \cos^{-1}\left(\frac{1}{4}\right)$$



We now compute the area using the formula for the area between two curves:

$$A = \frac{1}{2}\int_{-\theta_1}^{\theta_1}\left((4\cos\theta)^2 - 1^2\right) d\theta = \frac{1}{2}\int_{-\theta_1}^{\theta_1}\left(16\cos^2\theta - 1\right) d\theta$$

Using the identity $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$ we get

$$A = \frac{1}{2}\int_{-\theta_1}^{\theta_1}\left(\frac{16\,(\cos 2\theta + 1)}{2} - 1\right) d\theta = \frac{1}{2}\int_{-\theta_1}^{\theta_1} (8\cos 2\theta + 7)\ d\theta = \frac{1}{2}\,(4\sin 2\theta + 7\theta)\Bigg|_{-\theta_1}^{\theta_1}$$
$$= 4\sin 2\theta_1 + 7\theta_1 = 8\sin\theta_1\cos\theta_1 + 7\theta_1 = 8\sqrt{1 - \cos^2\theta_1}\,\cos\theta_1 + 7\theta_1$$

Using the fact that $\cos\theta_1 = \frac{1}{4}$ we get

$$A = \frac{\sqrt{15}}{2} + 7\cos^{-1}\left(\frac{1}{4}\right) \approx 11.163$$

14. Find the area of the shaded region in Figure 20, enclosed by the circle $r = \frac{1}{2}$ and a petal of the curve $r = \cos 3\theta$. *Hint:* Compute the area of both the petal and the region inside the petal and outside the circle.

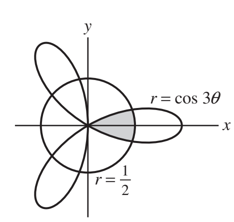
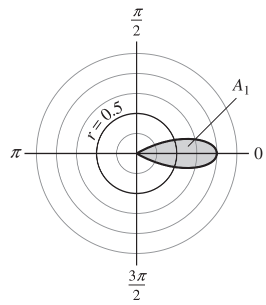
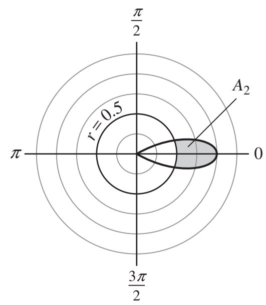


FIGURE 20

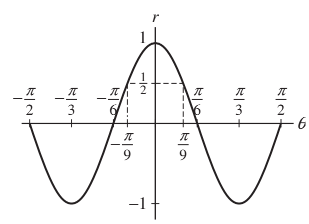
SOLUTION We compute the area A of the given region as the difference between the area A_1 of the leaf, shown here:



The area, A_2, of the region inside the leaf and outside the circle, shown here:



Computing A_1: To determine the limits of integration we use the following graph of $r = \cos 3\theta$:



$r = \cos 3\theta$

As θ varies from $-\frac{\pi}{6}$ to 0, r increases from 0 to 1. Then, as θ varies from 0 to $\frac{\pi}{6}$, r decreases from 1 back to zero. Hence the leaf is traced as θ varies from $-\frac{\pi}{6}$ to $\frac{\pi}{6}$. We use the formula for the area in polar coordinates to obtain

$$A_1 = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta \, d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left(\frac{1}{2} + \frac{1}{2}\cos 6\theta\right) d\theta = \frac{1}{4}\int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta)\, d\theta$$
$$= \frac{1}{4}\left(\theta + \frac{\sin 6\theta}{6}\right)\Bigg|_{-\pi/6}^{\pi/6} = \frac{1}{4}\left(\left(\frac{\pi}{6} + \frac{\sin \pi}{6}\right) - \left(-\frac{\pi}{6} + \frac{\sin(-\pi)}{6}\right)\right) = \frac{1}{4}\cdot\frac{2\pi}{6} = \frac{\pi}{12}$$

Computing A_2: The two curves intersect at the points where $\cos 3\theta = \frac{1}{2}$, that is, $\theta = \pm\frac{\pi}{9}$ (see the graph of $r = \cos 3\theta$ in the $r\theta$-plane). Using the formula for the area between two curves we get

$$A_2 = \frac{1}{2}\int_{-\pi/9}^{\pi/9} \left(\cos^2 3\theta - \left(\frac{1}{2}\right)^2\right) d\theta = \frac{1}{2}\int_{-\pi/9}^{\pi/9} \left(\frac{1}{2} + \frac{1}{2}\cos 6\theta - \frac{1}{4}\right) d\theta$$
$$= \frac{1}{8}\int_{-\pi/9}^{\pi/9} (1 + 2\cos 6\theta)\, d\theta = \frac{1}{8}\left(\theta + \frac{\sin 6\theta}{3}\right)\Bigg|_{-\pi/9}^{\pi/9}$$
$$= \frac{1}{8}\left(\left(\frac{\pi}{9} + \frac{\sin\frac{6\pi}{9}}{3}\right) - \left(-\frac{\pi}{9} + \frac{\sin\left(-\frac{6\pi}{9}\right)}{3}\right)\right) = \frac{1}{4}\left(\frac{\pi}{9} + \frac{\sqrt{3}}{6}\right) = \frac{\pi}{36} + \frac{\sqrt{3}}{24}$$

The required area is the difference between A_1 and A_2, that is,

$$A = A_1 - A_2 = \frac{\pi}{12} - \left(\frac{\pi}{36} + \frac{\sqrt{3}}{24}\right) = \frac{\pi}{18} - \frac{\sqrt{3}}{24} \approx 0.102.$$

15. Find the area of the inner loop of the limaçon with polar equation $r = 2\cos\theta - 1$ (Figure 21).

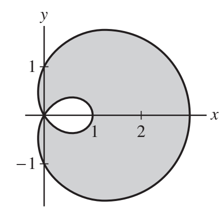
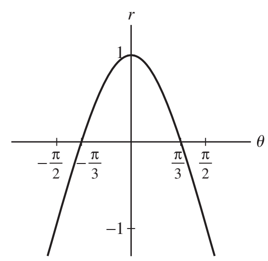
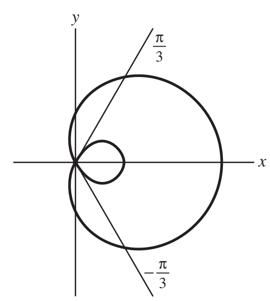


FIGURE 21 The limaçon $r = 2\cos\theta - 1$.

SOLUTION We consider the graph of $r = 2\cos\theta - 1$ in cartesian and in polar, for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$:



$r = 2\cos\theta - 1$



As θ varies from $-\frac{\pi}{3}$ to 0, r increases from 0 to 1. As θ varies from 0 to $\frac{\pi}{3}$, r decreases from 1 back to 0. Hence, the inner loop of the limaçon is traced as θ varies from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$. The area of the shaded region is thus

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} r^2\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta - 1)^2\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(4\cos^2\theta - 4\cos\theta + 1\right)\,d\theta$$

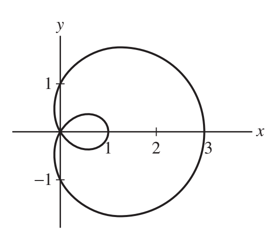
$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\,(\cos 2\theta + 1) - 4\cos\theta + 1)\;d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos 2\theta - 4\cos\theta + 3)\;d\theta$$

$$= \frac{1}{2}\,(\sin 2\theta - 4\sin\theta + 3\theta)\Big|_{-\pi/3}^{\pi/3} = \frac{1}{2}\left(\left(\sin\frac{2\pi}{3} - 4\sin\frac{\pi}{3} + \pi\right) - \left(\sin\left(-\frac{2\pi}{3}\right) - 4\sin\left(-\frac{\pi}{3}\right) - \pi\right)\right)$$

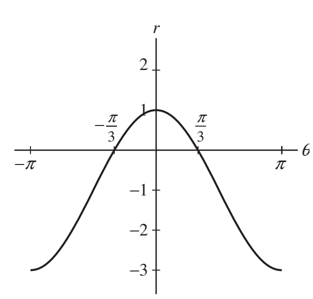
$$= \frac{\sqrt{3}}{2} - \frac{4\sqrt{3}}{2} + \pi = \pi - \frac{3\sqrt{3}}{2} \approx 0.54$$

16. Find the area of the shaded region in Figure 21 between the inner and outer loop of the limaçon $r = 2\cos\theta - 1$.

SOLUTION The region is shown in the figure below.

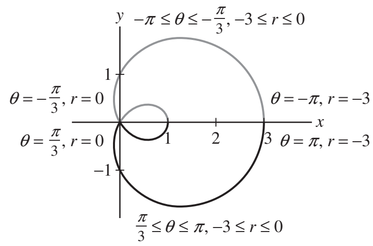


We use the following graph.



Graph of $r = 2\cos\theta - 1$

As θ varies from $\frac{\pi}{3}$ to π, r is negative and $|r|$ increases from 0 to 3. This gives the outer loop of the limaçon which is in the lower half plane. Similarly, the outer loop which is in the upper half plane is traced for $-\pi \le \theta \le -\frac{\pi}{3}$.



Using symmetry with respect to the x-axis, we obtain the following for the area of the outer loop:

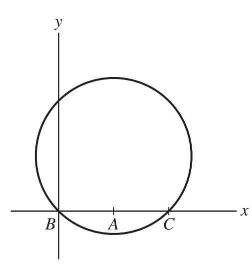
$$
\begin{aligned}
A &= 2 \cdot \frac{1}{2} \int_{\pi/3}^{\pi} r^2\, d\theta = \int_{\pi/3}^{\pi} (2\cos\theta - 1)^2\, d\theta = \int_{\pi/3}^{\pi} \left(4\cos^2\theta - 4\cos\theta + 1\right)\, d\theta \\
&= \int_{\pi/3}^{\pi} (2\,(1 + \cos 2\theta) - 4\cos\theta + 1)\ d\theta = \int_{\pi/3}^{\pi} (2\cos 2\theta - 4\cos\theta + 3)\ d\theta = \sin 2\theta - 4\sin\theta + 3\theta \Big|_{\pi/3}^{\pi} \\
&= (\sin 2\pi - 4\sin\pi + 3\pi) - \left(\sin\frac{2\pi}{3} - 4\sin\frac{\pi}{3} + \pi\right) = 3\pi - \left(\frac{\sqrt{3}}{2} - 2\sqrt{3} + \pi\right) = 2\pi + \frac{3\sqrt{3}}{2}
\end{aligned}
$$

Finally, to find the area of the region between the inner and outer loop of the limaçon, we subtract the area of the inner loop, obtained in the previous exercise, from the area of the outer loop:

$$\left(2\pi + \frac{3\sqrt{3}}{2}\right) - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}$$

17. Find the area of the part of the circle $r = \sin\theta + \cos\theta$ in the fourth quadrant (see Exercise 26 in Section 11.3).

SOLUTION The value of θ corresponding to the point B is the solution of $r = \sin\theta + \cos\theta = 0$ for $-\pi \le \theta \le \pi$.



That is,

$$\sin\theta + \cos\theta = 0 \Rightarrow \sin\theta = -\cos\theta \Rightarrow \tan\theta = -1 \Rightarrow \theta = -\frac{\pi}{4}$$

At the point C, we have $\theta = 0$. The part of the circle in the fourth quadrant is traced if θ varies between $-\frac{\pi}{4}$ and 0. This leads to the following area:

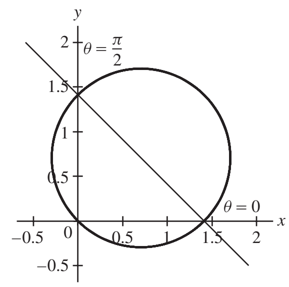
$$A = \frac{1}{2}\int_{-\pi/4}^{0} r^2\, d\theta = \frac{1}{2}\int_{-\pi/4}^{0} (\sin\theta + \cos\theta)^2\, d\theta = \frac{1}{2}\int_{-\pi/4}^{0} \left(\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\right)\, d\theta$$

Using the identities $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$ we get:

$$
\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/4}^{0} (1 + \sin 2\theta)\ d\theta = \frac{1}{2}\left(\theta - \frac{\cos 2\theta}{2}\right)\Big|_{-\pi/4}^{0} \\
&= \frac{1}{2}\left(\left(0 - \frac{1}{2}\right) - \left(-\frac{\pi}{4} - \frac{\cos\left(\frac{-\pi}{2}\right)}{2}\right)\right) = \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{8} - \frac{1}{4} \approx 0.14.
\end{aligned}
$$

18. Find the area of the region inside the circle $r = 2\sin\left(\theta + \frac{\pi}{4}\right)$ and above the line $r = \sec\left(\theta - \frac{\pi}{4}\right)$.

SOLUTION The line $r = \sec\left(\theta - \frac{\pi}{4}\right)$ intersects the circle $r = 2\sin\left(\theta + \frac{\pi}{4}\right)$ when $\theta = 0$ and $\theta = 2\pi$.



Thus the area of the region inside the circle and above the line is

$$\frac{1}{2}\int_0^{\pi/2}\left(\left(2\sin\left(\theta + \frac{\pi}{4}\right)\right)^2 - \left(\sec\left(\theta - \frac{\pi}{4}\right)\right)^2\right)d\theta = \frac{1}{2}\int_0^{\pi/2} 4\sin^2\left(\theta + \frac{\pi}{4}\right) - \sec^2\left(\theta - \frac{\pi}{4}\right)d\theta$$

$$= \frac{1}{2}\left(2t - 2\sin\left(t + \frac{\pi}{4}\right)\cos\left(t + \frac{\pi}{4}\right) - \tan\left(t - \frac{\pi}{4}\right)\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{2}\left(\pi - 2\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right) - \left(-2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right)\right)\right)$$

$$= \frac{1}{2}(\pi + 1 - 1 + 1 - 1) = \frac{\pi}{2}$$

19. Find the area between the two curves in Figure 22(A).

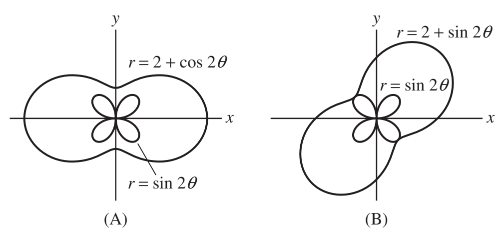
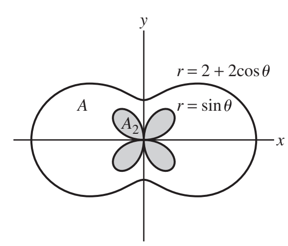


FIGURE 22

SOLUTION We compute the area A between the two curves as the difference between the area A_1 of the region enclosed in the outer curve $r = 2 + \cos 2\theta$ and the area A_2 of the region enclosed in the inner curve $r = \sin 2\theta$. That is,

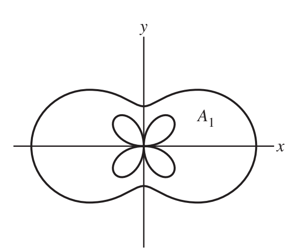
$$A = A_1 - A_2.$$



In Exercise 9 we showed that $A_2 = \frac{\pi}{2}$, hence,

$$A = A_1 - \frac{\pi}{2} \tag{1}$$

We compute the area A_1.



Using symmetry, the area is four times the area enclosed in the first quadrant. That is,

$$A_1 = 4 \cdot \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = 2\int_0^{\pi/2} (2+\cos 2\theta)^2\,d\theta = 2\int_0^{\pi/2}\left(4+4\cos 2\theta+\cos^2 2\theta\right)d\theta$$

Using the identity $\cos^2 2\theta = \frac{1}{2}\cos 4\theta + \frac{1}{2}$ we get

$$A_1 = 2\int_0^{\pi/2}\left(4+4\cos 2\theta+\frac{1}{2}\cos 4\theta+\frac{1}{2}\right)d\theta = 2\int_0^{\pi/2}\left(\frac{9}{2}+\frac{1}{2}\cos 4\theta+4\cos 2\theta\right)d\theta$$

$$= 2\left(\frac{9\theta}{2}+\frac{\sin 4\theta}{8}+2\sin 2\theta\right)\Big|_0^{\pi/2} = 2\left(\left(\frac{9\pi}{4}+\frac{\sin 2\pi}{8}+2\sin\pi\right)-0\right) = \frac{9\pi}{2} \tag{2}$$

Combining (1) and (2) we obtain

$$A = \frac{9\pi}{2} - \frac{\pi}{2} = 4\pi.$$

20. Find the area between the two curves in Figure 22(B).

SOLUTION Since

$$2+\cos 2\left(\theta-\frac{\pi}{4}\right) = 2+\cos\left(2\theta-\frac{\pi}{2}\right) = 2+\cos\left(\frac{\pi}{2}-2\theta\right) = 2+\sin 2\theta$$

it follows that the curve $r = 2+\sin 2\theta$ is obtained by rotating the curve $r = 2+\cos\theta$ by $\frac{\pi}{4}$ about the origin. Therefore the area between the curves $r = 2+\sin 2\theta$ and $r = \sin 2\theta$ is the same as the area between the curves $r = 2+\cos\theta$ and $r = \sin 2\theta$ computed in Exercise 19. That is, $A = 4\pi$. (Notice that if the inner curve remains inside the rotated curve, the area between the curves is not changed).

21. Find the area inside both curves in Figure 23.

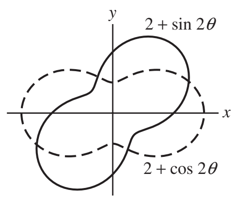
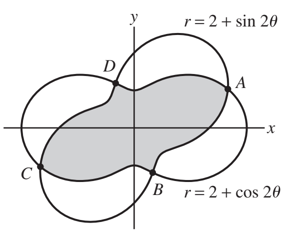


FIGURE 23

SOLUTION The area we need to find is the area of the shaded region in the figure.



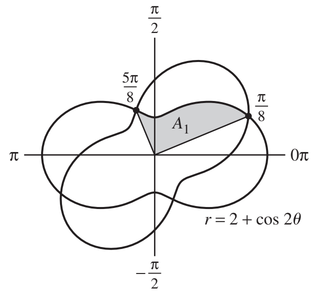
We first find the values of θ at the points of intersection A, B, C, and D of the two curves, by solving the following equation for $-\pi \le \theta \le \pi$:

$$2+\cos 2\theta = 2+\sin 2\theta$$
$$\cos 2\theta = \sin 2\theta$$
$$\tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4}+\pi k \Rightarrow \theta = \frac{\pi}{8}+\frac{\pi k}{2}$$

The solutions for $-\pi \le \theta \le \pi$ are

$$A:\quad \theta = \frac{\pi}{8}.$$
$$B:\quad \theta = -\frac{3\pi}{8}.$$
$$C:\quad \theta = -\frac{7\pi}{8}.$$
$$D:\quad \theta = \frac{5\pi}{8}.$$

Using symmetry, we compute the shaded area in the figure below and multiply it by 4:



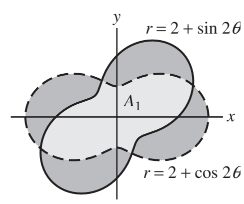
$$A = 4 \cdot A_1 = 4 \cdot \frac{1}{2} \cdot \int_{\pi/8}^{5\pi/8} (2+\cos 2\theta)^2\, d\theta = 2\int_{\pi/8}^{5\pi/8} \left(4 + 4\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= 2\int_{\pi/8}^{5\pi/8} \left(4 + 4\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta = \int_{\pi/8}^{5\pi/8} (9 + 8\cos 2\theta + \cos 4\theta)\, d\theta$$

$$= 9\theta + 4\sin 2\theta + \frac{\sin 4\theta}{4}\bigg|_{\pi/8}^{5\pi/8} = 9\left(\frac{5\pi}{8} - \frac{\pi}{8}\right) + 4\left(\sin\frac{5\pi}{4} - \sin\frac{\pi}{4}\right) + \frac{1}{4}\left(\sin\frac{5\pi}{2} - \sin\frac{\pi}{2}\right) = \frac{9\pi}{2} - 4\sqrt{2}$$

22. Find the area of the region that lies inside one but not both of the curves in Figure 23.

SOLUTION The area we need to find is the area of the shaded region in the following figure:



We denote by A_1 the area inside both curves. In Exercise 20 we showed that the curve $r = 2 + \sin 2\theta$ is obtained by rotating the curve $r = \cos 2\theta$ by $\frac{\pi}{4}$ around the origin. Hence, the areas enclosed in these curves are equal. We denote it by A_2. It follows that the area A that we need to find is

$$A = 2A_2 - 2A_1 = 2\,(A_2 - A_1) \qquad (1)$$

In Exercise 20 we found that $A_2 = \frac{9\pi}{2}$, and in Exercise 21 we showed that $A_1 = \frac{9\pi}{2} - 4\sqrt{2}$. Substituting in (1) we obtain

$$A = 2\left(\frac{9\pi}{2} - \left(\frac{9\pi}{2} - 4\sqrt{2}\right)\right) = 8\sqrt{2} \approx 11.3.$$

23. Calculate the total length of the circle $r = 4\sin\theta$ as an integral in polar coordinates.

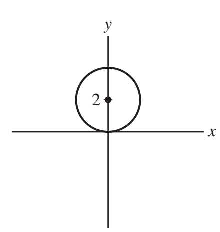
SOLUTION We use the formula for the arc length:

$$S = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta \qquad (1)$$

In this case, $f(\theta) = 4\sin\theta$ and $f'(\theta) = 4\cos\theta$, hence

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{(4\sin\theta)^2 + (4\cos\theta)^2} = \sqrt{16} = 4$$

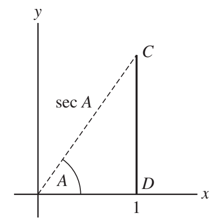
The circle is traced as θ is varied from 0 to π. Substituting $\alpha = 0$, $\beta = \pi$ in (1) yields $S = \int_0^\pi 4\, d\theta = 4\pi$.



The circle $r = 4\sin\theta$

24. Sketch the segment $r = \sec\theta$ for $0 \le \theta \le A$. Then compute its length in two ways: as an integral in polar coordinates and using trigonometry.

SOLUTION The line $r = \sec\theta$ has the rectangular equation $x = 1$. The segment AB for $0 \le \theta \le A$ is shown in the figure.



Using trigonometry, the length of the segment $\overline{AB}$ is

$$L = \overline{AB} = \overline{0B}\tan A = 1 \cdot \tan A = \tan A$$

Alternatively, we use the integral in polar coordinates with $f(\theta) = \sec(\theta)$ and $f'(\theta) = \tan\theta\sec\theta$. This gives

$$L = \int_0^A \sqrt{(\sec\theta)^2 + (\tan\theta\sec\theta)^2}\,d\theta = \int_0^A \sqrt{1+\tan^2\theta}\,\sec\theta\,d\theta = \int_0^A \sec^2\theta\,d\theta = \tan\theta\Big|_0^A = \tan A.$$

The two answers agree, as expected.

In Exercises 25–30, compute the length of the polar curve.

25. The length of $r = \theta^2$ for $0 \le \theta \le \pi$

SOLUTION We use the formula for the arc length. In this case $f(\theta) = \theta^2$, $f'(\theta) = 2\theta$, so we obtain

$$S = \int_0^\pi \sqrt{\left(\theta^2\right)^2 + (2\theta)^2}\,d\theta = \int_0^\pi \sqrt{\theta^4 + 4\theta^2}\,d\theta = \int_0^\pi \theta\sqrt{\theta^2+4}\,d\theta$$

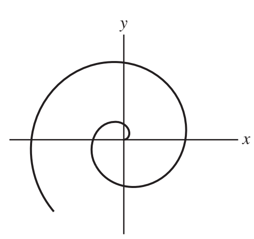
We compute the integral using the substitution $u = \theta^2 + 4$, $du = 2\theta\,d\theta$. This gives

$$S = \frac{1}{2}\int_4^{\pi^2+4} \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_4^{\pi^2+4} = \frac{1}{3}\left(\left(\pi^2+4\right)^{3/2} - 4^{3/2}\right) = \frac{1}{3}\left(\left(\pi^2+4\right)^{3/2} - 8\right) \approx 14.55$$

26. The spiral $r = \theta$ for $0 \le \theta \le A$

SOLUTION We use the formula for the arc length. In this case $f(\theta) = \theta$, $f'(\theta) = 1$. Using integration formulas we get:

$$S = \int_0^A \sqrt{\theta^2 + 1^2}\,d\theta = \int_0^A \sqrt{\theta^2+1}\,d\theta = \frac{\theta}{2}\sqrt{\theta^2+1} + \frac{1}{2}\ln\left|\theta + \sqrt{\theta^2+1}\right|\Big|_0^A$$
$$= \frac{A}{2}\sqrt{A^2+1} + \frac{1}{2}\ln\left|A + \sqrt{A^2+1}\right|$$



The spiral $r = \theta$

27. The equiangular spiral $r = e^\theta$ for $0 \le \theta \le 2\pi$

SOLUTION Since $f(\theta) = e^\theta$, by the formula for the arc length we have:

$$L = \int_0^{2\pi} \sqrt{f'(\theta)^2 + f(\theta)}\,d\theta + \int_0^{2\pi} \sqrt{\left(e^\theta\right)^2 + \left(e^\theta\right)^2}\,d\theta = \int_0^{2\pi} \sqrt{2e^{2\theta}}\,d\theta$$
$$= \sqrt{2}\int_0^{2\pi} e^\theta\,d\theta = \sqrt{2}e^\theta\Big|_0^{2\pi} = \sqrt{2}\left(e^{2\pi} - e^0\right) = \sqrt{2}\left(e^{2\pi} - 1\right) \approx 755.9$$

28. The inner loop of $r = 2\cos\theta - 1$ in Figure 21

SOLUTION In Exercise 15 it is shown that the inner loop of the limaçon $r = 2\cos\theta - 1$ is traced as θ varies from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$. Also,

$$f(\theta) = 2\cos\theta - 1 \quad \text{and} \quad f'(\theta) = -2\sin\theta.$$

Using the integral for the arc length we obtain

$$\begin{aligned} L &= \int_{-\pi/3}^{\pi/3} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_{-\pi/3}^{\pi/3} \sqrt{(2\cos\theta - 1)^2 + (-2\sin\theta)^2}\, d\theta \\ &= \int_{-\pi/3}^{\pi/3} \sqrt{4\cos^2\theta - 4\cos\theta + 1 + 4\sin^2\theta}\, d\theta = \int_{-\pi/3}^{\pi/3} \sqrt{5 - 4\cos\theta}\, d\theta \end{aligned}$$

29. The cardioid $r = 1 - \cos\theta$ in Figure 16

SOLUTION In the equation of the cardioid, $f(\theta) = 1 - \cos\theta$. Using the formula for arc length in polar coordinates we have:

$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta \tag{1}$$

We compute the integrand:

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{(1 - \cos\theta)^2 + (\sin\theta)^2} = \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} = \sqrt{2(1 - \cos\theta)}$$

We identify the interval of θ. Since $-1 \le \cos\theta \le 1$, every $0 \le \theta \le 2\pi$ corresponds to a nonnegative value of r. Hence, θ varies from 0 to 2π. By (1) we obtain

$$L = \int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta$$

Now, $1 - \cos\theta = 2\sin^2(\theta/2)$, and on the interval $0 \le \theta \le \pi$, $\sin(\theta/2)$ is nonnegative, so that $\sqrt{2(1 - \cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2\sin(\theta/2)$ there. The graph is symmetric, so it suffices to compute the integral for $0 \le \theta \le \pi$, and we have

$$L = 2\int_0^{\pi} \sqrt{2(1 - \cos\theta)}\, d\theta = 2\int_0^{\pi} 2\sin(\theta/2)\, d\theta = 8\sin\frac{\theta}{2}\Big|_0^{\pi} = 8$$

30. $r = \cos^2\theta$

SOLUTION Since $\cos\theta = \cos(-\theta)$ and $\cos^2(\pi - \theta) = \cos^2\theta$ the curve is symmetric with respect to the x and y-axis. Therefore, we may compute the length as four times the length of the part of the curve in the first quadrant. We use the formula for the arc length in polar coordinates. In this case, $f(\theta) = \cos^2\theta$, $f'(\theta) = 2\cos\theta\,(-\sin\theta)$, so we obtain

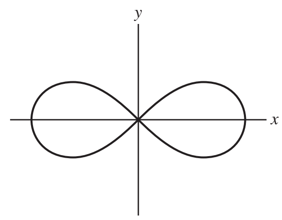
$$\begin{aligned} \sqrt{f(\theta)^2 + f'(\theta)^2} &= \sqrt{\cos^4\theta + 4\cos^2\theta\sin^2\theta} = \cos\theta\sqrt{\cos^2\theta + 4\sin^2\theta} \\ &= \cos\theta\sqrt{\cos^2\theta + \sin^2\theta + 3\sin^2\theta} = \cos\theta\sqrt{1 + 3\sin^2\theta} \end{aligned}$$

Thus,

$$L = \int_0^{\pi/2} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{\pi/2} \cos\theta\sqrt{1 + 3\sin^2\theta}\, d\theta.$$

We compute the integral using the substitution $u = \sqrt{3}\sin\theta$ we get

$$\begin{aligned} L &= \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} \sqrt{1 + u^2}\, du = \frac{1}{\sqrt{3}}\left(\frac{u}{2}\sqrt{1 + u^2} + \frac{1}{2}\ln|u + \sqrt{1 + u^2}|\right)\Big|_0^{\sqrt{3}} \\ &= \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\sqrt{1 + 3} + \frac{1}{2}\ln\left(\sqrt{3} + \sqrt{1 + 3}\right) - 0\right) = 1 + \frac{1}{2\sqrt{3}}\ln\left(2 + \sqrt{3}\right) \end{aligned}$$



Graph of $r = \cos^2\theta$

Thus the total length equals $4L = 4 + \frac{2}{\sqrt{3}} \ln\left(2+\sqrt{3}\right) \approx 5.52$.

In Exercises 31 and 32, express the length of the curve as an integral but do not evaluate it.

31. $r = (2-\cos\theta)^{-1}, \quad 0 \le \theta \le 2\pi$

SOLUTION We have $f(\theta) = (2-\cos\theta)^{-1}$, $f'(\theta) = -(2-\cos\theta)^{-2}\sin\theta$, hence,

$$\sqrt{f^2(\theta)+f'(\theta)^2} = \sqrt{(2-\cos\theta)^{-2}+(2-\cos\theta)^{-4}\sin^2\theta} = \sqrt{(2-\cos\theta)^{-4}\left((2-\cos\theta)^2+\sin^2\theta\right)}$$

$$= (2-\cos\theta)^{-2}\sqrt{4-4\cos\theta+\cos^2\theta+\sin^2\theta} = (2-\cos\theta)^{-2}\sqrt{5-4\cos\theta}$$

Using the integral for the arc length we get

$$L = \int_0^{2\pi} \sqrt{5-4\cos\theta}(2-\cos\theta)^{-2}\,d\theta.$$

32. $r = \sin^3 t, \quad 0 \le \theta \le 2\pi$

SOLUTION We have $f(t) = \sin^3 t$, $f'(t) = 3\sin^2 t\cos t$, so that

$$\sqrt{f(t)^2+f'(t)^2} = \sqrt{\sin^6 t + 9\sin^4 t\cos^2 t} = \sin^2 t\sqrt{\sin^2 t + 9\cos^2 t}$$

$$= \sin^2 t\sqrt{\sin^2 t + \cos^2 t + 8\cos^2 t} = \sin^2 t\sqrt{1+8\cos^2 t}$$

Using the formula for arc length integral we get

$$L = \int_0^{2\pi} \sin^2 t\sqrt{1+8\cos^2 t}\,dt$$

In Exercises 33–36, use a computer algebra system to calculate the total length to two decimal places.

33. CAS The three-petal rose $r = \cos 3\theta$ in Figure 20

SOLUTION We have $f(\theta) = \cos 3\theta$, $f'(\theta) = -3\sin 3\theta$, so that

$$\sqrt{f(\theta)^2+f'(\theta)^2} = \sqrt{\cos^2 3\theta + 9\sin^2 3\theta} = \sqrt{\cos^2 3\theta + \sin^2 3\theta + 8\sin^2 3\theta} = \sqrt{1+8\sin^2 3\theta}$$

Note that the curve is traversed completely for $0 \le \theta \le \pi$. Using the arc length formula and evaluating with Maple gives

$$L = \int_0^{\pi} \sqrt{f(\theta)^2+f'(\theta)^2}\,d\theta = \int_0^{\pi} \sqrt{1+8\sin^2 3\theta}\,d\theta \approx 6.682446608$$

34. CAS The curve $r = 2 + \sin 2\theta$ in Figure 23

SOLUTION We have $f(\theta) = 2+\sin 2\theta$, $f'(\theta) = 2\cos 2\theta$, so that

$$\sqrt{f(\theta)^2+f'(\theta)^2} = \sqrt{(2+\sin 2\theta)^2 + 4\cos^2 2\theta} = \sqrt{4+4\sin 2\theta + \sin^2 2\theta + 4\cos^2 2\theta}$$

$$= \sqrt{4+4\sin 2\theta+\sin^2 2\theta+\cos^2 2\theta+3\cos^2 2\theta}$$

$$= \sqrt{5+4\sin 2\theta+3\cos^2 2\theta}$$

The curve is traversed completely for $0 \le \theta \le 2\pi$. Using the arc length formula and evaluating with Maple gives

$$L = \int_0^{2\pi} \sqrt{f(\theta)^2+f'(\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{f(\theta)^2+f'(\theta)^2}\,d\theta \approx 15.40375907$$

35. CAS The curve $r = \theta \sin\theta$ in Figure 24 for $0 \le \theta \le 4\pi$

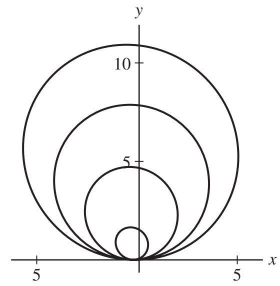


FIGURE 24 $r = \theta \sin\theta$ for $0 \le \theta \le 4\pi$.

SOLUTION We have $f(\theta) = \theta \sin\theta$, $f'(\theta) = \sin\theta + \theta\cos\theta$, so that

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{\theta^2 \sin^2\theta + (\sin\theta + \theta\cos\theta)^2} = \sqrt{\theta^2\sin^2\theta + \sin^2\theta + 2\theta\sin\theta\cos\theta + \theta^2\cos^2\theta}$$
$$= \sqrt{\theta^2 + \sin^2\theta + \theta\sin 2\theta}$$

using the identities $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$. Thus by the arc length formula and evaluating with Maple, we have

$$L = \int_0^{4\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{4\pi} \sqrt{\theta^2 + \sin^2\theta + \theta\sin 2\theta}\, d\theta \approx 79.56423976$$

36. CAS $r = \sqrt{\theta}$, $\quad 0 \le \theta \le 4\pi$

SOLUTION We have $f(\theta) = \sqrt{\theta}$, $f'(\theta) = \frac{1}{2}\theta^{-1/2}$, so that

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{\theta + \frac{1}{4\theta}}$$

so that by the arc length formula, evaluating with Maple, we have

$$L = \int_0^{4\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{4\pi} \sqrt{\theta + \frac{1}{4\theta}}\, d\theta \approx 30.50125041$$

Further Insights and Challenges

37. Suppose that the polar coordinates of a moving particle at time t are $(r(t), \theta(t))$. Prove that the particle's speed is equal to $\sqrt{(dr/dt)^2 + r^2(d\theta/dt)^2}$.

SOLUTION The speed of the particle in rectangular coordinates is:

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} \tag{1}$$

We need to express the speed in polar coordinates. The x and y coordinates of the moving particles as functions of t are

$$x(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t)$$

We differentiate $x(t)$ and $y(t)$, using the Product Rule for differentiation. We obtain (omitting the independent variable t)

$$x' = r'\cos\theta - r\,(\sin\theta)\,\theta'$$
$$y' = r'\sin\theta - r\,(\cos\theta)\,\theta'$$

Hence,

$$x'^2 + y'^2 = \left(r'\cos\theta - r\theta'\sin\theta\right)^2 + \left(r'\sin\theta + r\theta'\cos\theta\right)^2$$
$$= r'^2\cos^2\theta - 2r'r\theta'\cos\theta\sin\theta + r^2\theta'^2\sin^2\theta + r'^2\sin^2\theta + 2r'r\theta'\sin^2\theta\cos\theta + r^2\theta'^2\cos^2\theta$$
$$= r'^2\left(\cos^2\theta + \sin^2\theta\right) + r^2\theta'^2\left(\sin^2\theta + \cos^2\theta\right) = r'^2 + r^2\theta'^2 \tag{2}$$

Substituting (2) into (1) we get

$$\frac{ds}{dt} = \sqrt{r'^2 + r^2\theta'^2} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}$$

38. Compute the speed at time $t = 1$ of a particle whose polar coordinates at time t are $r = t, \theta = t$ (use Exercise 37). What would the speed be if the particle's rectangular coordinates were $x = t$, $y = t$? Why is the speed increasing in one case and constant in the other?

SOLUTION By Exercise 37 the speed of the particle is

$$\frac{ds}{dt} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2} \tag{1}$$

In this case $r = t$ and $\theta = t$ so $\frac{dr}{dt} = 1$ and $\frac{d\theta}{dt} = 1$. Substituting into (1) gives the following function of the speed:

$$\frac{ds}{dt} = \sqrt{1 + r(t)^2}$$

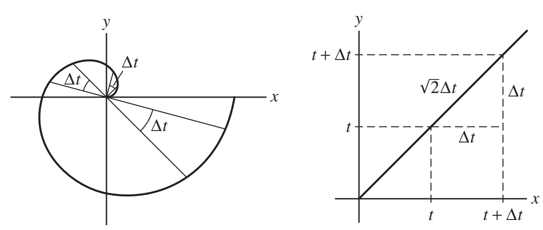
The speed expressed in rectangular coordinates is

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$$

If $x = t$ and $y = t$, then $x'(t) = 1$ and $y'(t) = 1$. So the speed of the particle at time t is

$$\frac{ds}{dt} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

On the curve $x = t$, $y = t$ the particle travels the same distance $\Delta t\sqrt{2}$ for all time intervals Δt, hence, it has a constant speed. However, on the spiral $r = t, \theta = t$ the particle travels greater distances for time intervals $(t, t + \Delta t)$ as t increases, hence the speed is an increasing function of t.



$x = t, y = t$ $\qquad$ $r = t, \theta = t$

11.5 Conic Sections

Preliminary Questions

1. Which of the following equations defines an ellipse? Which does not define a conic section?

(a) $4x^2 - 9y^2 = 12$ **(b)** $-4x + 9y^2 = 0$

(c) $4y^2 + 9x^2 = 12$ **(d)** $4x^3 + 9y^3 = 12$

SOLUTION

(a) This is the equation of the hyperbola $\left(\frac{x}{\sqrt{3}}\right)^2 - \left(\frac{y}{\frac{2}{\sqrt{3}}}\right)^2 = 1$, which is a conic section.

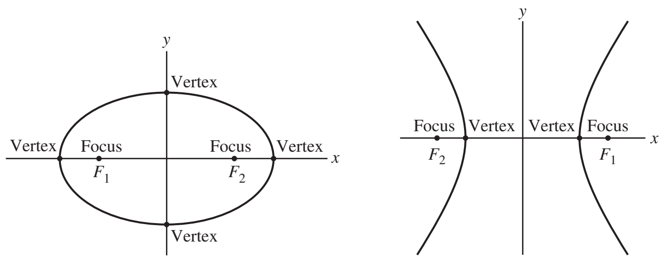
(b) The equation $-4x + 9y^2 = 0$ can be rewritten as $x = \frac{9}{4}y^2$, which defines a parabola. This is a conic section.

(c) The equation $4y^2 + 9x^2 = 12$ can be rewritten in the form $\left(\frac{y}{\sqrt{3}}\right)^2 + \left(\frac{x}{\frac{2}{\sqrt{3}}}\right)^2 = 1$, hence it is the equation of an ellipse, which is a conic section.

(d) This is not the equation of a conic section, since it is not an equation of degree two in x and y.

2. For which conic sections do the vertices lie between the foci?

SOLUTION If the vertices lie between the foci, the conic section is a hyperbola.

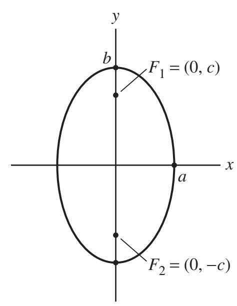


ellipse: foci between vertices hyperbola: vertices between foci

3. What are the foci of

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{if } a < b?$$

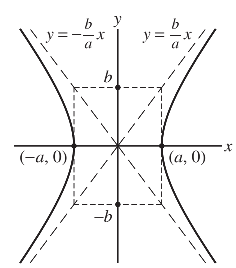
SOLUTION If $a < b$ the foci of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ are at the points $(0, c)$ and $(0, -c)$ on the y-axis, where $c = \sqrt{b^2 - a^2}$.



$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$; $a < b$

4. What is the geometric interpretation of b/a in the equation of a hyperbola in standard position?

SOLUTION The vertices, i.e., the points where the focal axis intersects the hyperbola, are at the points $(a, 0)$ and $(-a, 0)$. The values $\pm\frac{b}{a}$ are the slopes of the two asymptotes of the hyperbola.



Hyperbola in standard position

Exercises

In Exercises 1–6, find the vertices and foci of the conic section.

1. $\left(\frac{x}{9}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$

SOLUTION This is an ellipse in standard position with $a = 9$ and $b = 4$. Hence, $c = \sqrt{9^2 - 4^2} = \sqrt{65} \approx 8.06$. The foci are at $F_1 = (-8.06, 0)$ and $F_2 = (8.06, 0)$, and the vertices are $(9, 0)$, $(-9, 0)$, $(0, 4)$, $(0, -4)$.

2. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

SOLUTION Writing the equation in the from $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ we get an ellipse with $a = 3$ and $b = 2$. Hence $c = \sqrt{3^2 - 2^2} = \sqrt{5} \approx 2.24$. The foci are at $F_1 = (-2.24, 0)$ and $F_2 = (2.24, 0)$ and the vertices are $(3, 0)$, $(-3, 0)$, $(0, 2)$, $(0, -2)$.

3. $\left(\frac{x}{4}\right)^2 - \left(\frac{y}{9}\right)^2 = 1$

SOLUTION This is a hyperbola in standard position with $a = 4$ and $b = 9$. Hence, $c = \sqrt{a^2 + b^2} = \sqrt{97} \approx 9.85$. The foci are at $(\pm\sqrt{97}, 0)$ and the vertices are $(\pm 2, 0)$.

4. $\frac{x^2}{4} - \frac{y^2}{9} = 36$

SOLUTION Putting this equation in standard form gives

$$\left(\frac{x}{12}\right)^2 - \left(\frac{y}{18}\right)^2 = 1$$

so this is a hyperbola in standard position with $a = 12$ and $b = 18$. Thus

$$c = \sqrt{a^2 + b^2} = 6\sqrt{13} \approx 21.633$$

The foci are at $(\pm 6\sqrt{13}, 0)$ and the vertices are at $(\pm 12, 0)$.

5. $\left(\frac{x-3}{7}\right)^2 - \left(\frac{y+1}{4}\right)^2 = 1$

SOLUTION We first consider the hyperbola $\left(\frac{x}{7}\right)^2 - \left(\frac{y}{4}\right)^2 = 1$. For this hyperbola, $a = 7$, $b = 4$ and $c = \sqrt{7^2 + 4^2} \approx 8.06$. Hence, the foci are at $(8.06, 0)$ and $(-8.06, 0)$ and the vertices are at $(7, 0)$ and $(-7, 0)$. Since the given hyperbola is obtained by translating the center of the hyperbola $\left(\frac{x}{7}\right)^2 - \left(\frac{y}{4}\right)^2 = 1$ to the point $(3, -1)$, the foci are at $F_1 = (8.06 + 3, 0 - 1) = (11.06, -1)$ and $F_2 = (-8.06 + 3, 0 - 1) = (-5.06, -1)$ and the vertices are $A = (7 + 3, 0 - 1) = (10, -1)$ and $A' = (-7 + 3, 0 - 1) = (-4, -1)$.

6. $\left(\frac{x-3}{4}\right)^2 + \left(\frac{y+1}{7}\right)^2 = 1$

SOLUTION We first consider the ellipse $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{7}\right)^2 = 1$. Hence, $a = 4$ and $b = 7$ so $a < b$ and the focal axis is vertical. $c = \sqrt{7^2 - 4^2} \approx 5.74$ hence the foci are at $(0, 5.74)$ and $(0, -5.74)$. The vertices are $(4, 0)$, $(-4, 0)$, $(0, 7)$, $(0, -7)$. When we translate the ellipse so that its center is $(3, -1)$, the points above are translated so that the new vertices are $(4 + 3, 0 - 1) = (7, -1)$, $(-4 + 3, 0 - 1) = (-1, -1)$, $(0 + 3, 7 - 1) = (3, 6)$ and $(0 + 3, -7 - 1) = (3, -8)$. The new foci are at $(3, 4.74)$ and $(3, -6.74)$.

In Exercises 7–10, find the equation of the ellipse obtained by translating (as indicated) the ellipse

$$\left(\frac{x-8}{6}\right)^2 + \left(\frac{y+4}{3}\right)^2 = 1$$

7. Translated with center at the origin

SOLUTION Recall that the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

describes an ellipse with center (h, k). Thus, for our ellipse to be located at the origin, it must have equation

$$\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1$$

8. Translated with center at $(-2, -12)$

SOLUTION Recall that the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

describes an ellipse with center (h, k). Thus, for our ellipse to have center $(-2, -12)$, it must have equation

$$\frac{(x+2)^2}{6^2} + \frac{(y+12)^2}{3^2} = 1$$

9. Translated to the right six units

SOLUTION Recall that the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

describes an ellipse with center (h, k). The original ellipse has center at $(8, -4)$, so we want an ellipse with center $(14, -4)$. Thus its equation is

$$\frac{(x-14)^2}{6^2} + \frac{(y+4)^2}{3^2} = 1$$

10. Translated down four units

SOLUTION Recall that the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

describes an ellipse with center (h, k). The original ellipse has center at $(8, -4)$, so we want an ellipse with center $(8, -8)$. Thus its equation is

$$\frac{(x-8)^2}{6^2} + \frac{(y+8)^2}{3^2} = 1$$

In Exercises 11–14, find the equation of the given ellipse.

11. Vertices $(\pm 5, 0)$ and $(0, \pm 7)$

SOLUTION Since both sets of vertices are symmetric around the origin, the center of the ellipse is at $(0, 0)$. We have $a = 5$ and $b = 7$, so the equation of the ellipse is

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{7}\right)^2 = 1$$

12. Foci $(\pm 6, 0)$ and focal vertices $(\pm 10, 0)$

SOLUTION The equation is $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ with $a = 10$. The foci are $(\pm c, 0)$ with $c = 6$, so we use the relation $c = \sqrt{a^2 - b^2}$ to find b:

$$b^2 = a^2 - c^2 = 10^2 - 6^2 = 64 \Rightarrow b = 8$$

Therefore the equation of the ellipse is

$$\left(\frac{x}{10}\right)^2 + \left(\frac{y}{8}\right)^2 = 1.$$

13. Foci $(0, \pm 10)$ and eccentricity $e = \frac{3}{5}$

SOLUTION Since the foci are on the y axis, this ellipse has a vertical major axis with center $(0, 0)$, so its equation is

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

We have $a = \frac{c}{e} = \frac{10}{3/5} = \frac{50}{3}$ and

$$b = \sqrt{a^2 - c^2} = \sqrt{\frac{2500}{9} - 100} = \frac{1}{3}\sqrt{2500 - 900} = \frac{40}{3}$$

Thus the equation of the ellipse is

$$\left(\frac{x}{40/3}\right)^2 + \left(\frac{y}{50/3}\right)^2 = 1$$

14. Vertices $(4, 0)$, $(28, 0)$ and eccentricity $e = \frac{2}{3}$

SOLUTION This ellipse has a horizontal major axis with center midway between the vertices, at $(16, 0)$. Thus if the center were at $(0, 0)$, the ellipse would have vertices $(\pm 12, 0)$, so that $a = 12$ and $c = ae = 12 \cdot \frac{2}{3} = 8$. Then

$$b = \sqrt{a^2 - c^2} = \sqrt{12^2 - 8^2} = \sqrt{80} = 4\sqrt{5}$$

Finally, translating the center to (16, 0), the equation of the ellipse is

$$\left(\frac{(x-16)}{12}\right)^2 + \left(\frac{y}{4\sqrt{5}}\right)^2 = 1$$

In Exercises 15–20, find the equation of the given hyperbola.

15. Vertices $(\pm 3, 0)$ and foci $(\pm 5, 0)$

SOLUTION The equation is $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$. The vertices are $(\pm a, 0)$ with $a = 3$ and the foci $(\pm c, 0)$ with $c = 5$. We use the relation $c = \sqrt{a^2 + b^2}$ to find b:

$$b = \sqrt{c^2 - a^2} = \sqrt{5^2 - 3^2} = \sqrt{16} = 4$$

Therefore, the equation of the hyperbola is

$$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 = 1.$$

16. Vertices $(\pm 3, 0)$ and asymptotes $y = \pm\frac{1}{2}x$

SOLUTION The equation is $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$. The vertices are $(\pm a, 0)$ with $a = 3$ and the asymptotes are $y = \pm\frac{b}{a}x$ with $\frac{b}{a} = \frac{1}{2}$. Hence, $b = \frac{a}{2} = \frac{3}{2}$ so the equation of the hyperbola is

$$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{3/2}\right)^2 = 1$$

17. Foci $(\pm 4, 0)$ and eccentricity $e = 2$

SOLUTION We have $c = 4$ and $e = 2$; from $c = ae$ we get $a = 2$, and then

$$b = \sqrt{c^2 - a^2} = \sqrt{4^2 - 2^2} = 2\sqrt{3}$$

The hyperbola has center at (0, 0) and horizontal axis, so its equation is

$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{2\sqrt{3}}\right)^2 = 1$$

18. Vertices $(0, \pm 6)$ and eccentricity $e = 3$

SOLUTION The hyperbola has a vertical focal axis and center at (0, 0), so has equation

$$\left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1$$

$b = 6$ and $e = 3$ implies, since $be = c$, that $c = 18$, and

$$a = \sqrt{c^2 - b^2} = \sqrt{18^2 - 6^2} = \sqrt{288} = 12\sqrt{2}$$

Thus the equation of the hyperbola is

$$\left(\frac{y}{6}\right)^2 - \left(\frac{x}{12\sqrt{2}}\right)^2 = 1$$

19. Vertices $(-3, 0)$, $(7, 0)$ and eccentricity $e = 3$

SOLUTION The center is at $\frac{-3+7}{2} = 2$ with a horizontal focal axis, so the equation is

$$\left(\frac{x-2}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

Then $a = 7 - 2 = 5$, and $c = ae = 5 \cdot 3 = 15$. Finally,

$$b = \sqrt{c^2 - a^2} = \sqrt{15^2 - 5^2} = 10\sqrt{2}$$

so that the equation of the hyperbola is

$$\left(\frac{x-2}{5}\right)^2 - \left(\frac{y}{10\sqrt{2}}\right)^2 = 1$$

20. Vertices $(0, -6)$, $(0, 4)$ and foci $(0, -9)$, $(0, 7)$

SOLUTION The center of the hyperbola is at $\frac{-6+4}{2} = -1$ along the y axis; we write the equation as

$$\left(\frac{y+1}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1$$

$b = 5$ since it is the distance from the given vertex to the center, and $c = 8$ since it is the distance from the foci to the center. Then

$$a = \sqrt{c^2 - b^2} = \sqrt{64 - 25} = \sqrt{39}$$

so that the equation of the hyperbola is

$$\left(\frac{y+1}{5}\right)^2 - \left(\frac{x}{\sqrt{39}}\right)^2 = 1$$

In Exercises 21–28, find the equation of the parabola with the given properties.

21. Vertex $(0, 0)$, focus $\left(\frac{1}{12}, 0\right)$

SOLUTION Since the focus is on the x-axis rather than the y-axis, and the vertex is $(0, 0)$, the equation is $x = \frac{1}{4c}y^2$. The focus is $(0, c)$ with $c = \frac{1}{12}$, so the equation is

$$x = \frac{1}{4 \cdot \frac{1}{12}}y^2 = 3y^2$$

22. Vertex $(0, 0)$, focus $(0, 2)$

SOLUTION The vertex is at $(0, 0)$, so the equation is $y = \frac{1}{4c}x^2 = \frac{1}{8}x^2$.

23. Vertex $(0, 0)$, directrix $y = -5$

SOLUTION The equation is $y = \frac{1}{4c}x^2$. The directrix is $y = -c$ with $c = 5$, hence $y = \frac{1}{20}x^2$.

24. Vertex $(3, 4)$, directrix $y = -2$

SOLUTION If the graph were translated to the origin, the vertex would be $(0, 0)$ and the directrix would be translated down 4 units so would be $y = -6$. Then $c = 6$ so the equation is $y = \frac{1}{4c}x^2 = \frac{1}{24}x^2$. Translating back to $(3, 4)$ gives

$$y = \frac{1}{24}(x-3)^2 + 4$$

25. Focus $(0, 4)$, directrix $y = -4$

SOLUTION The focus is $(0, c)$ with $c = 4$ and the directrix is $y = -c$ with $c = 4$, hence the equation of the parabola is

$$y = \frac{1}{4c}x^2 = \frac{x^2}{16}.$$

26. Focus $(0, -4)$, directrix $y = 4$

SOLUTION The focus is at $(0, c)$ with $c = -4$ and the directrix is $y = -c$ with $c = -4$, hence the equation $y = \frac{x^2}{4c}$ of the parabola becomes $y = -\frac{x^2}{16}$. Since $c < 0$, the parabola is open downward.

27. Focus $(2, 0)$, directrix $x = -2$

SOLUTION The focus is on the x-axis rather than on the y-axis and the directrix is a vertical line rather than horizontal as in the parabola in standard position. Therefore, the equation of the parabola is obtained by interchanging x and y in $y = \frac{1}{4c}x^2$. Also, by the given information $c = 2$. Hence, $x = \frac{1}{4c}y^2 = \frac{1}{4 \cdot 2}y^2$ or $x = \frac{y^2}{8}$.

28. Focus $(-2, 0)$, vertex $(2, 0)$

SOLUTION The vertex is always midway between the focus and the directrix, so the directrix must be the vertical line $x = 6$, and $c = -2 - 2 = -4$. Since the directrix is a vertical line, the parabola is obtained by interchanging x and y in the equation for a parabola in standard position. Finally, $c = -2 - 2 = -4$ is the distance from the vertex to the focus, so the equation is

$$x - 2 = \frac{1}{4c}y^2 = -\frac{1}{16}y^2, \quad \text{so} \quad x = 2 - \frac{1}{16}y^2$$

In Exercises 29–38, find the vertices, foci, center (if an ellipse or a hyperbola), and asymptotes (if a hyperbola).

29. $x^2 + 4y^2 = 16$

SOLUTION We first divide the equation by 16 to convert it to the equation in standard form:

$$\frac{x^2}{16} + \frac{4y^2}{16} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1 \Rightarrow \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

For this ellipse, $a = 4$ and $b = 2$ hence $c = \sqrt{4^2 - 2^2} = \sqrt{12} \approx 3.5$. Since $a > b$ we have:

- The vertices are at $(\pm 4, 0)$, $(0, \pm 2)$.
- The foci are $F_1 = (-3.5, 0)$ and $F_2 = (3.5, 0)$.
- The focal axis is the x-axis and the conjugate axis is the y-axis.
- The ellipse is centered at the origin.

30. $4x^2 + y^2 = 16$

SOLUTION We divide the equation by 16 to rewrite it in the standard form:

$$\frac{4x^2}{16} + \frac{y^2}{16} = 1 \Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1 \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

This is the equation of an ellipse with $a = 2$, $b = 4$. Since $a < b$ the focal axis is the y-axis. Also, $c = \sqrt{4^2 - 2^2} = \sqrt{12} \approx 3.5$. We get:

- The vertices are at $(\pm 2, 0)$, $(0, \pm 4)$.
- The foci are $(0, \pm 3.5)$.
- The focal axis is the y-axis and the conjugate axis is the x-axis.
- The center is at the origin.

31. $\left(\frac{x-3}{4}\right)^2 - \left(\frac{y+5}{7}\right)^2 = 1$

SOLUTION For this hyperbola $a = 4$ and $b = 7$ so $c = \sqrt{4^2 + 7^2} = \sqrt{65} \approx 8.06$. For the standard hyperbola $\left(\frac{x}{4}\right)^2 - \left(\frac{y}{7}\right)^2 = 1$, we have

- The vertices are $A = (4, 0)$ and $A' = (-4, 0)$.
- The foci are $F = (\sqrt{65}, 0)$ and $F' = (-\sqrt{65}, 0)$.
- The focal axis is the x-axis $y = 0$, and the conjugate axis is the y-axis $x = 0$.
- The center is at the midpoint of $\overline{FF'}$; that is, at the origin.
- The asymptotes $y = \pm\frac{b}{a}x$ are $y = \pm\frac{7}{4}x$.

The given hyperbola is a translation of the standard hyperbola, 3 units to the right and 5 units downward. Hence the following holds:

- The vertices are at $A = (7, -5)$ and $A' = (-1, -5)$.
- The foci are at $F = (3 + \sqrt{65}, -5)$ and $F' = (3 - \sqrt{65}, -5)$.
- The focal axis is $y = -5$ and the conjugate axis is $x = 3$.
- The center is at $(3, -5)$.
- The asymptotes are $y + 5 = \pm\frac{7}{4}(x - 3)$.

32. $3x^2 - 27y^2 = 12$

SOLUTION We first rewrite the equation in the standard form:

$$\frac{3x^2}{12} - \frac{27y^2}{12} = 1 \Rightarrow \frac{x^2}{4} - \frac{y^2}{\frac{4}{9}} = 1 \Rightarrow \left(\frac{x}{2}\right)^2 - \left(\frac{y}{\frac{2}{3}}\right)^2 = 1$$

This is the equation of an hyperbola in standard position. We have $a = 2$, $b = \frac{2}{3}$ and $c = \sqrt{2^2 + \left(\frac{2}{3}\right)^2} \approx 2.1$. Hence:

- The vertices are $(\pm 2, 0)$.
- The foci are $(\pm 2.1, 0)$.

- The focal axis is the x-axis and the conjugate axis is the y-axis.
- The center is at the origin.
- The asymptotes are $y = \pm\frac{b}{a}x$, that is, $y = \pm\frac{1}{3}x$.

33. $4x^2 - 3y^2 + 8x + 30y = 215$

SOLUTION Since there is no cross term, we complete the square of the terms involving x and y separately:

$$4x^2 - 3y^2 + 8x + 30y = 4\left(x^2 + 2x\right) - 3\left(y^2 - 10y\right) = 4(x+1)^2 - 4 - 3(y-5)^2 + 75 = 215$$

Hence,

$$4(x+1)^2 - 3(y-5)^2 = 144$$
$$\frac{4(x+1)^2}{144} - \frac{3(y-5)^2}{144} = 1$$
$$\left(\frac{x+1}{6}\right)^2 - \left(\frac{y-5}{\sqrt{48}}\right)^2 = 1$$

This is the equation of the hyperbola obtained by translating the hyperbola $\left(\frac{x}{6}\right)^2 - \left(\frac{y}{\sqrt{48}}\right)^2 = 1$ one unit to the left and five units upwards. Since $a = 6$, $b = \sqrt{48}$, we have $c = \sqrt{36+48} = \sqrt{84} \sim 9.2$. We obtain the following table:

	Standard position	Translated hyperbola
vertices	$(6, 0), (-6, 0)$	$(5, 5), (-7, 5)$
foci	$(\pm 9.2, 0)$	$(8.2, 5), (-10.2, 5)$
focal axis	The x-axis	$y = 5$
conjugate axis	The y-axis	$x = -1$
center	The origin	$(-1, 5)$
asymptotes	$y = \pm 1.15x$	$y = -1.15x + 3.85$ $y = 1.15x + 6.15$

34. $y = 4x^2$

SOLUTION This is the parabola in standard position $y = \frac{1}{4c}x^2$ with $c = \frac{1}{16}$. The vertex of the parabola is at the origin, the focus is $F = \left(0, \frac{1}{16}\right)$ and the axis is the y-axis.

35. $y = 4(x-4)^2$

SOLUTION By Exercise 34, the parabola $y = 4x^2$ has the vertex at the origin, the focus at $\left(0, \frac{1}{16}\right)$ and its axis is the y-axis. Our parabola is a translation of the standard parabola four units to the right. Hence its vertex is at (4, 0), the focus is at $\left(4, \frac{1}{16}\right)$ and its axis is the vertical line $x = 4$.

36. $8y^2 + 6x^2 - 36x - 64y + 134 = 0$

SOLUTION We first identify the conic section. Since there is no cross term, we complete the square of the terms involving x and y terms separately:

$$\begin{aligned} 8y^2 + 6x^2 - 36x - 64y + 134 &= 6\left(x^2 - 6x\right) + 8\left(y^2 - 8y\right) + 134 \\ &= 6(x-3)^2 - 54 + 8(y-4)^2 - 128 + 134 \\ &= 6(x-3)^2 + 8(y-4)^2 - 48 \end{aligned}$$

We obtain the following equation:

$$6(x-3)^2 + 8(y-4)^2 - 48 = 0$$
$$3(x-3)^2 + 4(y-4)^2 = 24$$
$$\left(\frac{x-3}{\sqrt{8}}\right)^2 + \left(\frac{y-4}{\sqrt{6}}\right)^2 = 1$$

We identify the conic as a translation of the ellipse $\left(\frac{x}{\sqrt{8}}\right)^2 + \left(\frac{y}{\sqrt{6}}\right)^2 = 1$, so that the center is at $c = (3, 4)$. Since $a = \sqrt{8}$, $b = \sqrt{6}$ and $a > b$ the foci of the standard ellipse are $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, 0\right)$ for $\sqrt{2} = c = \sqrt{a^2 - b^2}$. Hence the foci of the translated ellipse are $\left(3-\sqrt{2},4\right)$ and $\left(3+\sqrt{2},4\right)$. The vertices $\left(\pm\sqrt{8},0\right)$ and $\left(0,\pm\sqrt{6}\right)$ of the standard ellipse are translated to the points $\left(3\pm\sqrt{8},4\right)$ and $\left(3,4\pm\sqrt{6}\right)$. The focal axis is the line $y=4$, and the conjugate axis is the line $x=3$.

37. $4x^2+25y^2-8x-10y=20$

SOLUTION Since there are no cross terms this conic section is obtained by translating a conic section in standard position. To identify the conic section we complete the square of the terms involving x and y separately:

$$\begin{aligned}4x^2+25y^2-8x-10y &= 4\left(x^2-2x\right)+25\left(y^2-\frac{2}{5}y\right)\\ &= 4(x-1)^2-4+25\left(y-\frac{1}{5}\right)^2-1\\ &= 4(x-1)^2+25\left(y-\frac{1}{5}\right)^2-5=20\end{aligned}$$

Hence,

$$\begin{aligned}4(x-1)^2+25\left(y-\frac{1}{5}\right)^2&=25\\ \frac{4}{25}(x-1)^2+\left(y-\frac{1}{5}\right)^2&=1\\ \left(\frac{x-1}{\frac{5}{2}}\right)^2+\left(y-\frac{1}{5}\right)^2&=1\end{aligned}$$

This is the equation of the ellipse obtained by translating the ellipse in standard position $\left(\frac{x}{\frac{5}{2}}\right)^2+y^2=1$ one unit to the right and $\frac{1}{5}$ unit upward. Since $a=\frac{5}{2}$, $b=1$ we have $c=\sqrt{\left(\frac{5}{2}\right)^2-1}\approx 2.3$, so we obtain the following table:

	Standard position	Translated ellipse
Vertices	$\left(\pm\frac{5}{2},0\right)$, $(0,\pm1)$	$\left(1\pm\frac{5}{2},\frac{1}{5}\right)$, $\left(1,\frac{1}{5}\pm1\right)$
Foci	$(-2.3,0)$, $(2.3,0)$	$\left(-1.3,\frac{1}{5}\right)$, $\left(3.3,\frac{1}{5}\right)$
Focal axis	The x-axis	$y=\frac{1}{5}$
Conjugate axis	The y-axis	$x=1$
Center	The origin	$\left(1,\frac{1}{5}\right)$

38. $16x^2+25y^2-64x-200y+64=0$

SOLUTION There is no cross term in this equation, so the conic section is obtained by translating a conic section in standard position. Complete the square in each variable:

$$\begin{aligned}-64=16x^2+25y^2-64x-200y&=16x^2-64x+64+25y^2-200y+400-64-400\\ &=16(x^2-4x+4)+25(y^2-8y+16)-464=16(x-2)^2+25(y-4)^2-464\end{aligned}$$

Collecting constants gives

$$16(x-2)^2+25(y-4)^2=400$$

and dividing through by 400 gives an ellipse whose equation in standard form is so that the curve is an ellipse whose equation in standard form is

$$\left(\frac{x-2}{5}\right)^2+\left(\frac{y-4}{4}\right)^2=1$$

Thus the center of the ellipse is $(2,4)$. The focal axis is $y=4$, because $a=5$ and $b=4$ imply that the focal axis is horizontal. Thus the conjugate axis is $x=2$. $c=\sqrt{a^2-b^2}=\sqrt{25-16}=3$. Thus

- The vertices are $(2\pm5,4)$ and $(2,4\pm4)$, so are $(-3,4)$, $(7,4)$, $(2,0)$, and $(2,8)$.
- The foci are $(2\pm3,4)$ so are $(5,4)$ and $(-1,4)$.

In Exercises 39–42, use the Discriminant Test to determine the type of the conic section (in each case, the equation is nondegenerate). Plot the curve if you have a computer algebra system.

39. $4x^2 + 5xy + 7y^2 = 24$

SOLUTION Here, $D = 25 - 4 \cdot 4 \cdot 7 = -87$, so the conic section is an ellipse.

40. $x^2 - 2xy + y^2 + 24x - 8 = 0$

SOLUTION Here, $D = 4 - 4 \cdot 1 \cdot 1 = 0$, giving us a parabola.

41. $2x^2 - 8xy + 3y^2 - 4 = 0$

SOLUTION Here, $D = 64 - 4 \cdot 2 \cdot 3 = 40$, giving us a hyperbola.

42. $2x^2 - 3xy + 5y^2 - 4 = 0$

SOLUTION Here, $D = 9 - 4 \cdot 2 \cdot (5) = -31$, giving us an ellipse or a circle. Since the coefficients of x^2 and y^2 are different, the curve is an ellipse.

43. Show that the "conic" $x^2 + 3y^2 - 6x + 12y + 23 = 0$ has no points.

SOLUTION Complete the square in each variable separately:

$$-23 = x^2 - 6x + 3y^2 + 12y = (x^2 - 6x + 9) + (3y^2 + 12y + 12) - 9 - 12 = (x-3)^2 + 3(y+2)^2 - 21$$

Collecting constants and reversing sides gives

$$(x-3)^2 + 3(y+2)^2 = -2$$

which has no solutions since the left-hand side is a sum of two squares so is always nonnegative.

44. For which values of a does the conic $3x^2 + 2y^2 - 16y + 12x = a$ have at least one point?

SOLUTION Complete the square in each variable:

$$a = 3x^2 + 2y^2 - 16y + 12x = 3x^2 + 12x + 12 + 2y^2 - 16y + 32 - 12 - 32 = 3(x+2)^2 + 2(x-4)^2 - 44$$

so that, collecting constants,

$$3(x+2)^2 + 2(x-4)^2 = a + 44$$

The left-hand side is a sum of two squares, so is always nonnegative, so in order for the conic (ellipse) to have at least one point, we must have $a + 44 \ge 0$, or $a \ge -44$.

45. Show that $\frac{b}{a} = \sqrt{1 - e^2}$ for a standard ellipse of eccentricity e.

SOLUTION By the definition of eccentricity:

$$e = \frac{c}{a} \tag{1}$$

For the ellipse in standard position, $c = \sqrt{a^2 - b^2}$. Substituting into (1) and simplifying yields

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

We square the two sides and solve for $\frac{b}{a}$:

$$e^2 = 1 - \left(\frac{b}{a}\right)^2 \Rightarrow \left(\frac{b}{a}\right)^2 = 1 - e^2 \Rightarrow \frac{b}{a} = \sqrt{1 - e^2}$$

46. Show that the eccentricity of a hyperbola in standard position is $e = \sqrt{1 + m^2}$, where $\pm m$ are the slopes of the asymptotes.

SOLUTION By the definition of eccentricity, we have:

$$e = \frac{c}{a} \tag{1}$$

For the hyperbola in standard position, $c = \sqrt{a^2 + b^2}$, by substituting in (1) we get

$$e = \frac{\sqrt{a^2 + b^2}}{a} = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{1 + \left(\frac{b}{a}\right)^2} \tag{2}$$

The slopes of the asymptotes are $\pm \frac{b}{a}$. Setting $m = \frac{b}{a}$ we get

$$e = \sqrt{1 + m^2}$$

47. Explain why the dots in Figure 23 lie on a parabola. Where are the focus and directrix located?

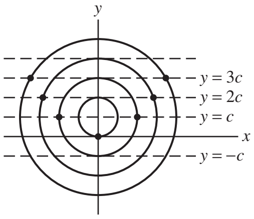
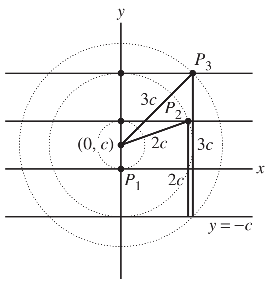


FIGURE 23

SOLUTION All the circles are centered at $(0, c)$ and the kth circle has radius kc. Hence the indicated point P_k on the kth circle has a distance kc from the point $F = (0, c)$. The point P_k also has distance kc from the line $y = -c$. That is, the indicated point on each circle is equidistant from the point $F = (0, c)$ and the line $y = -c$, hence it lies on the parabola with focus at $F = (0, c)$ and directrix $y = -c$.



48. Find the equation of the ellipse consisting of points P such that $PF_1 + PF_2 = 12$, where $F_1 = (4, 0)$ and $F_2 = (-2, 0)$.

SOLUTION This is a translation one unit to the right of an ellipse in standard position with foci $F_1 = (3, 0)$ and $F_2 = (-3, 0)$; points P on this ellipse therefore also satisfy the equation $PF_1 + PF_2 = 12$. But $PF_1 + PF_2 = 2a$ so that $a = 6$; since $(3, 0)$ is a focus, $c = 3$, so that $b = \sqrt{a^2 - c^2} = \sqrt{36 - 9} = 3\sqrt{3}$. The equation of the ellipse in standard position is therefore

$$\frac{x^2}{36} + \frac{y^2}{27} = 1$$

so that the equation of the desired ellipse is

$$\frac{(x-1)^2}{36} + \frac{y^2}{27} = 1$$

49. A **latus rectum** of a conic section is a chord through a focus parallel to the directrix. Find the area bounded by the parabola $y = x^2/(4c)$ and its latus rectum (refer to Figure 8).

SOLUTION The directrix is $y = -c$, and the focus is $(0, c)$. The chord through the focus parallel to $y = -c$ is clearly $y = c$; this line intersects the parabola when $c = x^2/(4c)$ or $4c^2 = x^2$, so when $x = \pm 2c$. The desired area is then

$$\int_{-2c}^{2c} c - \frac{1}{4c}x^2\,dx = \left(c\,x - \frac{1}{12c}x^3\right)\bigg|_{-2c}^{2c}$$
$$= 2c^2 - \frac{8c^3}{12c} - \left(-2c^2 - \frac{(-2c)^3}{12c}\right) = 4c^2 - \frac{4}{3}c^2 = \frac{8}{3}c^2$$

50. Show that the tangent line at a point $P = (x_0, y_0)$ on the hyperbola $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ has equation

$$Ax - By = 1$$

where $A = \dfrac{x_0}{a^2}$ and $B = \dfrac{y_0}{b^2}$.

SOLUTION The equation of the tangent line is

$$y - y_0 = m\,(x - x_0)\,;\; m = \frac{dy}{dx}\bigg|_{x=x_0,\,y=y_0} \tag{1}$$

To find the slope m we first implicitly differentiate the equation of the hyperbola with respect to x, which gives

$$2\left(\frac{x}{a}\right)\cdot\frac{1}{a}-2\left(\frac{y}{b}\right)\cdot\frac{1}{b}y'=0$$
$$\frac{x}{a^2}=\frac{y}{b^2}y' \Rightarrow y'=\frac{b^2}{a^2}\left(\frac{x}{y}\right)$$

We substitute $x=x_0$, $y=y_0$ to obtain the following slope of the tangent line:

$$m=\frac{b^2}{a^2}\frac{x_0}{y_0}=\frac{x_0}{a^2}\cdot\frac{b^2}{y_0}=A\cdot\frac{1}{B}=\frac{A}{B} \tag{2}$$

Substituting (2) in (1) gives

$$y-y_0=\frac{A}{B}(x-x_0)$$
$$By-By_0=Ax-Ax_0 \Rightarrow Ax-By=Ax_0-By_0 \tag{3}$$

Now,

$$Ax_0-By_0=\frac{x_0}{a^2}x_0-\frac{y_0}{b^2}y_0=\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}$$

and the point (x_0, y_0) lies on the hyperbola so

$$\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=1,$$

therefore $Ax_0-By_0=1$. Substituting in (3) we obtain $Ax-By=1$.

In Exercises 51–54, find the polar equation of the conic with the given eccentricity and directrix, and focus at the origin.

51. $e=\frac{1}{2}$, $x=3$

SOLUTION Substituting $e=\frac{1}{2}$ and $d=3$ in the polar equation of a conic section we obtain

$$r=\frac{ed}{1+e\cos\theta}=\frac{\frac{1}{2}\cdot 3}{1+\frac{1}{2}\cos\theta}=\frac{3}{2+\cos\theta}\Rightarrow r=\frac{3}{2+\cos\theta}$$

52. $e=\frac{1}{2}$, $x=-3$

SOLUTION We use the polar equation of a conic section with $e=\frac{1}{2}$ and $d=-3$ to obtain

$$r=\frac{ed}{1+e\cos\theta}=\frac{\frac{1}{2}\cdot(-3)}{1+\frac{1}{2}\cos\theta}=\frac{-3}{2+\cos\theta}\Rightarrow r=\frac{-3}{2+\cos\theta}$$

53. $e=1$, $x=4$

SOLUTION We substitute $e=1$ and $d=4$ in the polar equation of a conic section to obtain

$$r=\frac{ed}{1+e\cos\theta}=\frac{1\cdot 4}{1+1\cdot\cos\theta}=\frac{4}{1+\cos\theta}\Rightarrow r=\frac{4}{1+\cos\theta}$$

54. $e=\frac{3}{2}$, $x=-4$

SOLUTION Substituting $e=\frac{3}{2}$ and $d=-4$ in the polar equation of the conic section gives

$$r=\frac{ed}{1+e\cos\theta}=\frac{\frac{3}{2}\cdot(-4)}{1+\frac{3}{2}\cos\theta}=\frac{-12}{2+3\cos\theta}\Rightarrow r=\frac{-12}{2+3\cos\theta}$$

In Exercises 55–58, identify the type of conic, the eccentricity, and the equation of the directrix.

55. $r=\dfrac{8}{1+4\cos\theta}$

SOLUTION Matching with the polar equation $r=\frac{ed}{1+e\cos\theta}$ we get $ed=8$ and $e=4$ yielding $d=2$. Since $e>1$, the conic section is a hyperbola, having eccentricity $e=4$ and directrix $x=2$ (referring to the focus-directrix definition (11)).

56. $r = \dfrac{8}{4+\cos\theta}$

SOLUTION To identify the values of e and d we first rewrite the equation in the form $r = \frac{ed}{1+e\cos\theta}$:

$$r = \frac{8}{4+\cos\theta} = \frac{2}{1+\frac{1}{4}\cos\theta}$$

Thus, $ed = 2$ and $e = \frac{1}{4}$, yielding $d = 8$. Since $e < 1$, the conic is an ellipse, having eccentricity $e = \frac{1}{4}$ and directrix $x = 8$.

57. $r = \dfrac{8}{4+3\cos\theta}$

SOLUTION We first rewrite the equation in the form $r = \frac{ed}{1+e\cos\theta}$, obtaining

$$r = \frac{2}{1+\frac{3}{4}\cos\theta}$$

Hence, $ed = 2$ and $e = \frac{3}{4}$ yielding $d = \frac{8}{3}$. Since $e < 1$, the conic section is an ellipse, having eccentricity $e = \frac{3}{4}$ and directrix $x = \frac{8}{3}$.

58. $r = \dfrac{12}{4+3\cos\theta}$

SOLUTION We rewrite the equation in the form of the polar equation $r = \frac{ed}{1+e\cos\theta}$:

$$r = \frac{12}{4+3\cos\theta} = \frac{3}{1+\frac{3}{4}\cos\theta}$$

Hence, $ed = 3$ and $e = \frac{3}{4}$ which implies $d = 4$. Since $e < 1$, the conic section is an ellipse having eccentricity $e = \frac{3}{4}$ and directrix $x = 4$.

59. Find a polar equation for the hyperbola with focus at the origin, directrix $x = -2$, and eccentricity $e = 1.2$.

SOLUTION We substitute $d = -2$ and $e = 1.2$ in the polar equation $r = \frac{ed}{1+e\cos\theta}$ and use Exercise 40 to obtain

$$r = \frac{1.2\cdot(-2)}{1+1.2\cos\theta} = \frac{-2.4}{1+1.2\cos\theta} = \frac{-12}{5+6\cos\theta} = \frac{12}{5-6\cos\theta}$$

60. Let $\mathcal{C}$ be the ellipse $r = de/(1+e\cos\theta)$, where $e < 1$. Show that the x-coordinates of the points in Figure 24 are as follows:

Point	A	C	F_2	A'
x-coordinate	$\dfrac{de}{e+1}$	$-\dfrac{de^2}{1-e^2}$	$-\dfrac{2de^2}{1-e^2}$	$-\dfrac{de}{1-e}$

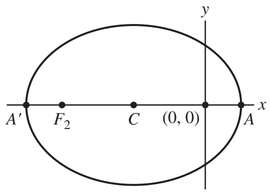


FIGURE 24

SOLUTION To find the x coordinate of A we substitute $\theta = 0$ in the polar equation $r = \frac{de}{1+e\cos\theta}$. This gives

$$x_A = r\cos 0 = r = \frac{de}{1+e\cos 0} = \frac{de}{1+e} \tag{1}$$

The point A' corresponds to $\theta = \pi$, so

$$x_{A'} = r\cos\pi = -r = -\frac{de}{1+e\cos\pi} = -\frac{de}{1-e} \tag{2}$$

The center C is the midpoint of $\overline{A'A}$. From (1) and (2) we obtain

$$x_C = \frac{x_A + x_{A'}}{2} = \frac{1}{2}\left(\frac{de}{1+e} - \frac{de}{1-e}\right) = \frac{de(1-e) - de(1+e)}{2(1+e)(1-e)} = \frac{-de^2}{1-e^2} \tag{3}$$

Finally, one focus is at the origin; the center C is the midpoint of $\overline{F_1F_2}$. Thus

$$x_C = \frac{x_{F_1} + x_{F_2}}{2} = \frac{0 + x_{F_2}}{2} = \frac{x_{F_2}}{2} \Rightarrow x_{F_2} = 2x_C$$

Using (3), we obtain

$$x_{F_2} = \frac{-2de^2}{1-e^2}$$

61. Find an equation in rectangular coordinates of the conic

$$r = \frac{16}{5 + 3\cos\theta}$$

Hint: Use the results of Exercise 60.

SOLUTION Put this equation in the form of the referenced exercise:

$$\frac{16}{5 + 3\cos\theta} = \frac{\frac{16}{5}}{1 + \frac{3}{5}\cos\theta} = \frac{\frac{16}{3}\cdot\frac{3}{5}}{1 + \frac{3}{5}\cos\theta}$$

so that $e = \frac{3}{5}$ and $d = \frac{16}{3}$. Then the center of the ellipse has x-coordinate

$$-\frac{de^2}{1-e^2} = -\frac{\frac{16}{3}\cdot\frac{9}{25}}{1 - \frac{9}{25}} = -\frac{16}{3}\cdot\frac{9}{25}\cdot\frac{25}{16} = -3$$

and y-coordinate 0, and A' has x-coordinate

$$-\frac{de}{1-e} = -\frac{\frac{16}{3}\cdot\frac{3}{5}}{1 - \frac{3}{5}} = -\frac{16}{3}\cdot\frac{3}{5}\cdot\frac{5}{2} = -8$$

and y-coordinate 0, so $a = -3 - (-8) = 5$, and the equation is

$$\left(\frac{x+3}{5}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

To find b, set $\theta = \frac{\pi}{2}$; then $r = \frac{16}{5}$. But the point corresponding to $\theta = \frac{\pi}{2}$ lies on the y-axis, so has coordinates $\left(0, \frac{16}{5}\right)$. This point is on the ellipse, so that

$$\left(\frac{0+3}{5}\right)^2 + \left(\frac{\frac{16}{5}}{b}\right)^2 = 1 \quad\Rightarrow\quad \frac{256}{25\cdot b^2} = \frac{16}{25} \quad\Rightarrow\quad \frac{256}{b^2} = 16 \quad\Rightarrow\quad b = 4$$

and the equation is

$$\left(\frac{x+3}{5}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

62. Let $e > 1$. Show that the vertices of the hyperbola $r = \dfrac{de}{1 + e\cos\theta}$ have x-coordinates $\dfrac{ed}{e+1}$ and $\dfrac{ed}{e-1}$.

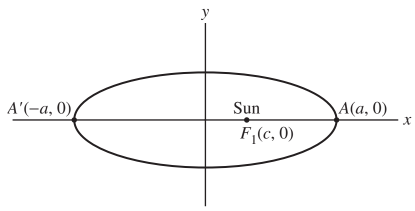
SOLUTION Since the focus is at the origin and the hyperbola is to the right (see figure), the two vertices have positive x coordinates. The corresponding values of θ at the vertices are $\theta = 0$ and $\theta = \pi$. Hence, since $e > 1$ we obtain

$$x_A = |r(0)| = \left|\frac{de}{1 + e\cos 0}\right| = \frac{de}{1+e}$$

$$x_{A'} = |r(\pi)| = \left|\frac{de}{1 + e\cos\pi}\right| = \left|\frac{de}{1-e}\right| = \frac{de}{e-1}$$

63. Kepler's First Law states that planetary orbits are ellipses with the sun at one focus. The orbit of Pluto has eccentricity $e \approx 0.25$. Its **perihelion** (closest distance to the sun) is approximately 2.7 billion miles. Find the **aphelion** (farthest distance from the sun).

SOLUTION We define an xy-coordinate system so that the orbit is an ellipse in standard position, as shown in the figure.



The aphelion is the length of $\overline{A'F_1}$, that is $a+c$. By the given data, we have

$$0.25 = e = \frac{c}{a} \Rightarrow c = 0.25a$$
$$a - c = 2.7 \Rightarrow c = a - 2.7$$

Equating the two expressions for c we get

$$0.25a = a - 2.7$$
$$0.75a = 2.7 \Rightarrow a = \frac{2.7}{0.75} = 3.6, \;\; c = 3.6 - 2.7 = 0.9$$

The aphelion is thus

$$\overline{A'F_0} = a + c = 3.6 + 0.9 = 4.5 \text{ billion miles.}$$

64. Kepler's Third Law states that the ratio $T/a^{3/2}$ is equal to a constant C for all planetary orbits around the sun, where T is the period (time for a complete orbit) and a is the semimajor axis.

(a) Compute C in units of days and kilometers, given that the semimajor axis of the earth's orbit is 150×10^6 km.

(b) Compute the period of Saturn's orbit, given that its semimajor axis is approximately 1.43×10^9 km.

(c) Saturn's orbit has eccentricity $e = 0.056$. Find the perihelion and aphelion of Saturn (see Exercise 63).

SOLUTION

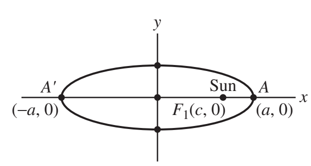
(a) By Kepler's Law, $\frac{T}{a^{3/2}} = C$. For the earth's orbit $a = 150 \times 10^6$ km and $T = 365$ days. Hence,

$$C = \frac{T}{a^{3/2}} = \frac{365}{(150 \times 10^6)^{3/2}} = \frac{365}{1837.12 \times 10^9} = 1.987 \cdot 10^{-10} \text{ days/km}$$

(b) By Kepler's Third Law and using the constant C computed in part (a) we get

$$\frac{T}{a^{3/2}} = C$$
$$\frac{T}{(1.43 \times 10^9)^{3/2}} = 1.987 \times 10^{-10}$$
$$T = (1.987 \times 10^{-10})(1.43 \times 10^9)^{3/2} = 10{,}745 \text{ days.}$$

(c) We define the xy-coordinate system so that the orbit is in standard position (see figure). (The sun is at one focus.)



The perihelion is $a - c$ and the aphelion is $a + c$. By the given information $a = 1.43 \times 10^9$ km and $e = 0.056$. Hence

$$e = \frac{c}{a} \Rightarrow 0.056 = \frac{c}{1.43 \times 10^9} \Rightarrow c = 0.08 \times 10^9 \text{ km}$$

We obtain the following solutions:

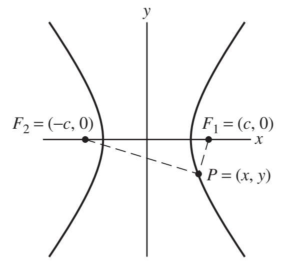
$$\text{perihelion} = a - c = 1.43 \times 10^9 - 0.08 \times 10^9 = 1.35 \times 10^9 \text{ km}$$
$$\text{aphelion} = a + c = 1.43 \times 10^9 + 0.08 \times 10^9 = 1.51 \times 10^9 \text{ km}$$

Further Insights and Challenges

65. Verify Theorem 2.

SOLUTION Let $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ and let $P(x, y)$ be an arbitrary point on the hyperbola. Then for some constant a,

$$\overline{PF_1} - \overline{PF_2} = \pm 2a$$



Using the distance formula we write this as

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a.$$

Moving the second term to the right and squaring both sides gives

$$\begin{aligned}
\sqrt{(x-c)^2 + y^2} &= \sqrt{(x+c)^2 + y^2} \pm 2a \\
(x-c)^2 + y^2 &= (x+c)^2 + y^2 \pm 4a\sqrt{(x+c)^2 + y^2} + 4a^2 \\
(x-c)^2 - (x+c)^2 - 4a^2 &= \pm 4a\sqrt{(x+c)^2 + y^2} \\
xc + a^2 &= \pm a\sqrt{(x+c)^2 + y^2}
\end{aligned}$$

We square and simplify to obtain

$$\begin{aligned}
x^2c^2 + 2xca^2 + a^4 &= a^2\left((x+c)^2 + y^2\right) \\
&= a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 \\
\left(c^2 - a^2\right)x^2 - a^2y^2 &= a^2\left(c^2 - a^2\right) \\
\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} &= 1
\end{aligned}$$

For $b = \sqrt{c^2 - a^2}$ (or $c = \sqrt{a^2 + b^2}$) we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

66. Verify Theorem 5 in the case $0 < e < 1$. *Hint:* Repeat the proof of Theorem 5, but set $c = d/(e^{-2} - 1)$.

SOLUTION We follow closely the proof of Theorem 5 in the book, which covered the case $e > 1$. This time, for $0 < e < 1$, we prove that $PF = ePD$ defines an ellipse. We choose our coordinate axes so that the focus F lies on the x-axis with coordinates $F = (c, 0)$ and so that the directrix is vertical, lying to the right of F at a distance d from F. As suggested by the hint, we set $c = \frac{d}{e^{-2}-1}$, but since we are working towards an ellipse, we will also need to let $b = \sqrt{a^2 - c^2}$ as opposed to the $\sqrt{c^2 - a^2}$ from the original proof of Theorem 5. Here's the complete list of definitions:

$$c = \frac{d}{e^{-2} - 1}, \quad a = \frac{c}{e}, \quad b = \sqrt{a^2 - c^2}$$

The directrix is the line

$$x = c + d = c + c(e^{-2} - 1) = ce^{-2} = \frac{a}{e}$$

Now, the equation

$$PF = e \cdot PD$$

for the points $P = (x, y)$, $F = (c, 0)$, and $D = (a/e, y)$ becomes

$$\sqrt{(x-c)^2 + y^2} = e \cdot \sqrt{(x-(a/e))^2}$$

Returning to the proof of Theorem 5, we see that this is the same equation that appears in the middle of the proof of the Theorem. As seen there, this equation can be transformed into

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

and this is equivalent to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

Since $a^2(1-e^2) = a^2 - a^2e^2 = a^2 - c^2 = b^2$, then we obtain the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

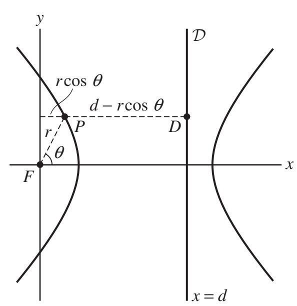
67. Verify that if $e > 1$, then Eq. (11) defines a hyperbola of eccentricity e, with its focus at the origin and directrix at $x = d$.

SOLUTION The points $P = (r, \theta)$ on the hyperbola satisfy $\overline{PF} = e\overline{PD}$, $e > 1$. Referring to the figure we see that

$$\overline{PF} = r,\ \overline{PD} = d - r\cos\theta \tag{1}$$

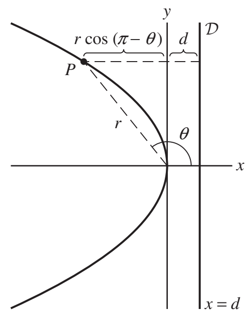
Hence

$$r = e(d - r\cos\theta)$$
$$r = ed - er\cos\theta$$
$$r(1 + e\cos\theta) = ed \Rightarrow r = \frac{ed}{1 + e\cos\theta}$$



Remark: Equality (1) holds also for $\theta > \frac{\pi}{2}$. For example, in the following figure, we have

$$\overline{PD} = d + r\cos(\pi - \theta) = d - r\cos\theta$$



Reflective Property of the Ellipse *In Exercises 68–70, we prove that the focal radii at a point on an ellipse make equal angles with the tangent line $\mathcal{L}$. Let $P = (x_0, y_0)$ be a point on the ellipse in Figure 25 with foci $F_1 = (-c, 0)$ and $F_2 = (c, 0)$, and eccentricity $e = c/a$.*

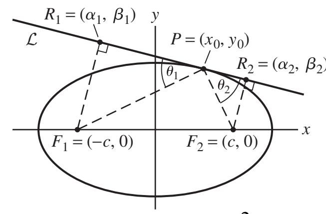


FIGURE 25 The ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

68. Show that the equation of the tangent line at P is $Ax + By = 1$, where $A = \dfrac{x_0}{a^2}$ and $B = \dfrac{y_0}{b^2}$.

SOLUTION The equation of the tangent line is

$$y - y_0 = m(x - x_0); \quad m = \left.\frac{dy}{dx}\right|_{x=x_0, y=y_0} \tag{1}$$

To find the slope m we implicitly differentiate the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to x. We get

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow \frac{yy'}{b^2} = -\frac{x}{a^2} \Rightarrow y' = -\frac{b^2}{a^2}\left(\frac{x}{y}\right)$$

We substitute $x = x_0$, $y = y_0$ to obtain the following slope of the tangent line:

$$m = -\frac{b^2}{a^2}\frac{x_0}{y_0} = -\frac{x_0}{a^2} \cdot \frac{b^2}{y_0} = -\frac{A}{B}$$

Substituting in (1) and simplifying gives

$$y - y_0 = -\frac{A}{B}(x - x_0)$$
$$By - By_0 = -Ax + Ax_0$$
$$Ax + By = Ax_0 + By_0$$

Now,

$$Ax_0 + By_0 = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2},$$

so we get $Ax + By = 1$.

69. Points R_1 and R_2 in Figure 25 are defined so that $\overline{F_1R_1}$ and $\overline{F_2R_2}$ are perpendicular to the tangent line.

(a) Show, with A and B as in Exercise 68, that

$$\frac{\alpha_1 + c}{\beta_1} = \frac{\alpha_2 - c}{\beta_2} = \frac{A}{B}$$

(b) Use (a) and the distance formula to show that

$$\frac{F_1R_1}{F_2R_2} = \frac{\beta_1}{\beta_2}$$

(c) Use (a) and the equation of the tangent line in Exercise 68 to show that

$$\beta_1 = \frac{B(1 + Ac)}{A^2 + B^2}, \qquad \beta_2 = \frac{B(1 - Ac)}{A^2 + B^2}$$

SOLUTION

(a) Since $R_1 = (\alpha_1, \beta_1)$ and $R_2 = (\alpha_2, \beta_2)$ lie on the tangent line at P, that is on the line $Ax + By = 1$, we have

$$A\alpha_1 + B\beta_1 = 1 \quad \text{and} \quad A\alpha_2 + B\beta_2 = 1$$

The slope of the line R_1F_1 is $\frac{\beta_1}{\alpha_1 + c}$ and it is perpendicular to the tangent line having slope $-\frac{A}{B}$. Similarly, the slope of the line R_2F_2 is $\frac{\beta_2}{\alpha_2 - c}$ and it is also perpendicular to the tangent line. Hence,

$$\frac{\alpha_1 + c}{\beta_1} = \frac{A}{B} \quad \text{and} \quad \frac{\alpha_2 - c}{\beta_2} = \frac{A}{B}.$$

(b) Using the distance formula, we have

$$\overline{R_1F_1}^2 = (\alpha_1 + c)^2 + \beta_1^2$$

Thus,

$$\overline{R_1F_1}^2 = \beta_1^2\left(\left(\frac{\alpha_1 + c}{\beta_1}\right)^2 + 1\right) \tag{1}$$

By part (a), $\frac{\alpha_1+c}{\beta_1} = \frac{A}{B}$. Substituting in (1) gives

$$\overline{R_1F_1}^2 = \beta_1^2\left(\frac{A^2}{B^2} + 1\right) \tag{2}$$

Likewise,

$$\overline{R_2F_2}^2 = (\alpha_2 - c)^2 + {\beta_2}^2 = {\beta_2}^2\left(\left(\frac{\alpha_2 - c}{\beta_2}\right)^2 + 1\right) \tag{3}$$

but since $\frac{\alpha_2-c}{\beta_2} = \frac{A}{B}$, substituting in (3) gives

$$\overline{R_2F_2}^2 = \beta_2^2\left(\frac{A^2}{B^2} + 1\right). \tag{4}$$

Dividing, we find that

$$\frac{\overline{R_1F_1}^2}{\overline{R_2F_2}^2} = \frac{\beta_1^2}{\beta_2^2} \quad \text{so} \quad \frac{\overline{R_1F_1}}{\overline{R_2F_2}} = \frac{\beta_1}{\beta_2},$$

as desired.

(c) In part (a) we showed that

$$\begin{cases} A\alpha_1 + B\beta_1 = 1 \\ \dfrac{\beta_1}{\alpha_1 + c} = \dfrac{B}{A} \end{cases}$$

Eliminating α_1 and solving for β_1 gives

$$\beta_1 = \frac{B(1 + Ac)}{A^2 + B^2}. \tag{5}$$

Similarly, we have

$$\begin{cases} A\alpha_2 + B\beta_2 = 1 \\ \dfrac{\beta_2}{\alpha_2 - c} = \dfrac{B}{A} \end{cases}$$

Eliminating α_2 and solving for β_2 yields

$$\beta_2 = \frac{B\,(1 - Ac)}{A^2 + B^2} \tag{6}$$

70. **(a)** Prove that $PF_1 = a + x_0e$ and $PF_2 = a - x_0e$. *Hint:* Show that $PF_1{}^2 - PF_2{}^2 = 4x_0c$. Then use the defining property $PF_1 + PF_2 = 2a$ and the relation $e = c/a$.

(b) Verify that $\dfrac{F_1R_1}{PF_1} = \dfrac{F_2R_2}{PF_2}$.

(c) Show that $\sin\theta_1 = \sin\theta_2$. Conclude that $\theta_1 = \theta_2$.

SOLUTION

(a) Using the distance formula we have

$$\overline{PF_1}^2 = (x_0 + c)^2 + y^2; \quad \overline{PF_2}^2 = (x_0 - c)^2 + y^2$$

Hence,

$$\overline{PF_1}^2 - \overline{PF_2}^2 = (x_0 + c)^2 + y^2 - (x_0 - c)^2 - y^2$$

$$= (x_0 + c)^2 - (x_0 - c)^2$$
$$= x_0{}^2 + 2x_0c + c^2 - x_0{}^2 + 2x_0c - c^2 = 4x_0c$$

That is, $\overline{PF_1}^2 - \overline{PF_2}^2 = 4x_0c$. Now use the identity $u^2 - v^2 = (u - v)(u + v)$ to write this as

$$\left(\overline{PF_1} - \overline{PF_2}\right)\left(\overline{PF_1} + \overline{PF_2}\right) = 4x_0c \tag{1}$$

Since P lies on the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ we have

$$\overline{PF_1} + \overline{PF_2} = 2a \tag{2}$$

Substituting in (1) gives

$$\left(\overline{PF_1} - \overline{PF_2}\right) \cdot 2a = 4x_0c$$

We divide by a and use the eccentricity $e = \frac{c}{a}$ to obtain

$$\overline{PF_1} - \overline{PF_2} = 2x_0e$$

Solve this equation together with equation (2) to see that

$$\overline{PF_1} = a + x_0e, \qquad \overline{PF_2} = a - x_0e$$

(b) Substituting the expression for β_1 from Eq. (5) in Exercise 69 into Eq. (2) in Exercise 69 yields

$$\overline{R_1F_1}^2 = \frac{B^2(1 + Ac)^2}{(A^2 + B^2)^2}\left(\frac{A^2}{B^2} + 1\right) = \frac{B^2(1 + Ac)^2(A^2 + B^2)}{(A^2 + B^2)^2B^2} = \frac{(1 + Ac)^2}{A^2 + B^2}$$

and similarly, substituting the expression for β_2 from Eq. (6) in Exercise 69 into Eq. (4) in Exercise 69 yields

$$\overline{R_2F_2}^2 = \frac{B^2(1 - Ac)^2}{(A^2 + B^2)^2}\left(\frac{A^2}{B^2} + 1\right) = \frac{B^2(1 - Ac)^2(A^2 + B^2)}{(A^2 + B^2)^2B^2} = \frac{(1 - Ac)^2}{A^2 + B^2}$$

Taking square roots and dividing these two formulas gives

$$\frac{\overline{R_1F_1}}{\overline{R_2F_2}} = \frac{\frac{1+Ac}{\sqrt{A^2+B^2}}}{\frac{1-Ac}{\sqrt{A^2+B^2}}} = \frac{1 + Ac}{1 - Ac}$$

Substitute $c = ea$ and $A = \frac{x_0}{a^2}$ (from Exercise 68) to get

$$\frac{\overline{R_1F_1}}{\overline{R_2F_2}} = \frac{1 + \frac{x_0ea}{a^2}}{1 - \frac{x_0ea}{a^2}} = \frac{1 + \frac{x_0e}{a}}{1 - \frac{x_0e}{a}} = \frac{a + x_0e}{a - x_0e}$$

But part (a) showed that $\overline{PF_1} = a + x_0e$ and $\overline{PF_2} = a - x_0e$, so that

$$\frac{\overline{R_1F_1}}{\overline{R_2F_2}} = \frac{\overline{PF_1}}{\overline{PF_2}} \qquad \Rightarrow \qquad \frac{\overline{R_1F_1}}{\overline{PF_1}} = \frac{\overline{R_2F_2}}{\overline{PF_2}}$$

(c) Since $\frac{\overline{R_1F_1}}{\overline{PF_1}} = \sin\theta_1$ and $\frac{\overline{R_2F_2}}{\overline{PF_2}} = \sin\theta_2$, we get $\sin\theta_1 = \sin\theta_2$, which implies that $\theta_1 = \theta_2$ since the two angles are acute.

71. Here is another proof of the Reflective Property.

(a) Figure 25 suggests that $\mathcal{L}$ is the unique line that intersects the ellipse only in the point P. Assuming this, prove that $QF_1 + QF_2 > PF_1 + PF_2$ for all points Q on the tangent line other than P.

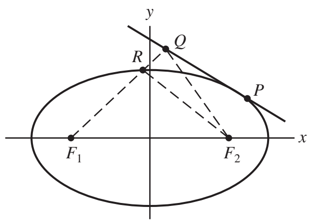
(b) Use the Principle of Least Distance (Example 6 in Section 4.7) to prove that $\theta_1 = \theta_2$.

SOLUTION

(a) Consider a point $Q \neq P$ on the line $\mathcal{L}$ (see figure). Since $\mathcal{L}$ intersects the ellipse in only one point, the remainder of the line lies outside the ellipse, so that QR does not have zero length, and F_2QR is a triangle. Thus

$$QF_1 + QF_2 = QR + RF_1 + QF_2 = RF_1 + (QR + QF_2) > RF_1 + RF_2$$

since the sum of lengths of two sides of a triangle exceeds the length of the third side. But since point R lies on the ellipse, $RF_2 + RF_2 = PF_1 + PF_2$, and we are done.



(b) Consider a beam of light traveling from F_1 to F_2 by reflection off of the line $\mathcal{L}$. By the principle of least distance, the light takes the shortest path, which by part (a) is the path through P. By Example 6 in Section 4.7, this shortest path has the property that the angle of incidence (θ_1) is equal to the angle of reflection (θ_2).

72. Show that the length QR in Figure 26 is independent of the point P.

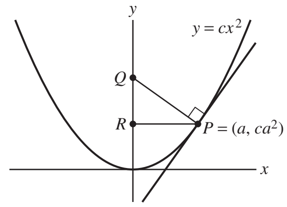


FIGURE 26

SOLUTION We find the slope m of the tangent line at $P = \left(a, ca^2\right)$:

$$m = (cx^2)'\Big|_{x=a} = 2cx\Big|_{x=a} = 2ca$$

The slope of the perpendicular line PQ is, thus, $-\frac{1}{2ca}$, and the equation of this line is

$$y - ca^2 = -\frac{1}{2ca}(x - a) \Rightarrow y = -\frac{x}{2ac} + ca^2 + \frac{1}{2c}$$

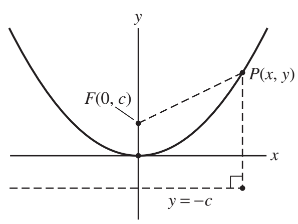
The y-intercept of the line PQ is $y = ca^2 + \frac{1}{2c}$. We now find the length $\overline{QR}$, by computing the distance between the points $Q(0, ca^2 + \frac{1}{2c})$ and $P(0, ca^2)$. This gives

$$\overline{QR} = ca^2 + \frac{1}{2c} - ca^2 = \frac{1}{2c}$$

Indeed, the length $\overline{QR}$ is independent of a, i.e. it is independent of the point P.

73. Show that $y = x^2/4c$ is the equation of a parabola with directrix $y = -c$, focus $(0, c)$, and the vertex at the origin, as stated in Theorem 3.

SOLUTION The points $P = (x, y)$ on the parabola are equidistant from $F = (0, c)$ and the line $y = -c$.



That is, by the distance formula, we have

$$\overline{PF} = \overline{PD}$$
$$\sqrt{x^2 + (y - c)^2} = |y + c|$$

Squaring and simplifying yields

$$x^2 + (y - c)^2 = (y + c)^2$$
$$x^2 + y^2 - 2yc + c^2 = y^2 + 2yc + c^2$$
$$x^2 - 2yc = 2yc$$

$$x^2 = 4yc \Rightarrow y = \frac{x^2}{4c}$$

Thus, we showed that the points that are equidistant from the focus $F = (0, c)$ and the directrix $y = -c$ satisfy the equation $y = \frac{x^2}{4c}$.

74. Consider two ellipses in standard position:

$$E_1: \quad \left(\frac{x}{a_1}\right)^2 + \left(\frac{y}{b_1}\right)^2 = 1$$

$$E_2: \quad \left(\frac{x}{a_2}\right)^2 + \left(\frac{y}{b_2}\right)^2 = 1$$

We say that E_1 is similar to E_2 under scaling if there exists a factor $r > 0$ such that for all (x, y) on E_1, the point (rx, ry) lies on E_2. Show that E_1 and E_2 are similar under scaling if and only if they have the same eccentricity. Show that any two circles are similar under scaling.

SOLUTION If E_1 and E_2 are similar under scaling, then since $(a_1, 0)$ and $(0, b_1)$ are points on the first ellipse, the scaled points $(ra_1, 0)$ and $(0, rb_1)$ must be on the second ellipse. This implies that $(ra_1/a_2)^2 + (0/b_2)^2 = 1$ and that $(0/a_1)^2 + (rb_1/b_2)^2 = 1$, which means that $ra_1 \pm a_2$ and $rb_1 = \pm b_2$. But, since r, a_1, and a_2 are all positive, then this implies that $a_2 = ra_1$ and $b_2 = rb_1$, and so

$$c_2 = \sqrt{a_2^2 - b_2^2} = r\sqrt{a_1^2 - b_1^2} = rc_1.$$

Thus,

$$e_2 = \frac{c_2}{a_2} = \frac{r_1 c_1}{r_1 a_1} = \frac{c_1}{a_1} = e_1$$

and so the two ellipses have the same eccentricity. On the other hand, if the two ellipses have the same eccentricity, then

$$\sqrt{1 - \frac{b_2^2}{a_2^2}} = \frac{c_2}{a_2} = e_2 = e_1 = \frac{c_1}{a_1} = \sqrt{1 - \frac{b_1^2}{a_1^2}}$$

which implies

$$\sqrt{1 - \frac{b_2^2}{a_2^2}} = \sqrt{1 - \frac{b_1^2}{a_1^2}}$$

and this implies that $b_2/a_2 = \pm b_1/a_1$ and so $b_2/a_2 = b_1/a_1$ (recall that all constants are positive). Define $r = b_2/b_1$. Then, $b_2 = rb_1$, but since $b_2/a_2 = b_1/a_1$ we get that $a_2 = ra_1$ as well. Thus, for the point (x, y) on the first ellipse, we have that

$$\left(\frac{x}{a_1}\right)^2 + \left(\frac{y}{b_1}\right)^2 = 1$$

If we put the scaled point (rx, ry) into the second ellipse, we get

$$\left(\frac{rx}{a_2}\right)^2 + \left(\frac{ry}{b_2}\right)^2 = \left(\frac{rx}{ra_1}\right)^2 + \left(\frac{ry}{rb_1}\right)^2 = \left(\frac{x}{a_1}\right)^2 + \left(\frac{y}{b_1}\right)^2 = 1$$

which implies that E_2 is a scaled version of E_1. Since all circles have eccentricity 0, then they are all similar under scaling.

75. Derive Eqs. (13) and (14) in the text as follows. Write the coordinates of P with respect to the rotated axes in Figure 21 in polar form $x' = r\cos\alpha$, $y' = r\sin\alpha$. Explain why P has polar coordinates $(r, \alpha + \theta)$ with respect to the standard x and y-axes and derive Eqs. (13) and (14) using the addition formulas for cosine and sine.

SOLUTION If the polar coordinates of P with respect to the rotated axes are (r, α), then the line from the origin to P has length r and makes an angle of α with the rotated x-axis (the x'-axis). Since the x'-axis makes an angle of θ with the x-axis, it follows that the line from the origin to P makes an angle of $\alpha + \theta$ with the x-axis, so that the polar coordinates of P with respect to the standard axes are $(r, \alpha + \theta)$. Write (x', y') for the rectangular coordinates of P with respect to the rotated axes and (x, y) for the rectangular coordinates of P with respect to the standard axes. Then

$$x = r\cos(\alpha + \theta) = (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta = x'\cos\theta - y'\sin\theta$$

$$y = r\sin(\alpha + \theta) = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta = (r\cos\alpha)\sin\theta + (r\sin\alpha)\cos\theta = x'\sin\theta + y'\cos\theta$$

76. If we rewrite the general equation of degree 2 (Eq. 12) in terms of variables x' and y' that are related to x and y by Eqs. (13) and (14), we obtain a new equation of degree 2 in x' and y' of the same form but with different coefficients:

$$a'x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0$$

(a) Show that $b' = b\cos 2\theta + (c - a)\sin 2\theta$.
(b) Show that if $b \neq 0$, then we obtain $b' = 0$ for

$$\theta = \frac{1}{2}\cot^{-1}\frac{a-c}{b}$$

This proves that it is always possible to eliminate the cross term bxy by rotating the axes through a suitable angle.

SOLUTION

(a) If we plug in $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ into the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$, we will get a very ugly mess. Fortunately, we only care about the $x'y'$ term, so we really only need to look at the $ax^2 + bxy + cy^2$ part of the formula. In fact, we only need to pull out those terms which have an $x'y'$ in them. Thus

$$\begin{aligned} ax^2 &\quad \text{becomes} \quad a(x'\cos\theta - y'\sin\theta)^2 = -2ax'y'\cos\theta\sin\theta + \dots \\ bxy &\quad \text{becomes} \quad b(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) = bx'y'(\cos^2\theta - \sin^2\theta) + \dots \\ cy^2 &\quad \text{becomes} \quad c(x'\sin\theta + y'\cos\theta)^2 = 2cx'y'\cos\theta\sin\theta + \dots \end{aligned}$$

so that

$$ax^2 + bxy + cy^2 = ((c-a)2\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta))x'y' + \dots = ((c-a)\sin 2\theta + b\cos 2\theta)x'y' + \dots$$

and thus b', the coefficient of $x'y'$, is $b\cos 2\theta + (c-a)\sin 2\theta$, as desired.
(b) Setting $b' = 0$, we get $0 = b\cos 2\theta + (c-a)\sin 2\theta$, so $b\cos 2\theta = (a-c)\sin 2\theta$, so $\cot 2\theta = \frac{a-c}{b}$, giving us $2\theta = \cot^{-1}\frac{a-c}{b}$, and thus $\theta = \frac{1}{2}\cot^{-1}\frac{a-c}{b}$.

CHAPTER REVIEW EXERCISES

1. Which of the following curves pass through the point $(1, 4)$?

(a) $c(t) = (t^2, t+3)$
(b) $c(t) = (t^2, t-3)$
(c) $c(t) = (t^2, 3-t)$
(d) $c(t) = (t-3, t^2)$

SOLUTION To check whether it passes through the point $(1, 4)$, we solve the equations $c(t) = (1, 4)$ for the given curves.

(a) Comparing the second coordinate of the curve and the point yields:

$$\begin{aligned} t + 3 &= 4 \\ t &= 1 \end{aligned}$$

We substitute $t = 1$ in the first coordinate, to obtain

$$t^2 = 1^2 = 1$$

Hence the curve passes through $(1, 4)$.

(b) Comparing the second coordinate of the curve and the point yields:

$$\begin{aligned} t - 3 &= 4 \\ t &= 7 \end{aligned}$$

We substitute $t = 7$ in the first coordinate to obtain

$$t^2 = 7^2 = 49 \neq 1$$

Hence the curve does not pass through $(1, 4)$.

(c) Comparing the second coordinate of the curve and the point yields

$$\begin{aligned} 3 - t &= 4 \\ t &= -1 \end{aligned}$$

We substitute $t = -1$ in the first coordinate, to obtain

$$t^2 = (-1)^2 = 1$$

Hence the curve passes through $(1, 4)$.

(d) Comparing the first coordinate of the curve and the point yields

$$t - 3 = 1$$
$$t = 4$$

We substitute $t = 4$ in the second coordinate, to obtain:

$$t^2 = 4^2 = 16 \neq 4$$

Hence the curve does not pass through $(1, 4)$.

2. Find parametric equations for the line through $P = (2, 5)$ perpendicular to the line $y = 4x - 3$.

SOLUTION The line perpendicular to $y = 4x - 3$ at $P = (2, 5)$ is the line of slope $-\frac{1}{4}$ passing through P. This line has the equation

$$y - 5 = -\frac{1}{4}(x - 2)$$

A bit of calculation shows that the parametric equations of the line are

$$c(t) = \left(2 + t, 5 - \frac{1}{4}t\right)$$

or

$$x = 2 + t$$
$$y = 5 - \frac{1}{4}t$$

3. Find parametric equations for the circle of radius 2 with center $(1, 1)$. Use the equations to find the points of intersection of the circle with the x- and y-axes.

SOLUTION Using the standard technique for parametric equations of curves, we obtain

$$c(t) = (1 + 2\cos t, 1 + 2\sin t)$$

We compare the x coordinate of $c(t)$ to 0:

$$1 + 2\cos t = 0$$
$$\cos t = -\frac{1}{2}$$
$$t = \pm\frac{2\pi}{3}$$

Substituting in the y coordinate yields

$$1 + 2\sin\left(\pm\frac{2\pi}{3}\right) = 1 \pm 2\frac{\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

Hence, the intersection points with the y-axis are $(0, 1 \pm \sqrt{3})$. We compare the y coordinate of $c(t)$ to 0:

$$1 + 2\sin t = 0$$
$$\sin t = -\frac{1}{2}$$
$$t = -\frac{\pi}{6} \quad \text{or} \quad \frac{7}{6}\pi$$

Substituting in the x coordinates yields

$$1 + 2\cos\left(-\frac{\pi}{6}\right) = 1 + 2\frac{\sqrt{3}}{2} = 1 + \sqrt{3}$$
$$1 + 2\cos\left(\frac{7}{6}\pi\right) = 1 - 2\cos\left(\frac{\pi}{6}\right) = 1 - 2\frac{\sqrt{3}}{2} = 1 - \sqrt{3}$$

Hence, the intersection points with the x-axis are $(1 \pm \sqrt{3}, 0)$.

4. Find a parametrization $c(t)$ of the line $y = 5 - 2x$ such that $c(0) = (2, 1)$.

SOLUTION The line is passing through $P = (0, 5)$ with slope -2, hence (by one of the examples in section 12.1) it has the parametrization

$$c(t) = (t, 5 - 2t)$$

This parametrization does not satisfy $c(0) = (2, 1)$. We replace the parameter t by a parameter s, so that $t = s + \beta$, to obtain another parametrization for the line:

$$c^*(s) = (s + \beta, 5 - 2(s + \beta)) = (s + \beta, 5 - 2\beta - 2s) \tag{1}$$

We require that $c^*(0) = (2, 1)$. That is,

$$c^*(0) = (\beta, 5 - 2\beta) = (2, 1)$$

or

$$\begin{aligned} \beta &= 2 \\ 5 - 2\beta &= 1 \end{aligned} \quad \Rightarrow \quad \beta = 2$$

Substituting in (1) gives the parametrization

$$c^*(s) = (s + 2, 1 - 2s)$$

5. Find a parametrization $c(\theta)$ of the unit circle such that $c(0) = (-1, 0)$.

SOLUTION The unit circle has the parametrization

$$c(t) = (\cos t, \sin t)$$

This parametrization does not satisfy $c(0) = (-1, 0)$. We replace the parameter t by a parameter θ so that $t = \theta + \alpha$, to obtain another parametrization for the circle:

$$c^*(\theta) = (\cos(\theta + \alpha), \sin(\theta + \alpha)) \tag{1}$$

We need that $c^*(0) = (1, 0)$, that is,

$$c^*(0) = (\cos \alpha, \sin \alpha) = (-1, 0)$$

Hence

$$\begin{aligned} \cos \alpha &= -1 \\ \sin \alpha &= 0 \end{aligned} \quad \Rightarrow \quad \alpha = \pi$$

Substituting in (1) we obtain the following parametrization:

$$c^*(\theta) = (\cos(\theta + \pi), \sin(\theta + \pi))$$

6. Find a path $c(t)$ that traces the parabolic arc $y = x^2$ from $(0, 0)$ to $(3, 9)$ for $0 \le t \le 1$.

SOLUTION The second coordinates of the points on the parabolic arc are the square of the first coordinates. Therefore the points on the arc have the form:

$$c(t) = (\alpha t, \alpha^2 t^2) \tag{1}$$

We need that $c(1) = (3, 9)$. That is,

$$c(1) = (\alpha, \alpha^2) = (3, 9) \Rightarrow \alpha = 3$$

Substituting in (1) gives the following parametrization:

$$c(t) = (3t, 9t^2)$$

7. Find a path $c(t)$ that traces the line $y = 2x + 1$ from $(1, 3)$ to $(3, 7)$ for $0 \le t \le 1$.

SOLUTION Solution 1: By one of the examples in section 12.1, the line through $P = (1, 3)$ with slope 2 has the parametrization

$$c(t) = (1 + t, 3 + 2t)$$

But this parametrization does not satisfy $c(1) = (3, 7)$. We replace the parameter t by a parameter s so that $t = \alpha s + \beta$. We get

$$c^*(s) = (1 + \alpha s + \beta, 3 + 2(\alpha s + \beta)) = (\alpha s + \beta + 1, 2\alpha s + 2\beta + 3)$$

We need that $c^*(0) = (1, 3)$ and $c^*(1) = (3, 7)$. Hence,

$$c^*(0) = (1 + \beta, 3 + 2\beta) = (1, 3)$$
$$c^*(1) = (\alpha + \beta + 1, 2\alpha + 2\beta + 3) = (3, 7)$$

We obtain the equations

$$\begin{aligned} 1 + \beta &= 1 \\ 3 + 2\beta &= 3 \\ \alpha + \beta + 1 &= 3 \\ 2\alpha + 2\beta + 3 &= 7 \end{aligned} \quad \Rightarrow \quad \beta = 0, \alpha = 2$$

Substituting in (1) gives

$$c^*(s) = (2s + 1, 4s + 3)$$

Solution 2: The segment from $(1, 3)$ to $(3, 7)$ has the following vector parametrization:

$$(1 - t)\langle 1, 3\rangle + t\langle 3, 7\rangle = \langle 1 - t + 3t, 3(1 - t) + 7t\rangle = \langle 1 + 2t, 3 + 4t\rangle$$

The parametrization is thus

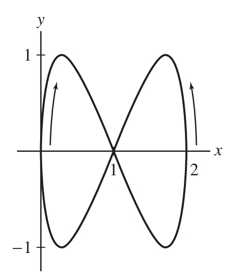
$$c(t) = (1 + 2t, 3 + 4t)$$

8. Sketch the graph $c(t) = (1 + \cos t, \sin 2t)$ for $0 \le t \le 2\pi$ and draw arrows specifying the direction of motion.

SOLUTION From $x = 1 + \cos t$ we have $x - 1 = \cos t$. We substitute this in the y coordinate to obtain

$$y = \sin 2t = 2 \sin t \cos t = \pm 2\sqrt{\sin^2 t} \cos t = \pm 2\sqrt{1 - \cos^2 t} \cos t = \pm 2\sqrt{1 - (x - 1)^2}(x - 1)$$

We can see that the graph is symmetric with respect to the x-axis, hence we plot the function $y = 2\sqrt{1 - (x - 1)^2}(x - 1)$ and reflect it with respect to the x-axis. When $t = 0$ we have $c(0) = (2, 0)$. when t increases near 0, $\cos t$ is decreasing and $\sin 2t$ is increasing, hence the general direction at the point $(2, 0)$ is upwards and left. As t approaches $\pi/2$, the x-coordinate decreases to 1 and the y-coordinate to 0. Likewise, as t moves from $\pi/2$ to π, the x-coordinate moves to 0 while the y-coordinate falls to -1 and then rises to 0. The resulting graph is seen here in the corresponding figure.



Plot of Exercise 8

In Exercises 9–12, express the parametric curve in the form $y = f(x)$.

9. $c(t) = (4t - 3, 10 - t)$

SOLUTION We use the given equation to express t in terms of x.

$$\begin{aligned} x &= 4t - 3 \\ 4t &= x + 3 \\ t &= \frac{x + 3}{4} \end{aligned}$$

Substituting in the equation of y yields

$$y = 10 - t = 10 - \frac{x + 3}{4} = -\frac{x}{4} + \frac{37}{4}$$

That is,

$$y = -\frac{x}{4} + \frac{37}{4}$$

10. $c(t) = (t^3 + 1, t^2 - 4)$

SOLUTION The parametric equations are $x = t^3 + 1$ and $y = t^2 - 4$. We express t in terms of x:

$$\begin{aligned} x &= t^3 + 1 \\ t^3 &= x - 1 \\ t &= (x-1)^{1/3} \end{aligned}$$

Substituting in the equation of y yields

$$y = t^2 - 4 = (x-1)^{2/3} - 4$$

That is,

$$y = (x-1)^{2/3} - 4$$

11. $c(t) = \left(3 - \frac{2}{t}, t^3 + \frac{1}{t}\right)$

SOLUTION We use the given equation to express t in terms of x:

$$\begin{aligned} x &= 3 - \frac{2}{t} \\ \frac{2}{t} &= 3 - x \\ t &= \frac{2}{3-x} \end{aligned}$$

Substituting in the equation of y yields

$$y = \left(\frac{2}{3-x}\right)^3 + \frac{1}{2/(3-x)} = \frac{8}{(3-x)^3} + \frac{3-x}{2}$$

12. $x = \tan t, \quad y = \sec t$

SOLUTION We use the trigonometric identity

$$1 + \tan^2 t = \sec^2 t$$

Substituting the parametric equations $x = \tan t$ and $y = \sec t$ we obtain

$$1 + x^2 = y^2 \quad \text{or} \quad y = \pm\sqrt{x^2 + 1}$$

In Exercises 13–16, calculate dy/dx at the point indicated.

13. $c(t) = (t^3 + t, t^2 - 1), \quad t = 3$

SOLUTION The parametric equations are $x = t^3 + t$ and $y = t^2 - 1$. We use the theorem on the slope of the tangent line to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2 + 1}$$

We now substitute $t = 3$ to obtain

$$\left.\frac{dy}{dx}\right|_{t=3} = \frac{2 \cdot 3}{3 \cdot 3^2 + 1} = \frac{3}{14}$$

14. $c(\theta) = (\tan^2\theta, \cos\theta), \quad \theta = \frac{\pi}{4}$

SOLUTION The parametric equations are $x = \tan^2\theta$, $y = \cos\theta$. We use the theorem on the slope of the tangent line to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\sin\theta}{2\tan\theta\sec^2\theta} = -\frac{\cos^3\theta}{2}$$

We now substitute $\theta = \frac{\pi}{4}$ to obtain

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -\frac{\cos^3\frac{\pi}{4}}{2} = -\frac{1}{4\sqrt{2}}$$

15. $c(t) = (e^t - 1, \sin t), \quad t = 20$

SOLUTION We use the theorem for the slope of the tangent line to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(\sin t)'}{(e^t - 1)'} = \frac{\cos t}{e^t}$$

We now substitute $t = 20$:

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{\cos 20}{e^{20}}$$

16. $c(t) = (\ln t, 3t^2 - t), \quad P = (0, 2)$

SOLUTION The parametric equations are $x = \ln t$, $y = 3t^2 - t$. We use the theorem for the slope of the tangent line to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t - 1}{\frac{1}{t}} = 6t^2 - t \tag{1}$$

We now must identify the value of t corresponding to the point $P = (0, 2)$ on the curve. We solve the following equations:

$$\begin{array}{r} \ln t = 0 \\ 3t^2 - t = 2 \end{array} \quad \Rightarrow \quad t = 1$$

Substituting $t = 1$ in (1) we obtain

$$\left.\frac{dy}{dx}\right|_P = 6 \cdot 1^2 - 1 = 5$$

17. CAS Find the point on the cycloid $c(t) = (t - \sin t, 1 - \cos t)$ where the tangent line has slope $\frac{1}{2}$.

SOLUTION Since $x = t - \sin t$ and $y = 1 - \cos t$, the theorem on the slope of the tangent line gives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1 - \cos t}$$

The points where the tangent line has slope $\frac{1}{2}$ are those where $\frac{dy}{dx} = \frac{1}{2}$. We solve for t:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} \\ \frac{\sin t}{1 - \cos t} &= \frac{1}{2} \\ 2\sin t &= 1 - \cos t \end{aligned} \tag{1}$$

We let $u = \sin t$. Then $\cos t = \pm\sqrt{1 - \sin^2 t} = \pm\sqrt{1 - u^2}$. Hence

$$2u = 1 \pm \sqrt{1 - u^2}$$

We transfer sides and square to obtain

$$\pm\sqrt{1 - u^2} = 2u - 1$$

$$1 - u^2 = 4u^2 - 4u + 1$$
$$5u^2 - 4u = u(5u - 4) = 0$$
$$u = 0,\ u = \frac{4}{5}$$

We find t by the relation $u = \sin t$:

$$u = 0: \quad \sin t = 0 \Rightarrow t = 0, t = \pi$$
$$u = \frac{4}{5}: \quad \sin t = \frac{4}{5} \Rightarrow t \approx 0.93, t \approx 2.21$$

These correspond to the points $(0, 1)$, $(\pi, 2)$, $(0.13, 0.40)$, and $(1.41, 1.60)$, respectively, for $0 < t < 2\pi$.

18. Find the points on $(t + \sin t, t - 2\sin t)$ where the tangent is vertical or horizontal.

SOLUTION We use the theorem for the slope of the tangent line to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - 2\cos t}{1 + \cos t}$$

We find the values of t for which the denominator is zero. We ignore the numerator, since when $1 + \cos t = 0$, $1 - 2\cos t = 3 \neq 0$.

$$1 + \cos t = 0$$
$$\cos t = -1$$
$$t = \pi + 2\pi k \quad \text{where } k \in \mathbb{Z}$$

We now find the values of t for which the numerator is 0:

$$1 - 2\cos t = 0$$
$$1 = 2\cos t$$
$$\frac{1}{2} = \cos t$$
$$t = \pm\frac{\pi}{3} + 2\pi k \quad \text{where } k \in \mathbb{Z}$$

Note that the denominator is not zero at these points. Thus, we have vertical tangents at $t = \pi + 2\pi k$ and horizontal tangents at $t = \pm\pi/3 + 2\pi k$.

19. Find the equation of the Bézier curve with control points

$$P_0 = (-1, -1), \quad P_1 = (-1, 1), \quad P_2 = (1, 1), \quad P_3(1, -1)$$

SOLUTION We substitute the given points in the appropriate formulas in the text to find the parametric equations of the Bézier curve. We obtain

$$\begin{aligned} x(t) &= -(1-t)^3 - 3t(1-t)^2 + t^2(1-t) + t^3 \\ &= -(1 - 3t + 3t^2 - t^3) - (3t - 6t^2 + 3t^3) + (t^2 - t^3) + t^3 \\ &= (-2t^3 + 4t^2 - 1) \\ y(t) &= -(1-t)^3 + 3t(1-t)^2 + t^2(1-t) - t^3 \\ &= -(1 - 3t + 3t^2 - t^3) + (3t - 6t^2 + 3t^3) + (t^2 - t^3) - t^3 \\ &= (2t^3 - 8t^2 + 6t - 1) \end{aligned}$$

20. Find the speed at $t = \frac{\pi}{4}$ of a particle whose position at time t seconds is $c(t) = (\sin 4t, \cos 3t)$.

SOLUTION We use the parametric definition to find the speed. We obtain

$$\frac{ds}{dt} = \sqrt{((\sin 4t)')^2 + ((\cos 3t)')^2} = \sqrt{(4\cos 4t)^2 + (-3\sin 3t)^2} = \sqrt{16\cos^2 4t + 9\sin^2 3t}$$

At time $t = \frac{\pi}{4}$ the speed is

$$\left.\frac{ds}{dt}\right|_{t=\pi/4} = \sqrt{16\cos^2 \pi + 9\sin^2 \frac{3\pi}{4}} = \sqrt{16 + 9 \cdot \frac{1}{2}} = \sqrt{20.5} \approx 4.53$$

21. Find the speed (as a function of t) of a particle whose position at time t seconds is $c(t) = (\sin t + t, \cos t + t)$. What is the particle's maximal speed?

SOLUTION We use the parametric definition to find the speed. We obtain

$$\frac{ds}{dt} = \sqrt{((\sin t + t)')^2 + ((\cos t + t)')^2} = \sqrt{(\cos t + 1)^2 + (1 - \sin t)^2}$$
$$= \sqrt{\cos^2 t + 2\cos t + 1 + 1 - 2\sin t + \sin^2 t} = \sqrt{3 + 2(\cos t - \sin t)}$$

We now differentiate the speed function to find its maximum:

$$\frac{d^2 s}{dt^2} = \left(\sqrt{3 + 2(\cos t - \sin t)}\right)' = \frac{-\sin t - \cos t}{\sqrt{3 + 2(\cos t - \sin t)}}$$

We equate the derivative to zero, to obtain the maximum point:

$$\begin{aligned}
\frac{d^2 s}{dt^2} &= 0 \\
\frac{-\sin t - \cos t}{\sqrt{3 + 2(\cos t - \sin t)}} &= 0 \\
-\sin t - \cos t &= 0 \\
-\sin t &= \cos t \\
\sin(-t) &= \cos(-t) \\
-t &= \frac{\pi}{4} + \pi k \\
t &= -\frac{\pi}{4} + \pi k
\end{aligned}$$

Substituting t in the function of speed we obtain the value of the maximal speed:

$$\sqrt{3 + 2\left(\cos -\frac{\pi}{4} - \sin -\frac{\pi}{4}\right)} = \sqrt{3 + 2\left(\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right)} = \sqrt{3 + 2\sqrt{2}}$$

22. Find the length of $(3e^t - 3, 4e^t + 7)$ for $0 \le t \le 1$.

SOLUTION We use the formula for arc length, to obtain

$$s = \int_0^1 \sqrt{((3e^t - 3)')^2 + ((4e^t + 7)')^2}\, dt = \int_0^1 \sqrt{(3e^t)^2 + (4e^t)^2}\, dt$$
$$= \int_0^1 \sqrt{9e^{2t} + 16e^{2t}}\, dt = \int_0^1 \sqrt{25e^{2t}}\, dt = \int_0^1 5e^t\, dt = 5e^t \Big|_0^1 = 5(e - 1)$$

In Exercises 23 and 24, let $c(t) = (e^{-t}\cos t, e^{-t}\sin t)$.

23. Show that $c(t)$ for $0 \le t < \infty$ has finite length and calculate its value.

SOLUTION We use the formula for arc length, to obtain:

$$\begin{aligned}
s &= \int_0^\infty \sqrt{((e^{-t}\cos t)')^2 + ((e^{-t}\sin t)')^2} dt \\
&= \int_0^\infty \sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2 + (-e^{-t}\sin t + e^{-t}\cos t)^2} dt \\
&= \int_0^\infty \sqrt{e^{-2t}(\cos t + \sin t)^2 + e^{-2t}(\cos t - \sin t)^2} dt \\
&= \int_0^\infty e^{-t}\sqrt{\cos^2 t + 2\sin t\cos t + \sin^2 t + \cos^2 t - 2\sin t\cos t + \sin^2 t}\, dt \\
&= \int_0^\infty e^{-t}\sqrt{2} dt = \sqrt{2}(-e^{-t})\Big|_0^\infty = -\sqrt{2}\left(\lim_{t\to\infty} e^{-t} - e^0\right) \\
&= -\sqrt{2}(0 - 1) = \sqrt{2}
\end{aligned}$$

24. Find the first positive value of t_0 such that the tangent line to $c(t_0)$ is vertical, and calculate the speed at $t = t_0$.

SOLUTION The curve has a vertical tangent where $\lim_{t\to t_0}\left|\frac{dy}{dx}\right| = \infty$. We first find $\frac{dy}{dx}$ using the theorem for the slope of a tangent line:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(e^{-t}\sin t)'}{(e^{-t}\cos t)'} = \frac{-e^{-t}\sin t + e^{-t}\cos t}{-e^{-t}\cos t - e^{-t}\sin t}$$
$$= -\frac{\cos t - \sin t}{\cos t + \sin t} = \frac{\sin t - \cos t}{\sin t + \cos t}$$

We now search for t_0 such that $\lim_{t\to t_0}\left|\frac{dy}{dx}\right| = \infty$. In our case, this happens when the denominator is 0, but the numerator is not, thus:

$$\sin t_0 + \cos t_0 = 0$$
$$\cos t_0 = -\sin t_0$$
$$\cos -t_0 = \sin -t_0$$
$$-t_0 = \frac{\pi}{4} - \pi$$
$$t_0 = \frac{3}{4}\pi$$

We now use the formula for the speed, to find the speed at t_0.

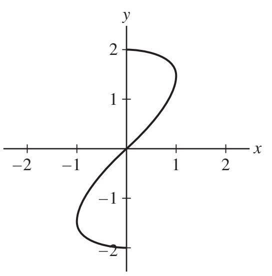
$$\frac{ds}{dt} = \sqrt{((e^{-t}\sin t)')^2 + ((e^{-t}\cos t)')^2}$$
$$= \sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2 + (-e^{-t}\sin t + e^{-t}\cos t)^2}$$
$$= \sqrt{e^{-2t}(\cos t + \sin t)^2 + e^{-2t}(\cos t - \sin t)^2}$$
$$= e^{-t}\sqrt{\cos^2 t + 2\sin t\cos t + \sin^2 t + \cos^2 t - 2\sin t\cos t + \sin^2 t} = e^{-t}\sqrt{2}$$

Next we substitute $t = \frac{3}{4}\pi$, to obtain

$$e^{-t_0}\sqrt{2} = e^{-3\pi/4}\sqrt{2}$$

25. CAS Plot $c(t) = (\sin 2t, 2\cos t)$ for $0 \le t \le \pi$. Express the length of the curve as a definite integral, and approximate it using a computer algebra system.

SOLUTION We use a CAS to plot the curve. The resulting graph is shown here.



Plot of the curve $(\sin 2t, 2\cos t)$

To calculate the arc length we use the formula for the arc length to obtain

$$s = \int_0^{\pi}\sqrt{(2\cos 2t)^2 + (-2\sin t)^2}\,dt = 2\int_0^{\pi}\sqrt{\cos^2 2t + \sin^2 t}\,dt$$

We use a CAS to obtain $s = 6.0972$.

26. Convert the points $(x, y) = (1, -3), (3, -1)$ from rectangular to polar coordinates.

SOLUTION We convert the given points from cartesian coordinates to polar coordinates. For the first point we have

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-3)^2} = \sqrt{10}$$
$$\theta = \arctan \frac{y}{x} = \arctan -3 = 5.034$$

For the second point we have

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-1)^2} = \sqrt{10}$$
$$\theta = \arctan \frac{y}{x} = \arctan \frac{-1}{3} = -0.321, \; 5.961$$

27. Convert the points $(r, \theta) = \left(1, \frac{\pi}{6}\right), \left(3, \frac{5\pi}{4}\right)$ from polar to rectangular coordinates.

SOLUTION We convert the points from polar coordinates to cartesian coordinates. For the first point we have

$$x = r\cos\theta = 1 \cdot \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$
$$y = r\sin\theta = 1 \cdot \sin\frac{\pi}{6} = \frac{1}{2}$$

For the second point we have

$$x = r\cos\theta = 3\cos\frac{5\pi}{4} = -\frac{3\sqrt{2}}{2}$$
$$y = r\sin\theta = 3\sin\frac{5\pi}{4} = -\frac{3\sqrt{2}}{2}$$

28. Write $(x + y)^2 = xy + 6$ as an equation in polar coordinates.

SOLUTION We use the formula for converting from cartesian coordinates to polar coordinates to substitute r and θ for x and y:

$$\begin{aligned}
(x+y)^2 &= xy + 6 \\
x^2 + 2xy + y^2 &= xy + 6 \\
x^2 + y^2 &= -xy + 6 \\
r^2 &= -(r\cos\theta)(r\sin\theta) + 6 \\
r^2 &= -r^2\cos\theta\sin\theta + 6 \\
r^2(1 + \sin\theta\cos\theta) &= 6 \\
r^2 &= \frac{6}{1 + \sin\theta\cos\theta} \\
r^2 &= \frac{6}{1 + \frac{\sin 2\theta}{2}} \\
r^2 &= \frac{12}{2 + \sin 2\theta}
\end{aligned}$$

29. Write $r = \dfrac{2\cos\theta}{\cos\theta - \sin\theta}$ as an equation in rectangular coordinates.

SOLUTION We use the formula for converting from polar coordinates to cartesian coordinates to substitute x and y for r and θ:

$$\begin{aligned}
r &= \frac{2\cos\theta}{\cos\theta - \sin\theta} \\
\sqrt{x^2 + y^2} &= \frac{2r\cos\theta}{r\cos\theta - r\sin\theta} \\
\sqrt{x^2 + y^2} &= \frac{2x}{x - y}
\end{aligned}$$

30. Show that $r = \dfrac{4}{7\cos\theta - \sin\theta}$ is the polar equation of a line.

SOLUTION We use the formula for converting from polar coordinates to cartesian coordinates to substitute x and y for r and θ:

$$\begin{aligned} r &= \frac{4}{7\cos\theta - \sin\theta} \\ 1 &= \frac{4}{7r\cos\theta - r\sin\theta} \\ 1 &= \frac{4}{7x - y} \\ 7x - y &= 4 \\ y &= 7x - 4 \end{aligned}$$

We obtained a linear function. Since the original equation in polar coordinates represents the same curve, it represents a straight line as well.

31. GU Convert the equation

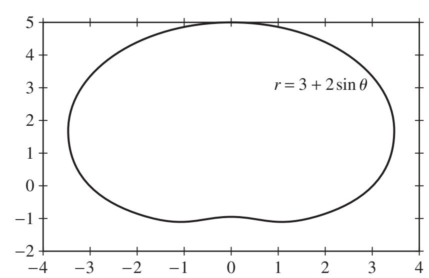
$$9(x^2 + y^2) = (x^2 + y^2 - 2y)^2$$

to polar coordinates, and plot it with a graphing utility.

SOLUTION We use the formula for converting from cartesian coordinates to polar coordinates to substitute r and θ for x and y:

$$\begin{aligned} 9(x^2 + y^2) &= (x^2 + y^2 - 2y)^2 \\ 9r^2 &= (r^2 - 2r\sin\theta)^2 \\ 3r &= r^2 - 2r\sin\theta \\ 3 &= r - 2\sin\theta \\ r &= 3 + 2\sin\theta \end{aligned}$$

The plot of $r = 3 + 2\sin\theta$ is shown here:



Plot of $r = 3 + 2\sin\theta$

32. Calculate the area of the circle $r = 3\sin\theta$ bounded by the rays $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$.

SOLUTION We use the formula for area in polar coordinates to obtain

$$A = \frac{1}{2}\int_{\pi/3}^{2\pi/3} (3\sin\theta)^2\,d\theta = \frac{9}{2}\int_{\pi/3}^{2\pi/3} \sin^2\theta\,d\theta = \frac{9}{4}\int_{\pi/3}^{2\pi/3} (1 - \cos 2\theta)\,d\theta = \frac{9}{4}\left(\theta - \frac{\sin 2\theta}{2}\Bigg|_{\pi/3}^{2\pi/3}\right)$$

$$= \frac{9}{4}\left(\frac{\pi}{3} - \frac{1}{2}\left(\sin\frac{4\pi}{3} - \sin\frac{2\pi}{3}\right)\right) = \frac{9}{4}\left(\frac{\pi}{3} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right) = \frac{9}{4}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$$

33. Calculate the area of one petal of $r = \sin 4\theta$ (see Figure 1).

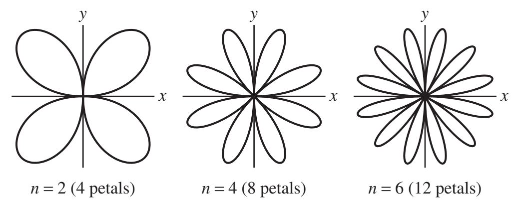
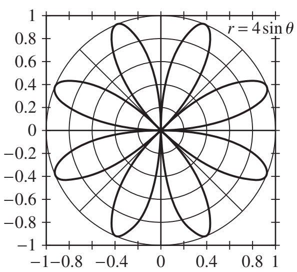


FIGURE 1 Plot of $r = \sin(n\theta)$.

SOLUTION We use a CAS to generate the plot, as shown here.



Plot of $r = \sin 4\theta$

We can see that one leaf lies between the rays $\theta = 0$ and $\theta = \dfrac{\theta}{4}$. We now use the formula for area in polar coordinates to obtain

$$A = \frac{1}{2}\int_0^{\pi/4} \sin^2 4\theta \, d\theta = \frac{1}{4}\int_0^{\pi/4} (1 - \cos 8\theta)\, d\theta = \frac{1}{4}\left(\theta - \frac{\sin 8\theta}{8}\bigg|_0^{\pi/4}\right)$$
$$= \frac{\pi}{16} - \frac{1}{32}(\sin 2\pi - \sin 0) = \frac{\pi}{16}$$

34. The equation $r = \sin(n\theta)$, where $n \geq 2$ is even, is a "rose" of $2n$ petals (Figure 1). Compute the total area of the flower, and show that it does not depend on n.

SOLUTION We calculate the total area of the flower, that is, the area between the rays $\theta = 0$ and $\theta = 2\pi$, using the formula for area in polar coordinates:

$$A = \frac{1}{2}\int_0^{2\pi} \sin^2 2n\theta \, d\theta = \frac{1}{4}\int_0^{2\pi} (1 - \cos 4n\theta)\, d\theta = \frac{1}{4}\left(\theta - \frac{\sin 4n\theta}{4n}\bigg|_0^{2\pi}\right)$$
$$= \frac{\pi}{2} - \frac{1}{16n}(\sin 8n\pi - \sin 0) = \frac{\pi}{2}$$

Since the area is $\dfrac{\pi}{2}$ for every $n \in \mathbb{Z}$, the area is independent of n.

35. Calculate the total area enclosed by the curve $r^2 = \cos\theta e^{\sin\theta}$ (Figure 2).

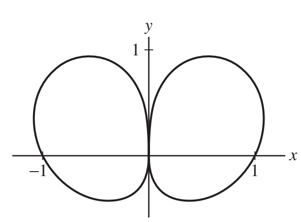


FIGURE 2 Graph of $r^2 = \cos\theta e^{\sin\theta}$.

SOLUTION Note that this is defined only for θ between $-\pi/2$ and $\pi/2$. We use the formula for area in polar coordinates to obtain:

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} r^2 \, d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos\theta e^{\sin\theta}\, d\theta$$

We evaluate the integral by making the substitution $x = \sin\theta\, dx = \cos\theta\, d\theta$:

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos\theta e^{\sin\theta} d\theta = \frac{1}{2}e^x\Big|_{-1}^{1} = \frac{1}{2}\left(e - e^{-1}\right)$$

36. Find the shaded area in Figure 3.

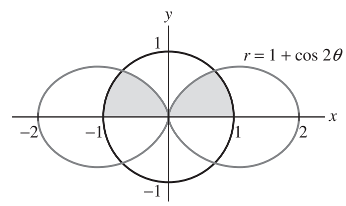


FIGURE 3

SOLUTION We first find the points of intersection between the unit circle and the function.

$$\begin{aligned} 1 &= 1 + \cos 2\theta \\ \cos 2\theta &= 0 \\ 2\theta &= \frac{\pi}{2} + \pi n \\ \theta &= \frac{\pi}{4} + \frac{\pi}{2}n \end{aligned}$$

We now find the area of the shaded figure in the first quadrant. This has two parts. The first, from 0 to $\pi/4$, is just an octant of the unit circle, and thus has area $\pi/8$. The second, from $\pi/4$ to $\pi/2$, is found as follows:

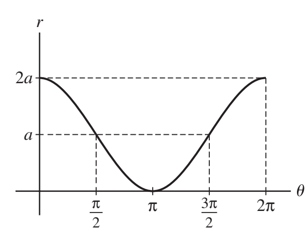
$$A = \frac{1}{2}\int_{\pi/4}^{\pi/2} (1 + \cos 2\theta)^2\, d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta\, d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\, d\theta$$

$$= \frac{1}{2}\left(\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right)\Big|_{\pi/4}^{\pi/2} = \frac{1}{2}\left(\frac{3\pi}{8} - 1\right)$$

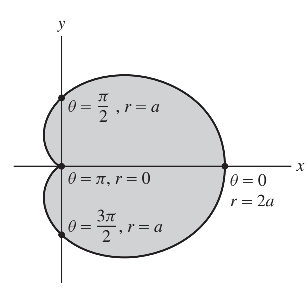
The total area in the first quadrant is thus $\frac{5\pi}{16} - \frac{1}{2}$; multiply by 2 to get the total area of $\frac{5\pi}{8} - 1$.

37. Find the area enclosed by the cardioid $r = a(1 + \cos\theta)$, where $a > 0$.

SOLUTION The graph of $r = a\,(1 + \cos\theta)$ in the $r\theta$-plane for $0 \le \theta \le 2\pi$ and the cardioid in the xy-plane are shown in the following figures:



$r = a\,(1 + \cos\theta)$



The cardioid $r = a\,(1 + \cos\theta)$, $a > 0$

As θ varies from 0 to π the radius r decreases from $2a$ to 0, and this gives the upper part of the cardioid.

The lower part is traced as θ varies from π to 2π and consequently r increases from 0 back to $2a$. We compute the area enclosed by the upper part of the cardioid and the x-axis, using the following integral (we use the identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$):

$$\begin{aligned} \frac{1}{2}\int_0^{\pi} r^2\, d\theta &= \frac{1}{2}\int_0^{\pi} a^2(1 + \cos\theta)^2\, d\theta = \frac{a^2}{2}\int_0^{\pi}\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta \\ &= \frac{a^2}{2}\int_0^{\pi}\left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = \frac{a^2}{2}\int_0^{\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta \\ &= \frac{a^2}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]\Big|_0^{\pi} = \frac{a^2}{2}\left[\frac{3\pi}{2} + 2\sin\pi + \frac{1}{4}\sin 2\pi - 0\right] = \frac{3\pi a^2}{4} \end{aligned}$$

Using symmetry, the total area A enclosed by the cardioid is

$$A = 2 \cdot \frac{3\pi a^2}{4} = \frac{3\pi a^2}{2}$$

38. Calculate the length of the curve with polar equation $r = \theta$ in Figure 4.

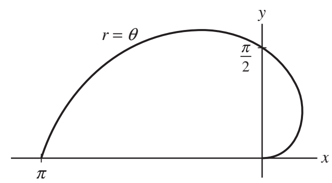


FIGURE 4

SOLUTION The interval of θ values is $0 \le \theta \le \pi$. We use the formula for the arc length in polar coordinates, with $r = f(\theta) = \theta$. We get

$$S = \int_0^{\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{\pi} \sqrt{\theta^2 + (\theta')^2}\, d\theta = \int_0^{\pi} \sqrt{\theta^2 + 1}\, d\theta$$
$$= \frac{\theta}{2}\sqrt{\theta^2+1} + \frac{1}{2}\ln\left|\theta + \sqrt{\theta^2+1}\right|\Bigg|_{\theta=0}^{\pi} = \frac{\pi}{2}\sqrt{\pi^2+1} + \frac{1}{2}\ln\left(\pi + \sqrt{\pi^2+1}\right)$$

39. CAS Figure 5 shows the graph of $r = e^{0.5\theta}\sin\theta$ for $0 \le \theta \le 2\pi$. Use a computer algebra system to approximate the difference in length between the outer and inner loops.

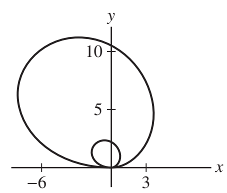


FIGURE 5

SOLUTION We note that the inner loop is the curve for $\theta \in [0, \pi]$, and the outer loop is the curve for $\theta \in [\pi, 2\pi]$. We express the length of these loops using the formula for the arc length. The length of the inner loop is

$$s_1 = \int_0^{\pi} \sqrt{(e^{0.5\theta}\sin\theta)^2 + ((e^{0.5\theta}\sin\theta)')^2} d\theta = \int_0^{\pi} \sqrt{e^{\theta}\sin^2\theta + \left(\frac{e^{0.5\theta}\sin\theta}{2} + e^{0.5\theta}\cos\theta\right)^2}\, d\theta$$

and the length of the outer loop is

$$s_2 = \int_{\pi}^{2\pi} \sqrt{e^{\theta}\sin^2\theta + \left(\frac{e^{0.5\theta}\sin\theta}{2} + e^{0.5\theta}\cos\theta\right)^2}\, d\theta$$

We now use the CAS to calculate the arc length of each of the loops. We obtain that the length of the inner loop is 7.5087 and the length of the outer loop is 36.121, hence the outer one is 4.81 times longer than the inner one.

40. Show that $r = f_1(\theta)$ and $r = f_2(\theta)$ define the same curves in polar coordinates if $f_1(\theta) = -f_2(\theta + \pi)$. Use this to show that the following define the same conic section:

$$r = \frac{de}{1 - e\cos\theta}, \qquad r = \frac{-de}{1 + e\cos\theta}$$

SOLUTION Suppose (r, θ) lies on the curve $r = f_2(\theta)$. Since (r, θ) and $(-r, \theta + \pi)$ define the same point in polar coordinates, we have $-r = f_2(\theta + \pi) = -f_1(\theta)$, so that $r = f_1(\theta)$, Thus (r, θ) lies on f_1 as well. Conversely, suppose (r, θ) lies on $r = f_1(\theta)$. Since (r, θ) and $(-r, \theta - \pi)$ define the same point in polar coordinates, we have $-r = f_1(\theta - \pi) = -f_2(\theta - \pi + \pi) = -f_2(\theta)$ so that $r = f_2(\theta)$ and (r, θ) lies on f_2 as well. Thus the two equations define exactly the same set of points.

Now set

$$f_1(\theta) = \frac{de}{1 - e\cos\theta} \qquad f_2(\theta) = -\frac{de}{1 + e\cos\theta}$$

and consider the polar equations $r = f_1(\theta)$ and $r = f_2(\theta)$. We have

$$-f_2(\theta + \pi) = -\frac{-de}{1 + e\cos(\theta + \pi)} = \frac{de}{1 - e\cos\theta} = f_1(\theta)$$

so that by the above, the two equations define the same conic section.

In Exercises 41–44, identify the conic section. Find the vertices and foci.

41. $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

SOLUTION This is an ellipse in standard position. Its foci are $(\pm\sqrt{3^2 - 2^2}, 0) = (\pm\sqrt{5}, 0)$ and its vertices are $(\pm 3, 0)$, $(0, \pm 2)$.

42. $x^2 - 2y^2 = 4$

SOLUTION We divide the equation by 4 to obtain

$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{\sqrt{2}}\right)^2 = 1$$

This is a hyperbola in standard position, its foci are $\left(\pm\sqrt{2^2 + \sqrt{2}^2}, 0\right) = (\pm\sqrt{6}, 0)$, and its vertices are $(\pm 2, 0)$.

43. $\left(2x + \frac{1}{2}y\right)^2 = 4 - (x - y)^2$

SOLUTION We simplify the equation:

$$\begin{aligned}
\left(2x + \frac{1}{2}y\right)^2 &= 4 - (x - y)^2 \\
4x^2 + 2xy + \frac{1}{4}y^2 &= 4 - x^2 + 2xy - y^2 \\
5x^2 + \frac{5}{4}y^2 &= 4 \\
\frac{5x^2}{4} + \frac{5y^2}{16} &= 1 \\
\left(\frac{x}{\frac{2}{\sqrt{5}}}\right)^2 + \left(\frac{y}{\frac{4}{\sqrt{5}}}\right)^2 &= 1
\end{aligned}$$

This is an ellipse in standard position, with foci $\left(0, \pm\sqrt{\left(\frac{4}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}}\right)^2}\right) = \left(0, \pm\sqrt{\frac{12}{5}}\right)$ and vertices $\left(\pm\frac{2}{\sqrt{5}}, 0\right)$, $\left(0, \pm\frac{4}{\sqrt{5}}\right)$.

44. $(y - 3)^2 = 2x^2 - 1$

SOLUTION We simplify the equation:

$$\begin{aligned}
(y - 3)^2 &= 2x^2 - 1 \\
2x^2 - (y - 3)^2 &= 1 \\
\left(\frac{x}{\frac{1}{\sqrt{2}}}\right)^2 - (y - 3)^2 &= 1
\end{aligned}$$

This is a hyperbola shifted 3 units on the y-axis. Therefore, its foci are $\left(\pm\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 1}, 3\right) = \left(\pm\sqrt{\frac{3}{2}}, 3\right)$ and its vertices are $\left(\pm\frac{1}{\sqrt{2}}, 3\right)$.

In Exercises 45–50, find the equation of the conic section indicated.

45. Ellipse with vertices $(\pm 8, 0)$ and foci $(\pm\sqrt{3}, 0)$

SOLUTION Since the foci of the desired ellipse are on the x-axis, we conclude that $a > b$. We are given that the points $(\pm 8, 0)$ are vertices of the ellipse, and since they are on the x-axis, $a = 8$. We are given that the foci are $(\pm\sqrt{3}, 0)$ and we have shown that $a > b$, hence we have that $\sqrt{a^2 - b^2} = \sqrt{3}$. Solving for b yields

$$\sqrt{a^2 - b^2} = \sqrt{3}$$
$$a^2 - b^2 = 3$$
$$8^2 - b^2 = 3$$
$$b^2 = 61$$
$$b = \sqrt{61}$$

Next we use a and b to construct the equation of the ellipse:

$$\left(\frac{x}{8}\right)^2 + \left(\frac{y}{\sqrt{61}}\right)^2 = 1.$$

46. Ellipse with foci $(\pm 8, 0)$, eccentricity $\frac{1}{8}$

SOLUTION If the foci are on the x-axis, then $a > b$, and $c = \sqrt{a^2 - b^2}$. We are given that $e = \frac{1}{8}$, and $c = 8$. Substituting and solving for a and b yields

$$e = \frac{c}{a}$$
$$c = \sqrt{a^2 - b^2}$$
$$\frac{1}{8} = \frac{8}{a}$$
$$64 = a$$
$$8 = \sqrt{64^2 - b^2}$$
$$64 = 64^2 - b^2$$
$$b^2 = 64 \cdot 63$$
$$b = 8\sqrt{63}$$

We use a and b to construct the equation of the ellipse:

$$\left(\frac{x}{64}\right)^2 + \left(\frac{y}{8\sqrt{63}}\right)^2 = 1.$$

47. Hyperbola with vertices $(\pm 8, 0)$, asymptotes $y = \pm\frac{3}{4}x$

SOLUTION Since the asymptotes of the hyperbola are $y = \pm\frac{3}{4}x$, and the equation of the asymptotes for a general hyperbola in standard position is $y = \pm\frac{b}{a}x$, we conclude that $\frac{b}{a} = \frac{3}{4}$. We are given that the vertices are $(\pm 8, 0)$, thus $a = 8$. We substitute and solve for b:

$$\frac{b}{a} = \frac{3}{4}$$
$$\frac{b}{8} = \frac{3}{4}$$
$$b = 6$$

Next we use a and b to construct the equation of the hyperbola:

$$\left(\frac{x}{8}\right)^2 - \left(\frac{y}{6}\right)^2 = 1.$$

48. Hyperbola with foci $(2, 0)$ and $(10, 0)$, eccentricity $e = 4$

SOLUTION Since the foci lie on the x axis, the x is the focal axis. The center of the hyperbola is midway between the foci, so lies at $(6, 0)$, and $c = 4$. Then $c = ae$ gives $a = 1$; then $b = \sqrt{c^2 - a^2} = \sqrt{15}$, so that the equation of the hyperbola is

$$(x - 6)^2 - \left(\frac{y}{\sqrt{15}}\right)^2 = 1$$

49. Parabola with focus $(8, 0)$, directrix $x = -8$

SOLUTION This is similar to the usual equation of a parabola, but we must use y as x, and x as y, to obtain

$$x = \frac{1}{32}y^2.$$

50. Parabola with vertex $(4, -1)$, directrix $x = 15$

SOLUTION The directrix is a vertical line and the vertex is $(4, -1)$, so the equation is of the form

$$x - 4 = \frac{1}{4c}(y+1)^2$$

The directrix is to the right of the vertex; the distance from the directrix to the vertex is -11, so $c = -11$ and the equation is

$$x = 4 - \frac{1}{44}(y+1)^2$$

51. Find the asymptotes of the hyperbola $3x^2 + 6x - y^2 - 10y = 1$.

SOLUTION We complete the squares and simplify:

$$\begin{aligned} 3x^2 + 6x - y^2 - 10y &= 1 \\ 3(x^2 + 2x) - (y^2 + 10y) &= 1 \\ 3(x^2 + 2x + 1 - 1) - (y^2 + 10y + 25 - 25) &= 1 \\ 3(x+1)^2 - 3 - (y+5)^2 + 25 &= 1 \\ 3(x+1)^2 - (y+5)^2 &= -21 \\ \left(\frac{y+5}{\sqrt{21}}\right)^2 - \left(\frac{x+1}{\sqrt{7}}\right)^2 &= 1 \end{aligned}$$

We obtained a hyperbola with focal axis that is parallel to the y-axis, and is shifted -5 units on the y-axis, and -1 units in the x-axis. Therefore, the asymptotes are

$$x + 1 = \pm\frac{\sqrt{7}}{\sqrt{21}}(y+5) \quad \text{or} \quad y + 5 = \pm\sqrt{3}(x+1).$$

52. Show that the "conic section" with equation $x^2 - 4x + y^2 + 5 = 0$ has no points.

SOLUTION We complete the squares in the given equation:

$$\begin{aligned} x^2 - 4x + 4y^2 + 5 &= 0 \\ x^2 - 4x + 4 - 4 + 4y^2 + 5 &= 0 \\ (x-2)^2 + 4y^2 &= -1 \end{aligned}$$

Since $(x-2)^2 \ge 0$ and $y^2 \ge 0$, there is no point satisfying the equation, hence it cannot represent a conic section.

53. Show that the relation $\frac{dy}{dx} = (e^2 - 1)\frac{x}{y}$ holds on a standard ellipse or hyperbola of eccentricity e.

SOLUTION We differentiate the equations of the standard ellipse and the hyperbola with respect to x:

$$\begin{array}{ll} \text{Ellipse:} & \text{Hyperbola:} \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 & \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \\ \dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0 & \dfrac{2x}{a^2} - \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0 \\ \dfrac{dy}{dx} = -\dfrac{b^2}{a^2}\dfrac{x}{y} & \dfrac{dy}{dx} = \dfrac{b^2}{a^2}\dfrac{x}{y} \end{array}$$

The eccentricity of the ellipse is $e = \frac{\sqrt{a^2-b^2}}{a}$, hence $e^2a^2 = a^2 - b^2$ or $e^2 = 1 - \frac{b^2}{a^2}$ yielding $\frac{b^2}{a^2} = 1 - e^2$.

The eccentricity of the hyperbola is $e = \frac{\sqrt{a^2+b^2}}{a}$, hence $e^2a^2 = a^2 + b^2$ or $e^2 = 1 + \frac{b^2}{a^2}$, giving $\frac{b^2}{a^2} = e^2 - 1$. Combining with the expressions for $\frac{dy}{dx}$ we get:

Ellipse: $$\frac{dy}{dx} = -(1-e^2)\frac{x}{y} = (e^2-1)\frac{x}{y}$$

Hyperbola: $$\frac{dy}{dx} = (e^2-1)\frac{x}{y}$$

We, thus, proved that the relation $\frac{dy}{dx} = (e^2-1)\frac{x}{y}$ holds on a standard ellipse or hyperbola of eccentricity e.

54. The orbit of Jupiter is an ellipse with the sun at a focus. Find the eccentricity of the orbit if the perihelion (closest distance to the sun) equals 740×10^6 km and the aphelion (farthest distance from the sun) equals 816×10^6 km.

SOLUTION For the sake of simplicity, we treat all numbers in units of 10^6 km. By Kepler's First Law we conclude that the sun is at one of the foci of the ellipse. Therefore, the closest and farthest points to the sun are vertices. Moreover, they are the vertices on the x-axis, hence we conclude that the distance between the two vertices is

$$2a = 740 + 816 = 1556$$

Since the distance between each focus and the vertex that is closest to it is the same distance, and since $a = 778$, we conclude that the distance between the foci is

$$c = a - 740 = 38$$

We substitute this in the formula for the eccentricity to obtain:

$$e = \frac{c}{a} = 0.0488.$$

55. Refer to Figure 25 in Section 11.5. Prove that the product of the perpendicular distances F_1R_1 and F_2R_2 from the foci to a tangent line of an ellipse is equal to the square b^2 of the semiminor axes.

SOLUTION We first consider the ellipse in standard position:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The equation of the tangent line at $P = (x_0, y_0)$ is

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

or

$$b^2x_0x + a^2y_0y - a^2b^2 = 0$$

The distances of the foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ from the tangent line are

$$\overline{F_1R_1} = \frac{|b^2x_0c - a^2b^2|}{\sqrt{b^4x_0^2 + a^4y_0^2}}; \quad \overline{F_2R_2} = \frac{|b^2x_0c + a^2b^2|}{\sqrt{b^4x_0^2 + a^4y_0^2}}$$

We compute the product of the distances:

$$\overline{F_1R_1} \cdot \overline{F_2R_2} = \left| \frac{\left(b^2x_0c - a^2b^2\right)\left(b^2x_0c + a^2b^2\right)}{b^4x_0^2 + a^4y_0^2} \right| = \left| \frac{b^4x_0^2c^2 - a^4b^4}{b^4x_0^2 + a^4y_0^2} \right| \tag{1}$$

The point $P = (x_0, y_0)$ lies on the ellipse, hence:

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1 \Rightarrow a^4y_0^2 = a^4b^2 - a^2b^2x_0^2$$

We substitute in (1) to obtain (notice that $b^2 - a^2 = -c^2$)

$$\overline{F_1R_1} \cdot \overline{F_2R_2} = \frac{|b^4x_0^2c^2 - a^4b^4|}{|b^4x_0^2 + a^4b^2 - a^2b^2x_0^2|} = \frac{|b^4x_0^2c^2 - a^4b^4|}{|b^2(b^2 - a^2)x_0^2 + a^4b^2|}$$
$$= \frac{|b^4x_0^2c^2 - a^4b^4|}{|-b^2x_0^2c^2 + a^4b^2|} = \frac{|b^2(x_0^2c^2 - a^4)|}{|-(x_0^2c^2 - a^4)|} = |-b^2| = b^2$$

The product $\overline{F_1R_1} \cdot \overline{F_2R_2}$ remains unchanged if we translate the standard ellipse.