9 INTRODUCTION TO DIFFERENTIAL EQUATIONS

9.1 Solving Differential Equations

Preliminary Questions

1. Determine the order of the following differential equations:

(a) $x^5 y' = 1$

(b) $(y')^3 + x = 1$

(c) $y''' + x^4 y' = 2$

(d) $\sin(y'') + x = y$

SOLUTION

(a) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

(b) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

(c) The highest order derivative that appears in this equation is a third derivative, so this is a third order equation.

(d) The highest order derivative that appears in this equation is a second derivative, so this is a second order equation.

2. Is $y'' = \sin x$ a linear differential equation?

SOLUTION Yes.

3. Give an example of a nonlinear differential equation of the form $y' = f(y)$.

SOLUTION One possibility is $y' = y^2$.

4. Can a nonlinear differential equation be separable? If so, give an example.

SOLUTION Yes. An example is $y' = y^2$.

5. Give an example of a linear, nonseparable differential equation.

SOLUTION One example is $y' + y = x$.

Exercises

1. Which of the following differential equations are first-order?

(a) $y' = x^2$

(b) $y'' = y^2$

(c) $(y')^3 + yy' = \sin x$

(d) $x^2 y' - e^x y = \sin y$

(e) $y'' + 3y' = \dfrac{y}{x}$

(f) $yy' + x + y = 0$

SOLUTION

(a) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

(b) The highest order derivative that appears in this equation is a second derivative, so this is not a first order equation.

(c) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

(d) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

(e) The highest order derivative that appears in this equation is a second derivative, so this is not a first order equation.

(f) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

2. Which of the equations in Exercise 1 are linear?

SOLUTION

(a) Linear; $(1)y' - x^2 = 0$.

(b) Not linear; y^2 is not a linear function of y.

(c) Not linear; $(y')^3$ is not a linear function of y'.

(d) Not linear; $\sin y$ is not a linear function of y.

(e) Linear; $(1)y'' + (3)y' - \frac{1}{x}y = 0$.

(f) Not linear. yy' cannot be expressed as $a(x)y^{(n)}$.

In Exercises 3–8, verify that the given function is a solution of the differential equation.

3. $y' - 8x = 0, \quad y = 4x^2$

SOLUTION Let $y = 4x^2$. Then $y' = 8x$ and

$$y' - 8x = 8x - 8x = 0.$$

4. $yy' + 4x = 0, \quad y = \sqrt{12 - 4x^2}$

SOLUTION Let $y = \sqrt{12 - 4x^2}$. Then

$$y' = \frac{-4x}{\sqrt{12 - 4x^2}},$$

and

$$yy' + 4x = \sqrt{12 - 4x^2}\,\frac{-4x}{\sqrt{12 - 4x^2}} + 4x = -4x + 4x = 0.$$

5. $y' + 4xy = 0, \quad y = 25e^{-2x^2}$

SOLUTION Let $y = 25e^{-2x^2}$. Then $y' = -100xe^{-2x^2}$, and

$$y' + 4xy = -100xe^{-2x^2} + 4x(25e^{-2x^2}) = 0.$$

6. $(x^2 - 1)y' + xy = 0, \quad y = 4(x^2 - 1)^{-1/2}$

SOLUTION Let $y = 4(x^2 - 1)^{-1/2}$. Then $y' = -4x(x^2 - 1)^{-3/2}$, and

$$\begin{aligned}(x^2 - 1)y' + xy &= (x^2 - 1)(-4x)(x^2 - 1)^{-3/2} + 4x(x^2 - 1)^{-1/2}\\ &= -4x(x^2 - 1)^{-1/2} + 4x(x^2 - 1)^{-1/2} = 0.\end{aligned}$$

7. $y'' - 2xy' + 8y = 0, \quad y = 4x^4 - 12x^2 + 3$

SOLUTION Let $y = 4x^4 - 12x^2 + 3$. Then $y' = 16x^3 - 24x$, $y'' = 48x^2 - 24$, and

$$\begin{aligned}y'' - 2xy' + 8y &= (48x^2 - 24) - 2x(16x^3 - 24x) + 8(4x^4 - 12x^2 + 3)\\ &= 48x^2 - 24 - 32x^4 + 48x^2 + 32x^4 - 96x^2 + 24 = 0.\end{aligned}$$

8. $y'' - 2y' + 5y = 0, \quad y = e^x \sin 2x$

SOLUTION Let $y = e^x \sin 2x$. Then

$$\begin{aligned}y' &= 2e^x \cos 2x + e^x \sin 2x,\\ y'' &= -4e^x \sin 2x + 2e^x \cos 2x + 2e^x \cos 2x + e^x \sin 2x = -3e^x \sin 2x + 4e^x \cos 2x,\end{aligned}$$

and

$$\begin{aligned}y'' - 2y' + 5y &= -3e^x \sin 2x + 4e^x \cos 2x - 4e^x \cos 2x - 2e^x \sin 2x + 5e^x \sin 2x\\ &= (-3e^x - 2e^x + 5e^x)\sin 2x + (4e^x - 4e^x)\cos 2x = 0.\end{aligned}$$

9. Which of the following equations are separable? Write those that are separable in the form $y' = f(x)g(y)$ (but do not solve).

(a) $xy' - 9y^2 = 0$

(b) $\sqrt{4 - x^2}\,y' = e^{3y} \sin x$

(c) $y' = x^2 + y^2$

(d) $y' = 9 - y^2$

SOLUTION

(a) $xy' - 9y^2 = 0$ is separable:

$$\begin{aligned}xy' - 9y^2 &= 0\\ xy' &= 9y^2\\ y' &= \frac{9}{x}y^2\end{aligned}$$

(b) $\sqrt{4-x^2}\,y' = e^{3y}\sin x$ is separable:

$$\sqrt{4-x^2}\,y' = e^{3y}\sin x$$
$$y' = e^{3y}\frac{\sin x}{\sqrt{4-x^2}}.$$

(c) $y' = x^2 + y^2$ is not separable; y' is already isolated, but is not equal to a product $f(x)g(y)$.
(d) $y' = 9 - y^2$ is separable: $y' = (1)(9-y^2)$.

10. The following differential equations appear similar but have very different solutions.

$$\frac{dy}{dx} = x, \qquad \frac{dy}{dx} = y$$

Solve both subject to the initial condition $y(1) = 2$.

SOLUTION For the first differential equation, we have $y' = x$ so that, integrating,

$$y = \frac{x^2}{2} + C$$

Since $y(1) = 2$, $C = \frac{3}{2}$, so that

$$y = \frac{x^2+3}{2}$$

The second equation is separable: $y^{-1}\,dy = 1\,dx$, so that $\ln|y| = x + C$ and $y = Ce^x$. Since $y(1) = 2$, we have $2 = Ce$ or $C = 2e^{-1}$. Thus $y = 2e^{x-1}$.

11. Consider the differential equation $y^3y' - 9x^2 = 0$.
(a) Write it as $y^3\,dy = 9x^2\,dx$.
(b) Integrate both sides to obtain $\frac{1}{4}y^4 = 3x^3 + C$.
(c) Verify that $y = (12x^3 + C)^{1/4}$ is the general solution.
(d) Find the particular solution satisfying $y(1) = 2$.

SOLUTION Solving $y^3y' - 9x^2 = 0$ for y' gives $y' = 9x^2y^{-3}$.
(a) Separating variables in the equation above yields

$$y^3\,dy = 9x^2\,dx$$

(b) Integrating both sides gives

$$\frac{y^4}{4} = 3x^3 + C$$

(c) Simplify the equation above to get $y^4 = 12x^3 + C$, or $y = (12x^3 + C)^{1/4}$.
(d) Solve $2 = (12\cdot 1^3 + C)^{1/4}$ to get $16 = 12 + C$, or $C = 4$. Thus the particular solution is $y = (12x^3 + 4)^{1/4}$.

12. Verify that $x^2y' + e^{-y} = 0$ is separable.
(a) Write it as $e^y\,dy = -x^{-2}\,dx$.
(b) Integrate both sides to obtain $e^y = x^{-1} + C$.
(c) Verify that $y = \ln(x^{-1} + C)$ is the general solution.
(d) Find the particular solution satisfying $y(2) = 4$.

SOLUTION Solving $x^2y' + e^{-y} = 0$ for y' yields

$$y' = -x^{-2}e^{-y}.$$

(a) Separating variables in the last equation yields

$$e^y dy = -x^{-2}dx.$$

(b) Integrating both sides of the result of part (a), we find

$$\int e^y dy = -\int x^{-2}dx$$
$$e^y + C_1 = x^{-1} + C_2$$
$$e^y = x^{-1} + C$$

(c) Solving the last expression from part (b) for y, we find

$$y = \ln\left|x^{-1} + C\right|$$

(d) $y(2) = 4$ yields $4 = \ln\left|\frac{1}{2} + C\right|$, or $e^4 = C + \frac{1}{2}$. Thus the particular solution is

$$y = \ln\left|\frac{1}{x} - \frac{1}{2} + e^4\right|$$

In Exercises 13–28, use separation of variables to find the general solution.

13. $y' + 4xy^2 = 0$

SOLUTION Rewrite

$$y' + 4xy^2 = 0 \quad \text{as} \quad \frac{dy}{dx} = -4xy^2 \quad \text{and then as} \quad y^{-2}\,dy = -4x\,dx$$

Integrating both sides of this equation gives

$$\int y^{-2}\,dy = -4\int x\,dx$$
$$-y^{-1} = -2x^2 + C$$
$$y^{-1} = 2x^2 + C$$

Solving for y gives

$$y = \frac{1}{2x^2 + C}$$

where C is an arbitrary constant.

14. $y' + x^2y = 0$

SOLUTION Rewrite

$$y' + x^2y = 0 \quad \text{as} \quad \frac{dy}{dx} = -x^2y \quad \text{and then as} \quad y^{-1}\,dy = -x^2\,dx$$

Integrating both sides of this equation gives

$$\int y^{-1}\,dy = -\int x^2\,dx$$
$$\ln|y| = -\frac{x^3}{3} + C_1$$

Solve for y to get

$$y = \pm e^{-x^3/3 + C_1} = Ce^{-x^3/3}$$

where $C = \pm e^{C_1}$ is an arbitrary constant.

15. $\dfrac{dy}{dt} - 20t^4e^{-y} = 0$

SOLUTION Rewrite

$$\frac{dy}{dt} - 20t^4e^{-y} = 0 \quad \text{as} \quad \frac{dy}{dt} = 20t^4e^{-y} \quad \text{and then as} \quad e^y\,dy = 20t^4\,dt$$

Integrating both sides of this equation gives

$$\int e^y\,dy = \int 20t^4\,dt$$
$$e^y = 4t^5 + C$$

Solve for y to get $y = \ln(4t^5 + C)$, where C is an arbitrary constant.

16. $t^3y' + 4y^2 = 0$

SOLUTION Rewrite

$$t^3y' + 4y^2 = 0 \quad \text{as} \quad \frac{dy}{dt} = -4y^2t^{-3} \quad \text{and then as} \quad y^{-2}\,dy = -4t^{-3}\,dt$$

Integrating both sides of this equation gives

$$\int y^{-2}\,dy = -4\int t^{-3}\,dt$$
$$-y^{-1} = 2t^{-2} + C$$

Solve for y to get

$$y = \frac{-1}{2t^{-2} + C} = \frac{-t^2}{2 + Ct^2}$$

where C is an arbitrary constant.

17. $2y' + 5y = 4$

SOLUTION Rewrite

$$2y' + 5y = 4 \quad \text{as} \quad y' = 2 - \frac{5}{2}y \quad \text{and then as} \quad (4 - 5y)^{-1}\,dy = \frac{1}{2}\,dx$$

Integrating both sides and solving for y gives

$$\begin{aligned} \int \frac{dy}{4 - 5y} &= \frac{1}{2}\int 1\,dx \\ -\frac{1}{5}\ln|4 - 5y| &= \frac{1}{2}x + C_1 \\ \ln|4 - 5y| &= C_2 - \frac{5}{2}x \\ 4 - 5y &= C_3 e^{-5x/2} \\ 5y &= 4 - C_3 e^{-5x/2} \\ y &= Ce^{-5x/2} + \frac{4}{5} \end{aligned}$$

where C is an arbitrary constant.

18. $\dfrac{dy}{dt} = 8\sqrt{y}$

SOLUTION Rewrite

$$\frac{dy}{dt} = 8\sqrt{y} \qquad \text{as} \qquad \frac{dy}{\sqrt{y}} = 8\,dt.$$

Integrating both sides of this equation yields

$$\begin{aligned} \int \frac{dy}{\sqrt{y}} &= 8\int dt \\ 2\sqrt{y} &= 8t + C. \end{aligned}$$

Solving for y, we find

$$\begin{aligned} \sqrt{y} &= 4t + C \\ y &= (4t + C)^2, \end{aligned}$$

where C is an arbitrary constant.

19. $\sqrt{1 - x^2}\,y' = xy$

SOLUTION Rewrite

$$\sqrt{1 - x^2}\,\frac{dy}{dx} = xy \qquad \text{as} \qquad \frac{dy}{y} = \frac{x}{\sqrt{1 - x^2}}\,dx.$$

Integrating both sides of this equation yields

$$\begin{aligned} \int \frac{dy}{y} &= \int \frac{x}{\sqrt{1 - x^2}}\,dx \\ \ln|y| &= -\sqrt{1 - x^2} + C. \end{aligned}$$

Solving for y, we find

$$\begin{aligned} |y| &= e^{-\sqrt{1-x^2}+C} = e^C e^{-\sqrt{1-x^2}} \\ y &= \pm e^C e^{-\sqrt{1-x^2}} = Ae^{-\sqrt{1-x^2}}, \end{aligned}$$

where A is an arbitrary constant.

20. $y' = y^2(1 - x^2)$

SOLUTION Rewrite

$$\frac{dy}{dx} = y^2(1 - x^2) \qquad \text{as} \qquad \frac{dy}{y^2} = (1 - x^2)\,dx.$$

Integrating both sides of this equation yields

$$\int \frac{dy}{y^2} = \int (1 - x^2)\,dx$$
$$-y^{-1} = x - \frac{1}{3}x^3 + C.$$

Solving for y, we find

$$y^{-1} = \frac{1}{3}x^3 - x + C$$
$$y = \frac{1}{\frac{1}{3}x^3 - x + C},$$

where C is an arbitrary constant.

21. $yy' = x$

SOLUTION Rewrite

$$y\frac{dy}{dx} = x \qquad \text{as} \qquad y\,dy = x\,dx.$$

Integrating both sides of this equation yields

$$\int y\,dy = \int x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C.$$

Solving for y, we find

$$y^2 = x^2 + 2C$$
$$y = \pm\sqrt{x^2 + A},$$

where $A = 2C$ is an arbitrary constant.

22. $(\ln y)y' - ty = 0$

SOLUTION Rewrite

$$(\ln y)y' - ty = 0 \quad \text{as} \quad (\ln y)\frac{dy}{dt} = ty \quad \text{and then as} \quad \frac{\ln y}{y}\,dy = t\,dt$$

Integrating both sides of this equation gives

$$\int \frac{\ln y}{y}\,dy = \int t\,dt$$
$$\frac{1}{2}\ln^2 y = \frac{1}{2}t^2 + C_1$$
$$\ln^2 y = t^2 + C$$
$$\ln y = \pm\sqrt{t^2 + C}$$
$$y = e^{\pm\sqrt{t^2+C}}$$

23. $\dfrac{dx}{dt} = (t + 1)(x^2 + 1)$

SOLUTION Rewrite

$$\frac{dx}{dt} = (t + 1)(x^2 + 1) \qquad \text{as} \qquad \frac{1}{x^2 + 1}\,dx = (t + 1)\,dt.$$

Integrating both sides of this equation yields

$$\int \frac{1}{x^2+1}\,dx = \int (t+1)\,dt$$
$$\tan^{-1} x = \frac{1}{2}t^2 + t + C.$$

Solving for x, we find

$$x = \tan\left(\frac{1}{2}t^2 + t + C\right).$$

where $A = \tan C$ is an arbitrary constant.

24. $(1+x^2)y' = x^3 y$

SOLUTION Rewrite

$$(1+x^2)\frac{dy}{dx} = x^3 y \qquad \text{as} \qquad \frac{1}{y}\,dy = \frac{x^3}{1+x^2}\,dx.$$

Integrating both sides of this equation yields

$$\int \frac{1}{y}\,dy = \int \frac{x^3}{1+x^2}\,dx.$$

To integrate $\frac{x^3}{1+x^2}$, note

$$\frac{x^3}{1+x^2} = \frac{(x^3+x)-x}{1+x^2} = x - \frac{x}{1+x^2}.$$

Thus,

$$\ln|y| = \frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + C$$
$$|y| = e^C\,\frac{e^{x^2/2}}{\sqrt{x^2+1}}$$
$$y = \pm e^C\,\frac{e^{x^2/2}}{\sqrt{x^2+1}} = A\,\frac{e^{x^2/2}}{\sqrt{x^2+1}},$$

where $A = \pm e^C$ is an arbitrary constant.

25. $y' = x \sec y$

SOLUTION Rewrite

$$\frac{dy}{dx} = x \sec y \qquad \text{as} \qquad \cos y\,dy = x\,dx.$$

Integrating both sides of this equation yields

$$\int \cos y\,dy = \int x\,dx$$
$$\sin y = \frac{1}{2}x^2 + C.$$

Solving for y, we find

$$y = \sin^{-1}\left(\frac{1}{2}x^2 + C\right),$$

where C is an arbitrary constant.

26. $\dfrac{dy}{d\theta} = \tan y$

SOLUTION Rewrite

$$\frac{dy}{d\theta} = \tan y \qquad \text{as} \qquad \cot y\,dy = d\theta.$$

Integrating both sides of this equation yields

$$\int \frac{\cos y}{\sin y}\, dy = \int d\theta$$
$$\ln|\sin y| = \theta + C.$$

Solving for y, we have

$$|\sin y| = e^{\theta+C} = e^C e^\theta$$
$$\sin y = \pm e^C e^\theta$$
$$y = \sin^{-1}\left(Ae^\theta\right),$$

where $A = \pm e^C$ is an arbitrary constant.

27. $\dfrac{dy}{dt} = y\tan t$

SOLUTION Rewrite

$$\frac{dy}{dt} = y\tan t \qquad \text{as} \qquad \frac{1}{y}\, dy = \tan t\, dt.$$

Integrating both sides of this equation yields

$$\int \frac{1}{y}\, dy = \int \tan t\, dt$$
$$\ln|y| = \ln|\sec t| + C.$$

Solving for y, we find

$$|y| = e^{\ln|\sec t|+C} = e^C\,|\sec t|$$
$$y = \pm e^C \sec t = A\sec t,$$

where $A = \pm e^C$ is an arbitrary constant.

28. $\dfrac{dx}{dt} = t\tan x$

SOLUTION Rewrite

$$\frac{dx}{dt} = t\tan x \qquad \text{as} \qquad \cot x\, dx = t\, dt.$$

Integrating both sides of this equation yields

$$\int \cot x\, dx = \int t\, dt$$
$$\ln|\sin x| = \frac{1}{2}t^2 + C.$$

Solving for y, we find

$$|\sin x| = e^{\frac{1}{2}t^2+C} = e^C e^{\frac{1}{2}t^2}$$
$$\sin x = \pm e^C e^{\frac{1}{2}t^2}$$
$$x = \sin^{-1}\left(Ae^{\frac{1}{2}t^2}\right),$$

where $A = \pm e^C$ is an arbitrary constant.

In Exercises 29–42, solve the initial value problem.

29. $y' + 2y = 0, \quad y(\ln 5) = 3$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} + 2y = 0 \qquad \text{as} \qquad \frac{1}{y}\, dy = -2\, dx,$$

and then integrate to obtain

$$\ln|y| = -2x + C.$$

Thus,

$$y = Ae^{-2x},$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(\ln 5) = 3$ allows us to determine the value of A.

$$3 = Ae^{-2(\ln 5)}; \quad 3 = A\frac{1}{25}; \quad \text{so} \quad 75 = A.$$

Finally,

$$y = 75e^{-2x}.$$

30. $y' - 3y + 12 = 0, \quad y(2) = 1$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} - 3y + 12 = 0 \qquad \text{as} \qquad \frac{1}{3y-12}\,dy = 1\,dx,$$

and then integrate to obtain

$$\frac{1}{3}\ln|3y - 12| = x + C.$$

Thus,

$$y = Ae^{3x} + 4,$$

where $A = \pm\frac{1}{3}e^{3C}$ is an arbitrary constant. The initial condition $y(2) = 1$ allows us to determine the value of A.

$$1 = Ae^6 + 4; \quad -3 = Ae^6; \quad \text{so} \quad -3e^{-6} = A.$$

Finally,

$$y = -3e^{-6}e^{3x} + 4 = -3e^{3x-6} + 4$$

31. $yy' = xe^{-y^2}, \quad y(0) = -2$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$y\frac{dy}{dx} = xe^{-y^2} \qquad \text{as} \qquad ye^{y^2}\,dy = x\,dx,$$

and then integrate to obtain

$$\frac{1}{2}e^{y^2} = \frac{1}{2}x^2 + C.$$

Thus,

$$y = \pm\sqrt{\ln(x^2 + A)},$$

where $A = 2C$ is an arbitrary constant. The initial condition $y(0) = -2$ allows us to determine the value of A. Since $y(0) < 0$, we have $y = -\sqrt{\ln(x^2 + A)}$, and

$$-2 = -\sqrt{\ln(A)}; \quad 4 = \ln(A); \quad \text{so} \quad e^4 = A.$$

Finally,

$$y = -\sqrt{\ln(x^2 + e^4)}.$$

32. $y^2\frac{dy}{dx} = x^{-3}, \quad y(1) = 0$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$y^2\frac{dy}{dx} = x^{-3} \qquad \text{as} \qquad y^2\,dy = x^{-3}\,dx,$$

and then integrate to obtain

$$\frac{1}{3}y^3 = -\frac{1}{2}x^{-2} + C.$$

Thus,

$$y = \left(A - \frac{3}{2}x^{-2}\right)^{1/3},$$

where $A = 3C$ is an arbitrary constant. The initial condition $y(1) = 0$ allows us to determine the value of A.

$$0 = \left(A - \frac{3}{2}1^{-2}\right)^{1/3}; \quad 0 = \left(A - \frac{3}{2}\right)^{1/3}; \quad \text{so} \quad A = \frac{3}{2}.$$

Finally,

$$y = \left(\frac{3}{2} - \frac{3}{2}x^{-2}\right)^{1/3}.$$

33. $y' = (x-1)(y-2), \quad y(2) = 4$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = (x-1)(y-2) \qquad \text{as} \qquad \frac{1}{y-2}\,dy = (x-1)\,dx,$$

and then integrate to obtain

$$\ln|y-2| = \frac{1}{2}x^2 - x + C.$$

Thus,

$$y = Ae^{(1/2)x^2 - x} + 2,$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(2) = 4$ allows us to determine the value of A.

$$4 = Ae^0 + 2 \quad \text{so} \quad A = 2.$$

Finally,

$$y = 2e^{(1/2)x^2 - x} + 2.$$

34. $y' = (x-1)(y-2), \quad y(2) = 2$

SOLUTION First (as in the previous problem), we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = (x-1)(y-2) \qquad \text{as} \qquad \frac{1}{y-2}\,dy = (x-1)\,dx,$$

and then integrate to obtain

$$\ln|y-2| = \frac{1}{2}x^2 - x + C.$$

Thus,

$$y = Ae^{(1/2)x^2 - x} + 2,$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(2) = 2$ allows us to determine the value of A.

$$2 = Ae^0 + 2 \quad \text{so} \quad A = 0.$$

Finally,

$$y = 2.$$

35. $y' = x(y^2 + 1), \quad y(0) = 0$

SOLUTION First, find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = x(y^2+1) \qquad \text{as} \qquad \frac{1}{y^2+1}\,dy = x\,dx$$

and integrate to obtain

$$\tan^{-1} y = \frac{1}{2}x^2 + C$$

so that

$$y = \tan\left(\frac{1}{2}x^2 + C\right)$$

where C is an arbitrary constant. The initial condition $y(0) = 0$ allows us to determine the value of C: $0 = \tan(C)$, so $C = 0$. Finally,

$$y = \tan\left(\frac{1}{2}x^2\right)$$

36. $(1-t)\frac{dy}{dt} - y = 0, \quad y(2) = -4$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$(1-t)\frac{dy}{dt} = y \qquad \text{as} \qquad \frac{1}{y}\,dy = \frac{-1}{t-1}\,dt,$$

and then integrate to obtain

$$\ln|y| = -\ln|t-1| + C.$$

Thus,

$$y = \frac{A}{t-1},$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(2) = -4$ allows us to determine the value of A.

$$-4 = \frac{A}{2-1} = A.$$

Finally,

$$y = \frac{-4}{t-1}.$$

37. $\frac{dy}{dt} = ye^{-t}, \quad y(0) = 1$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dt} = ye^{-t} \qquad \text{as} \qquad \frac{1}{y}\,dy = e^{-t}\,dt,$$

and then integrate to obtain

$$\ln|y| = -e^{-t} + C.$$

Thus,

$$y = Ae^{-e^{-t}},$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(0) = 1$ allows us to determine the value of A.

$$1 = Ae^{-1} \quad \text{so} \quad A = e.$$

Finally,

$$y = (e)e^{-e^{-t}} = e^{1-e^{-t}}.$$

38. $\frac{dy}{dt} = te^{-y}, \quad y(1) = 0$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dt} = te^{-y} \qquad \text{as} \qquad e^y\,dy = t\,dt,$$

and then integrate to obtain

$$e^y = \frac{1}{2}t^2 + C.$$

Thus,

$$y = \ln\left(\frac{1}{2}t^2 + C\right),$$

where C is an arbitrary constant. The initial condition $y(1) = 0$ allows us to determine the value of C.

$$0 = \ln\left(\frac{1}{2} + C\right); \quad 1 = \frac{1}{2} + C; \quad \text{so} \quad C = \frac{1}{2}.$$

Finally,

$$y = \ln\left(\frac{1}{2}t^2 + \frac{1}{2}\right).$$

39. $t^2\dfrac{dy}{dt} - t = 1 + y + ty, \quad y(1) = 0$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$t^2\frac{dy}{dt} = 1 + t + y + ty = (1+t)(1+y)$$

as

$$\frac{1}{1+y}\,dy = \frac{1+t}{t^2}\,dt,$$

and then integrate to obtain

$$\ln|1+y| = -t^{-1} + \ln|t| + C.$$

Thus,

$$y = A\frac{t}{e^{1/t}} - 1,$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(1) = 0$ allows us to determine the value of A.

$$0 = A\left(\frac{1}{e}\right) - 1 \quad \text{so} \quad A = e.$$

Finally,

$$y = \frac{et}{e^{1/t}} - 1.$$

40. $\sqrt{1-x^2}\,y' = y^2 + 1, \quad y(0) = 0$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\sqrt{1-x^2}\,\frac{dy}{dx} = y^2 + 1 \qquad \text{as} \qquad \frac{1}{y^2+1}\,dy = \frac{1}{\sqrt{1-x^2}}\,dx,$$

and then integrate to obtain

$$\tan^{-1} y = \sin^{-1} x + C.$$

Thus,

$$y = \tan(\sin^{-1} x + C),$$

where C is an arbitrary constant. The initial condition $y(0) = 0$ allows us to determine the value of C.

$$0 = \tan\left(\sin^{-1} 0 + C\right) = \tan C \quad \text{so} \quad 0 = C.$$

Finally,

$$y = \tan\left(\sin^{-1} x\right).$$

41. $y' = \tan y, \quad y(\ln 2) = \dfrac{\pi}{2}$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = \tan y \qquad \text{as} \qquad \frac{dy}{\tan y} = dx,$$

and then integrate to obtain

$$\ln|\sin y| = x + C.$$

Thus,

$$y = \sin^{-1}(Ae^x),$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(\ln 2) = \frac{\pi}{2}$ allows us to determine the value of A.

$$\frac{\pi}{2} = \sin^{-1}(2A); \quad 1 = 2A \quad \text{so} \quad A = \frac{1}{2}.$$

Finally,

$$y = \sin^{-1}\left(\frac{1}{2}e^x\right).$$

42. $y' = y^2 \sin x, \quad y(\pi) = 2$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = y^2 \sin x \qquad \text{as} \qquad y^{-2}\,dy = \sin x\,dx,$$

and then integrate to obtain

$$-y^{-1} = -\cos x + C.$$

Thus,

$$y = \frac{1}{A + \cos x},$$

where $A = -C$ is an arbitrary constant. The initial condition $y(\pi) = 2$ allows us to determine the value of A.

$$2 = \frac{1}{A-1}; \quad A - 1 = \frac{1}{2} \quad \text{so} \quad A = \frac{1}{2} + 1 = \frac{3}{2}.$$

Finally,

$$y = \frac{1}{\cos x + (3/2)} = \frac{2}{3 + 2\cos x}.$$

43. Find all values of a such that $y = x^a$ is a solution of

$$y'' - 12x^{-2}y = 0$$

SOLUTION Let $y = x^a$. Then

$$y' = ax^{a-1} \quad \text{and} \quad y'' = a(a-1)x^{a-2}.$$

Substituting into the differential equation, we find

$$y'' - 12x^{-2}y = a(a-1)x^{a-2} - 12x^{a-2} = x^{a-2}(a^2 - a - 12).$$

Thus, $y'' - 12x^{-2}y = 0$ if and only if

$$a^2 - a - 12 = (a-4)(a+3) = 0.$$

Hence, $y = x^a$ is a solution of the differential equation $y'' - 12x^{-2}y = 0$ provided $a = 4$ or $a = -3$.

44. Find all values of a such that $y = e^{ax}$ is a solution of

$$y'' + 4y' - 12y = 0$$

SOLUTION Let $y = e^{ax}$. Then

$$y' = ae^{ax} \quad \text{and} \quad y'' = a^2e^{ax}.$$

Substituting into the differential equation, we find

$$y'' + 4y' - 12y = e^{ax}(a^2 + 4a - 12).$$

Because e^{ax} is never zero, $y'' + 4y' - 12y = 0$ if only if $a^2 + 4a - 12 = (a+6)(a-2) = 0$. Hence, $y = e^{ax}$ is a solution of the differential equation $y'' + 4y' - 12y = 0$ provided $a = -6$ or $a = 2$.

In Exercises 45 and 46, let $y(t)$ *be a solution of* $(\cos y + 1)\dfrac{dy}{dt} = 2t$ *such that* $y(2) = 0$.

45. Show that $\sin y + y = t^2 + C$. We cannot solve for y as a function of t, but, assuming that $y(2) = 0$, find the values of t at which $y(t) = \pi$.

SOLUTION Rewrite

$$(\cos y + 1)\frac{dy}{dt} = 2t \quad \text{as} \quad (\cos y + 1)\,dy = 2t\,dt$$

and integrate to obtain

$$\sin y + y = t^2 + C$$

where C is an arbitrary constant. Since $y(2) = 0$, we have $\sin 0 + 0 = 4 + C$ so that $C = -4$ and the particular solution we seek is $\sin y + y = t^2 - 4$. To find values of t at which $y(t) = \pi$, we must solve $\sin \pi + \pi = t^2 - 4$, or $t^2 - 4 = \pi$; thus $t = \pm\sqrt{\pi + 4}$.

46. Assuming that $y(6) = \pi/3$, find an equation of the tangent line to the graph of $y(t)$ at $(6, \pi/3)$.

SOLUTION At $(6, \pi/3)$, we have

$$\left(\cos\frac{\pi}{3} + 1\right)\frac{dy}{dt} = 2(6) = 12 \quad \Rightarrow \quad \frac{3}{2}y' = 12$$

and hence $y' = 8$. The tangent line has equation

$$(y - \pi/3) = 8(x - 6)$$

In Exercises 47–52, use Eq. (4) and Torricelli's Law [Eq. (5)].

47. Water leaks through a hole of area 0.002 m^2 at the bottom of a cylindrical tank that is filled with water and has height 3 m and a base of area 10 m^2. How long does it take (a) for half of the water to leak out and (b) for the tank to empty?

SOLUTION Because the tank has a constant cross-sectional area of 10 m^2 and the hole has an area of 0.002 m^2, the differential equation for the height of the water in the tank is

$$\frac{dy}{dt} = \frac{0.002v}{10} = 0.0002v.$$

By Torricelli's Law,

$$v = -\sqrt{2gy} = -\sqrt{19.6y},$$

using $g = 9.8\ \text{m/s}^2$. Thus,

$$\frac{dy}{dt} = -0.0002\sqrt{19.6y} = -0.0002\sqrt{19.6}\cdot\sqrt{y}.$$

Separating variables and then integrating yields

$$y^{-1/2}\,dy = -0.0002\sqrt{19.6}\,dt$$
$$2y^{1/2} = -0.0002\sqrt{19.6}t + C$$

Solving for y, we find

$$y(t) = \left(C - 0.0001\sqrt{19.6}t\right)^2.$$

Since the tank is originally full, we have the initial condition $y(0) = 10$, whence $\sqrt{10} = C$. Therefore,

$$y(t) = \left(\sqrt{10} - 0.0001\sqrt{19.6}t\right)^2.$$

When half of the water is out of the tank, $y = 1.5$, so we solve:

$$1.5 = \left(\sqrt{10} - 0.0001\sqrt{19.6}t\right)^2$$

for t, finding

$$t = \frac{1}{0.0002\sqrt{19.6}}(2\sqrt{10} - \sqrt{6}) \approx 4376.44 \text{ sec.}$$

When all of the water is out of the tank, $y = 0$, so

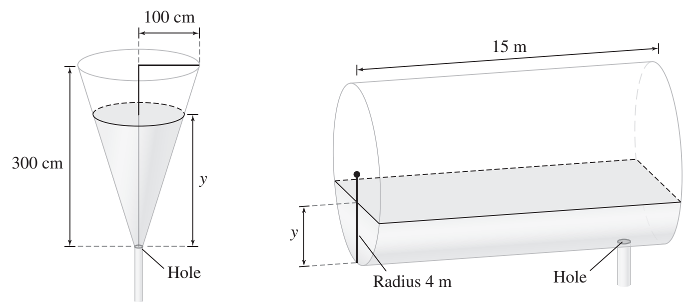
$$\sqrt{10} - 0.0001\sqrt{19.6}t = 0 \quad \text{and} \quad t = \frac{\sqrt{10}}{0.0001\sqrt{19.6}} \approx 7142.86 \text{ sec.}$$

48. At $t = 0$, a conical tank of height 300 cm and top radius 100 cm [Figure 7(A)] is filled with water. Water leaks through a hole in the bottom of area 3 cm^2. Let $y(t)$ be the water level at time t.

(a) Show that the tank's cross-sectional area at height y is $A(y) = \frac{\pi}{9}y^2$.

(b) Find and solve the differential equation satisfied by $y(t)$

(c) How long does it take for the tank to empty?



(A) Conical tank (B) Horizontal tank

FIGURE 7

SOLUTION

(a) By similar triangles, the radius r at height y satisfies

$$\frac{r}{y} = \frac{100}{300} = \frac{1}{3},$$

so $r = y/3$ and

$$A(y) = \pi r^2 = \frac{\pi}{9}y^2.$$

(b) The area of the hole is $B = 3$ cm^2, so the differential equation for the height of the water in the tank becomes:

$$\frac{dy}{dt} = -\frac{3\sqrt{19.6}\sqrt{y}}{A(y)} = -\frac{27\sqrt{19.6}}{\pi}y^{-3/2}.$$

Separating variables and integrating then yields

$$y^{3/2}\,dy = -\frac{27\sqrt{19.6}}{\pi}\,dt$$

$$\frac{2}{5}y^{5/2} = C - \frac{27\sqrt{19.6}}{\pi}t$$

When $t = 0$, $y = 300$, so we find $C = \frac{2}{5}(300)^{5/2}$. Therefore,

$$y(t) = \left(300^{5/2} - \frac{135\sqrt{19.6}}{2\pi}t\right)^{2/5}.$$

(c) The tank is empty when $y = 0$. Using the result from part (b), $y = 0$ when

$$t = \frac{4000\pi\sqrt{300}}{3\sqrt{19.6}} \approx 16{,}387.82 \text{ seconds.}$$

Thus, it takes roughly 4 hours, 33 minutes for the tank to empty.

49. The tank in Figure 7(B) is a cylinder of radius 4 m and height 15 m. Assume that the tank is half-filled with water and that water leaks through a hole in the bottom of area $B = 0.001\ \text{m}^2$. Determine the water level $y(t)$ and the time t_e when the tank is empty.

SOLUTION When the water is at height y over the bottom, the top cross section is a rectangle with length 15 m, and with width x satisfying the equation:

$$(x/2)^2 + (y-4)^2 = 16.$$

Thus, $x = 2\sqrt{8y - y^2}$, and

$$A(y) = 15x = 30\sqrt{8y - y^2}.$$

With $B = 0.001\ \text{m}^2$ and $v = -\sqrt{2gy} = -\sqrt{19.6}\sqrt{y}$, it follows that

$$\frac{dy}{dt} = -\frac{0.001\sqrt{19.6}\sqrt{y}}{30\sqrt{8y - y^2}} = -\frac{0.001\sqrt{19.6}}{30\sqrt{8-y}}.$$

Separating variables and integrating then yields:

$$\sqrt{8-y}\,dy = -\frac{0.001\sqrt{19.6}}{30}\,dt = -\frac{0.0001\sqrt{19.6}}{3}\,dt$$

$$-\frac{2}{3}(8-y)^{3/2} = -\frac{0.0001\sqrt{19.6}}{3}t + C$$

When $t = 0$, $y = 4$, so $C = -\frac{2}{3}4^{3/2} = -\frac{16}{3}$, and

$$-\frac{2}{3}(8-y)^{3/2} = -\frac{0.0001\sqrt{19.6}}{3}t - \frac{16}{3}$$

$$y(t) = 8 - \left(\frac{0.0001\sqrt{19.6}}{2}t + 8\right)^{2/3}.$$

The tank is empty when $y = 0$. Thus, t_e satisfies the equation

$$8 - \left(\frac{0.0001\sqrt{19.6}}{2}t + 8\right)^{2/3} = 0.$$

It follows that

$$t_e = \frac{2(8^{3/2} - 8)}{0.0001\sqrt{19.6}} \approx 66{,}079.9 \text{ seconds}.$$

50. A tank has the shape of the parabola $y = x^2$, revolved around the y-axis. Water leaks from a hole of area $B = 0.0005\ \text{m}^2$ at the bottom of the tank. Let $y(t)$ be the water level at time t. How long does it take for the tank to empty if it is initially filled to height $y_0 = 1$ m.

SOLUTION When the water is at height y, the surface of the water is a circle with radius $\sqrt{y}$, so the cross-sectional area is $A(y) = \pi y$. With $B = 0.0005$ m and $v = -\sqrt{2gy} = -\sqrt{19.6}\sqrt{y}$, it follows that

$$\frac{dy}{dt} = -\frac{0.0005\sqrt{19.6}\sqrt{y}}{A(y)} = -\frac{0.0005\sqrt{19.6}\sqrt{y}}{\pi y} = -\frac{0.0005\sqrt{19.6}}{\pi\sqrt{y}}$$

Separating variables and integrating yields

$$\pi y^{1/2}\,dy = -0.0005\sqrt{19.6}\,dt$$

$$\frac{2}{3}\pi y^{3/2} = -0.0005\sqrt{19.6}t + C$$

$$y^{3/2} = -\frac{0.00075\sqrt{19.6}}{\pi}t + C$$

Since $y(0) = 1$, we have $C = 1$, so that

$$y = \left(1 - \frac{0.00075\sqrt{19.6}}{\pi}t\right)^{2/3}$$

The tank is empty when $y = 0$, so when $1 - \frac{0.00075\sqrt{19.6}}{\pi}t = 0$ and thus

$$t = \frac{\pi}{0.00075\sqrt{19.6}} \approx 946.15 \text{ s}$$

51. A tank has the shape of the parabola $y = ax^2$ (where a is a constant) revolved around the y-axis. Water drains from a hole of area B m^2 at the bottom of the tank.

(a) Show that the water level at time t is

$$y(t) = \left(y_0^{3/2} - \frac{3aB\sqrt{2g}}{2\pi}t\right)^{2/3}$$

where y_0 is the water level at time $t = 0$.

(b) Show that if the total volume of water in the tank has volume V at time $t = 0$, then $y_0 = \sqrt{2aV/\pi}$. *Hint:* Compute the volume of the tank as a volume of rotation.

(c) Show that the tank is empty at time

$$t_e = \left(\frac{2}{3B\sqrt{g}}\right)\left(\frac{2\pi V^3}{a}\right)^{1/4}$$

We see that for fixed initial water volume V, the time t_e is proportional to $a^{-1/4}$. A large value of a corresponds to a tall thin tank. Such a tank drains more quickly than a short wide tank of the same initial volume.

SOLUTION

(a) When the water is at height y, the surface of the water is a circle of radius $\sqrt{y/a}$, so that the cross-sectional area is $A(y) = \pi y/a$. With $v = -\sqrt{2gy} = -\sqrt{2g}\sqrt{y}$, we have

$$\frac{dy}{dt} = -\frac{B\sqrt{2g}\sqrt{y}}{A} = -\frac{aB\sqrt{2g}\sqrt{y}}{\pi y} = -\frac{aB\sqrt{2g}}{\pi}y^{-1/2}$$

Separating variables and integrating gives

$$\sqrt{y}\,dy = -\frac{aB\sqrt{2g}}{\pi}\,dt$$

$$\frac{2}{3}y^{3/2} = -\frac{aB\sqrt{2g}}{\pi}t + C_1$$

$$y^{3/2} = -\frac{3aB\sqrt{2g}}{2\pi}t + C$$

Since $y(0) = y_0$, we have $C = y_0^{3/2}$; solving for y gives

$$y = \left(y_0^{3/2} - \frac{3aB\sqrt{2g}}{2\pi}t\right)^{2/3}$$

(b) The volume of the tank can be computed as a volume of rotation. Using the disk method and applying it to the function $x = \sqrt{y/a}$, we have

$$V = \int_0^{y_0} \pi\sqrt{\frac{y}{a}}^2\,dy = \frac{\pi}{a}\int_0^{y_0} y\,dy = \frac{\pi}{2a}y^2\Big|_0^{y_0} = \frac{\pi}{2a}y_0^2$$

Solving for y_0 gives

$$y_0 = \sqrt{2aV/\pi}$$

(c) The tank is empty when $y = 0$; this occurs when

$$y_0^{3/2} - \frac{3aB\sqrt{2g}}{2\pi}t = 0$$

From part (b), we have

$$y_0^{3/2} = \sqrt{2aV/\pi}^{3/2} = ((2aV/\pi)^{1/2})^{3/2} = (2aV/\pi)^{3/4}$$

so that

$$t_e = \frac{2\pi y_0^{3/2}}{3aB\sqrt{2g}} = \frac{2\pi\sqrt[4]{8a^3V^3}}{3\pi^{3/4}B\sqrt[4]{a^4}\sqrt[4]{4}\sqrt{g}} = \frac{2\pi^{1/4}\sqrt[4]{2V^3a^{-1}}}{3B\sqrt{g}} = \left(\frac{2}{3B\sqrt{g}}\right)\left(\frac{2\pi V^3}{a}\right)^{1/4}$$

52. A cylindrical tank filled with water has height h and a base of area A. Water leaks through a hole in the bottom of area B.

(a) Show that the time required for the tank to empty is proportional to $A\sqrt{h}/B$.
(b) Show that the emptying time is proportional to $Vh^{-1/2}$, where V is the volume of the tank.
(c) Two tanks have the same volume and a hole of the same size, but they have different heights and bases. Which tank empties first: the taller or the shorter tank?

SOLUTION Torricelli's law gives the differential equation for the height of the water in the tank as

$$\frac{dy}{dt} = -\sqrt{2g}\frac{B\sqrt{y}}{A}$$

Separating variables and integrating then yields:

$$y^{-1/2}\,dy = -\sqrt{2g}\frac{B}{A}\,dt$$

$$2y^{1/2} = -\sqrt{2g}\frac{Bt}{A} + C$$

$$y^{1/2} = -\sqrt{g/2}\frac{Bt}{A} + C$$

When $t = 0$, $y = h$, so $C = h^{1/2}$ and

$$y^{1/2} = \sqrt{h} - \sqrt{g/2}\frac{Bt}{A}.$$

(a) When the tank is empty, $y = 0$. Thus, the time required for the tank to empty, t_e, satisfies the equation

$$0 = \sqrt{h} - \sqrt{g/2}\frac{Bt_e}{A}.$$

It follows that

$$t_e = \frac{A}{B}\sqrt{2h/g} = \sqrt{2/g}\left(\frac{A\sqrt{h}}{B}\right);$$

that is, the time required for the tank to empty is proportional to $A\sqrt{h}/B$.
(b) The volume of the tank is $V = Ah$; therefore

$$\frac{A\sqrt{h}}{B} = \frac{1}{B}\frac{V}{\sqrt{h}},$$

and

$$t_e = \sqrt{2/g}\left(\frac{A\sqrt{h}}{B}\right) = \frac{\sqrt{2/g}}{B}\left(\frac{V}{\sqrt{h}}\right);$$

that is, the time required for the tank to empty is proportional to $Vh^{-1/2}$.
(c) By part (b), with V and B held constant, the emptying time decreases with height. The taller tank therefore empties first.

53. Figure 8 shows a circuit consisting of a resistor of R ohms, a capacitor of C farads, and a battery of voltage V. When the circuit is completed, the amount of charge $q(t)$ (in coulombs) on the plates of the capacitor varies according to the differential equation (t in seconds)

$$R\frac{dq}{dt} + \frac{1}{C}q = V$$

where R, C, and V are constants.
(a) Solve for $q(t)$, assuming that $q(0) = 0$.
(b) Show that $\lim_{t\to\infty} q(t) = CV$.
(c) Show that the capacitor charges to approximately 63% of its final value CV after a time period of length $\tau = RC$ (τ is called the time constant of the capacitor).

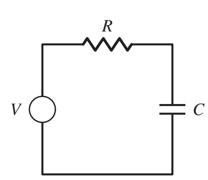


FIGURE 8 An RC circuit.

SOLUTION

(a) Upon rearranging the terms of the differential equation, we have

$$\frac{dq}{dt} = -\frac{q - CV}{RC}.$$

Separating the variables and integrating both sides, we obtain

$$\frac{dq}{q - CV} = -\frac{dt}{RC}$$

$$\int \frac{dq}{q - CV} = -\int \frac{dt}{RC}$$

and

$$\ln |q - CV| = -\frac{t}{RC} + k,$$

where k is an arbitrary constant. Solving for $q(t)$ yields

$$q(t) = CV + Ke^{-\frac{1}{RC}t},$$

where $K = \pm e^k$. We use the initial condition $q(0) = 0$ to solve for K:

$$0 = CV + K \quad \Rightarrow \quad K = -CV$$

so that the particular solution is

$$q(t) = CV(1 - e^{-\frac{1}{RC}t})$$

(b) Using the result from part (a), we calculate

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} CV(1 - e^{-\frac{1}{RC}t}) = CV(1 - \lim_{t\to\infty} 1 - e^{-\frac{1}{RC}t}) = CV.$$

(c) We have

$$q(\tau) = q(RC) = CV(1 - e^{-\frac{1}{RC}RC}) = CV(1 - e^{-1}) \approx 0.632CV.$$

54. Assume in the circuit of Figure 8 that $R = 200\ \Omega$, $C = 0.02$ F, and $V = 12$ V. How many seconds does it take for the charge on the capacitor plates to reach half of its limiting value?

SOLUTION From Exercise 53, we know that

$$q(t) = CV\left(1 - e^{-t/(RC)}\right) = 0.24(1 - e^{-t/4}),$$

and the limiting value of $q(t)$ is $CV = 0.24$. If the charge on the capacitor plates has reached half its limiting value, then

$$\begin{aligned} \frac{0.24}{2} &= 0.24(1 - e^{-t/4}) \\ 1 - e^{-t/4} &= 1/2 \\ e^{-t/4} &= 1/2 \\ t &= 4\ln 2 \end{aligned}$$

Therefore, the charge on the capacitor plates reaches half of its limiting value after $4\ln 2 \approx 2.773$ seconds.

55. According to one hypothesis, the growth rate dV/dt of a cell's volume V is proportional to its surface area A. Since V has cubic units such as cm^3 and A has square units such as cm^2, we may assume roughly that $A \propto V^{2/3}$, and hence $dV/dt = kV^{2/3}$ for some constant k. If this hypothesis is correct, which dependence of volume on time would we expect to see (again, roughly speaking) in the laboratory?

(a) Linear **(b)** Quadratic **(c)** Cubic

SOLUTION Rewrite

$$\frac{dV}{dt} = kV^{2/3} \qquad \text{as} \qquad V^{-2/3}\,dv = k\,dt,$$

and then integrate both sides to obtain

$$3V^{1/3} = kt + C$$
$$V = (kt/3 + C)^3.$$

Thus, we expect to see V increasing roughly like the cube of time.

56. We might also guess that the volume V of a melting snowball decreases at a rate proportional to its surface area. Argue as in Exercise 55 to find a differential equation satisfied by V. Suppose the snowball has volume 1000 cm^3 and that it loses half of its volume after 5 min. According to this model, when will the snowball disappear?

SOLUTION Since the volume is decreasing, we write (as in Exercise 55) $V' = -kV^{2/3}$ where k is positive, so $V(t) = (C - kt/3)^3$. $V(0) = 1000$ implies that $C = 10$ so $V(t) = (10 - kt/3)^3$. Since it loses half of its volume after 5 minutes, we have $V(5) = \frac{1}{2}V(0)$, so that

$$(10 - 5k/3)^3 = 500 \qquad \text{so that} \quad k = 6 - 3 \cdot 2^{2/3} \approx 1.2378$$

and finally the equation is

$$V(t) = \left(10 - \frac{1.2378t}{3}\right)^3$$

The snowball is melted when its volume is zero, so when

$$10 - \frac{1.2378t}{3} = 0 \quad \Rightarrow \quad t = \frac{30}{1.2378} \approx 24.24 \text{ minutes}$$

57. In general, $(fg)'$ is not equal to $f'g'$, but let $f(x) = e^{3x}$ and find a function $g(x)$ such that $(fg)' = f'g'$. Do the same for $f(x) = x$.

SOLUTION If $(fg)' = f'g'$, we have

$$f'(x)g(x) + g'(x)f(x) = f'(x)g'(x)$$
$$g'(x)(f(x) - f'(x)) = -g(x)f'(x)$$
$$\frac{g'(x)}{g(x)} = \frac{f'(x)}{f'(x) - f(x)}$$

Now, let $f(x) = e^{3x}$. Then $f'(x) = 3e^{3x}$ and

$$\frac{g'(x)}{g(x)} = \frac{3e^{3x}}{3e^{3x} - e^{3x}} = \frac{3}{2}.$$

Integrating and solving for $g(x)$, we find

$$\frac{dg}{g} = \frac{3}{2}\,dx$$
$$\ln|g| = \frac{3}{2}x + C$$
$$g(x) = Ae^{(3/2)x},$$

where $A = \pm e^C$ is an arbitrary constant.

If $f(x) = x$, then $f'(x) = 1$, and

$$\frac{g'(x)}{g(x)} = \frac{1}{1-x}.$$

Thus,

$$\frac{dg}{g} = \frac{1}{1-x}\,dx$$
$$\ln|g| = -\ln|1-x| + C$$
$$g(x) = \frac{A}{1-x},$$

where $A = \pm e^C$ is an arbitrary constant.

58. A boy standing at point B on a dock holds a rope of length ℓ attached to a boat at point A [Figure 9(A)]. As the boy walks along the dock, holding the rope taut, the boat moves along a curve called a **tractrix** (from the Latin *tractus*, meaning "to pull"). The segment from a point P on the curve to the x-axis along the tangent line has constant length ℓ. Let $y = f(x)$ be the equation of the tractrix.

(a) Show that $y^2 + (y/y')^2 = \ell^2$ and conclude $y' = -\dfrac{y}{\sqrt{\ell^2 - y^2}}$. Why must we choose the negative square root?

(b) Prove that the tractrix is the graph of

$$x = \ell \ln\left(\frac{\ell + \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2}$$

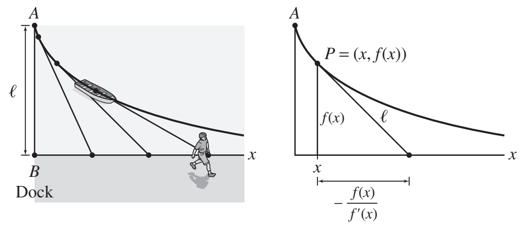


FIGURE 9

SOLUTION

(a) From the diagram on the right in Figure 9, we see that

$$f(x)^2 + \left(-\frac{f(x)}{f'(x)}\right)^2 = \ell^2.$$

If we let $y = f(x)$, this last equation reduces to $y^2 + (y/y')^2 = \ell^2$. Solving for y', we find

$$y' = -\frac{y}{\sqrt{\ell^2 - y^2}},$$

where we must choose the negative sign because y is a decreasing function of x.

(b) Rewrite

$$\frac{dy}{dx} = -\frac{y}{\sqrt{\ell^2 - y^2}} \qquad \text{as} \qquad \frac{\sqrt{\ell^2 - y^2}}{y}\,dy = -dx,$$

and then integrate both sides to obtain

$$-x + C = \int \frac{\sqrt{\ell^2 - y^2}}{y}\,dy.$$

For the remaining integral, we use the trigonometric substitution $y = \ell \sin\theta$, $dy = \ell\cos\theta\,d\theta$. Then

$$\int \frac{\sqrt{\ell^2 - y^2}}{y}\,dy = \ell \int \frac{\cos^2\theta}{\sin\theta}\,d\theta = \ell \int \frac{1 - \sin^2\theta}{\sin\theta}\,d\theta = \ell \int (\csc\theta - \sin\theta)\,d\theta$$

$$= \ell\left[\ln|\csc\theta - \cot\theta| + \cos\theta\right] + C = \ell \ln\left(\frac{\ell}{y} - \frac{\sqrt{\ell^2 - y^2}}{y}\right) + \sqrt{\ell^2 - y^2} + C$$

Therefore,

$$x = -\ell \ln\left(\frac{\ell - \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2} + C = \ell \ln\left(\frac{y}{\ell - \sqrt{\ell^2 - y^2}}\right) - \sqrt{\ell^2 - y^2} + C$$

$$= \ell \ln\left(\frac{\ell + \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2} + C$$

Now, when $x = 0$, $y = \ell$, so we find $C = 0$. Finally, the equation for the tractrix is

$$x = \ell \ln\left(\frac{\ell + \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2}.$$

59. Show that the differential equations $y' = 3y/x$ and $y' = -x/3y$ define **orthogonal families** of curves; that is, the graphs of solutions to the first equation intersect the graphs of the solutions to the second equation in right angles (Figure 10). Find these curves explicitly.

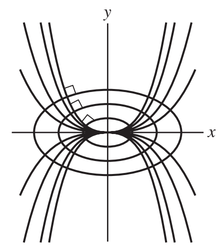


FIGURE 10 Two orthogonal families of curves.

SOLUTION Let y_1 be a solution to $y' = \frac{3y}{x}$ and let y_2 be a solution to $y' = -\frac{x}{3y}$. Suppose these two curves intersect at a point (x_0, y_0). The line tangent to the curve $y_1(x)$ at (x_0, y_0) has a slope of $\frac{3y_0}{x_0}$ and the line tangent to the curve $y_2(x)$ has a slope of $-\frac{x_0}{3y_0}$. The slopes are negative reciprocals of one another; hence the tangent lines are perpendicular.

Separation of variables and integration applied to $y' = \frac{3y}{x}$ gives

$$\frac{dy}{y} = 3\frac{dx}{x}$$
$$\ln|y| = 3\ln|x| + C$$
$$y = Ax^3$$

On the other hand, separation of variables and integration applied to $y' = -\frac{x}{3y}$ gives

$$3y\,dy = -x\,dx$$
$$3y^2/2 = -x^2/2 + C$$
$$y = \pm\sqrt{C - x^2/3}$$

60. Find the family of curves satisfying $y' = x/y$ and sketch several members of the family. Then find the differential equation for the orthogonal family (see Exercise 59), find its general solution, and add some members of this orthogonal family to your plot.

SOLUTION Separation of variables and integration applied to $y' = x/y$ gives

$$y\,dy = x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$
$$y = \pm\sqrt{x^2 + C}$$

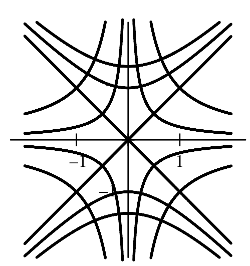
If $y(x)$ is a curve of the family orthogonal to these, it must have tangent lines of slope $-y/x$ at every point (x, y). This gives

$$y' = -y/x$$

Separation of variables and integration give

$$\frac{dy}{y} = -\frac{dx}{x}$$
$$\ln|y| = -\ln|x| + C$$
$$y = \frac{A}{x}$$

Several solution curves of both differential equations appear below:



61. A 50-kg model rocket lifts off by expelling fuel at a rate of $k = 4.75$ kg/s for 10 s. The fuel leaves the end of the rocket with an exhaust velocity of $b = -100$ m/s. Let $m(t)$ be the mass of the rocket at time t. From the law of conservation of momentum, we find the following differential equation for the rocket's velocity $v(t)$ (in meters per second):

$$m(t)v'(t) = -9.8m(t) + b\frac{dm}{dt}$$

(a) Show that $m(t) = 50 - 4.75t$ kg.
(b) Solve for $v(t)$ and compute the rocket's velocity at rocket burnout (after 10 s).

SOLUTION

(a) For $0 \le t \le 10$, the rocket is expelling fuel at a constant rate of 4.75 kg/s, giving $m'(t) = -4.75$. Hence, $m(t) = -4.75t + C$. Initially, the rocket has a mass of 50 kg, so $C = 50$. Therefore, $m(t) = 50 - 4.75t$.

(b) With $m(t) = 50 - 4.75t$ and $\dfrac{dm}{dt} = -4.75$, the equation for v becomes

$$\frac{dv}{dt} = -9.8 + \frac{b\frac{dm}{dt}}{50 - 4.75t} = -9.8 + \frac{(100)(-4.75)}{50 - 4.75t}$$

and therefore

$$v(t) = -9.8t + 100\int \frac{4.75\,dt}{50 - 4.75t} = -9.8t - 100\ln(50 - 4.75t) + C$$

Because $v(0) = 0$, we find $C = 100\ln 50$ and

$$v(t) = -9.8t - 100\ln(50 - 4.75t) + 100\ln(50).$$

After 10 seconds the velocity is:

$$v(10) = -98 - 100\ln(2.5) + 100\ln(50) \approx 201.573 \text{ m/s}.$$

62. Let $v(t)$ be the velocity of an object of mass m in free fall near the earth's surface. If we assume that air resistance is proportional to v^2, then v satisfies the differential equation $m\frac{dv}{dt} = -g + kv^2$ for some constant $k > 0$.

(a) Set $\alpha = (g/k)^{1/2}$ and rewrite the differential equation as

$$\frac{dv}{dt} = -\frac{k}{m}(\alpha^2 - v^2)$$

Then solve using separation of variables with initial condition $v(0) = 0$.
(b) Show that the terminal velocity $\lim_{t\to\infty} v(t)$ is equal to $-\alpha$.

SOLUTION

(a) Let $\alpha = (g/k)^{1/2}$. Then

$$\frac{dv}{dt} = -\frac{g}{m} + \frac{k}{m}v^2 = -\frac{k}{m}\left(\frac{g}{k} - v^2\right) = -\frac{k}{m}\left(\alpha^2 - v^2\right)$$

Separating variables and integrating yields

$$\int \frac{dv}{\alpha^2 - v^2} = -\frac{k}{m}\int dt = -\frac{k}{m}t + C$$

We now use partial fraction decomposition for the remaining integral to obtain

$$\int \frac{dv}{\alpha^2 - v^2} = \frac{1}{2\alpha}\int\left(\frac{1}{\alpha + v} + \frac{1}{\alpha - v}\right)dv = \frac{1}{2\alpha}\ln\left|\frac{\alpha + v}{\alpha - v}\right|$$

Therefore,

$$\frac{1}{2\alpha}\ln\left|\frac{\alpha + v}{\alpha - v}\right| = -\frac{k}{m}t + C.$$

The initial condition $v(0) = 0$ allows us to determine the value of C:

$$\frac{1}{2\alpha}\ln\left|\frac{\alpha + 0}{\alpha - 0}\right| = -\frac{k}{m}(0) + C$$

$$C = \frac{1}{2\alpha}\ln 1 = 0.$$

Finally, solving for v, we find

$$v(t) = -\alpha\left(\frac{1 - e^{-2(\sqrt{gk}/m)t}}{1 + e^{-2(\sqrt{gk}/m)t}}\right).$$

(b) As $t \to \infty$, $e^{-2(\sqrt{gk}/m)t} \to 0$, so

$$v(t) \to -\alpha\left(\frac{1-0}{1+0}\right) = -\alpha.$$

63. If a bucket of water spins about a vertical axis with constant angular velocity ω (in radians per second), the water climbs up the side of the bucket until it reaches an equilibrium position (Figure 11). Two forces act on a particle located at a distance x from the vertical axis: the gravitational force $-mg$ acting downward and the force of the bucket on the particle (transmitted indirectly through the liquid) in the direction perpendicular to the surface of the water. These two forces must combine to supply a centripetal force $m\omega^2 x$, and this occurs if the diagonal of the rectangle in Figure 11 is normal to the water's surface (that is, perpendicular to the tangent line). Prove that if $y = f(x)$ is the equation of the curve obtained by taking a vertical cross section through the axis, then $-1/y' = -g/(\omega^2 x)$. Show that $y = f(x)$ is a parabola.

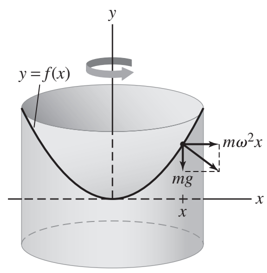


FIGURE 11

SOLUTION At any point along the surface of the water, the slope of the tangent line is given by the value of y' at that point; hence, the slope of the line perpendicular to the surface of the water is given by $-1/y'$. The slope of the resultant force generated by the gravitational force and the centrifugal force is

$$\frac{-mg}{m\omega^2 x} = -\frac{g}{\omega^2 x}.$$

Therefore, the curve obtained by taking a vertical cross-section of the water surface is determined by the equation

$$-\frac{1}{y'} = -\frac{g}{\omega^2 x} \qquad \text{or} \qquad y' = \frac{\omega^2}{g}x.$$

Performing one integration yields

$$y = f(x) = \frac{\omega^2}{2g}x^2 + C,$$

where C is a constant of integration. Thus, $y = f(x)$ is a parabola.

Further Insights and Challenges

64. In Section 6.2, we computed the volume V of a solid as the integral of cross-sectional area. Explain this formula in terms of differential equations. Let $V(y)$ be the volume of the solid up to height y, and let $A(y)$ be the cross-sectional area at height y as in Figure 12.

(a) Explain the following approximation for small Δy:

$$V(y + \Delta y) - V(y) \approx A(y)\,\Delta y \qquad \boxed{8}$$

(b) Use Eq. (8) to justify the differential equation $dV/dy = A(y)$. Then derive the formula

$$V = \int_a^b A(y)\,dy$$

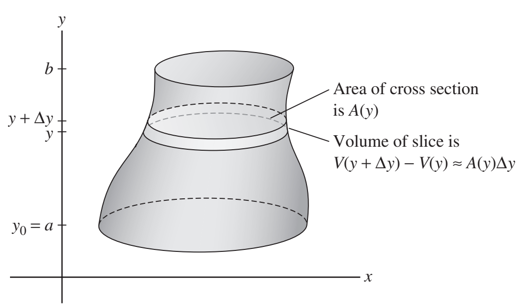


FIGURE 12

SOLUTION

(a) If Δy is very small, then the slice between y and $y + \Delta y$ is very similar to the *prism* formed by thickening the cross-sectional area $A(y)$ by a thickness of Δy. A prism with cross-sectional area A and height Δy has volume $A\Delta y$. This gives

$$V(y + \Delta y) - V(y) \approx A(y)\Delta y.$$

(b) Dividing Eq. (8) by Δy, we obtain

$$\frac{V(y + \Delta y) - V(y)}{\Delta y} \approx A(y).$$

In the limit as $\Delta y \to 0$, this becomes

$$\frac{dV}{dy} = A(y).$$

Integrating this last equation yields

$$V = \int_a^b A(y)\,dy.$$

65. A basic theorem states that a *linear* differential equation of order n has a general solution that depends on n arbitrary constants. There are, however, nonlinear exceptions.

(a) Show that $(y')^2 + y^2 = 0$ is a first-order equation with only one solution $y = 0$.

(b) Show that $(y')^2 + y^2 + 1 = 0$ is a first-order equation with no solutions.

SOLUTION

(a) $(y')^2 + y^2 \geq 0$ and equals zero if and only if $y' = 0$ and $y = 0$

(b) $(y')^2 + y^2 + 1 \geq 1 > 0$ for all y' and y, so $(y')^2 + y^2 + 1 = 0$ has no solution

66. Show that $y = Ce^{rx}$ is a solution of $y'' + ay' + by = 0$ if and only if r is a root of $P(r) = r^2 + ar + b$. Then verify directly that $y = C_1e^{3x} + C_2e^{-x}$ is a solution of $y'' - 2y' - 3y = 0$ for any constants C_1, C_2.

SOLUTION

(a) Let $y(x) = Ce^{rx}$. Then $y' = rCe^{rx}$, and $y'' = r^2Ce^{rx}$. Thus

$$y'' + ay' + by = r^2Ce^{rx} + arCe^{rx} + bCe^{rx} = Ce^{rx}\left(r^2 + ar + b\right) = Ce^{rx}P(r).$$

Hence, Ce^{rx} is a solution of the differential equation $y'' + ay' + by = 0$ if and only if $P(r) = 0$.

(b) Let $y(x) = C_1e^{3x} + C_2e^{-x}$. Then

$$y'(x) = 3C_1e^{3x} - C_2e^{-x}$$
$$y''(x) = 9C_1e^{3x} + C_2e^{-x}$$

and

$$\begin{aligned} y'' - 2y' - 3y &= 9C_1e^{3x} + C_2e^{-x} - 6C_1e^{3x} + 2C_2e^{-x} - 3C_1e^{3x} - 3C_2e^{-x} \\ &= (9 - 6 - 3)\,C_1e^{3x} + (1 + 2 - 3)\,C_2e^{-x} = 0. \end{aligned}$$

67. A spherical tank of radius R is half-filled with water. Suppose that water leaks through a hole in the bottom of area B. Let $y(t)$ be the water level at time t (seconds).

(a) Show that $\dfrac{dy}{dt} = \dfrac{-\sqrt{2g}B\sqrt{y}}{\pi(2Ry - y^2)}$.

(b) Show that for some constant C,

$$\frac{2\pi}{15B\sqrt{2g}}\left(10Ry^{3/2} - 3y^{5/2}\right) = C - t$$

(c) Use the initial condition $y(0) = R$ to compute C, and show that $C = t_e$, the time at which the tank is empty.

(d) Show that t_e is proportional to $R^{5/2}$ and inversely proportional to B.

SOLUTION

(a) At height y above the bottom of the tank, the cross section is a circle of radius

$$r = \sqrt{R^2 - (R - y)^2} = \sqrt{2Ry - y^2}.$$

The cross-sectional area function is then $A(y) = \pi(2Ry - y^2)$. The differential equation for the height of the water in the tank is then

$$\frac{dy}{dt} = -\frac{\sqrt{2g}B\sqrt{y}}{\pi(2Ry - y^2)}$$

by Torricelli's law.

(b) Rewrite the differential equation as

$$\frac{\pi}{\sqrt{2g}B}\left(2Ry^{1/2} - y^{3/2}\right) dy = -\,dt,$$

and then integrate both sides to obtain

$$\frac{2\pi}{\sqrt{2g}B}\left(\frac{2}{3}Ry^{3/2} - \frac{1}{5}y^{5/2}\right) = C - t,$$

where C is an arbitrary constant. Simplifying gives

$$\frac{2\pi}{15B\sqrt{2g}}(10Ry^{3/2} - 3y^{5/2}) = C - t \qquad (*)$$

(c) From Equation (*) we see that $y = 0$ when $t = C$. It follows that $C = t_e$, the time at which the tank is empty. Moreover, the initial condition $y(0) = R$ allows us to determine the value of C:

$$\frac{2\pi}{15B\sqrt{2g}}(10R^{5/2} - 3R^{5/2}) = \frac{14\pi}{15B\sqrt{2g}}R^{5/2} = C$$

(d) From part (c),

$$t_e = \frac{14\pi}{15\sqrt{2g}} \cdot \frac{R^{5/2}}{B},$$

from which it is clear that t_e is proportional to $R^{5/2}$ and inversely proportional to B.

9.2 Models Involving $y' = k(y - b)$

Preliminary Questions

1. Write down a solution to $y' = 4(y - 5)$ that tends to $-\infty$ as $t \to \infty$.

SOLUTION The general solution is $y(t) = 5 + Ce^{4t}$ for any constant C; thus the solution tends to $-\infty$ as $t \to \infty$ whenever $C < 0$. One specific example is $y(t) = 5 - e^{4t}$.

2. Does $y' = -4(y - 5)$ have a solution that tends to ∞ as $t \to \infty$?

SOLUTION The general solution is $y(t) = 5 + Ce^{-4t}$ for any constant C. As $t \to \infty$, $y(t) \to 5$. Thus, there is no solution of $y' = -4(y - 5)$ that tends to ∞ as $t \to \infty$.

3. True or false? If $k > 0$, then all solutions of $y' = -k(y - b)$ approach the same limit as $t \to \infty$.

SOLUTION True. The general solution of $y' = -k(y - b)$ is $y(t) = b + Ce^{-kt}$ for any constant C. If $k > 0$, then $y(t) \to b$ as $t \to \infty$.

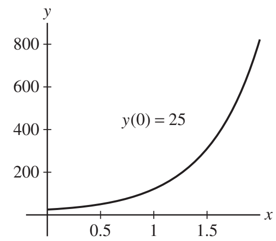
4. As an object cools, its rate of cooling slows. Explain how this follows from Newton's Law of Cooling.

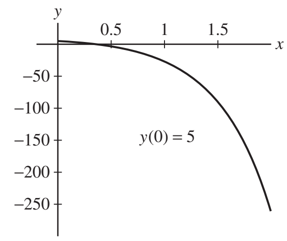
SOLUTION Newton's Law of Cooling states that $y' = -k(y - T_0)$ where $y(t)$ is the temperature and T_0 is the ambient temperature. Thus as $y(t)$ gets closer to T_0, $y'(t)$, the rate of cooling, gets smaller and the rate of cooling slows.

Exercises

1. Find the general solution of $y' = 2(y - 10)$. Then find the two solutions satisfying $y(0) = 25$ and $y(0) = 5$, and sketch their graphs.

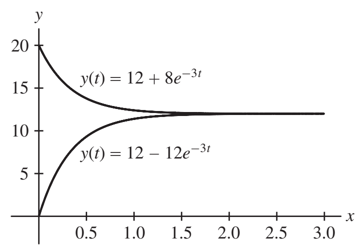
SOLUTION The general solution of $y' = 2(y - 10)$ is $y(t) = 10 + Ce^{2t}$ for any constant C. If $y(0) = 25$, then $10 + C = 25$, or $C = 15$; therefore, $y(t) = 10 + 15e^{2t}$. On the other hand, if $y(0) = 5$, then $10 + C = 5$, or $C = -5$; therefore, $y(t) = 10 - 5e^{2t}$. Graphs of these two functions are given below.





2. Verify directly that $y = 12 + Ce^{-3t}$ satisfies $y' = -3(y - 12)$ for all C. Then find the two solutions satisfying $y(0) = 20$ and $y(0) = 0$, and sketch their graphs.

SOLUTION The general solution of $y' = -3(y - 12)$ is $y(t) = 12 + Ce^{-3t}$ for any constant C. If $y(0) = 20$, then $12 + C = 20$, or $C = 8$; therefore, $y(t) = 12 + 8e^{-3t}$. On the other hand, if $y(0) = 0$, then $12 + C = 0$, or $C = -12$; therefore, $y(t) = 12 - 12e^{-3t}$. Graphs of these two functions are given below.



3. Solve $y' = 4y + 24$ subject to $y(0) = 5$.

SOLUTION Rewrite

$$y' = 4y + 24 \quad \text{as} \quad \frac{1}{4y + 24}\, dy = 1\, dt$$

Integrating gives

$$\begin{aligned} \frac{1}{4} \ln|4y + 24| &= t + C \\ \ln|4y + 24| &= 4t + C \\ 4y + 24 &= \pm e^{4t+C} \\ y &= Ae^{4t} - 6 \end{aligned}$$

where $A = \pm e^C/4$ is any constant. Since $y(0) = 5$ we have $5 = A - 6$ so that $A = 11$, and the solution is $y = 11e^{4t} - 6$.

4. Solve $y' + 6y = 12$ subject to $y(2) = 10$.

SOLUTION Rewrite

$$y' + 6y = 12 \quad \text{as} \quad \frac{dy}{dt} = 12 - 6y \quad \text{and then as} \quad \frac{1}{12 - 6y}\, dy = 1\, dt$$

Integrate both sides:

$$\begin{aligned} -\frac{1}{6} \ln|12 - 6y| &= t + C \\ \ln|12 - 6y| &= -6t + C \\ 12 - 6y &= \pm e^{-6t+C} \\ y &= Ae^{-6t} + 2 \end{aligned}$$

where $A = \pm e^C/6$ is any constant. Since $y(2) = 10$ we have $10 = Ae^{-12} + 2$ so that $A = 8e^{12}$, and the solution is $y = 8e^{12-6t} + 2$.

In Exercises 5–12, use Newton's Law of Cooling.

5. A hot anvil with cooling constant $k = 0.02\ \text{s}^{-1}$ is submerged in a large pool of water whose temperature is 10°C. Let $y(t)$ be the anvil's temperature t seconds later.

(a) What is the differential equation satisfied by $y(t)$?
(b) Find a formula for $y(t)$, assuming the object's initial temperature is 100°C.
(c) How long does it take the object to cool down to 20°?

SOLUTION

(a) By Newton's Law of Cooling, the differential equation is

$$y' = -0.02(y - 10)$$

(b) Separating variables gives

$$\frac{1}{y-10}\,dy = -0.02\,dt$$

Integrate to get

$$\begin{aligned} \ln|y - 10| &= -0.02t + C \\ y - 10 &= \pm e^{-0.02t+C} \\ y &= 10 + Ae^{-0.02t} \end{aligned}$$

where $A = \pm e^C$ is a constant. Since the initial temperature is 100°C, we have $y(0) = 100 = 10 + A$ so that $A = 90$, and $y = 10 + 90e^{-0.02t}$.

(c) We must find the value of t such that $y(t) = 20$, so we need to solve $20 = 10 + 90e^{-0.02t}$. Thus

$$10 = 90e^{-0.02t} \quad \Rightarrow \quad \frac{1}{9} = e^{-0.02t} \quad \Rightarrow \quad -\ln 9 = -0.02t \quad \Rightarrow \quad t = 50 \ln 9 \approx 109.86 \text{ s}$$

6. Frank's automobile engine runs at 100°C. On a day when the outside temperature is 21°C, he turns off the ignition and notes that five minutes later, the engine has cooled to 70°C.

(a) Determine the engine's cooling constant k.

(b) What is the formula for $y(t)$?

(c) When will the engine cool to 40°C?

SOLUTION

(a) The differential equation is

$$y' = -k(y - 21)$$

Rewriting gives $\frac{1}{y-21}\,dy = -k\,dt$. Integrate to get

$$\begin{aligned} \ln|y - 21| &= -kt + C \\ y - 21 &= \pm e^{C-kt} \\ y &= 21 + Ae^{-kt} \end{aligned}$$

where $A = \pm e^C$ is a constant. The initial temperature is 100°C, so $y(0) = 100$. Thus $100 = 21 + A$ and $A = 79$, so that $y = 21 + 79e^{-kt}$. The second piece of information tells us that $y(5) = 70 = 21 + 79e^{-5k}$. Solving for k gives

$$k = -\frac{1}{5} \ln \frac{49}{79} \approx 0.0955$$

(b) From part (b), the equation is $y = 21 + 79e^{-0.0955t}$.

(c) The engine has cooled to 40°C when $y(t) = 40$; solving gives

$$40 = 21 + 79e^{-0.0955t} \quad \Rightarrow \quad e^{-0.0955t} = \frac{19}{79} \quad \Rightarrow \quad t = -\frac{1}{0.0955} \ln \frac{19}{79} \approx 14.92 \text{ m}$$

7. At 10:30 AM, detectives discover a dead body in a room and measure its temperature at 26°C. One hour later, the body's temperature had dropped to 24.8°C. Determine the time of death (when the body temperature was a normal 37°C), assuming that the temperature in the room was held constant at 20°C.

SOLUTION Let $t = 0$ be the time when the person died, and let t_0 denote 10:30AM. The differential equation satisfied by the body temperature, $y(t)$, is

$$y' = -k(y - 20)$$

by Newton's Law of Cooling. Separating variables gives $\frac{1}{y-20}\,dy = -k\,dt$. Integrate to get

$$\ln|y-20| = -kt + C$$
$$y - 20 = \pm e^{-kt+C}$$
$$y = 20 + Ae^{-kt}$$

where $A = \pm e^C$ is a constant. Since normal body temperature is 37°C, we have $y(0) = 37 = 20 + A$ so that $A = 17$. To determine k, note that

$$26 = 20 + 17e^{-kt_0} \qquad \text{and} \qquad 24.8 = 20 + 17e^{-k(t_0+1)}$$
$$kt_0 = -\ln\frac{6}{17} \qquad\qquad kt_0 + k = -\ln\frac{4.8}{17}$$

Subtracting these equations gives

$$k = \ln\frac{6}{17} - \ln\frac{4.8}{17} = \ln\frac{6}{4.8} \approx 0.223$$

We thus have

$$y = 20 + 17e^{-0.223t}$$

as the equation for the body temperature at time t. Since $y(t_0) = 26$, we have

$$26 = 20 + 17e^{-0.223t} \quad\Rightarrow\quad e^{-0.223t} = \frac{6}{17} \quad\Rightarrow\quad t = -\frac{1}{0.223}\ln\frac{6}{17} \approx 4.667 \text{ h}$$

so that the time of death was approximately 4 hours and 40 minutes ago.

8. A cup of coffee with cooling constant $k = 0.09$ min^{-1} is placed in a room at temperature 20°C.
(a) How fast is the coffee cooling (in degrees per minute) when its temperature is $T = 80$°C?
(b) Use the Linear Approximation to estimate the change in temperature over the next 6 s when $T = 80$°C.
(c) If the coffee is served at 90°C, how long will it take to reach an optimal drinking temperature of 65°C?

SOLUTION
(a) According to Newton's Law of Cooling, the coffee will cool at the rate $k(T - T_0)$, where k is the cooling constant of the coffee, T is the current temperature of the coffee and T_0 is the temperature of the surroundings. With $k = 0.09$ min^{-1}, $T = 80$°C and $T_0 = 20$°C, the coffee is cooling at the rate

$$0.09(80 - 20) = 5.4\text{°C/min}.$$

(b) Using the result from part (a) and the Linear Approximation, we estimate that the coffee will cool

$$(5.4\text{°C/min})(0.1 \text{ min}) = 0.54\text{°C}$$

over the next 6 seconds.
(c) With $T_0 = 20$°C and an initial temperature of 90°C, the temperature of the coffee at any time t is $T(t) = 20 + 70e^{-0.09t}$. Solving $20 + 70e^{-0.09t} = 65$ for t yields

$$t = -\frac{1}{0.09}\ln\left(\frac{45}{70}\right) \approx 4.91 \text{ minutes.}$$

9. A cold metal bar at −30°C is submerged in a pool maintained at a temperature of 40°C. Half a minute later, the temperature of the bar is 20°C. How long will it take for the bar to attain a temperature of 30°C?

SOLUTION With $T_0 = 40$°C, the temperature of the bar is given by $F(t) = 40 + Ce^{-kt}$ for some constants C and k. From the initial condition, $F(0) = 40 + C = -30$, so $C = -70$. After 30 seconds, $F(30) = 40 - 70e^{-30k} = 20$, so

$$k = -\frac{1}{30}\ln\left(\frac{20}{70}\right) \approx 0.0418 \text{ seconds}^{-1}.$$

To attain a temperature of 30°C we must solve $40 - 70e^{-0.0418t} = 30$ for t. This yields

$$t = \frac{\ln\left(\frac{10}{70}\right)}{-0.0418} \approx 46.55 \text{ seconds.}$$

10. When a hot object is placed in a water bath whose temperature is 25°C, it cools from 100°C to 50°C in 150 s. In another bath, the same cooling occurs in 120 s. Find the temperature of the second bath.

SOLUTION With $T_0 = 25°C$, the temperature of the object is given by $F(t) = 25 + Ce^{-kt}$ for some constants C and k. From the initial condition, $F(0) = 25 + C = 100$, so $C = 75$. After 150 seconds, $F(150) = 25 + 75e^{-150k} = 50$, so

$$k = -\frac{1}{150}\ln\left(\frac{25}{75}\right) \approx 0.0073 \text{ seconds}^{-1}.$$

If we place the same object with a temperature of 100°C into a second bath whose temperature is T_0, then the temperature of the object is given by

$$F(t) = T_0 + (100 - T_0)e^{-0.0073t}.$$

To cool from 100°C to 50°C in 120 seconds, T_0 must satisfy

$$T_0 + (100 - T_0)e^{-0.0073(120)} = 50.$$

Thus, $T_0 = 14.32°C$.

11. GU Objects A and B are placed in a warm bath at temperature $T_0 = 40°C$. Object A has initial temperature $-20°C$ and cooling constant $k = 0.004\ s^{-1}$. Object B has initial temperature 0°C and cooling constant $k = 0.002\ s^{-1}$. Plot the temperatures of A and B for $0 \le t \le 1000$. After how many seconds will the objects have the same temperature?

SOLUTION With $T_0 = 40°C$, the temperature of A and B are given by

$$A(t) = 40 + C_A e^{-0.004t} \qquad B(t) = 40 + C_B e^{-0.002t}$$

Since $A(0) = -20$ and $B(0) = 0$, we have

$$A(t) = 40 - 60e^{-0.004t} \qquad B(t) = 40 - 40e^{-0.002t}$$

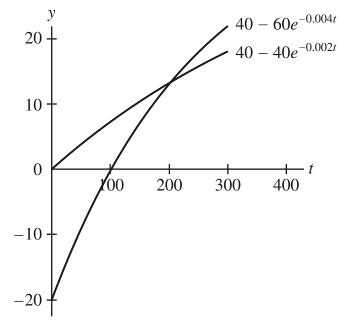
The two objects will have the same temperature whenever $A(t) = B(t)$, so we must solve

$$40 - 60e^{-0.004t} = 40 - 40e^{-0.002t} \quad \Rightarrow \quad 3e^{-0.004t} = 2e^{-0.002t}$$

Take logs to get

$$-0.004t + \ln 3 = -0.002t + \ln 2 \quad \Rightarrow \quad t = \frac{\ln 3 - \ln 2}{0.002} \approx 202.7 \text{ s}$$

or about 3 minutes 22 seconds.



12. In Newton's Law of Cooling, the constant $\tau = 1/k$ is called the "characteristic time." Show that τ is the time required for the temperature difference $(y - T_0)$ to decrease by the factor $e^{-1} \approx 0.37$. For example, if $y(0) = 100°C$ and $T_0 = 0°C$, then the object cools to $100/e \approx 37°C$ in time τ, to $100/e^2 \approx 13.5°C$ in time 2τ, and so on.

SOLUTION If $y' = -k(y - T_0)$, then $y(t) = T_0 + Ce^{-kt}$. But then

$$\frac{y(t+\tau) - T_0}{y(t) - T_0} = \frac{Ce^{-k(t+\tau)}}{Ce^{-kt}} = e^{-k\tau} = e^{-k \cdot 1/k} = e^{-1}$$

Thus after time τ starting from any time t, the temperature difference will have decreased by a factor of e^{-1}.

In Exercises 13–16, use Eq. (3) as a model for free-fall with air resistance.

13. A 60-kg skydiver jumps out of an airplane. What is her terminal velocity, in meters per second, assuming that $k = 10$ kg/s for free-fall (no parachute)?

SOLUTION The free-fall terminal velocity is

$$\frac{-gm}{k} = \frac{-9.8(60)}{10} = -58.8 \text{ m/s.}$$

14. Find the terminal velocity of a skydiver of weight $w = 192$ lb if $k = 1.2$ lb-s/ft. How long does it take him to reach half of his terminal velocity if his initial velocity is zero? Mass and weight are related by $w = mg$, and Eq. (3) becomes $v' = -(kg/w)(v + w/k)$ with $g = 32$ ft/s^2.

SOLUTION The skydiver's velocity $v(t)$ satisfies the differential equation

$$v' = -\frac{kg}{w}\left(v + \frac{w}{k}\right),$$

where

$$\frac{kg}{w} = \frac{(1.2)(32)}{192} = 0.2 \text{ sec}^{-1} \quad \text{and} \quad \frac{w}{k} = \frac{192}{1.2} = 160 \text{ ft/sec.}$$

The general solution to this equation is $v(t) = -160 + Ce^{-0.2t}$, for some constant C. From the initial condition $v(0) = 0$, we find $0 = -160 + C$, or $C = 160$. Therefore,

$$v(t) = -160 + 160e^{-0.2t} = -160(1 - e^{-0.2t}).$$

Now, the terminal velocity of the skydiver is

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} -160(1 - e^{-0.2t}) = -160 \text{ ft/sec.}$$

To determine how long it takes for the skydiver to reach half this terminal velocity, we must solve the equation $v(t) = -80$ for t:

$$\begin{aligned} -160(1 - e^{-0.2t}) &= -80 \\ 1 - e^{-0.2t} &= \frac{1}{2} \\ e^{-0.2t} &= \frac{1}{2} \\ t &= -\frac{1}{0.2} \ln \frac{1}{2} \approx 3.47 \text{ sec.} \end{aligned}$$

15. A 80-kg skydiver jumps out of an airplane (with zero initial velocity). Assume that $k = 12$ kg/s with a closed parachute and $k = 70$ kg/s with an open parachute. What is the skydiver's velocity at $t = 25$ s if the parachute opens after 20 s of free fall?

SOLUTION We first compute the skydiver's velocity after 20 s of free fall, then use that as the initial velocity to calculate her velocity after an additional 5 s of restrained fall. We have $m = 80$ and $g = 9.8$; for free fall, $k = 12$, so

$$\frac{k}{m} = \frac{12}{80} = 0.15, \qquad \frac{-mg}{k} = \frac{-80 \cdot 9.8}{12} \approx -65.33$$

The general solution is thus $v(t) = -65.33 + Ce^{-0.15t}$. Since $v(0) = 0$, we have $C = 65.33$, so that

$$v(t) = -65.33(1 - e^{-0.15t})$$

After 20 s of free fall, the diver's velocity is thus

$$v(20) = -65.33(1 - e^{-0.15 \cdot 20}) \approx -62.08 \text{ m/s}$$

Once the parachute opens, $k = 70$, so

$$\frac{k}{m} = \frac{70}{80} = 0.875, \qquad \frac{mg}{k} = \frac{80 \cdot 9.8}{70} = 11.2$$

so that the general solution for the restrained fall model is $v_r(t) = -11.2 + Ce^{-0.875t}$. Here $v_r(0) = -62.08$, so that $C = 11.2 - 62.08 = -50.88$ and $v_r(t) = -11.20 - 50.88e^{-0.875t}$. After 5 additional seconds, the diver's velocity is therefore

$$v_r(5) = -11.20 - 50.88e^{-0.875 \cdot 5} \approx -11.84 \text{ m/s}$$

16. Does a heavier or a lighter skydiver reach terminal velocity faster?

SOLUTION The velocity of a skydiver is

$$v(t) = -\frac{gm}{k} + Ce^{-kt/m}.$$

As m decreases, the fraction $-k/m$ becomes more negative and $e^{-(k/m)t}$ approaches zero more rapidly. Thus, a lighter skydiver approaches terminal velocity faster.

17. A continuous annuity with withdrawal rate $N = \$5000$/year and interest rate $r = 5\%$ is funded by an initial deposit of $P_0 = \$50{,}000$.

(a) What is the balance in the annuity after 10 years?

(b) When will the annuity run out of funds?

SOLUTION

(a) From Equation , the value of the annuity is given by

$$P(t) = \frac{5000}{0.05} + Ce^{0.05t} = 100{,}000 + Ce^{0.05t}$$

for some constant C. Since $P(0) = 50{,}000$, we have $C = -50{,}000$ and $P(t) = 100{,}000 - 50{,}000e^{0.05t}$. After ten years, then, the balance in the annuity is

$$P(10) = 100{,}000 - 50{,}000e^{0.05\cdot 10} = 100{,}000 - 50{,}000e^{0.5} \approx \$17{,}563.94$$

(b) The annuity will run out of funds when $P(t) = 0$:

$$0 = 100{,}000 - 50{,}000e^{0.05t} \quad \Rightarrow \quad e^{0.05t} = 2 \quad \Rightarrow \quad t = \frac{\ln 2}{0.05} \approx 13.86$$

The annuity will run out of funds after approximately 13 years 10 months.

18. Show that a continuous annuity with withdrawal rate $N = \$5000$/year and interest rate $r = 8\%$, funded by an initial deposit of $P_0 = \$75{,}000$, never runs out of money.

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{5000}{0.08} + Ce^{0.08t} = 62500 + Ce^{0.08t}$$

for some constant C. If $P_0 = 75{,}000$, then $75{,}000 = 62{,}500 + C$ and $C = 12{,}500$. Thus, $P(t) = 62{,}500 + 12{,}500e^{0.08t}$. As $t \to \infty$, $P(t) \to \infty$, so the annuity lives forever. Note the annuity will live forever for any $P_0 \geq \$62{,}500$.

19. Find the minimum initial deposit P_0 that will allow an annuity to pay out \$6000/year indefinitely if it earns interest at a rate of 5%.

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{6000}{0.05} + Ce^{0.05t} = 120{,}000 + Ce^{0.05t}$$

for some constant C. To fund the annuity indefinitely, we must have $C \geq 0$. If the initial deposit is P_0, then $P_0 = 120{,}000 + C$ and $C = P_0 - 120{,}000$. Thus, to fund the annuity indefinitely, we must have $P_0 \geq \$120{,}000$.

20. Find the minimum initial deposit P_0 necessary to fund an annuity for 20 years if withdrawals are made at a rate of \$10,000/year and interest is earned at a rate of 7%.

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{10{,}000}{0.07} + Ce^{0.07t} = 142{,}857.14 + Ce^{0.07t}$$

for some constant C. If the initial deposit is P_0, then $P_0 = 142{,}857.14 + C$ and $C = 142{,}857.14 - P_0$. To fund the annuity for 20 years, we need

$$P(20) = 142{,}857.14 + (P_0 - 142{,}857.14)e^{0.07(20)} \geq 0.$$

Hence,

$$P_0 \geq 142{,}857.14(1 - e^{-1.4}) = \$107{,}629.00.$$

21. An initial deposit of 100,000 euros are placed in an annuity with a French bank. What is the minimum interest rate the annuity must earn to allow withdrawals at a rate of 8000 euros/year to continue indefinitely?

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{8000}{r} + Ce^{rt}$$

for some constant C. To fund the annuity indefinitely, we need $C \geq 0$. If the initial deposit is 100,000 euros, then $100{,}000 = \frac{8000}{r} + C$ and $C = 100{,}000 - \frac{8000}{r}$. Thus, to fund the annuity indefinitely, we need $100{,}000 - \frac{8000}{r} \geq 0$, or $r \geq 0.08$. The bank must pay at least 8%.

22. Show that a continuous annuity never runs out of money if the initial balance is greater than or equal to N/r, where N is the withdrawal rate and r the interest rate.

SOLUTION With a withdrawal rate of N and an interest rate of r, the balance in the annuity is $P(t) = \frac{N}{r} + Ce^{rt}$ for some constant C. Let P_0 denote the initial balance. Then $P_0 = P(0) = \frac{N}{r} + C$ and $C = P_0 - \frac{N}{r}$. If $P_0 \geq \frac{N}{r}$, then $C \geq 0$ and the annuity lives forever.

23. Sam borrows \$10,000 from a bank at an interest rate of 9% and pays back the loan continuously at a rate of N dollars per year. Let $P(t)$ denote the amount still owed at time t.

(a) Explain why $P(t)$ satisfies the differential equation

$$y' = 0.09y - N$$

(b) How long will it take Sam to pay back the loan if $N = \$1200$?

(c) Will the loan ever be paid back if $N = \$800$?

SOLUTION

(a)

$$\text{Rate of Change of Loan} = (\text{Amount still owed})(\text{Interest rate}) - (\text{Payback rate})$$
$$= P(t) \cdot r - N = r\left(P - \frac{N}{r}\right).$$

Therefore, if $y = P(t)$,

$$y' = r\left(y - \frac{N}{r}\right) = ry - N$$

(b) From the differential equation derived in part (a), we know that $P(t) = \frac{N}{r} + Ce^{rt} = 13{,}333.33 + Ce^{0.09t}$. Since \$10,000 was initially borrowed, $P(0) = 13{,}333.33 + C = 10{,}000$, and $C = -3333.33$. The loan is paid off when $P(t) = 13{,}333.33 - 3333.33e^{0.09t} = 0$. This yields

$$t = \frac{1}{0.09}\ln\left(\frac{13{,}333.33}{3333.33}\right) \approx 15.4 \text{ years.}$$

(c) If the annual rate of payment is \$800, then $P(t) = 800/0.09 + Ce^{0.09t} = 8888.89 + Ce^{0.09t}$. With $P(0) = 8888.89 + C = 10{,}000$, it follows that $C = 1111.11$. Since $C > 0$ and $e^{0.09t} \to \infty$ as $t \to \infty$, $P(t) \to \infty$, and the loan will never be paid back.

24. April borrows \$18,000 at an interest rate of 5% to purchase a new automobile. At what rate (in dollars per year) must she pay back the loan, if the loan must be paid off in 5 years? *Hint:* Set up the differential equation as in Exercise 23).

SOLUTION As in Exercise 23, the differential equation is

$$P(t)' = rP(t) - N = r\left(P(t) - \frac{N}{r}\right)$$

where r is the interest rate and N is the payment amount, so that here

$$P(t)' = 0.05\left(P(t) - \frac{N}{0.05}\right) \quad \Rightarrow \quad P(t) = \frac{N}{0.05} + Ce^{0.05t}$$

Since $P(0) = 18{,}000$, we have $C = 18{,}000 - \frac{N}{0.05}$, so that

$$P(t) = \frac{N}{0.05} + \left(18{,}000 - \frac{N}{0.05}\right)e^{0.05t}$$

If the loan is to be paid back in 5 years, we must have

$$P(5) = 0 = \frac{N}{0.05} + \left(18{,}000 - \frac{N}{0.05}\right) e^{0.05 \cdot 5}$$

Solving for N gives

$$N = \frac{900}{1 - e^{-0.25}} \approx 4068.73$$

so the payments must be at least \$4068.73 per year.

25. Let $N(t)$ be the fraction of the population who have heard a given piece of news t hours after its initial release. According to one model, the rate $N'(t)$ at which the news spreads is equal to k times the fraction of the population that has not yet heard the news, for some constant $k > 0$.

(a) Determine the differential equation satisfied by $N(t)$.

(b) Find the solution of this differential equation with the initial condition $N(0) = 0$ in terms of k.

(c) Suppose that half of the population is aware of an earthquake 8 hours after it occurs. Use the model to calculate k and estimate the percentage that will know about the earthquake 12 hours after it occurs.

SOLUTION

(a) $N'(t) = k(1 - N(t)) = -k(N(t) - 1)$.

(b) The general solution of the differential equation from part (a) is $N(t) = 1 + Ce^{-kt}$. The initial condition determines the value of C: $N(0) = 1 + C = 0$ so $C = -1$. Thus, $N(t) = 1 - e^{-kt}$.

(c) Knowing that $N(8) = 1 - e^{-8k} = \frac{1}{2}$, we find that

$$k = -\frac{1}{8} \ln\left(\frac{1}{2}\right) \approx 0.0866 \text{ hours}^{-1}.$$

With the value of k determined, we estimate that

$$N(12) = 1 - e^{-0.0866(12)} \approx 0.6463 = 64.63\%$$

of the population will know about the earthquake after 12 hours.

26. Current in a Circuit When the circuit in Figure 6 (which consists of a battery of V volts, a resistor of R ohms, and an inductor of L henries) is connected, the current $I(t)$ flowing in the circuit satisfies

$$L\frac{dI}{dt} + RI = V$$

with the initial condition $I(0) = 0$.

(a) Find a formula for $I(t)$ in terms of L, V, and R.

(b) Show that $\lim_{t\to\infty} I(t) = V/R$.

(c) Show that $I(t)$ reaches approximately 63% of its maximum value at the "characteristic time" $\tau = L/R$.

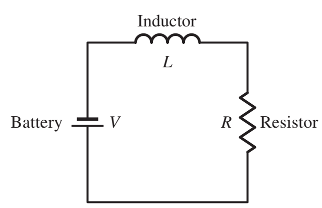


FIGURE 6 Current flow approaches the level $I_{\max} = V/R$.

SOLUTION

(a) Solve the differential equation for $\frac{dI}{dt}$:

$$\frac{dI}{dt} = -\frac{1}{L}(RI - V) = -\frac{R}{L}\left(I - \frac{V}{R}\right)$$

so that the general solution is

$$I(t) = \frac{V}{R} + Ce^{-(R/L)t}$$

The initial condition $I(0) = 0$ gives $C = -\frac{V}{R}$, so that

$$I(t) = \frac{V}{R}(1 - e^{-(R/L)t})$$

(b) As $t \to \infty$, $e^{-(R/L)t} \to 0$, so that $I(t) \to \frac{V}{R}$.
(c) When $t = \tau = L/R$,

$$I(\tau) = \frac{V}{R}(1 - e^{-(R/L)\tau}) = \frac{V}{R}(1 - e^{-(R/L)(L/R)}) = \frac{V}{R}(1 - e^{-1}) \approx 0.63\frac{V}{R}$$

which is 63% of the maximum value of V/R.

Further Insights and Challenges

27. Show that the cooling constant of an object can be determined from two temperature readings $y(t_1)$ and $y(t_2)$ at times $t_1 \neq t_2$ by the formula

$$k = \frac{1}{t_1 - t_2} \ln\left(\frac{y(t_2) - T_0}{y(t_1) - T_0}\right)$$

SOLUTION We know that $y(t_1) = T_0 + Ce^{-kt_1}$ and $y(t_2) = T_0 + Ce^{-kt_2}$. Thus, $y(t_1) - T_0 = Ce^{-kt_1}$ and $y(t_2) - T_0 = Ce^{-kt_2}$. Dividing the latter equation by the former yields

$$e^{-kt_2 + kt_1} = \frac{y(t_2) - T_0}{y(t_1) - T_0},$$

so that

$$k(t_1 - t_2) = \ln\left(\frac{y(t_2) - T_0}{y(t_1) - T_0}\right) \quad \text{and} \quad k = \frac{1}{t_1 - t_2} \ln\left(\frac{y(t_2) - T_0}{y(t_1) - T_0}\right).$$

28. Show that by Newton's Law of Cooling, the time required to cool an object from temperature A to temperature B is

$$t = \frac{1}{k} \ln\left(\frac{A - T_0}{B - T_0}\right)$$

where T_0 is the ambient temperature.

SOLUTION At any time t, the temperature of the object is $y(t) = T_0 + Ce^{-kt}$ for some constant C. Suppose the object is initially at temperature A and reaches temperature B at time t. Then $A = T_0 + C$, so $C = A - T_0$. Moreover,

$$B = T_0 + Ce^{-kt} = T_0 + (A - T_0)e^{-kt}.$$

Solving this last equation for t yields

$$t = \frac{1}{k} \ln\left(\frac{A - T_0}{B - T_0}\right).$$

29. Air Resistance A projectile of mass $m = 1$ travels straight up from ground level with initial velocity v_0. Suppose that the velocity v satisfies $v' = -g - kv$.
(a) Find a formula for $v(t)$.
(b) Show that the projectile's height $h(t)$ is given by

$$h(t) = C(1 - e^{-kt}) - \frac{g}{k}t$$

where $C = k^{-2}(g + kv_0)$.
(c) Show that the projectile reaches its maximum height at time $t_{\max} = k^{-1} \ln(1 + kv_0/g)$.
(d) In the absence of air resistance, the maximum height is reached at time $t = v_0/g$. In view of this, explain why we should expect that

$$\lim_{k \to 0} \frac{\ln(1 + \frac{kv_0}{g})}{k} = \frac{v_0}{g} \qquad \boxed{8}$$

(e) Verify Eq. (8). *Hint:* Use Theorem 2 in Section 5.8 to show that $\lim_{k \to 0}\left(1 + \frac{kv_0}{g}\right)^{1/k} = e^{v_0/g}$ or use L'Hôpital's Rule.

SOLUTION

(a) Since $v' = -g - kv = -k\left(v - \frac{-g}{k}\right)$ it follows that $v(t) = \frac{-g}{k} + Be^{-kt}$ for some constant B. The initial condition $v(0) = v_0$ determines B: $v_0 = -\frac{g}{k} + B$, so $B = v_0 + \frac{g}{k}$. Thus,

$$v(t) = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}.$$

(b) $v(t) = h'(t)$ so

$$h(t) = \int \left(-\frac{g}{k} + \left(v_0 + \frac{g}{k}\right) e^{-kt}\right) dt = -\frac{g}{k}t - \frac{1}{k}\left(v_0 + \frac{g}{k}\right) e^{-kt} + D.$$

The initial condition $h(0) = 0$ determines

$$D = \frac{1}{k}\left(v_0 + \frac{g}{k}\right) = \frac{1}{k^2}(v_0 k + g).$$

Let $C = \frac{1}{k^2}(v_0 k + g)$. Then

$$h(t) = C(1 - e^{-kt}) - \frac{g}{k}t.$$

(c) The projectile reaches its maximum height when $v(t) = 0$. This occurs when

$$-\frac{g}{k} + \left(v_0 + \frac{g}{k}\right) e^{-kt} = 0,$$

or

$$t = \frac{1}{-k} \ln\left(\frac{g}{kv_0 + g}\right) = \frac{1}{k} \ln\left(1 + \frac{kv_0}{g}\right).$$

(d) Recall that k is the proportionality constant for the force due to air resistance. Thus, as $k \to 0$, the effect of air resistance disappears. We should therefore expect that, as $k \to 0$, the time at which the maximum height is achieved from part (c) should approach v_0/g. In other words, we should expect

$$\lim_{k\to 0} \frac{1}{k} \ln\left(1 + \frac{kv_0}{g}\right) = \frac{v_0}{g}.$$

(e) Recall that

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

If we substitute $x = v_0/g$ and $k = 1/n$, we find

$$e^{v_0/g} = \lim_{k\to 0} \left(1 + \frac{v_0 k}{g}\right)^{1/k}.$$

Then

$$\lim_{k\to 0} \frac{1}{k} \ln\left(1 + \frac{kv_0}{g}\right) = \lim_{k\to 0} \ln\left(1 + \frac{v_0 k}{g}\right)^{1/k} = \ln\left(\lim_{k\to 0}\left(1 + \frac{v_0 k}{g}\right)^{1/k}\right) = \ln(e^{v_0/g}) = \frac{v_0}{g}.$$

9.3 Graphical and Numerical Methods

Preliminary Questions

1. What is the slope of the segment in the slope field for $\dot{y} = ty + 1$ at the point $(2, 3)$?

SOLUTION The slope of the segment in the slope field for $\dot{y} = ty + 1$ at the point $(2, 3)$ is $(2)(3) + 1 = 7$.

2. What is the equation of the isocline of slope $c = 1$ for $\dot{y} = y^2 - t$?

SOLUTION The isocline of slope $c = 1$ has equation $y^2 - t = 1$, or $y = \pm\sqrt{1 + t}$.

3. For which of the following differential equations are the slopes at points on a vertical line $t = C$ all equal?

(a) $\dot{y} = \ln y$

(b) $\dot{y} = \ln t$

SOLUTION Only for the equation in part (b). The slope at a point is simply the value of $\dot{y}$ at that point, so for part (a), the slope depends on y, while for part (b), the slope depends only on t.

4. Let $y(t)$ be the solution to $\dot{y} = F(t, y)$ with $y(1) = 3$. How many iterations of Euler's Method are required to approximate $y(3)$ if the time step is $h = 0.1$?

SOLUTION The initial condition is specified at $t = 1$ and we want to obtain an approximation to the value of the solution at $t = 3$. With a time step of $h = 0.1$,

$$\frac{3 - 1}{0.1} = 20$$

iterations of Euler's method are required.

Exercises

1. Figure 8 shows the slope field for $\dot{y} = \sin y \sin t$. Sketch the graphs of the solutions with initial conditions $y(0) = 1$ and $y(0) = -1$. Show that $y(t) = 0$ is a solution and add its graph to the plot.

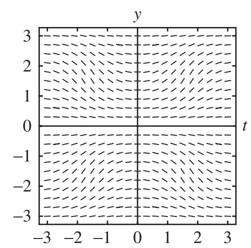
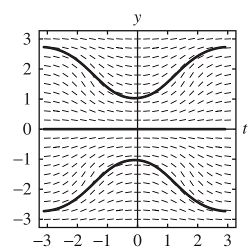


FIGURE 8 Slope field for $\dot{y} = \sin y \sin t$.

SOLUTION The sketches of the solutions appear below.



If $y(t) = 0$, then $y' = 0$; moreover, $\sin 0 \sin t = 0$. Thus, $y(t) = 0$ is a solution of $\dot{y} = \sin y \sin t$.

2. Figure 9 shows the slope field for $\dot{y} = y^2 - t^2$. Sketch the integral curve passing through the point $(0, -1)$, the curve through $(0, 0)$, and the curve through $(0, 2)$. Is $y(t) = 0$ a solution?

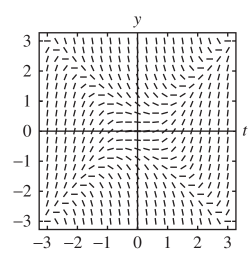
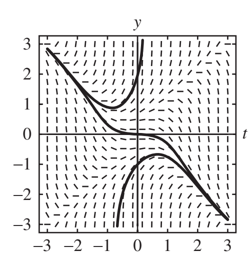


FIGURE 9 Slope field for $\dot{y} = y^2 - t^2$.

SOLUTION The sketches of the solutions appear below.



Let $y(t) = 0$. Because $\dot{y} = 0$ but $y^2 - t^2 = -t^2 \neq 0$, it follows that $y(t) = 0$ is not a solution of $\dot{y} = y^2 - t^2$.

3. Show that $f(t) = \frac{1}{2}\left(t - \frac{1}{2}\right)$ is a solution to $\dot{y} = t - 2y$. Sketch the four solutions with $y(0) = \pm 0.5, \pm 1$ on the slope field in Figure 10. The slope field suggests that every solution approaches $f(t)$ as $t \to \infty$. Confirm this by showing that $y = f(t) + Ce^{-2t}$ is the general solution.

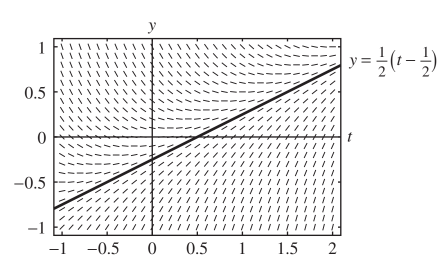
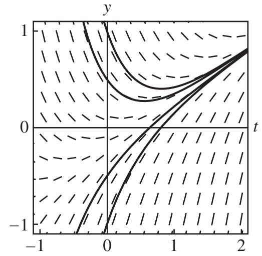


FIGURE 10 Slope field for $\dot{y} = t - 2y$.

SOLUTION Let $y = f(t) = \frac{1}{2}(t - \frac{1}{2})$. Then $\dot{y} = \frac{1}{2}$ and

$$\dot{y} + 2y = \frac{1}{2} + t - \frac{1}{2} = t,$$

so $f(t) = \frac{1}{2}(t - \frac{1}{2})$ is a solution to $\dot{y} = t - 2y$. The slope field with the four required solutions is shown below.



Now, let $y = f(t) + Ce^{-2t} = \frac{1}{2}(t - \frac{1}{2}) + Ce^{-2t}$. Then

$$\dot{y} = \frac{1}{2} - 2Ce^{-2t},$$

and

$$\dot{y} + 2y = \frac{1}{2} - 2Ce^{-2t} + \left(t - \frac{1}{2}\right) + 2Ce^{-2t} = t.$$

Thus, $y = f(t) + Ce^{-2t}$ is the general solution to the equation $\dot{y} = t - 2y$.

4. One of the slope fields in Figures 11(A) and (B) is the slope field for $\dot{y} = t^2$. The other is for $\dot{y} = y^2$. Identify which is which. In each case, sketch the solutions with initial conditions $y(0) = 1$, $y(0) = 0$, and $y(0) = -1$.

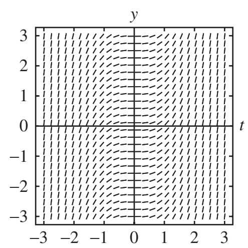


FIGURE 11(A)

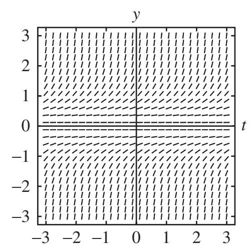
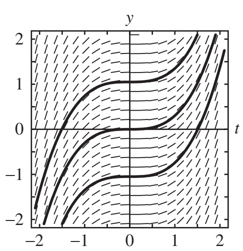
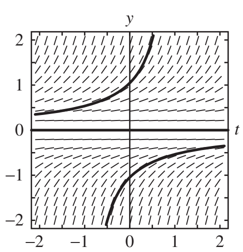


FIGURE 11(B)

SOLUTION For $y' = t^2$, y' only depends on t. The isoclines of any slope c will be the two vertical lines $t = \pm\sqrt{c}$. This indicates that the slope field will be the one given in Figure 11(A). The solutions are sketched below:



For $y' = y^2$, y' only depends on y. The isoclines of any slope c will be the two *horizontal* lines $y = \pm\sqrt{c}$. This indicates that the slope field will be the one given in Figure 11(B). The solutions are sketched below:



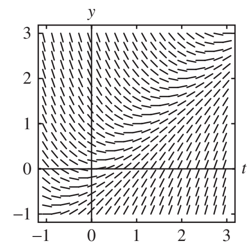
5. Consider the differential equation $\dot{y} = t - y$.

(a) Sketch the slope field of the differential equation $\dot{y} = t - y$ in the range $-1 \le t \le 3$, $-1 \le y \le 3$. As an aid, observe that the isocline of slope c is the line $t - y = c$, so the segments have slope c at points on the line $y = t - c$.

(b) Show that $y = t - 1 + Ce^{-t}$ is a solution for all C. Since $\lim_{t\to\infty} e^{-t} = 0$, these solutions approach the particular solution $y = t - 1$ as $t \to \infty$. Explain how this behavior is reflected in your slope field.

SOLUTION

(a) Here is a sketch of the slope field:



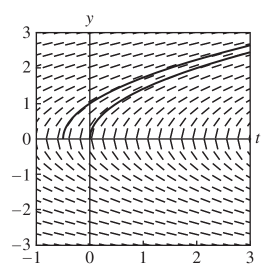
(b) Let $y = t - 1 + Ce^{-t}$. Then $\dot{y} = 1 - C^{-t}$, and

$$t - y = t - (t - 1 + Ce^{-t}) = 1 - Ce^{-t}.$$

Thus, $y = t - 1 + Ce^{-t}$ is a solution of $\dot{y} = t - y$. On the slope field, we can see that the isoclines of 1 all lie along the line $y = t - 1$. Whenever $y > t - 1$, $\dot{y} = t - y < 1$, so the solution curve will converge downward towards the line $y = t - 1$. On the other hand, if $y < t - 1$, $\dot{y} = t - y > 1$, so the solution curve will converge upward towards $y = t - 1$. In either case, the solution is approaching $t - 1$.

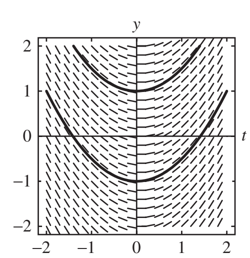
6. Show that the isoclines of $\dot{y} = 1/y$ are horizontal lines. Sketch the slope field for $-2 \le t \le 2$, $-2 \le y \le 2$ and plot the solutions with initial conditions $y(0) = 0$ and $y(0) = 1$.

SOLUTION The isocline of slope c is defined by $\frac{1}{y} = c$. This is equivalent to $y = \frac{1}{c}$, which is a horizontal line. The slope field and the solutions are shown below.



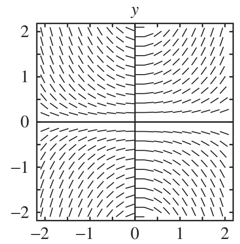
7. Show that the isoclines of $\dot{y} = t$ are vertical lines. Sketch the slope field for $-2 \le t \le 2$, $-2 \le y \le 2$ and plot the integral curves passing through $(0, -1)$ and $(0, 1)$.

SOLUTION The isocline of slope c for the differential equation $\dot{y} = t$ has equation $t = c$, which is the equation of a vertical line. The slope field and the required solution curves are shown below.



8. Sketch the slope field of $\dot{y} = ty$ for $-2 \le t \le 2$, $-2 \le y \le 2$. Based on the sketch, determine $\lim_{t\to\infty} y(t)$, where $y(t)$ is a solution with $y(0) > 0$. What is $\lim_{t\to\infty} y(t)$ if $y(0) < 0$?

SOLUTION The slope field for $\dot{y} = ty$ is shown below.



With $y(0) > 0$, the slope field indicates that y is an always increasing, always concave up function; consequently, $\lim_{t\to\infty} y = \infty$. On the other hand, when $y(0) < 0$, the slope field indicates that y is an always decreasing, always concave down function; consequently, $\lim_{t\to\infty} y = -\infty$.

9. Match each differential equation with its slope field in Figures 12(A)–(F).

(i) $\dot{y} = -1$
(ii) $\dot{y} = \dfrac{y}{t}$
(iii) $\dot{y} = t^2 y$
(iv) $\dot{y} = ty^2$
(v) $\dot{y} = t^2 + y^2$
(vi) $\dot{y} = t$

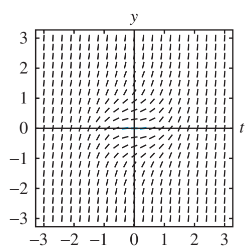


FIGURE 12(A)

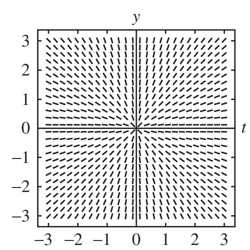


FIGURE 12(B)

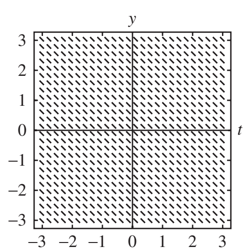


FIGURE 12(C)

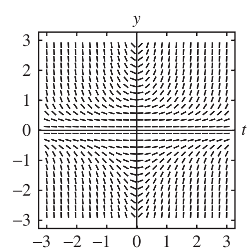


FIGURE 12(D)

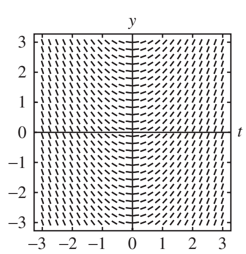


FIGURE 12(E)

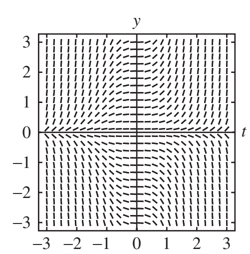


FIGURE 12(F)

SOLUTION

(i) Every segment in the slope field for $\dot{y} = -1$ will have slope -1; this matches Figure 12(C).

(ii) The segments in the slope field for $\dot{y} = \dfrac{y}{t}$ will have positive slope in the first and third quadrants and negative slopes in the second and fourth quadrant; this matches Figure 12(B).

(iii) The segments in the slope field for $\dot{y} = t^2 y$ will have positive slope in the upper half of the plane and negative slopes in the lower half of the plane; this matches Figure 12(F).
(iv) The segments in the slope field for $\dot{y} = ty^2$ will have positive slope on the right side of the plane and negative slopes on the left side of the plane; this matches Figure 12(D).
(v) Every segment in the slope field for $\dot{y} = t^2 + y^2$, except at the origin, will have positive slope; this matches Figure 12(A).
(vi) The isoclines for $\dot{y} = t$ are vertical lines; this matches Figure 12(E).

10. Sketch the solution of $\dot{y} = ty^2$ satisfying $y(0) = 1$ in the appropriate slope field of Figure 12(A)–(F). Then show, using separation of variables, that if $y(t)$ is a solution such that $y(0) > 0$, then $y(t)$ tends to infinity as $t \to \sqrt{2/y(0)}$.

SOLUTION Rewrite

$$\dot{y} = ty^2 \quad \text{as} \quad \frac{1}{y^2}\,dy = t\,dt$$

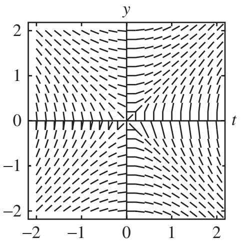
Integrate both sides:

$$\int \frac{1}{y^2}\,dy = \int t\,dt$$
$$-y^{-1} = \frac{1}{2}t^2 + C_1$$
$$-y = \frac{2}{t^2 + C}$$
$$y = \frac{2}{C - t^2}$$

where $C = -C_1$ is an arbitrary constant. Then $y(0) = 2/C$ so that $C = 2/y(0)$, and then the denominator of y approaches 0 as $t \to \sqrt{2/y(0)}$, so that y tends to infinity.

11. (a) Sketch the slope field of $\dot{y} = t/y$ in the region $-2 \le t \le 2$, $-2 \le y \le 2$.
(b) Check that $y = \pm\sqrt{t^2 + C}$ is the general solution.
(c) Sketch the solutions on the slope field with initial conditions $y(0) = 1$ and $y(0) = -1$.

SOLUTION
(a) The slope field is shown below:



(b) Rewrite

$$\frac{dy}{dt} = \frac{t}{y} \qquad \text{as} \qquad y\,dy = t\,dt,$$

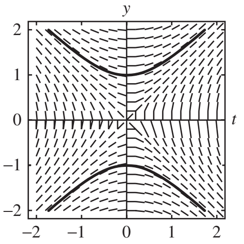
and then integrate both sides to obtain

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + C.$$

Solving for y, we find that the general solution is

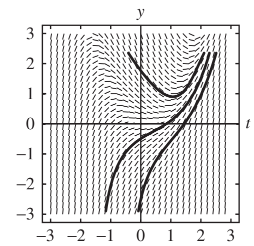
$$y = \pm\sqrt{t^2 + C}.$$

(c) The sketches of the two solutions are shown below:



12. Sketch the slope field of $\dot{y} = t^2 - y$ in the region $-3 \le t \le 3$, $-3 \le y \le 3$ and sketch the solutions satisfying $y(1) = 0$, $y(1) = 1$, and $y(1) = -1$.

SOLUTION The slope field for $\dot{y} = t^2 - y$, together with the required solution curves, is shown below.



13. Let $F(t, y) = t^2 - y$ and let $y(t)$ be the solution of $\dot{y} = F(t, y)$ satisfying $y(2) = 3$. Let $h = 0.1$ be the time step in Euler's Method, and set $y_0 = y(2) = 3$.

(a) Calculate $y_1 = y_0 + hF(2, 3)$.

(b) Calculate $y_2 = y_1 + hF(2.1, y_1)$.

(c) Calculate $y_3 = y_2 + hF(2.2, y_2)$ and continue computing y_4, y_5, and y_6.

(d) Find approximations to $y(2.2)$ and $y(2.5)$.

SOLUTION

(a) With $y_0 = 3$, $t_0 = 2$, $h = 0.1$, and $F(t, y) = t^2 - y$, we find

$$y_1 = y_0 + hF(t_0, y_0) = 3 + 0.1(1) = 3.1.$$

(b) With $y_1 = 3.1$, $t_1 = 2.1$, $h = 0.1$, and $F(t, y) = t^2 - y$, we find

$$y_2 = y_1 + hF(t_1, y_1) = 3.1 + 0.1(4.41 - 3.1) = 3.231.$$

(c) Continuing as in the previous two parts, we find

$$y_3 = y_2 + hF(t_2, y_2) = 3.3919;$$
$$y_4 = y_3 + hF(t_3, y_3) = 3.58171;$$
$$y_5 = y_4 + hF(t_4, y_4) = 3.799539;$$
$$y_6 = y_5 + hF(t_5, y_5) = 4.0445851.$$

(d) $y(2.2) \approx y_2 = 3.231$, and $y(2.5) \approx y_5 = 3.799539$.

14. Let $y(t)$ be the solution to $\dot{y} = te^{-y}$ satisfying $y(0) = 0$.

(a) Use Euler's Method with time step $h = 0.1$ to approximate $y(0.1), y(0.2), \ldots, y(0.5)$.

(b) Use separation of variables to find $y(t)$ exactly.

(c) Compute the errors in the approximations to $y(0.1)$ and $y(0.5)$.

SOLUTION

(a) With $y_0 = 0$, $t_0 = 0$, $h = 0.1$, and $F(t, y) = te^{-y}$, we compute

n	t_n	y_n
0	0	0
1	0.1	$y_0 + hF(t_0, y_0) = 0$
2	0.2	$y_1 + hF(t_1, y_1) = 0.01$
3	0.3	$y_2 + hF(t_2, y_2) = 0.029801$
4	0.4	$y_3 + hF(t_3, y_3) = 0.058920$
5	0.5	$y_4 + hF(t_4, y_4) = 0.096631$
6	0.6	$y_5 + hF(t_5, y_5) = 0.142026$
7	0.7	$y_6 + hF(t_6, y_6) = 0.194082$
8	0.8	$y_7 + hF(t_7, y_7) = 0.251733$
9	0.9	$y_8 + hF(t_8, y_8) = 0.313929$
10	1.0	$y_9 + hF(t_9, y_9) = 0.379681$

(b) Rewrite

$$\frac{dy}{dt} = te^{-y} \qquad \text{as} \qquad e^y\,dy = t\,dt,$$

and then integrate both sides to obtain

$$e^y = \frac{1}{2}t^2 + C.$$

Thus,

$$y = \ln\left|\frac{1}{2}t^2 + C\right|.$$

Applying the initial condition $y(0) = 0$ yields $0 = \ln|C|$, so $C = 1$. The exact solution to the initial value problem is then $y = \ln\left(\frac{1}{2}t^2 + 1\right)$.

(c) The two errors requested are computed here:

$$|y(0.1) - y_1| = |0.00498754 - 0| = 0.00498754;$$
$$|y(0.5) - y_5| = |0.117783 - 0.0966314| = 0.021152$$

In Exercises 15–20, use Euler's Method to approximate the given value of $y(t)$ with the time step h indicated.

15. $y(0.5)$; $\dot{y} = y + t$, $y(0) = 1$, $h = 0.1$

SOLUTION With $y_0 = 1$, $t_0 = 0$, $h = 0.1$, and $F(t, y) = y + t$, we compute

n	t_n	y_n
0	0	1
1	0.1	$y_0 + hF(t_0, y_0) = 1.1$
2	0.2	$y_1 + hF(t_1, y_1) = 1.22$
3	0.3	$y_2 + hF(t_2, y_2) = 1.362$
4	0.4	$y_3 + hF(t_3, y_3) = 1.5282$
5	0.5	$y_4 + hF(t_4, y_4) = 1.72102$

16. $y(0.7)$; $\dot{y} = 2y$, $y(0) = 3$, $h = 0.1$

SOLUTION With $y_0 = 3$, $t_0 = 0$, $h = 0.1$, and $F(t, y) = 2y$, we compute

n	t_n	y_n
0	0	3
1	0.1	$y_0 + hF(t_0, y_0) = 3.6$
2	0.2	$y_1 + hF(t_1, y_1) = 4.32$
3	0.3	$y_2 + hF(t_2, y_2) = 5.184$
4	0.4	$y_3 + hF(t_3, y_3) = 6.2208$
5	0.5	$y_4 + hF(t_4, y_4) = 7.464960$
6	0.6	$y_5 + hF(t_5, y_5) = 8.957952$
7	0.7	$y_6 + hF(t_6, y_6) = 10.749542$

17. $y(3.3)$; $\dot{y} = t^2 - y$, $y(3) = 1$, $h = 0.05$

SOLUTION With $y_0 = 1$, $t_0 = 3$, $h = 0.05$, and $F(t, y) = t^2 - y$, we compute

n	t_n	y_n
0	3	1
1	3.05	$y_0 + hF(t_0, y_0) = 1.4$
2	3.1	$y_1 + hF(t_1, y_1) = 1.795125$
3	3.15	$y_2 + hF(t_2, y_2) = 2.185869$
4	3.2	$y_3 + hF(t_3, y_3) = 2.572700$
5	3.25	$y_4 + hF(t_4, y_4) = 2.956065$
6	3.3	$y_5 + hF(t_5, y_5) = 3.336387$

18. $y(3)$; $\dot{y} = \sqrt{t + y}$, $y(2.7) = 5$, $h = 0.05$

SOLUTION With $y_0 = 5$, $t_0 = 2.7$, $h = 0.05$, and $F(t, y) = \sqrt{t + y}$, we compute

n	t_n	y_n
0	2.7	5
1	2.75	$y_0 + hF(t_0, y_0) = 5.138744$
2	2.8	$y_1 + hF(t_1, y_1) = 5.279179$
3	2.85	$y_2 + hF(t_2, y_2) = 5.421298$
4	2.9	$y_3 + hF(t_3, y_3) = 5.565098$
5	2.95	$y_4 + hF(t_4, y_4) = 5.710572$
6	3.0	$y_5 + hF(t_5, y_5) = 5.857716$

19. $y(2)$; $\dot{y} = t \sin y$, $y(1) = 2$, $h = 0.2$

SOLUTION Let $F(t, y) = t \sin y$. With $t_0 = 1$, $y_0 = 2$ and $h = 0.2$, we compute

n	t_n	y_n
0	1	2
1	1.2	$y_0 + hF(t_0, y_0) = 2.181859$
2	1.4	$y_1 + hF(t_1, y_1) = 2.378429$
3	1.6	$y_2 + hF(t_2, y_2) = 2.571968$
4	1.8	$y_3 + hF(t_3, y_3) = 2.744549$
5	2.0	$y_4 + hF(t_4, y_4) = 2.883759$

20. $y(5.2)$; $\dot{y} = t - \sec y$, $y(4) = -2$, $h = 0.2$

SOLUTION With $t_0 = 4$, $y_0 = -2$, $F(t, y) = t - \sec y$, and $h = 0.2$, we compute

n	t_n	y_n
0	4	-2
1	4.2	$y_0 + hF(t_0, y_0) = -0.7194$
2	4.4	$y_1 + hF(t_1, y_1) = -0.142587$
3	4.6	$y_2 + hF(t_2, y_2) = 0.532584$
4	4.8	$y_3 + hF(t_3, y_3) = 1.220430$
5	5.0	$y_4 + hF(t_4, y_4) = 1.597751$
6	5.2	$y_5 + hF(t_5, y_5) = 10.018619$

Note that $\sec y$ has a discontinuity at $y = \pi/2 \approx 1.57$ and at $y = 3\pi/2 \approx 4.71$, so this numerical solution should be regarded with some skepticism.

Further Insights and Challenges

21. If $f(t)$ is continuous on $[a, b]$, then the solution to $\dot{y} = f(t)$ with initial condition $y(a) = 0$ is $y(t) = \int_a^t f(u)\,du$. Show that Euler's Method with time step $h = (b-a)/N$ for N steps yields the Nth left-endpoint approximation to $y(b) = \int_a^b f(u)\,du$.

SOLUTION For a differential equation of the form $\dot{y} = f(t)$, the equation for Euler's method reduces to

$$y_k = y_{k-1} + hf(t_{k-1}).$$

With a step size of $h = (b-a)/N$, $y(b) =\approx y_N$. Starting from $y_0 = 0$, we compute

$$\begin{aligned}
y_1 &= y_0 + hf(t_0) = hf(t_0)\\
y_2 &= y_1 + hf(t_1) = h\left[f(t_0) + f(t_1)\right]\\
y_3 &= y_2 + hf(t_2) = h\left[f(t_0) + f(t_1) + f(t_2)\right]\\
&\vdots\\
y_N &= y_{N_1} + hf(t_{N-1}) = h\left[f(t_0) + f(t_1) + f(t_2) + \ldots + f(t_{N-1})\right] = h\sum_{k=0}^{N-1} f(t_k)
\end{aligned}$$

Observe this last expression is exactly the Nth left-endpoint approximation to $y(b) = \displaystyle\int_a^b f(u)\,du$.

Exercises 22–27: ***Euler's Midpoint Method*** *is a variation on Euler's Method that is significantly more accurate in general. For time step h and initial value $y_0 = y(t_0)$, the values y_k are defined successively by*

$$y_k = y_{k-1} + hm_{k-1}$$

where $m_{k-1} = F\left(t_{k-1} + \dfrac{h}{2}, y_{k-1} + \dfrac{h}{2}F(t_{k-1}, y_{k-1})\right)$.

22. Apply both Euler's Method and the Euler Midpoint Method with $h = 0.1$ to estimate $y(1.5)$, where $y(t)$ satisfies $\dot{y} = y$ with $y(0) = 1$. Find $y(t)$ exactly and compute the errors in these two approximations.

SOLUTION Let $F(t, y) = y$. With $t_0 = 0$, $y_0 = 1$, and $h = 0.1$, fifteen iterations of Euler's method yield

$$y(1.5) \approx y_{15} = 4.177248.$$

The Euler midpoint approximation with $F(t, y) = y$ is

$$m_{k-1} = F\left(t_{k-1} + \frac{h}{2}, y_{k-1} + \frac{h}{2}F(t_{k-1}, y_{k-1})\right) = y_{k-1} + \frac{h}{2}y_{k-1}$$

$$y_k = y_{k-1} + h\left(y_{k-1} + \frac{h}{2}y_{k-1}\right) = y_{k-1} + hy_{k-1} + \frac{h^2}{2}y_{k-1}$$

Fifteen iterations of Euler's midpoint method yield:

$$y(1.5) \approx y_{15} = 4.471304.$$

The exact solution to $y' = y,\ y(0) = 1$ is $y(t) = e^t$; therefore $y(1.5) = 4.481689$. The error from Euler's method is $|4.177248 - 4.481689| = 0.304441$, while the error from Euler's midpoint method is $|4.471304 - 4.481689| = 0.010385$.

In Exercises 23–26, use Euler's Midpoint Method with the time step indicated to approximate the given value of $y(t)$.

23. $y(0.5)$; $\quad \dot{y} = y + t, \quad y(0) = 1, \quad h = 0.1$

SOLUTION With $t_0 = 0$, $y_0 = 1$, $F(t, y) = y + t$, and $h = 0.1$ we compute

n	t_n	y_n
0	0	1
1	0.1	$y_0 + hF(t_0 + h/2, y_0 + (h/2)F(t_0, y_0)) = 1.11$
2	0.2	$y_1 + hF(t_1 + h/2, y_1 + (h/2)F(t_1, y_1)) = 1.242050$
3	0.3	$y_2 + hF(t_2 + h/2, y_2 + (h/2)F(t_2, y_2)) = 1.398465$
4	0.4	$y_3 + hF(t_3 + h/2, y_3 + (h/2)F(t_3, y_3)) = 1.581804$
5	0.5	$y_4 + hF(t_4 + h/2, y_4 + (h/2)F(t_4, y_4)) = 1.794894$

24. $y(2)$; $\dot{y} = t^2 - y$, $y(1) = 3$, $h = 0.2$

SOLUTION With $t_0 = 1$, $y_0 = 3$, $F(t, y) = t^2 - y$, and $h = 0.2$ we compute

n	t_n	y_n
0	1	3
1	1.2	$y_0 + hF(t_0 + h/2, y_0 + (h/2)F(t_0, y_0)) = 2.682$
2	1.4	$y_1 + hF(t_1 + h/2, y_1 + (h/2)F(t_1, y_1)) = 2.50844$
3	1.6	$y_2 + hF(t_2 + h/2, y_2 + (h/2)F(t_2, y_2)) = 2.467721$
4	1.8	$y_3 + hF(t_3 + h/2, y_3 + (h/2)F(t_3, y_3)) = 2.550331$
5	2.0	$y_4 + hF(t_4 + h/2, y_4 + (h/2)F(t_4, y_4)) = 2.748471$

25. $y(0.25)$; $\dot{y} = \cos(y + t)$, $y(0) = 1$, $h = 0.05$

SOLUTION With $t_0 = 0$, $y_0 = 1$, $F(t, y) = \cos(y + t)$, and $h = 0.05$ we compute

n	t_n	y_n
0	0	1
1	0.05	$y_0 + hF(t_0 + h/2, y_0 + (h/2)F(t_0, y_0)) = 1.025375$
2	0.10	$y_1 + hF(t_1 + h/2, y_1 + (h/2)F(t_1, y_1)) = 1.047507$
3	0.15	$y_2 + hF(t_2 + h/2, y_2 + (h/2)F(t_2, y_2)) = 1.066425$
4	0.20	$y_3 + hF(t_3 + h/2, y_3 + (h/2)F(t_3, y_3)) = 1.082186$
5	0.25	$y_4 + hF(t_4 + h/2, y_4 + (h/2)F(t_4, y_4)) = 1.094871$

26. $y(2.3)$; $\dot{y} = y + t^2$, $y(2) = 1$, $h = 0.05$

SOLUTION With $t_0 = 2$, $y_0 = 1$, $F(t, y) = y + t^2$, and $h = 0.05$ we compute

n	t_n	y_n
0	2.00	1
1	2.05	$y_0 + hF(t_0 + h/2, y_0 + (h/2)F(t_0, y_0)) = 1.261281$
2	2.10	$y_1 + hF(t_1 + h/2, y_1 + (h/2)F(t_1, y_1)) = 1.546456$
3	2.15	$y_2 + hF(t_2 + h/2, y_2 + (h/2)F(t_2, y_2)) = 1.857006$
4	2.20	$y_3 + hF(t_3 + h/2, y_3 + (h/2)F(t_3, y_3)) = 2.194487$
5	2.25	$y_4 + hF(t_4 + h/2, y_4 + (h/2)F(t_4, y_4)) = 2.560536$
6	2.30	$y_5 + hF(t_5 + h/2, y_5 + (h/2)F(t_5, y_5)) = 2.956872$

27. Assume that $f(t)$ is continuous on $[a, b]$. Show that Euler's Midpoint Method applied to $\dot{y} = f(t)$ with initial condition $y(a) = 0$ and time step $h = (b - a)/N$ for N steps yields the Nth midpoint approximation to

$$y(b) = \int_a^b f(u)\,du$$

SOLUTION For a differential equation of the form $\dot{y} = f(t)$, the equations for Euler's midpoint method reduce to

$$m_{k-1} = f\left(t_{k-1} + \frac{h}{2}\right) \qquad \text{and} \qquad y_k = y_{k-1} + hf\left(t_{k-1} + \frac{h}{2}\right).$$

With a step size of $h = (b - a)/N$, $y(b) =\approx y_N$. Starting from $y_0 = 0$, we compute

$$y_1 = y_0 + hf\left(t_0 + \frac{h}{2}\right) = hf\left(t_0 + \frac{h}{2}\right)$$

$$y_2 = y_1 + hf\left(t_1 + \frac{h}{2}\right) = h\left[f\left(t_0 + \frac{h}{2}\right) + f\left(t_1 + \frac{h}{2}\right)\right]$$

$$y_3 = y_2 + hf\left(t_2 + \frac{h}{2}\right) = h\left[f\left(t_0 + \frac{h}{2}\right) + f\left(t_1 + \frac{h}{2}\right) + f\left(t_2 + \frac{h}{2}\right)\right]$$

$$\vdots$$

$$y_N = y_{N_1} + hf\left(t_{N-1} + \frac{h}{2}\right) = h\left[f\left(t_0 + \frac{h}{2}\right) + f\left(t_1 + \frac{h}{2}\right) + f\left(t_2 + \frac{h}{2}\right) + \ldots + f\left(t_{N-1} + \frac{h}{2}\right)\right]$$

$$= h\sum_{k=0}^{N-1} f\left(t_k + \frac{h}{2}\right)$$

Observe this last expression is exactly the Nth midpoint approximation to $y(b) = \int_a^b f(u)\,du$.

9.4 The Logistic Equation

Preliminary Questions

1. Which of the following differential equations is a logistic differential equation?

(a) $\dot{y} = 2y(1 - y^2)$

(b) $\dot{y} = 2y\left(1 - \frac{y}{3}\right)$

(c) $\dot{y} = 2y\left(1 - \frac{t}{4}\right)$

(d) $\dot{y} = 2y(1 - 3y)$

SOLUTION The differential equations in **(b)** and **(d)** are logistic equations. The equation in **(a)** is not a logistic equation because of the y^2 term inside the parentheses on the right-hand side; the equation in **(c)** is not a logistic equation because of the presence of the independent variable on the right-hand side.

2. Is the logistic equation a linear differential equation?

SOLUTION No, the logistic equation is not linear.

$$\dot{y} = ky\left(1 - \frac{y}{A}\right) \quad \text{can be rewritten} \quad \dot{y} = ky - \frac{k}{A}y^2$$

and we see that a term involving y^2 occurs.

3. Is the logistic equation separable?

SOLUTION Yes, the logistic equation is a separable differential equation.

Exercises

1. Find the general solution of the logistic equation

$$\dot{y} = 3y\left(1 - \frac{y}{5}\right)$$

Then find the particular solution satisfying $y(0) = 2$.

SOLUTION $\dot{y} = 3y(1 - y/5)$ is a logistic equation with $k = 3$ and $A = 5$; therefore, the general solution is

$$y = \frac{5}{1 - e^{-3t}/C}.$$

The initial condition $y(0) = 2$ allows us to determine the value of C:

$$2 = \frac{5}{1 - 1/C}; \quad 1 - \frac{1}{C} = \frac{5}{2}; \quad \text{so} \quad C = -\frac{2}{3}.$$

The particular solution is then

$$y = \frac{5}{1 + \frac{3}{2}e^{-3t}} = \frac{10}{2 + 3e^{-3t}}.$$

2. Find the solution of $\dot{y} = 2y(3 - y)$, $y(0) = 10$.

SOLUTION By rewriting

$$2y(3 - y) \qquad \text{as} \qquad 6y\left(1 - \frac{y}{3}\right),$$

we identify the given differential equation as a logistic equation with $k = 6$ and $A = 3$. The general solution is therefore

$$y = \frac{3}{1 - e^{-6t}/C}.$$

The initial condition $y(0) = 10$ allows us to determine the value of C:

$$10 = \frac{3}{1 - 1/C}; \quad 1 - \frac{1}{C} = \frac{3}{10}; \quad \text{so} \quad C = \frac{10}{7}.$$

The particular solution is then

$$y = \frac{3}{1 - \frac{7}{10}e^{-6t}} = \frac{30}{10 - 7e^{-6t}}.$$

3. Let $y(t)$ be a solution of $\dot{y} = 0.5y(1 - 0.5y)$ such that $y(0) = 4$. Determine $\lim_{t\to\infty} y(t)$ without finding $y(t)$ explicitly.

SOLUTION This is a logistic equation with $k = \frac{1}{2}$ and $A = 2$, so the carrying capacity is 2. Thus the required limit is 2.

4. Let $y(t)$ be a solution of $\dot{y} = 5y(1 - y/5)$. State whether $y(t)$ is increasing, decreasing, or constant in the following cases:

(a) $y(0) = 2$ **(b)** $y(0) = 5$ **(c)** $y(0) = 8$

SOLUTION This is a logistic equation with $k = A = 5$.

(a) $0 < y(0) < A$, so $y(t)$ is increasing and approaches A asymptotically.

(b) $y(0) = A$; this represents a stable equilibrium and $y(t)$ is constant.

(c) $y(0) > A$, so $y(t)$ is decreasing and approaches A asymptotically.

5. A population of squirrels lives in a forest with a carrying capacity of 2000. Assume logistic growth with growth constant $k = 0.6$ yr^{-1}.

(a) Find a formula for the squirrel population $P(t)$, assuming an initial population of 500 squirrels.

(b) How long will it take for the squirrel population to double?

SOLUTION

(a) Since $k = 0.6$ and the carrying capacity is $A = 2000$, the population $P(t)$ of the squirrels satisfies the differential equation

$$P'(t) = 0.6P(t)(1 - P(t)/2000),$$

with general solution

$$P(t) = \frac{2000}{1 - e^{-0.6t}/C}.$$

The initial condition $P(0) = 500$ allows us to determine the value of C:

$$500 = \frac{2000}{1 - 1/C}; \quad 1 - \frac{1}{C} = 4; \quad \text{so} \quad C = -\frac{1}{3}.$$

The formula for the population is then

$$P(t) = \frac{2000}{1 + 3e^{-0.6t}}.$$

(b) The squirrel population will have doubled at the time t where $P(t) = 1000$. This gives

$$1000 = \frac{2000}{1 + 3e^{-0.6t}}; \quad 1 + 3e^{-0.6t} = 2; \quad \text{so} \quad t = \frac{5}{3}\ln 3 \approx 1.83.$$

It therefore takes approximately 1.83 years for the squirrel population to double.

6. The population $P(t)$ of mosquito larvae growing in a tree hole increases according to the logistic equation with growth constant $k = 0.3$ day^{-1} and carrying capacity $A = 500$.

(a) Find a formula for the larvae population $P(t)$, assuming an initial population of $P_0 = 50$ larvae.

(b) After how many days will the larvae population reach 200?

SOLUTION

(a) Since $k = 0.3$ and $A = 500$, the population of the larvae satisfies the differential equation

$$P'(t) = 0.3P(t)(1 - P(t)/500),$$

with general solution

$$P(t) = \frac{500}{1 - e^{-0.3t}/C}.$$

The initial condition $P(0) = 50$ allows us to determine the value of C:

$$50 = \frac{500}{1 - 1/C}; \quad 1 - \frac{1}{C} = 10; \quad \text{so} \quad C = -\frac{1}{9}.$$

The particular solution is then

$$P(t) = \frac{500}{1 + 9e^{-0.3t}}.$$

(b) The population will reach 200 after t days, where $P(t) = 200$. This gives

$$200 = \frac{500}{1 + 9e^{-0.3t}}; \quad 1 + 9e^{-0.3t} = 2.5; \quad \text{so} \quad t = \frac{10}{3}\ln 6 \approx 5.97.$$

It therefore takes approximately 5.97 days for the larvae to reach 200 in number.

7. Sunset Lake is stocked with 2000 rainbow trout, and after 1 year the population has grown to 4500. Assuming logistic growth with a carrying capacity of 20,000, find the growth constant k (specify the units) and determine when the population will increase to 10,000.

SOLUTION Since $A = 20{,}000$, the trout population $P(t)$ satisfies the logistic equation

$$P'(t) = kP(t)(1 - P(t)/20{,}000),$$

with general solution

$$P(t) = \frac{20{,}000}{1 - e^{-kt}/C}.$$

The initial condition $P(0) = 2000$ allows us to determine the value of C:

$$2000 = \frac{20{,}000}{1 - 1/C}; \quad 1 - \frac{1}{C} = 10; \quad \text{so} \quad C = -\frac{1}{9}.$$

After one year, we know the population has grown to 4500. Let's measure time in years. Then

$$4500 = \frac{20{,}000}{1 + 9e^{-k}}$$
$$1 + 9e^{-k} = \frac{40}{9}$$
$$e^{-k} = \frac{31}{81}$$
$$k = \ln\frac{81}{31} \approx 0.9605 \text{ years}^{-1}.$$

The population will increase to 10,000 at time t where $P(t) = 10{,}000$. This gives

$$10{,}000 = \frac{20{,}000}{1 + 9e^{-0.9605t}}$$
$$1 + 9e^{-0.9605t} = 2$$
$$e^{-0.9605t} = \frac{1}{9}$$
$$t = \frac{1}{0.9605}\ln 9 \approx 2.29 \text{ years.}$$

8. Spread of a Rumor A rumor spreads through a small town. Let $y(t)$ be the fraction of the population that has heard the rumor at time t and assume that the rate at which the rumor spreads is proportional to the product of the fraction y of the population that has heard the rumor and the fraction $1 - y$ that has not yet heard the rumor.

(a) Write down the differential equation satisfied by y in terms of a proportionality factor k.

(b) Find k (in units of day^{-1}), assuming that 10% of the population knows the rumor at $t = 0$ and 40% knows it at $t = 2$ days.

(c) Using the assumptions of part (b), determine when 75% of the population will know the rumor.

SOLUTION

(a) $y'(t)$ is the rate at which the rumor is spreading, in percentage of the population per day. By the description given, the rate satisfies:

$$y'(t) = ky(1 - y),$$

where k is a constant of proportionality.

(b) The equation in part (a) is a logistic equation with constant k and capacity 1 (no more than 100% of the population can hear the rumor). Thus, y takes the form

$$y(t) = \frac{1}{1 - e^{-kt}/C}.$$

The initial condition $y(0) = \frac{1}{10}$ allows us to determine the value of C:

$$\frac{1}{10} = \frac{1}{1 - 1/C}; \quad 1 - \frac{1}{C} = 10; \quad \text{so} \quad C = -\frac{1}{9}.$$

The condition $y(2) = \frac{2}{5}$ now allows us to determine the value of k:

$$\frac{2}{5} = \frac{1}{1 + 9e^{-2k}}; \quad 1 + 9e^{-2k} = \frac{5}{2}; \quad \text{so} \quad k = \frac{1}{2}\ln 6 \approx 0.896 \text{ days}^{-1}.$$

The particular solution of the differential equation for y is then

$$y(t) = \frac{1}{1 + 9e^{-0.896t}}.$$

(c) If 75% of the population knows the rumor at time t, we have

$$\frac{3}{4} = \frac{1}{1 + 9e^{-0.896t}}$$

$$1 + 9e^{-0.896t} = \frac{4}{3}$$

$$t = \frac{\ln 27}{0.896} \approx 3.67839$$

Thus, 75% of the population knows the rumor after approximately 3.67 days.

9. A rumor spreads through a school with 1000 students. At 8 AM, 80 students have heard the rumor, and by noon, half the school has heard it. Using the logistic model of Exercise 8, determine when 90% of the students will have heard the rumor.

SOLUTION Let $y(t)$ be the proportion of students that have heard the rumor at a time t hours after 8 AM. In the logistic model of Exercise 8, we have a capacity of $A = 1$ (100% of students) and an unknown growth factor of k. Hence,

$$y(t) = \frac{1}{1 - e^{-kt}/C}.$$

The initial condition $y(0) = 0.08$ allows us to determine the value of C:

$$\frac{2}{25} = \frac{1}{1 - 1/C}; \quad 1 - \frac{1}{C} = \frac{25}{2}; \quad \text{so} \quad C = -\frac{2}{23}.$$

so that

$$y(t) = \frac{2}{2 + 23e^{-kt}}.$$

The condition $y(4) = 0.5$ now allows us to determine the value of k:

$$\frac{1}{2} = \frac{2}{2 + 23e^{-4k}}; \quad 2 + 23e^{-4k} = 4; \quad \text{so} \quad k = \frac{1}{4} \ln \frac{23}{2} \approx 0.6106 \text{ hours}^{-1}.$$

90% of the students have heard the rumor when $y(t) = 0.9$. Thus

$$\frac{9}{10} = \frac{2}{2 + 23e^{-0.6106t}}$$

$$2 + 23e^{-0.6106t} = \frac{20}{9}$$

$$t = \frac{1}{0.6106} \ln \frac{207}{2} \approx 7.6 \text{ hours.}$$

Thus, 90% of the students have heard the rumor after 7.6 hours, or at 3:36 PM.

10. GU A simpler model for the spread of a rumor assumes that the rate at which the rumor spreads is proportional (with factor k) to the fraction of the population that has not yet heard the rumor.

(a) Compute the solutions to this model and the model of Exercise 8 with the values $k = 0.9$ and $y_0 = 0.1$.

(b) Graph the two solutions on the same axis.

(c) Which model seems more realistic? Why?

SOLUTION

(a) Let $y(t)$ denote the fraction of a population that has heard a rumor, and suppose the rumor spreads at a rate proportional to the fraction of the population that has not yet heard the rumor. Then

$$y' = k(1 - y),$$

for some constant of proportionality k. Separating variables and integrating both sides yields

$$\frac{dy}{1 - y} = k\,dt$$

$$-\ln|1 - y| = kt + C.$$

Thus,

$$y(t) = 1 - Ae^{-kt},$$

where $A = \pm e^{-C}$ is an arbitrary constant. The initial condition $y(0) = 0.1$ allows us to determine the value of A:

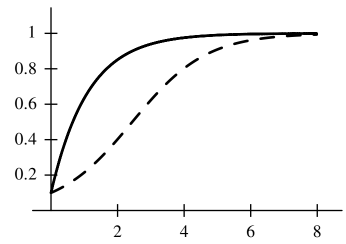
$$0.1 = 1 - A \qquad \text{so} \qquad A = 0.9.$$

With $k = 0.9$, we have $y(t) = 1 - 0.9e^{-0.9t}$.

Using the model from Exercise 8 with $k = 0.9$ and $y(0) = 0.1$, we find

$$y(t) = \frac{1}{1 + 9e^{-0.9t}}.$$

(b) The figure below shows the solutions from part (a): the solid curve corresponds to the model presented in this exercise while the dashed curve corresponds to the model from Exercise 8.



(c) The model from Exercise 8 seems more realistic because it predicts the rumor starts spreading slowly, picks up speed and then levels off as we near the time when the entire population has heard the rumor.

11. Let $k = 1$ and $A = 1$ in the logistic equation.

(a) Find the solutions satisfying $y_1(0) = 10$ and $y_2(0) = -1$.

(b) Find the time t when $y_1(t) = 5$.

(c) When does $y_2(t)$ become infinite?

SOLUTION The general solution of the logistic equation with $k = 1$ and $A = 1$ is

$$y(t) = \frac{1}{1 - e^{-t}/C}.$$

(a) Given $y_1(0) = 10$, we find $C = \frac{10}{9}$, and

$$y_1(t) = \frac{1}{1 - \frac{10}{9}e^{-t}} = \frac{10}{10 - 9e^{-t}}.$$

On the other hand, given $y_2(0) = -1$, we find $C = \frac{1}{2}$, and

$$y_2(t) = \frac{1}{1 - 2e^{-t}}.$$

(b) From part (a), we have

$$y_1(t) = \frac{10}{10 - 9e^{-t}}.$$

Thus, $y_1(t) = 5$ when

$$5 = \frac{10}{10 - 9e^{-t}}; \quad 10 - 9e^{-t} = 2; \quad \text{so} \quad t = \ln\frac{9}{8}.$$

(c) From part (a), we have

$$y_2(t) = \frac{1}{1 - 2e^{-t}}.$$

Thus, $y_2(t)$ becomes infinite when

$$1 - 2e^{-t} = 0 \qquad \text{or} \qquad t = \ln 2.$$

12. A tissue culture grows until it has a maximum area of M cm^2. The area $A(t)$ of the culture at time t may be modeled by the differential equation

$$\dot{A} = k\sqrt{A}\left(1 - \frac{A}{M}\right) \tag{7}$$

where k is a growth constant.

(a) Show that if we set $A = u^2$, then

$$\dot{u} = \frac{1}{2}k\left(1 - \frac{u^2}{M}\right)$$

Then find the general solution using separation of variables.

(b) Show that the general solution to Eq. (7) is

$$A(t) = M\left(\frac{Ce^{(k/\sqrt{M})t} - 1}{Ce^{(k/\sqrt{M})t} + 1}\right)^2$$

SOLUTION

(a) Let $A = u^2$. This gives $\dot{A} = 2u\dot{u}$, so that Eq. (7) becomes:

$$2u\dot{u} = ku\left(1 - \frac{u^2}{M}\right)$$

$$\dot{u} = \frac{k}{2}\left(1 - \frac{u^2}{M}\right)$$

Now, rewrite

$$\frac{du}{dt} = \frac{k}{2}\left(1 - \frac{u^2}{M}\right) \qquad \text{as} \qquad \frac{du}{1 - u^2/M} = \frac{1}{2}k\,dt.$$

The partial fraction decomposition for the term on the left-hand side is

$$\frac{1}{1 - u^2/M} = \frac{\sqrt{M}}{2}\left(\frac{1}{\sqrt{M} + u} + \frac{1}{\sqrt{M} - u}\right),$$

so after integrating both sides, we obtain

$$\frac{\sqrt{M}}{2}\ln\left|\frac{\sqrt{M}+u}{\sqrt{M}-u}\right| = \tfrac{1}{2}kt + C.$$

Thus,

$$\frac{\sqrt{M}+u}{\sqrt{M}-u} = Ce^{(k/\sqrt{M})t}$$
$$u(Ce^{(k/\sqrt{M})t}+1) = \sqrt{M}(Ce^{(k/\sqrt{M})t}-1)$$

and

$$u = \sqrt{M}\frac{Ce^{(k/\sqrt{M})t}-1}{Ce^{(k/\sqrt{M})t}+1}.$$

(b) Recall $A = u^2$. Therefore,

$$A(t) = M\left(\frac{Ce^{(k/\sqrt{M})t}-1}{Ce^{(k/\sqrt{M})t}+1}\right)^2.$$

13. GU In the model of Exercise 12, let $A(t)$ be the area at time t (hours) of a growing tissue culture with initial size $A(0) = 1$ cm^2, assuming that the maximum area is $M = 16$ cm^2 and the growth constant is $k = 0.1$.

(a) Find a formula for $A(t)$. *Note:* The initial condition is satisfied for two values of the constant C. Choose the value of C for which $A(t)$ is increasing.

(b) Determine the area of the culture at $t = 10$ hours.

(c) GU Graph the solution using a graphing utility.

SOLUTION

(a) From the values for M and k we have

$$A(t) = 16\left(\frac{Ce^{t/40}-1}{Ce^{t/40}+1}\right)^2$$

and the initial condition then gives us

$$A(0) = 1 = 16\left(\frac{Ce^{0/40}-1}{Ce^{0/40}+1}\right)^2$$

so, simplifying,

$$1 = 16\left(\frac{C-1}{C+1}\right)^2 \quad\Rightarrow\quad C^2+2C+1 = 16C^2-32C+16 \quad\Rightarrow\quad 15C^2-34C+15 = 0$$

and thus $C = \frac{5}{3}$ or $C = \frac{3}{5}$. The derivative of $A(t)$ is

$$A'(t) = \frac{16Ce^{t/40}}{(Ce^{t/40}+1)^3}\cdot(Ce^{t/40}-1)$$

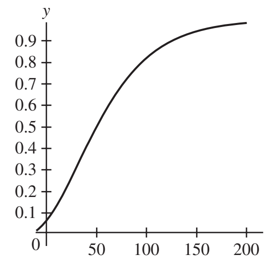
For $C = 3/5$, $A'(t)$ can be negative, while for $C = 5/3$, it is always positive. So let $C = 5/3$.

(b) From part (a), we have

$$A(t) = 16\left(\frac{\frac{5}{3}e^{t/40}-1}{\frac{5}{3}e^{t/40}+1}\right)^2$$

and $A(10) \approx 2.11$.

(c)



14. Show that if a tissue culture grows according to Eq. (7), then the growth rate reaches a maximum when $A = M/3$.

SOLUTION According to Eq. (7), the growth rate of the tissue culture is $k\sqrt{A}(1 - \frac{A}{M})$. Therefore

$$\frac{d}{dA}\left(k\sqrt{A}\left(1 - \frac{A}{M}\right)\right) = \frac{1}{2}kA^{-1/2} - \frac{3}{2}kA^{1/2}/M = \frac{1}{2}kA^{-1/2}\left(1 - \frac{3A}{M}\right) = 0$$

when $A = M/3$. Because the growth rate is zero for $A = 0$ and for $A = M$ and is positive for $0 < A < M$, it follows that the maximum growth rate occurs when $A = M/3$.

15. In 1751, Benjamin Franklin predicted that the U.S. population $P(t)$ would increase with growth constant $k = 0.028$ year^{-1}. According to the census, the U.S. population was 5 million in 1800 and 76 million in 1900. Assuming logistic growth with $k = 0.028$, find the predicted carrying capacity for the U.S. population. *Hint:* Use Eqs. (3) and (4) to show that

$$\frac{P(t)}{P(t) - A} = \frac{P_0}{P_0 - A}e^{kt}$$

SOLUTION Assuming the population grows according to the logistic equation,

$$\frac{P(t)}{P(t) - A} = Ce^{kt}.$$

But

$$C = \frac{P_0}{P_0 - A},$$

so

$$\frac{P(t)}{P(t) - A} = \frac{P_0}{P_0 - A}e^{kt}.$$

Now, let $t = 0$ correspond to the year 1800. Then the year 1900 corresponds to $t = 100$, and with $k = 0.028$, we have

$$\frac{76}{76 - A} = \frac{5}{5 - A}e^{(0.028)(100)}.$$

Solving for A, we find

$$A = \frac{5(e^{2.8} - 1)}{\frac{5}{76}e^{2.8} - 1} \approx 943.07.$$

Thus, the predicted carrying capacity for the U.S. population is approximately 943 million.

16. **Reverse Logistic Equation** Consider the following logistic equation (with $k, B > 0$):

$$\frac{dP}{dt} = -kP\left(1 - \frac{P}{B}\right) \qquad \boxed{8}$$

(a) Sketch the slope field of this equation.

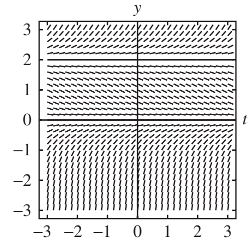
(b) The general solution is $P(t) = B/(1 - e^{kt}/C)$, where C is a nonzero constant. Show that $P(0) > B$ if $C > 1$ and $0 < P(0) < B$ if $C < 0$.

(c) Show that Eq. (8) models an "extinction–explosion" population. That is, $P(t)$ tends to zero if the initial population satisfies $0 < P(0) < B$, and it tends to ∞ after a finite amount of time if $P(0) > B$.

(d) Show that $P = 0$ is a stable equilibrium and $P = B$ an unstable equilibrium.

SOLUTION

(a) The slope field of this equation is shown below.



(b) Suppose that $C > 0$. Then $1 - \frac{1}{C} < 1$, $\left(1 - \frac{1}{C}\right)^{-1} > 1$, and

$$P(0) = \frac{B}{1 - \frac{1}{C}} > B.$$

On the other hand, if $C < 0$, then $1 - \frac{1}{C} > 1$, $0 < \left(1 - \frac{1}{C}\right)^{-1} < 1$, and

$$0 < P(0) = \frac{B}{1 - \frac{1}{C}} < B.$$

(c) From part (b), $0 < P(0) < B$ when $C < 0$. In this case, $1 - e^{kt}/C$ is never zero, but

$$1 - \frac{e^{kt}}{C} \to \infty$$

as $t \to \infty$. Thus, $P(t) \to 0$ as $t \to \infty$. On the other hand, $P(0) > B$ when $C > 0$. In this case $1 - e^{kt}/C = 0$ when $t = \frac{1}{k} \ln C$. Thus,

$$P(t) \to \infty \qquad \text{as} \qquad t \to \frac{1}{k} \ln C.$$

(d) Let

$$F(P) = -kP\left(1 - \frac{P}{B}\right).$$

Then, $F'(P) = -k + \frac{2kP}{B}$. Thus, $F'(0) = -k < 0$, and $F'(B) = -k + 2k = k > 0$, so $P = 0$ is a stable equilibrium and $P = B$ is an unstable equilibrium.

Further Insights and Challenges

In Exercises 17 and 18, let $y(t)$ be a solution of the logistic equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right) \qquad \boxed{9}$$

where $A > 0$ and $k > 0$.

17. (a) Differentiate Eq. (9) with respect to t and use the Chain Rule to show that

$$\frac{d^2y}{dt^2} = k^2 y\left(1 - \frac{y}{A}\right)\left(1 - \frac{2y}{A}\right)$$

(b) Show that $y(t)$ is concave up if $0 < y < A/2$ and concave down if $A/2 < y < A$.

(c) Show that if $0 < y(0) < A/2$, then $y(t)$ has a point of inflection at $y = A/2$ (Figure 6).

(d) Assume that $0 < y(0) < A/2$. Find the time t when $y(t)$ reaches the inflection point.

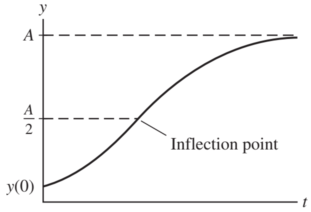


FIGURE 6 An inflection point occurs at $y = A/2$ in the logistic curve.

SOLUTION

(a) The derivative of Eq. (9) with respect to t is

$$y'' = ky' - \frac{2kyy'}{A} = ky'\left(1 - \frac{2y}{A}\right) = k\left(1 - \frac{y}{A}\right)ky\left(1 - \frac{2y}{A}\right) = k^2 y\left(1 - \frac{y}{A}\right)\left(1 - \frac{2y}{A}\right).$$

(b) If $0 < y < A/2$, $1 - \frac{y}{A}$ and $1 - \frac{2y}{A}$ are both positive, so $y'' > 0$. Therefore, y is concave up. If $A/2 < y < A$, $1 - \frac{y}{A} > 0$, but $1 - \frac{2y}{A} < 0$, so $y'' < 0$, so y is concave down.

(c) If $y_0 < A$, y grows and $\lim_{t\to\infty} y(t) = A$. If $0 < y < A/2$, y is concave up at first. Once y passes $A/2$, y becomes concave down, so y has an inflection point at $y = A/2$.

(d) The general solution to Eq. (9) is

$$y = \frac{A}{1 - e^{-kt}/C};$$

thus, $y = A/2$ when

$$\frac{A}{2} = \frac{A}{1 - e^{-kt}/C}$$

$$1 - e^{-kt}/C = 2$$

$$t = -\frac{1}{k}\ln(-C)$$

Now, $C = y_0/(y_0 - A)$, so

$$t = -\frac{1}{k}\ln\frac{y_0}{A - y_0} = \frac{1}{k}\ln\frac{A - y_0}{y_0}.$$

18. Let $y = \dfrac{A}{1 - e^{-kt}/C}$ be the general nonequilibrium Eq. (9). If $y(t)$ has a vertical asymptote at $t = t_b$, that is, if $\lim_{t\to t_b-} y(t) = \pm\infty$, we say that the solution "blows up" at $t = t_b$.

(a) Show that if $0 < y(0) < A$, then y does not blow up at any time t_b.

(b) Show that if $y(0) > A$, then y blows up at a time t_b, which is negative (and hence does not correspond to a real time).

(c) Show that y blows up at some positive time t_b if and only if $y(0) < 0$ (and hence does not correspond to a real population).

SOLUTION

(a) Let $y(0) = y_0$. From the general solution, we find

$$y_0 = \frac{A}{1 - 1/C}; \quad 1 - \frac{1}{C} = \frac{A}{y_0}; \quad \text{so} \quad C = \frac{y_0}{y_0 - A}.$$

If $y_0 < A$, then $C < 0$, and the denominator in the general solution, $1 - e^{-kt}/C$, is always positive. Thus, when $0 < y(0) < A$, y does not blow up at any time.

(b) $1 - e^{-kt}/C = 0$ when $C = e^{-kt}$. Solving for t we find

$$t = -\frac{1}{k}\ln C.$$

Because $C = \frac{y_0}{y_0 - A}$ and $y_0 > A$, it follows that $C > 1$, and thus, $\ln C > 0$. Therefore, y blows up at a time which is negative.

(c) Suppose that y blows up at some $t_b > 0$. From part (b), we know that

$$t_b = -\frac{1}{k}\ln C.$$

Thus, in order for t_b to be positive, we must have $\ln C < 0$, which requires $C < 1$. Now,

$$C = \frac{y_0}{y_0 - A},$$

so $t_b > 0$ if and only if

$$\frac{y_0}{y_0 - A} < 1 \qquad \text{or equivalently} \qquad \frac{y_0 - A}{y_0} = 1 - \frac{A}{y_0} > 1.$$

This last inequality holds if and only if $y_0 = y(0) < 0$.

9.5 First-Order Linear Equations

Preliminary Questions

1. Which of the following are first-order linear equations?

(a) $y' + x^2 y = 1$

(b) $y' + xy^2 = 1$

(c) $x^5 y' + y = e^x$

(d) $x^5 y' + y = e^y$

SOLUTION The equations in **(a)** and **(c)** are first-order linear differential equations. The equation in **(b)** is not linear because of the y^2 factor in the second term on the left-hand side of the equation; the equation in **(d)** is not linear because of the e^y term on the right-hand side of the equation.

2. If $\alpha(x)$ is an integrating factor for $y' + A(x)y = B(x)$, then $\alpha'(x)$ is equal to (choose the correct answer):

(a) $B(x)$

(b) $\alpha(x)A(x)$

(c) $\alpha(x)A'(x)$

(d) $\alpha(x)B(x)$

SOLUTION The correct answer is **(b)**: $\alpha(x)A(x)$.

Exercises

1. Consider $y' + x^{-1}y = x^3$.

(a) Verify that $\alpha(x) = x$ is an integrating factor.

(b) Show that when multiplied by $\alpha(x)$, the differential equation can be written $(xy)' = x^4$.

(c) Conclude that xy is an antiderivative of x^4 and use this information to find the general solution.

(d) Find the particular solution satisfying $y(1) = 0$.

SOLUTION

(a) The equation is of the form

$$y' + A(x)y = B(x)$$

for $A(x) = x^{-1}$ and $B(x) = x^3$. By Theorem 1, $\alpha(x)$ is defined by

$$\alpha(x) = e^{\int A(x)\,dx} = e^{\ln x} = x.$$

(b) When multiplied by $\alpha(x)$, the equation becomes:

$$xy' + y = x^4.$$

Now, $xy' + y = xy' + (x)'y = (xy)'$, so

$$(xy)' = x^4.$$

(c) Since $(xy)' = x^4$, $(xy) = \frac{x^5}{5} + C$ and

$$y = \frac{x^4}{5} + \frac{C}{x}$$

(d) If $y(1) = 0$, we find

$$0 = \frac{1}{5} + C \quad \text{so} \quad -\frac{1}{5} = C.$$

The solution, therefore, is

$$y = \frac{x^4}{5} - \frac{1}{5x}.$$

2. Consider $\frac{dy}{dt} + 2y = e^{-3t}$.

(a) Verify that $\alpha(t) = e^{2t}$ is an integrating factor.

(b) Use Eq. (4) to find the general solution.

(c) Find the particular solution with initial condition $y(0) = 1$.

SOLUTION

(a) The equation is of the form

$$y' + A(t)y = B(t)$$

for $A(t) = 2$ and $B(t) = e^{-3t}$. Thus,

$$\alpha(t) = e^{\int A(t)\,dt} = e^{2t}.$$

(b) According to Eq. (4),

$$y(t) = \frac{1}{\alpha(t)}\left(\int \alpha(t)B(t)\,dt + C\right).$$

With $\alpha(t) = e^{2t}$ and $B(t) = e^{-3t}$, this yields

$$y(t) = e^{-2t}\left(\int e^{-t}\,dt + C\right) = e^{-2t}\left(C - e^{-t}\right) = Ce^{-2t} - e^{-3t}.$$

(c) Using the initial condition $y(0) = 1$, we find

$$1 = -1 + C \quad \text{so} \quad 2 = C.$$

The particular solution is therefore

$$y = -e^{-3t} + 2e^{-2t}.$$

3. Let $\alpha(x) = e^{x^2}$. Verify the identity

$$(\alpha(x)y)' = \alpha(x)(y' + 2xy)$$

and explain how it is used to find the general solution of

$$y' + 2xy = x$$

SOLUTION Let $\alpha(x) = e^{x^2}$. Then

$$(\alpha(x)y)' = (e^{x^2}y)' = 2xe^{x^2}y + e^{x^2}y' = e^{x^2}\left(2xy + y'\right) = \alpha(x)\left(y' + 2xy\right).$$

If we now multiply both sides of the differential equation $y' + 2xy = x$ by $\alpha(x)$, we obtain

$$\alpha(x)(y' + 2xy) = x\alpha(x) = xe^{x^2}.$$

But $\alpha(x)(y' + 2xy) = (\alpha(x)y)'$, so by integration we find

$$\alpha(x)y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C.$$

Finally,

$$y(x) = \frac{1}{2} + Ce^{-x^2}.$$

4. Find the solution of $y' - y = e^{2x}$, $y(0) = 1$.

SOLUTION We first find the general solution of the differential equation $y' - y = e^{2x}$. This is of the standard linear form

$$y' + A(x)y = B(x)$$

with $A(x) = -1$, $B(x) = e^{2x}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{-x}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$e^{-x}y' - e^{-x}y = e^x \qquad \text{or} \qquad (e^{-x}y)' = e^x.$$

Integration of both sides now yields

$$e^{-x}y = \int e^x\,dx = e^x + C.$$

Therefore,

$$y(x) = e^{2x} + Ce^x.$$

Using the initial condition $y(0) = 1$, we find

$$1 = 1 + C \quad \text{so} \quad 0 = C.$$

Therefore,

$$y = e^{2x}.$$

In Exercises 5–18, find the general solution of the first-order linear differential equation.

5. $xy' + y = x$

SOLUTION Rewrite the equation as

$$y' + \frac{1}{x}y = 1,$$

which is in standard linear form with $A(x) = \frac{1}{x}$ and $B(x) = 1$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{\ln x} = x.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$xy' + y = x \qquad \text{or} \qquad (xy)' = x.$$

Integration of both sides now yields

$$xy = \frac{1}{2}x^2 + C.$$

Finally,

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

6. $xy' - y = x^2 - x$

SOLUTION Rewrite the equation as

$$y' - \frac{1}{x}y = x - 1,$$

which is in standard linear form with $A(x) = -\frac{1}{x}$ and $B(x) = x - 1$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{-\ln x} = x^{-1}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$\frac{1}{x}y' - \frac{1}{x^2}y = 1 - \frac{1}{x} \qquad \text{or} \qquad \left(\frac{y}{x}\right)' = 1 - \frac{1}{x}.$$

Integration of both sides now yields

$$\frac{y}{x} = x - \ln x + C.$$

Finally,

$$y(x) = x^2 - x\ln x + Cx.$$

7. $3xy' - y = x^{-1}$

SOLUTION Rewrite the equation as

$$y' - \frac{1}{3x}y = \frac{1}{3x^2},$$

which is in standard form with $A(x) = -\frac{1}{3}x^{-1}$ and $B(x) = \frac{1}{3}x^{-2}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{-(1/3)\ln x} = x^{-1/3}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$x^{-1/3}y' - \frac{1}{3}x^{-4/3} = \frac{1}{3}x^{-7/3} \qquad \text{or} \qquad (x^{-1/3}y)' = \frac{1}{3}x^{-7/3}.$$

Integration of both sides now yields

$$x^{-1/3}y = -\frac{1}{4}x^{-4/3} + C.$$

Finally,

$$y(x) = -\frac{1}{4}x^{-1} + Cx^{1/3}.$$

8. $y' + xy = x$

SOLUTION This equation is in standard form with $A(x) = x$ and $B(x) = x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int x\,dx} = e^{(1/2)x^2}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$e^{(1/2)x^2}y' + xe^{(1/2)x^2}y = xe^{(1/2)x^2} \qquad \text{or} \qquad (e^{(1/2)x^2}y)' = xe^{(1/2)x^2}.$$

Integration of both sides now yields

$$e^{(1/2)x^2}y = e^{(1/2)x^2} + C.$$

Finally,

$$y(x) = 1 + Ce^{-(1/2)x^2}.$$

9. $y' + 3x^{-1}y = x + x^{-1}$

SOLUTION This equation is in standard form with $A(x) = 3x^{-1}$ and $B(x) = x + x^{-1}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int 3x^{-1}} = e^{3\ln x} = x^3.$$

When multiplied by the integrating factor, the original differential equation becomes

$$x^3y' + 3x^2y = x^4 + x^2 \qquad \text{or} \qquad (x^3y)' = x^4 + x^3.$$

Integration of both sides now yields

$$x^3y = \frac{1}{5}x^5 + \frac{1}{3}x^3 + C.$$

Finally,

$$y(x) = \frac{1}{5}x^2 + \frac{1}{3} + Cx^{-3}.$$

10. $y' + x^{-1}y = \cos(x^2)$

SOLUTION This equation is in standard form with $A(x) = x^{-1}$ and $B(x) = \cos(x^2)$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int x^{-1}\,dx} = e^{\ln x} = x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$xy' + y = x\cos(x^2) \qquad \text{or} \qquad (xy)' = x\cos(x^2).$$

Integration of both sides now yields

$$xy = \frac{1}{2}\sin(x^2) + C.$$

Finally,

$$y(x) = \frac{1}{2}x^{-1}\sin(x^2) + Cx^{-1}.$$

11. $xy' = y - x$

SOLUTION Rewrite the equation as

$$y' - \frac{1}{x}y = -1,$$

which is in standard form with $A(x) = -\frac{1}{x}$ and $B(x) = -1$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int -(1/x)\,dx} = e^{-\ln x} = x^{-1}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$\frac{1}{x}y' - \frac{1}{x^2}y = -\frac{1}{x} \qquad \text{or} \qquad \left(\frac{1}{x}y\right)' = -\frac{1}{x}.$$

Integration on both sides now yields

$$\frac{1}{x}y = -\ln x + C.$$

Finally,

$$y(x) = -x\ln x + Cx.$$

12. $xy' = x^{-2} - \dfrac{3y}{x}$

SOLUTION Rewrite the equation is

$$y' + \frac{3}{x^2}y = \frac{1}{x^3}$$

which is in standard form with $A(x) = \frac{3}{x^2}$ and $B(x) = \frac{1}{x^3}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int (3/x^2)\,dx} = e^{-3x^{-1}}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$e^{-3/x}y' + \frac{3}{x^2}e^{-3/x}y = \frac{1}{x^3}e^{-3/x}$$

Integration on both sides now yields

$$e^{-3/x}y = \frac{x+3}{9x}e^{-3/x} + C \qquad \text{or} \qquad y = \frac{x+3}{9x} + Ce^{3/x}$$

13. $y' + y = e^x$

SOLUTION This equation is in standard form with $A(x) = 1$ and $B(x) = e^x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int 1\,dx} = e^x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$e^x y' + e^x y = e^{2x} \qquad \text{or} \qquad (e^x y)' = e^{2x}.$$

Integration on both sides now yields

$$e^x y = \frac{1}{2}e^{2x} + C.$$

Finally,

$$y(x) = \frac{1}{2}e^x + Ce^{-x}.$$

14. $y' + (\sec x)y = \cos x$

SOLUTION This equation is in standard form with $A(x) = \sec x$ and $B(x) = \cos x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int \sec x\,dx} = e^{\ln(\sec x + \tan x)} = \sec x + \tan x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(\sec x + \tan x)y' + (\sec^2 x + \sec x \tan x)y = 1 + \sin x$$

or

$$((\sec x + \tan x)y)' = 1 + \sin x.$$

Integration on both sides now yields

$$(\sec x + \tan x)y = x - \cos x + C.$$

Finally,

$$y(x) = \frac{x - \cos x + C}{\sec x + \tan x}.$$

15. $y' + (\tan x)y = \cos x$

SOLUTION This equation is in standard form with $A(x) = \tan x$ and $B(x) = \cos x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int \tan x\,dx} = e^{\ln \sec x} = \sec x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$\sec x y' + \sec x \tan x y = 1 \qquad \text{or} \qquad (y \sec x)' = 1.$$

Integration on both sides now yields

$$y \sec x = x + C.$$

Finally,

$$y(x) = x \cos x + C \cos x.$$

16. $e^{2x} y' = 1 - e^x y$

SOLUTION Rewrite the equation as

$$y' + e^{-x} y = e^{-2x},$$

which is in standard form with $A(x) = e^{-x}$ and $B(x) = e^{-2x}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int e^{-x}\,dx} = e^{-e^{-x}}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$e^{-e^{-x}} y' + e^{-x-e^{-x}} y = e^{-2x} e^{-e^{-x}} \qquad \text{or} \qquad (e^{-e^{-x}} y)' = e^{-2x} e^{-e^{-x}}.$$

Integration on both sides now yields

$$(e^{-e^{-x}} y) = \int e^{-2x} e^{-e^{-x}}\,dx.$$

To handle the remaining integral, make the substitution $u = -e^{-x}$, $du = e^{-x}\,dx$. Then

$$\int e^{-2x} e^{-e^{-x}}\,dx = -\int u e^u\,du = -ue^u + e^u + C = e^{-x} e^{-e^{-x}} + e^{-e^{-x}} + C.$$

Finally,

$$y(x) = 1 + e^{-x} + Ce^{e^{-x}}.$$

17. $y' - (\ln x)y = x^x$

SOLUTION This equation is in standard form with $A(x) = -\ln x$ and $B(x) = x^x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int -\ln x\,dx} = e^{x - x\ln x} = \frac{e^x}{x^x}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$x^{-x}e^x y' - (\ln x)x^{-x}e^x y = e^x \qquad \text{or} \qquad (x^{-x}e^x y)' = e^x.$$

Integration on both sides now yields

$$x^{-x}e^x y = e^x + C.$$

Finally,

$$y(x) = x^x + Cx^x e^{-x}.$$

18. $y' + y = \cos x$

SOLUTION This equation is in standard form with $A(x) = 1$ and $B(x) = \cos x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int 1\,dx} = e^x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$e^x y' + e^x y = e^x \cos x \qquad \text{or} \qquad (e^x y)' = e^x \cos x.$$

Integration on both sides (integration by parts is needed on the right-hand side of the equation) now yields

$$e^x y = \frac{1}{2}e^x (\sin x + \cos x) + C.$$

Finally,

$$y(x) = \frac{1}{2}(\sin x + \cos x) + Ce^{-x}.$$

In Exercises 19–26, solve the initial value problem.

19. $y' + 3y = e^{2x}, \quad y(0) = -1$

SOLUTION First, we find the general solution of the differential equation. This linear equation is in standard form with $A(x) = 3$ and $B(x) = e^{2x}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{3x}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^{3x}y)' = e^{5x}.$$

Integration on both sides now yields

$$(e^{3x}y) = \frac{1}{5}e^{5x} + C;$$

hence,

$$y(x) = \frac{1}{5}e^{2x} + Ce^{-3x}.$$

The initial condition $y(0) = -1$ allows us to determine the value of C:

$$-1 = \frac{1}{5} + C \qquad \text{so} \qquad C = -\frac{6}{5}.$$

The solution to the initial value problem is therefore

$$y(x) = \frac{1}{5}e^{2x} - \frac{6}{5}e^{-3x}.$$

20. $xy' + y = e^x, \quad y(1) = 3$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' + \frac{1}{x}y = \frac{1}{x}e^x,$$

which is in standard form with $A(x) = x^{-1}$ and $B(x) = x^{-1}e^x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int x^{-1}\,dx} = e^{\ln x} = x.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(xy)' = e^x.$$

Integration on both sides now yields

$$xy = e^x + C;$$

hence,

$$y(x) = \frac{1}{x}e^x + \frac{C}{x}.$$

The initial condition $y(1) = 3$ allows us to determine the value of C:

$$3 = e + \frac{C}{1} \qquad \text{so} \qquad C = 3 - e.$$

The solution to the initial value problem is therefore

$$y(x) = \frac{1}{x}e^x + \frac{3-e}{x}.$$

21. $y' + \frac{1}{x+1}y = x^{-2}, \quad y(1) = 2$

SOLUTION First, we find the general solution of the differential equation. This linear equation is in standard form with $A(x) = \frac{1}{x+1}$ and $B(x) = x^{-2}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int 1/(x+1)\,dx} = e^{\ln(x+1)} = x + 1.$$

When multiplied by the integrating factor, the original differential equation becomes

$$((x+1)y)' = x^{-1} + x^{-2}.$$

Integration on both sides now yields

$$(x+1)y = \ln x - x^{-1} + C;$$

hence,

$$y(x) = \frac{1}{x+1}\left(C + \ln x - \frac{1}{x}\right).$$

The initial condition $y(1) = 2$ allows us to determine the value of C:

$$2 = \frac{1}{2}(C - 1) \qquad \text{so} \qquad C = 5.$$

The solution to the initial value problem is therefore

$$y(x) = \frac{1}{x+1}\left(5 + \ln x - \frac{1}{x}\right).$$

22. $y' + y = \sin x, \quad y(0) = 1$

SOLUTION First, we find the general solution of the differential equation. This equation is in standard form with $A(x) = 1$ and $B(x) = \sin x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int 1\,dx} = e^x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^x y)' = e^x \sin x.$$

Integration on both sides (integration by parts is needed on the right-hand side of the equation) now yields

$$(e^x y) = \frac{1}{2}e^x(\sin x - \cos x) + C;$$

hence,

$$y(x) = \frac{1}{2}(\sin x - \cos x) + Ce^{-x}.$$

The initial condition $y(0) = 1$ allows us to determine the value of C:

$$1 = -\frac{1}{2} + C \qquad \text{so} \qquad C = \frac{3}{2}.$$

The solution to the initial value problem is therefore

$$y(x) = \frac{1}{2}(\sin x - \cos x) + \frac{3}{2}e^{-x}.$$

23. $(\sin x)y' = (\cos x)y + 1, \quad y\left(\frac{\pi}{4}\right) = 0$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' - (\cot x)y = \csc x,$$

which is in standard form with $A(x) = -\cot x$ and $B(x) = \csc x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int -\cot x\,dx} = e^{-\ln \sin x} = \csc x.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(\csc x y)' = \csc^2 x.$$

Integration on both sides now yields

$$(\csc x)y = -\cot x + C;$$

hence,

$$y(x) = -\cos x + C\sin x.$$

The initial condition $y(\pi/4) = 0$ allows us to determine the value of C:

$$0 = -\frac{\sqrt{2}}{2} + C\frac{\sqrt{2}}{2} \qquad \text{so} \qquad C = 1.$$

The solution to the initial value problem is therefore

$$y(x) = -\cos x + \sin x.$$

24. $y' + (\sec t)y = \sec t, \quad y\left(\frac{\pi}{4}\right) = 1$

SOLUTION First, we find the general solution of the differential equation. This equation is in standard form with $A(t) = \sec t$ and $B(t) = \sec t$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{\int \sec t\,dt} = e^{\ln(\sec t + \tan t)} = \sec t + \tan t.$$

When multiplied by the integrating factor, the original differential equation becomes

$$((\sec t + \tan t)y)' = \sec^2 t + \sec t \tan t.$$

Integration on both sides now yields

$$(\sec t + \tan t)y = \tan t + \sec t + C;$$

hence,

$$y(t) = 1 + \frac{C}{\sec t + \tan t}.$$

The initial condition $y(\pi/4) = 1$ allows us to determine the value of C:

$$1 = 1 + \frac{C}{\sqrt{2} + 1} \qquad \text{so} \qquad C = 0.$$

The solution to the initial value problem is therefore

$$y(x) = 1.$$

25. $y' + (\tanh x)y = 1, \quad y(0) = 3$

SOLUTION First, we find the general solution of the differential equation. This equation is in standard form with $A(x) = \tanh x$ and $B(x) = 1$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int \tanh x\,dx} = e^{\ln \cosh x} = \cosh x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(\cosh xy)' = \cosh x.$$

Integration on both sides now yields

$$(\cosh xy) = \sinh x + C;$$

hence,

$$y(x) = \tanh x + C \operatorname{sech} x.$$

The initial condition $y(0) = 3$ allows us to determine the value of C:

$$3 = C.$$

The solution to the initial value problem is therefore

$$y(x) = \tanh x + 3 \operatorname{sech} x.$$

26. $y' + \dfrac{x}{1+x^2}y = \dfrac{1}{(1+x^2)^{3/2}}, \quad y(1) = 0$

SOLUTION First, we find the general solution of the differential equation. This equation is in standard form with $A(x) = \frac{x}{1+x^2}$ and $B(x) = \frac{1}{(1+x^2)^{3/2}}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int (x/(1+x^2))\,dx} = e^{(1/2)\ln(1+x^2)} = \sqrt{1+x^2}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$\left(\sqrt{1+x^2}y\right)' = \frac{1}{1+x^2}.$$

Integration on both sides now yields

$$\sqrt{1+x^2}y = \tan^{-1} x + C;$$

hence,

$$y(x) = \frac{\tan^{-1} x}{\sqrt{1+x^2}} + \frac{C}{\sqrt{1+x^2}}.$$

The initial condition $y(1) = 0$ allows us to determine the value of C:

$$0 = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} + C\right) \qquad \text{so} \qquad C = -\frac{\pi}{4}.$$

The solution to the initial value problem is therefore

$$y(x) = \frac{1}{\sqrt{1+x^2}}\left(\tan^{-1} x - \frac{\pi}{4}\right).$$

27. Find the general solution of $y' + ny = e^{mx}$ for all m, n. *Note:* The case $m = -n$ must be treated separately.

SOLUTION For any m, n, Theorem 1 gives us the formula for $\alpha(x)$:

$$\alpha(x) = e^{\int n\,dx} = e^{nx}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^{nx}y)' = e^{(m+n)x}.$$

If $m \neq -n$, integration on both sides yields

$$e^{nx}y = \frac{1}{m+n}e^{(m+n)x} + C,$$

so

$$y(x) = \frac{1}{m+n}e^{mx} + Ce^{-nx}.$$

However, if $m = -n$, then $m + n = 0$ and the equation reduces to

$$(e^{nx} y)' = 1,$$

so integration yields

$$e^{nx} y = x + C \qquad \text{or} \qquad y(x) = (x + C)e^{-nx}.$$

28. Find the general solution of $y' + ny = \cos x$ for all n.

SOLUTION This equation is in standard form with $A(x) = n$ and $B(x) = \cos x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int n\,dx} = e^{nx}$$

When multiplied by the integrating factor, the differential equation becomes

$$e^{nx} y' + ne^{nx} y = e^{nx} \cos x$$

Integrating both sides gives

$$e^{nx} y = \frac{e^{nx}}{n^2 + 1}(\sin x + n \cos x) + C$$

(To integrate the right hand side, apply integration by parts twice with $u = e^{nx}$). Finally

$$y = Ce^{-nx} + \frac{\sin x + n \cos x}{n^2 + 1}$$

In Exercises 29–32, a 1000 L *tank contains* 500 L *of water with a salt concentration of* 10 g/L*. Water with a salt concentration of* 50 g/L *flows into the tank at a rate of* 80 L/min*. The fluid mixes instantaneously and is pumped out at a specified rate* R_{out}*. Let* $y(t)$ *denote the quantity of salt in the tank at time* t*.*

29. Assume that $R_{\text{out}} = 40$ L/min.

(a) Set up and solve the differential equation for $y(t)$.

(b) What is the salt concentration when the tank overflows?

SOLUTION Because water flows into the tank at the rate of 80 L/min but flows out at the rate of $R_{\text{out}} = 40$ L/min, there is a net inflow of 40 L/min. Therefore, at any time t, there are $500 + 40t$ liters of water in the tank.

(a) The net flow of salt into the tank at time t is

$$\frac{dy}{dt} = \text{salt rate in} - \text{salt rate out} = \left(80\frac{\text{L}}{\text{min}}\right)\left(50\frac{\text{g}}{\text{L}}\right) - \left(40\frac{\text{L}}{\text{min}}\right)\left(\frac{y\text{ g}}{500 + 40t\text{ L}}\right) = 4000 - 40 \cdot \frac{y}{500 + 40t}$$

Rewriting this linear equation in standard form, we have

$$\frac{dy}{dt} + \frac{4}{50 + 4t} y = 4000,$$

so $A(t) = \frac{4}{50+4t}$ and $B(t) = 4000$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{\int 4(50+4t)^{-1}\,dt} = e^{\ln(50+4t)} = 50 + 4t.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$((50 + 4t)y)' = 4000(50 + 4t).$$

Integration on both sides now yields

$$(50 + 4t)y = 200{,}000t + 16{,}000t^2 + C;$$

hence,

$$y(t) = \frac{200{,}000t + 8000t^2 + C}{50 + 4t}.$$

The initial condition $y(0) = 10$ allows us to determine the value of C:

$$10 = \frac{C}{50} \qquad \text{so} \qquad C = 500.$$

The solution to the initial value problem is therefore

$$y(t) = \frac{200{,}000t + 8000t^2 + 500}{50 + 4t} = \frac{250 + 4000t^2 + 100{,}000t}{25 + 2t}.$$

(b) The tank overflows when $t = 25/2 = 12.5$. The amount of salt in the tank at that time is

$$y(12.5) = 37{,}505 \text{ g},$$

so the concentration of salt is

$$\frac{37{,}505 \text{ g}}{1000 \text{ L}} = 37.505 \text{ g/L}.$$

30. Find the salt concentration when the tank overflows, assuming that $R_{\text{out}} = 60$ L/min.

SOLUTION We work as in Exercise 29, but with $R_{\text{out}} = 60$. There is a net inflow of 20 L/min, so at time t, there are $500 + 20t$ liters of water in the tank. The net flow of salt into the tank at time t is

$$\frac{dy}{dt} = \text{salt rate in} - \text{salt rate out} = \left(80\frac{\text{L}}{\text{min}}\right)\left(50\frac{\text{g}}{\text{L}}\right) - \left(60\frac{\text{L}}{\text{min}}\right)\left(\frac{y \text{ g}}{500 + 20t \text{ L}}\right) = 4000 - 6 \cdot \frac{y}{50 + 2t}$$

Rewriting this linear equation in standard form, we have

$$\frac{dy}{dt} + \frac{6}{50 + 2t}y = 4000,$$

so $A(t) = \frac{6}{50+2t}$ and $B(t) = 4000$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{\int 6(50+2t)^{-1}\,dt} = e^{3\ln(50+2t)} = (50 + 2t)^3.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$((50 + 2t)^3 y)' = 4000(50 + 2t)^3.$$

Integration on both sides now yields

$$(50 + 2t)^3 y = 500(50 + 2t)^4 + C;$$

hence,

$$y(t) = 25{,}000 + 1000t + \frac{C}{(50 + 2t)^3}.$$

The initial condition $y(0) = 10$ allows us to determine the value of C:

$$10 = 25{,}000 + \frac{C}{50^3} \qquad \text{so} \qquad C = -3123.75 \times 10^6.$$

The solution to the initial value problem is therefore

$$y(t) = 25{,}000 + 1000t - \frac{390{,}468{,}750}{(25 + t)^2}.$$

The tank overflows when $t = 25$. The amount of salt in the tank at that time is

$$y(25) = 46{,}876.25 \text{ g},$$

so the concentration of salt is

$$\frac{46{,}876.25 \text{ g}}{1000 \text{ L}} \approx 46.876 \text{ g/L}.$$

31. Find the limiting salt concentration as $t \to \infty$ assuming that $R_{\text{out}} = 80$ L/min.

SOLUTION The total volume of water is now constant at 500 liters, so the net flow of salt at time t is

$$\frac{dy}{dt} = \text{salt rate in} - \text{salt rate out} = \left(80\frac{\text{L}}{\text{min}}\right)\left(50\frac{\text{g}}{\text{L}}\right) - \left(80\frac{\text{L}}{\text{min}}\right)\left(\frac{y \text{ g}}{500 \text{ L}}\right) = 4000 - \frac{8}{50}y$$

Rewriting this equation in standard form gives

$$\frac{dy}{dt} + \frac{8}{50}y = 4000$$

so that the integrating factor is

$$e^{\int (8/50)\,dt} = e^{0.16t}$$

Multiplying both sides by the integrating factor gives

$$(e^{0.16t}y)' = 4000e^{0.16t}$$

Integrate both sides to get

$$e^{0.16t}y = 25{,}000e^{0.16t} + C \quad \text{so that} \quad y = 25{,}000 + Ce^{-0.16t}$$

As $t \to \infty$, the exponential term tends to zero, so that the amount of salt tends to 25,000g, or 50 g/L. (Note that this is precisely what would be expected naïvely, since the salt concentration flowing in is also 50 g/L).

32. Assuming that $R_{\text{out}} = 120$ L/min. Find $y(t)$. Then calculate the tank volume and the salt concentration at $t = 10$ minutes.

SOLUTION We work as in Exercise 29, but with $R_{\text{out}} = 120$. There is a net outflow of 40 L/min, so at time t, there are $500 - 40t$ liters of water in the tank. Note that after ten minutes, the volume of water in the tank is 100 liters.

The net flow of salt into the tank at time t is

$$\frac{dy}{dt} = \text{salt rate in} - \text{salt rate out} = \left(80\frac{\text{L}}{\text{min}}\right)\left(50\frac{\text{g}}{\text{L}}\right) - \left(120\frac{\text{L}}{\text{min}}\right)\left(\frac{y\ \text{g}}{500 - 40t\ \text{L}}\right) = 4000 - 12 \cdot \frac{y}{50 - 4t}$$

Rewriting this linear equation in standard form, we have

$$\frac{dy}{dt} + \frac{6}{25 - 2t}y = 4000,$$

so $A(t) = \frac{6}{25-2t}$ and $B(t) = 4000$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{\int 6(25-2t)^{-1}\,dt} = e^{-3\ln(25-2t)} = (25 - 2t)^{-3}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$((25 - 2t)^{-3}y)' = 4000(25 - 2t)^{-3}.$$

Integration on both sides now yields

$$(25 - 2t)^{-3}y = 1000(25 - 2t)^{-2} + C;$$

hence,

$$y(t) = 25{,}000 - 2000t + C(25 - 2t)^3.$$

The initial condition $y(0) = 10$ allows us to determine the value of C:

$$10 = 25{,}000 + C \cdot 50^3 \qquad \text{so} \qquad C = -1.599.$$

The solution to the initial value problem is therefore

$$y(t) = 25{,}000 - 2000t - 1.599(25 - 2t)^3.$$

The amount of salt in the tank at time $t = 10$ is then

$$y(10) = 4800.08 \text{ g},$$

so the concentration of salt is

$$\frac{4800.08 \text{ g}}{100 \text{ L}} \approx 48 \text{ g/L}.$$

33. Water flows into a tank at the variable rate of $R_{\text{in}} = 20/(1 + t)$ gal/min and out at the constant rate $R_{\text{out}} = 5$ gal/min. Let $V(t)$ be the volume of water in the tank at time t.

(a) Set up a differential equation for $V(t)$ and solve it with the initial condition $V(0) = 100$.

(b) Find the maximum value of V.

(c) CAS Plot $V(t)$ and estimate the time t when the tank is empty.

SOLUTION

(a) The rate of change of the volume of water in the tank is given by

$$\frac{dV}{dt} = R_{\text{in}} - R_{\text{out}} = \frac{20}{1 + t} - 5.$$

Because the right-hand side depends only on the independent variable t, we integrate to obtain

$$V(t) = 20\ln(1+t) - 5t + C.$$

The initial condition $V(0) = 100$ allows us to determine the value of C:

$$100 = 20\ln 1 - 0 + C \qquad \text{so} \qquad C = 100.$$

Therefore

$$V(t) = 20\ln(1+t) - 5t + 100.$$

(b) Using the result from part (a),

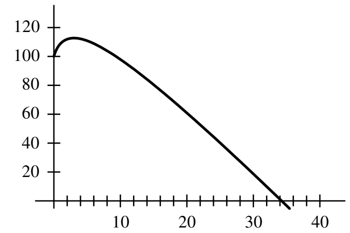
$$\frac{dV}{dt} = \frac{20}{1+t} - 5 = 0$$

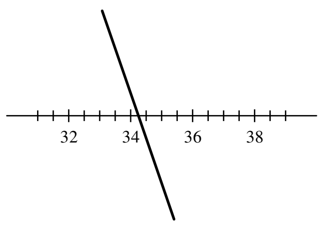
when $t = 3$. Because $\frac{dV}{dt} > 0$ for $t < 3$ and $\frac{dV}{dt} < 0$ for $t > 3$, it follows that

$$V(3) = 20\ln 4 - 15 + 100 \approx 112.726 \text{ gal}$$

is the maximum volume.

(c) $V(t)$ is plotted in the figure below at the left. On the right, we zoom in near the location where the curve crosses the t-axis. From this graph, we estimate that the tank is empty after roughly 34.25 minutes.





34. A stream feeds into a lake at a rate of 1000 m^3/day. The stream is polluted with a toxin whose concentration is 5 g/m^3. Assume that the lake has volume 10^6 m^3 and that water flows out of the lake at the same rate of 1000 m^3/day.

(a) Set up a differential equation for the concentration $c(t)$ of toxin in the lake and solve for $c(t)$, assuming that $c(0) = 0$. *Hint:* Find the differential equation for the quantity of toxin $y(t)$, and observe that $c(t) = y(t)/10^6$.

(b) What is the limiting concentration for large t?

SOLUTION

(a) Let $M(t)$ denote the amount of toxin, in grams, in the lake at time t. The rate at which toxin enters the lake is given by

$$5\frac{\text{g}}{\text{m}^3} \cdot 1000\frac{\text{m}^3}{\text{day}} = 5000\frac{\text{g}}{\text{day}},$$

while the rate at which toxin exits the lake is given by

$$\frac{M(t)\ \text{g}}{10^6\ \text{m}^3} \cdot 1000\frac{\text{m}^3}{\text{day}} = \frac{M(t)}{1000}\frac{\text{g}}{\text{day}},$$

where we have assumed that any toxin in the lake is spread uniformly throughout the lake. The differential equation for $M(t)$ is then

$$\frac{dM}{dt} = 5000 - \frac{M}{1000}.$$

The concentration of the toxin in the lake is given by $c(t) = \frac{M(t)}{10^6}$, so $c'(t) = \frac{1}{10^6}M'(t)$, giving

$$\frac{dc}{dt} = \frac{1}{200} - \frac{1}{1000}c.$$

Rewriting this linear equation in standard form, we have

$$\frac{dc}{dt} + \frac{1}{1000}c = \frac{1}{200},$$

so $A(t) = \frac{1}{1000}$ and $B(t) = \frac{1}{200}$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{t/1000}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(e^{t/1000}c)' = \frac{1}{200}e^{t/1000}.$$

Integration on both sides now yields

$$e^{t/1000}c = 5e^{t/1000} + A;$$

hence,

$$c(t) = 5 + Ae^{-t/1000}.$$

The initial condition $c(0) = 0$ allows us to determine the value of A:

$$0 = 5 + A \qquad \text{so} \qquad A = -5.$$

Therefore

$$c(t) = 5\left(1 - e^{-t/1000}\right) \text{ grams/m}^3.$$

(b) As $t \to \infty$, $c(t) \to 5$, so the limiting concentration of pollution is $5\ \frac{\text{grams}}{\text{m}^3}$.

In Exercises 35–38, consider a series circuit (Figure 4) consisting of a resistor of R ohms, an inductor of L henries, and a variable voltage source of V(t) volts (time t in seconds). The current through the circuit I(t) (in amperes) satisfies the differential equation

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}V(t) \qquad \boxed{10}$$

35. Find the solution to Eq. (10) with initial condition $I(0) = 0$, assuming that $R = 100\ \Omega$, $L = 5$ H, and $V(t)$ is constant with $V(t) = 10$ V.

SOLUTION If $R = 100$, $V(t) = 10$, and $L = 5$, the differential equation becomes

$$\frac{dI}{dt} + 20I = 2,$$

which is a linear equation in standard form with $A(t) = 20$ and $B(t) = 2$. The integrating factor is $\alpha(t) = e^{20t}$, and when multiplied by the integrating factor, the differential equation becomes

$$(e^{20t}I)' = 2e^{20t}.$$

Integration of both sides now yields

$$e^{20t}I = \frac{1}{10}e^{20t} + C;$$

hence,

$$I(t) = \frac{1}{10} + Ce^{-20t}.$$

The initial condition $I(0) = 0$ allows us to determine the value of C:

$$0 = \frac{1}{10} + C \qquad \text{so} \qquad C = -\frac{1}{10}.$$

Finally,

$$I(t) = \frac{1}{10}\left(1 - e^{-20t}\right).$$

36. Assume that $R = 110\ \Omega$, $L = 10$ H, and $V(t) = e^{-t}$.

(a) Solve Eq. (10) with initial condition $I(0) = 0$.

(b) Calculate t_m and $I(t_m)$, where t_m is the time at which $I(t)$ has a maximum value.

(c) GU Use a computer algebra system to sketch the graph of the solution for $0 \le t \le 3$.

SOLUTION

(a) If $R = 110$, $V(t) = e^{-t}$, and $L = 10$, the differential equation becomes

$$\frac{dI}{dt} + 11I = \frac{1}{10}e^{-t},$$

which is a linear equation in standard form with $A(t) = 11$ and $B(t) = \frac{1}{10}e^{-t}$. The integrating factor is $\alpha(t) = e^{11t}$, and when multiplied by the integrating factor, the differential equation becomes

$$(e^{11t}I)' = \frac{1}{10}e^{10t}.$$

Integration of both sides now yields

$$e^{11t}I = \frac{1}{100}e^{10t} + C;$$

hence,

$$I(t) = \frac{1}{100}e^{-t} + Ce^{-11t}.$$

The initial condition $I(0) = 0$ allows us to determine the value of C:

$$0 = \frac{1}{100} + C \qquad \text{so} \qquad C = -\frac{1}{100}.$$

Finally,

$$I(t) = \frac{1}{100}\left(e^{-t} - e^{-11t}\right).$$

(b) Using the result from part (a),

$$\frac{dI}{dt} = \frac{1}{100}\left(-e^{-t} + 11e^{-11t}\right) = 0$$

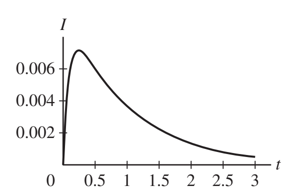
when

$$t = t_m = \frac{1}{10}\ln 11 \text{ seconds.}$$

Now,

$$I(t_m) = \frac{1}{100}\left(e^{-(1/10)\ln 11} - e^{-(11/10)\ln 11}\right) = \frac{1}{100}\left(11^{-1/10} - 11^{-11/10}\right) \approx 0.00715 \text{ amperes.}$$

(c) The graph of $I(t)$ is shown below.



37. Assume that $V(t) = V$ is constant and $I(0) = 0$.

(a) Solve for $I(t)$.

(b) Show that $\lim_{t\to\infty} I(t) = V/R$ and that $I(t)$ reaches approximately 63% of its limiting value after L/R seconds.

(c) How long does it take for $I(t)$ to reach 90% of its limiting value if $R = 500\ \Omega$, $L = 4$ H, and $V = 20$ V?

SOLUTION

(a) The equation

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}V$$

is a linear equation in standard form with $A(t) = \frac{R}{L}$ and $B(t) = \frac{1}{L}V(t)$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{\int (R/L)\,dt} = e^{(R/L)\,t}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^{(R/L)t}I)' = e^{(R/L)t}\frac{V}{L}.$$

Integration on both sides now yields

$$(e^{(R/L)t}I) = \frac{V}{R}e^{(R/L)t} + C;$$

hence,

$$I(t) = \frac{V}{R} + Ce^{-(R/L)t}.$$

The initial condition $I(0) = 0$ allows us to determine the value of C:

$$0 = \frac{V}{R} + C \qquad \text{so} \qquad C = -\frac{V}{R}.$$

Therefore the current is given by

$$I(t) = \frac{V}{R}\left(1 - e^{-(R/L)t}\right).$$

(b) As $t \to \infty$, $e^{-(R/L)t} \to 0$, so $I(t) \to \frac{V}{R}$. Moreover, when $t = (L/R)$ seconds, we have

$$I\left(\frac{L}{R}\right) = \frac{V}{R}\left(1 - e^{-(R/L)(L/R)}\right) = \frac{V}{R}\left(1 - e^{-1}\right) \approx 0.632\frac{V}{R}.$$

(c) Using the results from part (a) and part (b), $I(t)$ reaches 90% of its limiting value when

$$\frac{9}{10} = 1 - e^{-(R/L)t},$$

or when

$$t = \frac{L}{R}\ln 10.$$

With $L = 4$ and $R = 500$, this takes approximately 0.0184 seconds.

38. Solve for $I(t)$, assuming that $R = 500\ \Omega$, $L = 4$ H, and $V = 20\cos(80t)$ volts.

SOLUTION With $R = 500$, $L = 4$, and $V = 20\cos(80t)$, Eq. (10) becomes

$$\frac{dI}{dt} + 125I = 5\cos(80t)$$

which is a linear equation in standard form with $A(t) = 125$ and $B(t) = 5\cos(80t)$. The integrating factor is e^{125t}; when multiplied by the integrating factor, the differential equation becomes

$$(e^{125t}I)' = 5e^{125t}\cos(80t)$$

To integrate the right side, apply integration by parts twice and solve the resulting formula for the desired integral, giving

$$\int 5e^{125t}\cos(80t)\,dt = \frac{1}{881}e^{125t}(25\cos(80t) + 16\sin(80t) + C$$

so that the solution is

$$e^{125t}I = \frac{1}{881}e^{125t}(25\cos(80t) + 16\sin(80t) + C$$

Multiply through by e^{-125t} to get

$$I = \frac{1}{881}(25\cos(80t) + 16\sin(80t) + Ce^{-125t}$$

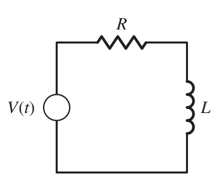


FIGURE 4 *RL* circuit.

39. Tank 1 in Figure 5 is filled with V_1 liters of water containing blue dye at an initial concentration of c_0 g/L. Water flows into the tank at a rate of R L/min, is mixed instantaneously with the dye solution, and flows out through the bottom at the same rate R. Let $c_1(t)$ be the dye concentration in the tank at time t.

(a) Explain why c_1 satisfies the differential equation $\dfrac{dc_1}{dt} = -\dfrac{R}{V_1}c_1$.

(b) Solve for $c_1(t)$ with $V_1 = 300$ L, $R = 50$, and $c_0 = 10$ g/L.

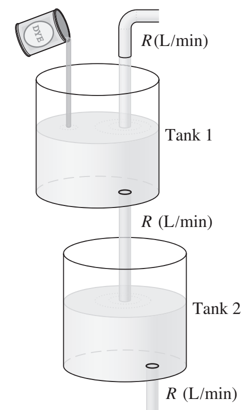


FIGURE 5

SOLUTION

(a) Let $g_1(t)$ be the number of grams of dye in the tank at time t. Then $g_1(t) = V_1 c_1(t)$ and $g_1'(t) = V_1 c_1'(t)$. Now,

$$g_1'(t) = \text{grams of dye in} - \text{grams of dye out} = 0 - \frac{g(t)}{V_1}\text{ g/L} \cdot R \text{ L/min} = -\frac{R}{V_1}g(t)$$

Substituting gives

$$V_1 c_1'(t) = -\frac{R}{V_1}c_1(t)V_1 \quad \text{and simplifying yields} \quad c_1'(t) = -\frac{R}{V_1}c_1(t)$$

(b) In standard form, the equation is

$$c_1'(t) + \frac{R}{V_1}c_1(t) = 0$$

so that $A(t) = \dfrac{R}{V_1}$ and $B(t) = 0$. The integrating factor is $e^{(R/V_1)t}$; multiplying through gives

$$(e^{(R/V_1)t}c_1(t))' = 0 \quad \text{so, integrating,} \quad e^{(R/V_1)t}c_1(t) = C$$

and thus $c_1(t) = Ce^{-(R/V_1)t}$. With $R = 50$ and $V_1 = 300$ we have $c_1(t) = Ce^{-t/6}$; the initial condition $c_1(0) = c_0 = 10$ gives $C = 10$. Finally,

$$c_1(t) = 10e^{-t/6}$$

40. Continuing with the previous exercise, let Tank 2 be another tank filled with V_2 gal of water. Assume that the dye solution from Tank 1 empties into Tank 2 as in Figure 5, mixes instantaneously, and leaves Tank 2 at the same rate R. Let $c_2(t)$ be the dye concentration in Tank 2 at time t.

(a) Explain why c_2 satisfies the differential equation

$$\frac{dc_2}{dt} = \frac{R}{V_2}(c_1 - c_2)$$

(b) Use the solution to Exercise 39 to solve for $c_2(t)$ if $V_1 = 300$, $V_2 = 200$, $R = 50$, and $c_0 = 10$.

(c) Find the maximum concentration in Tank 2.

(d) GU Plot the solution.

SOLUTION

(a) Let $g_2(t)$ be the amount in grams of dye in Tank 2 at time t. At time t, the concentration of dye in Tank 1, and thus the concentration of dye coming into Tank 2, is $c_1(t)$. Thus

$$\begin{aligned} g_2'(t) &= \text{grams of dye in} - \text{grams of dye out} \\ &= c_1(t)\text{ g/L} \cdot R \text{ L/min} - c_2(t)\text{g/L} \cdot R \text{ L/min} = R(c_1(t) - c_2(t)) \end{aligned}$$

Since $g_2'(t) = V_2 c_2'(t)$, we get

$$c_2'(t) = \frac{R}{V_2}(c_1(t) - c_2(t))$$

(b) With $V_1 = 300$, $R = 50$, and $c_0 = 10$, part (a) tells us that

$$c_1(t) = 10e^{-t/6}$$

Since $V_2 = 200$, we have

$$c_2'(t) = \frac{1}{4}(10e^{-t/6} - c_2(t))$$

Putting this linear equation in standard form gives

$$c_2'(t) + \frac{1}{4}c_2(t) = \frac{5}{2}e^{-t/6}$$

The integrating factor is $e^{t/4}$; multiplying through gives

$$(e^{t/4}c_2(t))' = \frac{5}{2}e^{t/12}$$

Integrate to get

$$e^{t/4}c_2(t) = 30e^{t/12} + C \quad \text{so that} \quad c_2(t) = 30e^{-t/6} + Ce^{-t/4}$$

Since Tank 2 starts out filled entirely with water, we have $c_2(0) = 0$ so that $C = -30$ and

$$c_2(t) = 30(e^{-t/6} - e^{-t/4})$$

(c) The maximum concentration in Tank 2 occurs when $c_2'(t) = 0$.

$$c_2'(t) = 0 = -5e^{-t/6} + \frac{15}{2}e^{-t/4}$$

Solve this equation for t as follows:

$$\begin{aligned}
5e^{-t/6} &= \frac{15}{2}e^{-t/4} \\
2e^{-t/6} &= 3e^{-t/4} \\
-\frac{t}{6} + \ln 2 &= -\frac{t}{4} + \ln 3 \\
\frac{t}{12} &= \ln 3 - \ln 2 = \ln(3/2) \\
t &= 12\ln(3/2) \approx 4.866
\end{aligned}$$

When $t = 12\ln(3/2)$,

$$c_2(t) = 30(e^{-2\ln(3/2)} - e^{-3\ln(3/2)}) = 30\left(\frac{4}{9} - \frac{8}{27}\right) = \frac{40}{9}$$

41. Let a, b, r be constants. Show that

$$y = Ce^{-kt} + a + bk\left(\frac{k\sin rt - r\cos rt}{k^2 + r^2}\right)$$

is a general solution of

$$\frac{dy}{dt} = -k\Big(y - a - b\sin rt\Big)$$

SOLUTION This is a linear differential equation; in standard form, it is

$$\frac{dy}{dt} + ky = k(a + b\sin rt)$$

The integrating factor is then e^{kt}; multiplying through gives

$$(e^{kt}y)' = kae^{kt} + kbe^{kt}\sin rt \qquad (*)$$

The first term on the right-hand side has integral ae^{kt}. To integrate the second term, use integration by parts twice; this result in an equation of the form

$$\int kbe^{kt}\sin rt = F(t) + A\int kbe^{kt}\sin rt$$

for some function $F(t)$ and constant A. Solving for the integral gives

$$\int kbe^{kt}\sin rt = kbe^{kt}\frac{k\sin rt - r\cos rt}{k^2+r^2}$$

so that integrating equation (*) gives

$$e^{kt}y = ae^{kt} + kbe^{kt}\frac{k\sin rt - r\cos rt}{k^2+r^2} + C$$

Divide through by e^{kt} to get

$$y = a + bk\left(\frac{k\sin rt - r\cos rt}{k^2+r^2}\right) + Ce^{-kt}$$

42. Assume that the outside temperature varies as

$$T(t) = 15 + 5\sin(\pi t/12)$$

where $t = 0$ is 12 noon. A house is heated to 25°C at $t = 0$ and after that, its temperature $y(t)$ varies according to Newton's Law of Cooling (Figure 6):

$$\frac{dy}{dt} = -0.1\Big(y(t) - T(t)\Big)$$

Use Exercise 41 to solve for $y(t)$.

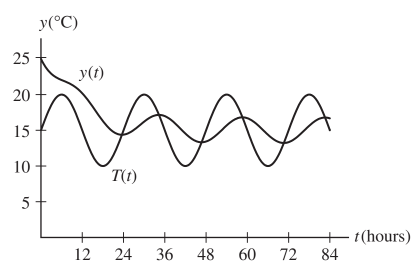


FIGURE 6 House temperature $y(t)$

SOLUTION The differential equation is

$$\frac{dy}{dt} = -0.1\Big(y(t) - 15 - 5\sin(\frac{\pi t}{12})\Big)$$

This differential equation is of the form considered in Exercise 41, with $a = 15$, $b = 5$, $r = \pi/12$, and $k = 0.1$. Thus the general solution is

$$y(t) = Ce^{-0.1t} + 15 + 0.5\left(\frac{0.1\sin(\pi t/12) - (\pi/12)\cos(\pi t/12)}{0.01 + \pi^2/144}\right)$$

Since $y(0) = 25$, we have

$$25 = C + 15 + 0.5\left(\frac{0 - \pi/12}{0.01 + \pi^2/144}\right) \approx C + 15 - 1.667$$

so that $C \approx 11.667$ and

$$y(t) = 11.667e^{-0.1t} + 15 + 0.5\left(\frac{0.1\sin(\pi t/12) - (\pi/12)\cos(\pi t/12)}{0.01 + \pi^2/144}\right)$$

Further Insights and Challenges

43. Let $\alpha(x)$ be an integrating factor for $y' + A(x)y = B(x)$. The differential equation $y' + A(x)y = 0$ is called the associated **homogeneous equation**.

(a) Show that $1/\alpha(x)$ is a solution of the associated homogeneous equation.

(b) Show that if $y = f(x)$ is a particular solution of $y' + A(x)y = B(x)$, then $f(x) + C/\alpha(x)$ is also a solution for any constant C.

SOLUTION

(a) Remember that $\alpha'(x) = A(x)\alpha(x)$. Now, let $y(x) = (\alpha(x))^{-1}$. Then

$$y' + A(x)y = -\frac{1}{(\alpha(x))^2}\alpha'(x) + \frac{A(x)}{\alpha(x)} = -\frac{1}{(\alpha(x))^2}A(x)\alpha(x) + \frac{A(x)}{\alpha(x)} = 0.$$

(b) Suppose $f(x)$ satisfies $f'(x) + A(x)f(x) = B(x)$. Now, let $y(x) = f(x) + C/\alpha(x)$, where C is an arbitrary constant. Then

$$\begin{aligned} y' + A(x)y &= f'(x) - \frac{C}{(\alpha(x))^2}\alpha'(x) + A(x)f(x) + \frac{CA(x)}{\alpha(x)} \\ &= \left(f'(x) + A(x)f(x)\right) + \frac{C}{\alpha(x)}\left(A(x) - \frac{\alpha'(x)}{\alpha(x)}\right) = B(x) + 0 = B(x). \end{aligned}$$

44. Use the Fundamental Theorem of Calculus and the Product Rule to verify directly that for any x_0, the function

$$f(x) = \alpha(x)^{-1}\int_{x_0}^{x} \alpha(t)B(t)\,dt$$

is a solution of the initial value problem

$$y' + A(x)y = B(x), \qquad y(x_0) = 0$$

where $\alpha(x)$ is an integrating factor [a solution to Eq. (3)].

SOLUTION Remember that $\alpha'(x) = A(x)\alpha(x)$. Now, let

$$y(x) = \frac{1}{\alpha(x)}\int_{x_0}^{x} \alpha(t)B(t)\,dt.$$

Then,

$$y(x_0) = \frac{1}{\alpha(x)}\int_{x_0}^{x_0} \alpha(t)B(t)\,dt = 0,$$

and

$$\begin{aligned} y' + A(x)y &= -\frac{\alpha'(x)}{(\alpha(x))^2}\int_{x_0}^{x} \alpha(t)B(t)\,dt + B(x) + \frac{A(x)}{\alpha(x)}\int_{x_0}^{x} \alpha(t)B(t)\,dt \\ &= B(x) + \left(-\frac{A(x)}{\alpha(x)} + \frac{A(x)}{\alpha(x)}\right)\int_{x_0}^{x} \alpha(t)B(t)\,dt = B(x). \end{aligned}$$

45. Transient Currents Suppose the circuit described by Eq. (10) is driven by a sinusoidal voltage source $V(t) = V\sin\omega t$ (where V and ω are constant).

(a) Show that

$$I(t) = \frac{V}{R^2 + L^2\omega^2}(R\sin\omega t - L\omega\cos\omega t) + Ce^{-(R/L)t}$$

(b) Let $Z = \sqrt{R^2 + L^2\omega^2}$. Choose θ so that $Z\cos\theta = R$ and $Z\sin\theta = L\omega$. Use the addition formula for the sine function to show that

$$I(t) = \frac{V}{Z}\sin(\omega t - \theta) + Ce^{-(R/L)t}$$

This shows that the current in the circuit varies sinusoidally apart from a DC term (called the **transient current** in electronics) that decreases exponentially.

SOLUTION

(a) With $V(t) = V \sin \omega t$, the equation

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{1}{L} V(t)$$

becomes

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{V}{L} \sin \omega t.$$

This is a linear equation in standard form with $A(t) = \frac{R}{L}$ and $B(t) = \frac{V}{L} \sin \omega t$. By Theorem 1, the integrating factor is

$$\alpha(t) = \int e^{\int A(t)\, dt} = e^{(R/L)\, t}.$$

When multiplied by the integrating factor, the equation becomes

$$(e^{(R/L)\, t} I)' = \frac{V}{L} e^{(R/L)\, t} \sin \omega t.$$

Integration on both sides (integration by parts is needed for the integral on the right-hand side) now yields

$$(e^{(R/L)\, t} I) = \frac{V}{R^2 + L^2\omega^2} e^{(R/L)\, t} (R \sin \omega t - L\omega \cos \omega t) + C;$$

hence,

$$I(t) = \frac{V}{R^2 + L^2\omega^2} (R \sin \omega t - L\omega \cos \omega t) + Ce^{-(R/L)\, t}.$$

(b) Let $Z = \sqrt{R^2 + L^2\omega^2}$, and choose θ so that $Z \cos \theta = R$ and $Z \sin \theta = L\omega$. Then

$$\begin{aligned}\frac{V}{R^2 + L^2\omega^2} (R \sin \omega t - L\omega \cos \omega t) &= \frac{V}{Z^2} (Z \cos \theta \sin \omega t - Z \sin \theta \cos \omega t) \\ &= \frac{V}{Z} (\cos \theta \sin \omega t - \sin \theta \cos \omega t) = \frac{V}{Z} \sin(\omega t - \theta).\end{aligned}$$

Thus,

$$I(t) = \frac{V}{Z} \sin(\omega t - \theta) + Ce^{-(R/L)\, t}.$$

CHAPTER REVIEW EXERCISES

1. Which of the following differential equations are linear? Determine the order of each equation.

(a) $y' = y^5 - 3x^4 y$

(b) $y' = x^5 - 3x^4 y$

(c) $y = y''' - 3x\sqrt{y}$

(d) $\sin x \cdot y'' = y - 1$

SOLUTION

(a) y^5 is a nonlinear term involving the dependent variable, so this is not a linear equation; the highest order derivative that appears in the equation is a first derivative, so this is a first-order equation.

(b) This is linear equation; the highest order derivative that appears in the equation is a first derivative, so this is a first-order equation.

(c) $\sqrt{y}$ is a nonlinear term involving the dependent variable, so this is not a linear equation; the highest order derivative that appears in the equation is a third derivative, so this is a third-order equation.

(d) This is linear equation; the highest order derivative that appears in the equation is a second derivative, so this is a second-order equation.

2. Find a value of c such that $y = x - 2 + e^{cx}$ is a solution of $2y' + y = x$.

SOLUTION Let $y = x - 2 + e^{cx}$. Then

$$y' = 1 + ce^{cx},$$

and

$$2y' + y = 2\left(1 + ce^{cx}\right) + \left(x - 2 + e^{cx}\right) = 2 + 2ce^{cx} + x - 2 + e^{cx} = (2c + 1)e^{cx} + x.$$

For this to equal x, we must have $2c + 1 = 0$, or $c = -\frac{1}{2}$ (remember that e^{cx} is never zero).

In Exercises 3–6, solve using separation of variables.

3. $\dfrac{dy}{dt} = t^2 y^{-3}$

SOLUTION Rewrite the equation as

$$y^3\,dy = t^2\,dt.$$

Upon integrating both sides of this equation, we obtain:

$$\int y^3\,dy = \int t^2\,dt$$
$$\frac{y^4}{4} = \frac{t^3}{3} + C.$$

Thus,

$$y = \pm\left(\frac{4}{3}t^3 + C\right)^{1/4},$$

where C is an arbitrary constant.

4. $xyy' = 1 - x^2$

SOLUTION Rewrite the equation

$$xy\frac{dy}{dx} = 1 - x^2 \qquad \text{as} \qquad y\,dy = \left(\frac{1}{x} - x\right)dx.$$

Upon integrating both sides of this equation, we obtain

$$\int y\,dy = \int \left(\frac{1}{x} - x\right)dx$$
$$\frac{y^2}{2} = \ln|x| - \frac{x^2}{2} + C.$$

Thus,

$$y = \pm\sqrt{\ln x^2 + A - x^2},$$

where $A = 2C$ is an arbitrary constant.

5. $x\dfrac{dy}{dx} - y = 1$

SOLUTION Rewrite the equation as

$$\frac{dy}{1+y} = \frac{dx}{x}.$$

upon integrating both sides of this equation, we obtain

$$\int \frac{dy}{1+y} = \int \frac{dx}{x}$$
$$\ln|1+y| = \ln|x| + C.$$

Thus,

$$y = -1 + Ax,$$

where $A = \pm e^C$ is an arbitrary constant.

6. $y' = \dfrac{xy^2}{x^2+1}$

SOLUTION Rewrite

$$\frac{dy}{dx} = \frac{xy^2}{x^2+1} \qquad \text{as} \qquad \frac{dy}{y^2} = \frac{x}{x^2+1}\,dx.$$

Upon integrating both sides of this equation, we obtain

$$\int \frac{dy}{y^2} = \int \frac{x}{x^2+1}\,dx$$
$$-\frac{1}{y} = \frac{1}{2}\ln\left(x^2+1\right) + C.$$

Thus,

$$y = -\frac{1}{\frac{1}{2}\ln\left(x^2+1\right)+C},$$

where C is an arbitrary constant.

In Exercises 7–10, solve the initial value problem using separation of variables.

7. $y' = \cos^2 x, \quad y(0) = \frac{\pi}{4}$

SOLUTION First, we find the general solution of the differential equation. Because the variables are already separated, we integrate both sides to obtain

$$y = \int \cos^2 x\,dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

The initial condition $y(0) = \frac{\pi}{4}$ allows us to determine $C = \frac{\pi}{4}$. Thus, the solution is:

$$y(x) = \frac{x}{2} + \frac{\sin 2x}{4} + \frac{\pi}{4}.$$

8. $y' = \cos^2 y, \quad y(0) = \frac{\pi}{4}$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = \cos^2 y \qquad \text{as} \qquad \frac{dy}{\cos^2 y} = dx.$$

Upon integrating both sides of this equation, we find

$$\tan y = x + C;$$

thus,

$$y = \tan^{-1}(x+C).$$

The initial condition $y(0) = \frac{\pi}{4}$ allows us to determine the value of C:

$$\frac{\pi}{4} = \tan^{-1} C \qquad \text{so} \qquad C = 1.$$

Hence, the solution is $y = \tan^{-1}(x+1)$.

9. $y' = xy^2, \quad y(1) = 2$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = xy^2 \qquad \text{as} \qquad \frac{dy}{y^2} = x\,dx.$$

Upon integrating both sides of this equation, we find

$$\int \frac{dy}{y^2} = \int x\,dx$$
$$-\frac{1}{y} = \frac{1}{2}x^2 + C.$$

Thus,

$$y = -\frac{1}{\frac{1}{2}x^2 + C}.$$

The initial condition $y(1) = 2$ allows us to determine the value of C:

$$2 = -\frac{1}{\frac{1}{2}\cdot 1^2 + C} = -\frac{2}{1+2C}$$
$$1 + 2C = -1$$
$$C = -1$$

Hence, the solution to the initial value problem is

$$y = -\frac{1}{\frac{1}{2}x^2 - 1} = -\frac{2}{x^2-2}.$$

10. $xyy' = 1, \quad y(3) = 2$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$xy\frac{dy}{dx} = 1 \qquad \text{as} \qquad y\,dy = \frac{dx}{x}.$$

Next we integrate both sides of the equation to obtain

$$\int y\,dy = \int \frac{dx}{x}$$
$$\frac{1}{2}y^2 = \ln|x| + C.$$

Thus,

$$y = \pm\sqrt{2(\ln|x| + C)}.$$

To satisfy the initial condition $y(3) = 2$ we must choose the positive square root; moreover,

$$2 = \sqrt{2(\ln 3 + C)} \qquad \text{so} \qquad C = 2 - \ln 3.$$

Hence, the solution to the initial value problem is

$$y = \sqrt{2(\ln|x| + 2 - \ln 3)} = \sqrt{\ln\left(\frac{x^2}{9}\right) + 4}.$$

11. Figure 1 shows the slope field for $\dot{y} = \sin y + ty$. Sketch the graphs of the solutions with the initial conditions $y(0) = 1$, $y(0) = 0$, and $y(0) = -1$.

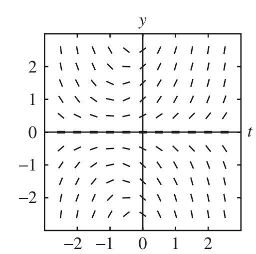
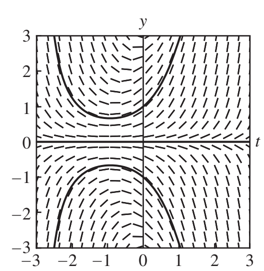


FIGURE 1

SOLUTION



12. Which of the equations (i)–(iii) corresponds to the slope field in Figure 2?

(i) $\dot{y} = 1 - y^2$
(ii) $\dot{y} = 1 + y^2$
(iii) $\dot{y} = y^2$

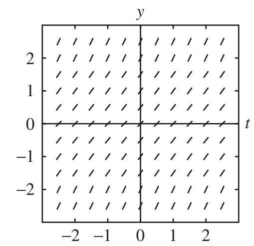
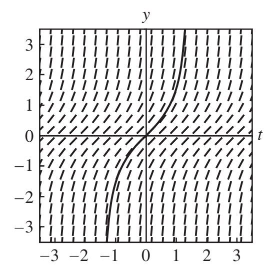


FIGURE 2

SOLUTION From the figure we see that the the slope is positive even for $y > 1$, thus, the slope field does not correspond to the equation $\dot{y} = 1 - y^2$. Moreover, the slope at $y = 0$ is positive, so the slope field also does not correspond to the equation $\dot{y} = y^2$. The slope field must therefore correspond to **(ii)**: $\dot{y} = 1 + y^2$.

13. Let $y(t)$ be the solution to the differential equation with slope field as shown in Figure 2, satisfying $y(0) = 0$. Sketch the graph of $y(t)$. Then use your answer to Exercise 12 to solve for $y(t)$.

SOLUTION As explained in the previous exercise, the slope field in Figure 2 corresponds to the equation $\dot{y} = 1 + y^2$. The graph of the solution satisfying $y(0) = 0$ is:



To solve the initial value problem $\dot{y} = 1 + y^2$, $y(0) = 0$, we first find the general solution of the differential equation. Separating variables yields:

$$\frac{dy}{1 + y^2} = dt.$$

Upon integrating both sides of this equation, we find

$$\tan^{-1} y = t + C \qquad \text{or} \qquad y = \tan(t + C).$$

The initial condition gives $C = 0$, so the solution is $y = \tan x$.

14. Let $y(t)$ be the solution of $4\dot{y} = y^2 + t$ satisfying $y(2) = 1$. Carry out Euler's Method with time step $h = 0.05$ for $n = 6$ steps.

SOLUTION Rewrite the differential equation as $\dot{y} = \frac{1}{4}(y^2 + t)$ to identify $F(t, y) = \frac{1}{4}\left(y^2 + t\right)$. With $t_0 = 2$, $y_0 = 1$, and $h = 0.05$, we calculate

$$\begin{aligned}
y_1 &= y_0 + hF(t_0, y_0) = 1.0375 \\
y_2 &= y_1 + hF(t_1, y_1) = 1.076580 \\
y_3 &= y_2 + hF(t_2, y_2) = 1.117318 \\
y_4 &= y_3 + hF(t_3, y_3) = 1.159798 \\
y_5 &= y_4 + hF(t_4, y_4) = 1.204112 \\
y_6 &= y_5 + hF(t_5, y_5) = 1.250361
\end{aligned}$$

15. Let $y(t)$ be the solution of $(x^3 + 1)\dot{y} = y$ satisfying $y(0) = 1$. Compute approximations to $y(0.1)$, $y(0.2)$, and $y(0.3)$ using Euler's Method with time step $h = 0.1$.

SOLUTION Rewriting the equation as $\dot{y} = \frac{y}{x^3+1}$ we have $F(x, y) = \frac{y}{x^3+1}$. Using Euler's Method with $x_0 = 0$, $y_0 = 1$ and $h = 0.1$, we calculate

$$\begin{aligned}
y(0.1) &\approx y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1 \cdot \frac{1}{0^3 + 1} = 1.1 \\
y(0.2) &\approx y_2 = y_1 + hF(x_1, y_1) = 1.209890 \\
y(0.3) &\approx y_3 = y_2 + hF(x_2, y_2) = 1.329919
\end{aligned}$$

In Exercises 16–19, solve using the method of integrating factors.

16. $\frac{dy}{dt} = y + t^2, \quad y(0) = 4$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' - y = t^2,$$

which is in standard form with $A(t) = -1$ and $B(t) = t^2$. The integrating factor is

$$\alpha(t) = e^{\int -1\,dt} = e^{-t}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(e^{-t}y)' = t^2 e^{-t}.$$

Integration on both sides (integration by parts is needed for the integral on the right-hand side of the equation) now yields

$$e^{-t}y = -e^{-t}(t^2 + 2t + 2) + C;$$

hence,

$$y(t) = Ce^t - t^2 - 2t - 2.$$

The initial condition $y(0) = 4$ allows us to determine the value of C:

$$4 = -2 + C \qquad \text{so} \qquad C = 6.$$

The solution to the initial value problem is then

$$y = 6e^t - t^2 - 2t - 2.$$

17. $\frac{dy}{dx} = \frac{y}{x} + x, \quad y(1) = 3$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' - \frac{1}{x}y = x,$$

which is in standard form with $A(x) = -\frac{1}{x}$ and $B(x) = x$. The integrating factor is

$$\alpha(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$\left(\frac{1}{x}y\right)' = 1.$$

Integration on both sides now yields

$$\frac{1}{x}y = x + C;$$

hence,

$$y(x) = x^2 + Cx.$$

The initial condition $y(1) = 3$ allows us to determine the value of C:

$$3 = 1 + C \qquad \text{so} \qquad C = 2.$$

The solution to the initial value problem is then

$$y = x^2 + 2x.$$

18. $\frac{dy}{dt} = y - 3t, \quad y(-1) = 2$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' - y = -3t,$$

which is in standard form with $A(t) = -1$ and $B(t) = -3t$. The integrating factor is

$$\alpha(t) = e^{\int A(t)\,dt} = e^{\int -dt} = e^{-t}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(e^{-t}y)' = -3te^{-t}.$$

Integration on both sides (integration by parts is needed for the integral on the right-hand side of the equation) now yields

$$e^{-t}y = (3t+3)e^{-t} + C;$$

hence,

$$y(t) = 3t + 3 + Ce^{t}.$$

The initial condition $y(-1) = 2$ allows us to determine the value of C;

$$2 = Ce^{-1} + 3(-1) + 3 \qquad \text{so} \qquad C = 2e.$$

The solution to the initial value problem is then

$$y = 2e \cdot e^{t} + 3t + 3 = 2e^{t+1} + 3t + 3.$$

19. $y' + 2y = 1 + e^{-x}, \quad y(0) = -4$

SOLUTION The equation is already in standard form with $A(x) = 2$ and $B(x) = 1 + e^{-x}$. The integrating factor is

$$\alpha(x) = e^{\int 2\,dx} = e^{2x}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^{2x}y)' = e^{2x} + e^{x}.$$

Integration on both sides now yields

$$e^{2x}y = \frac{1}{2}e^{2x} + e^{x} + C;$$

hence,

$$y(x) = \frac{1}{2} + e^{-x} + Ce^{-2x}.$$

The initial condition $y(0) = -4$ allows us to determine the value of C:

$$-4 = \frac{1}{2} + 1 + C \qquad \text{so} \qquad C = -\frac{11}{2}.$$

The solution to the initial value problem is then

$$y(x) = \frac{1}{2} + e^{-x} - \frac{11}{2}e^{-2x}.$$

In Exercises 20–27, solve using the appropriate method.

20. $x^2y' = x^2 + 1, \quad y(1) = 10$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' = 1 + \frac{1}{x^2}.$$

Because the variables have already been separated, we integrate both sides to obtain

$$y = \int \left(1 + \frac{1}{x^2}\right) dx = x - \frac{1}{x} + C.$$

The initial condition $y(1) = 10$ allows us to determine the value of C:

$$10 = 1 - 1 + C \qquad \text{so} \qquad C = 10.$$

The solution to the initial value problem is then

$$y = x - \frac{1}{x} + 10.$$

21. $y' + (\tan x)y = \cos^2 x, \quad y(\pi) = 2$

SOLUTION First, we find the general solution of the differential equation. As this is a first order linear equation with $A(x) = \tan x$ and $B(x) = \cos^2 x$, we compute the integrating factor

$$\alpha(x) = e^{\int A(x)\,dx} = e^{\int \tan x\,dx} = e^{-\ln\cos x} = \frac{1}{\cos x}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$\left(\frac{1}{\cos x}y\right)' = \cos x.$$

Integration on both sides now yields

$$\frac{1}{\cos x}y = \sin x + C;$$

hence,

$$y(x) = \sin x\cos x + C\cos x = \frac{1}{2}\sin 2x + C\cos x.$$

The initial condition $y(\pi) = 2$ allows us to determine the value of C:

$$2 = 0 + C(-1) \qquad \text{so} \qquad C = -2.$$

The solution to the initial value problem is then

$$y = \frac{1}{2}\sin 2x - 2\cos x.$$

22. $xy' = 2y + x - 1, \quad y\left(\frac{3}{2}\right) = 9$

SOLUTION First, we find the general solution of the differential equation. This is a linear equation which can be rewritten as

$$y' - \frac{2}{x}y = 1 - \frac{1}{x}.$$

Thus, $A(x) = -\frac{2}{x}$, $B(x) = 1 - \frac{1}{x}$ and the integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = \frac{1}{x^2}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$\left(\frac{1}{x^2}y\right)' = \frac{1}{x^2} - \frac{1}{x^3}.$$

Integration on both sides now yields

$$\frac{1}{x^2}y = -\frac{1}{x} + \frac{1}{2x^2} + C;$$

hence,

$$y(x) = -x + \frac{1}{2} + Cx^2.$$

The initial condition $y\left(\frac{3}{2}\right) = 9$ allows us to determine the value of C:

$$9 = -\frac{3}{2} + \frac{1}{2} + \frac{9}{4}C \qquad \text{so} \qquad C = \frac{40}{9}.$$

The solution to the initial value problem is then

$$y = \frac{40}{9}x^2 - x + \frac{1}{2}.$$

23. $(y-1)y' = t, \quad y(1) = -3$

SOLUTION First, we find the general solution of the differential equation. This is a separable equation that we rewrite as

$$(y-1)\,dy = t\,dt.$$

Upon integrating both sides of this equation, we find

$$\int (y-1)\,dy = \int t\,dt$$
$$\frac{y^2}{2} - y = \frac{1}{2}t^2 + C$$
$$y^2 - 2y + 1 = t^2 + C$$
$$(y-1)^2 = t^2 + C$$
$$y(t) = \pm\sqrt{t^2 + C} + 1$$

To satisfy the initial condition $y(1) = -3$ we must choose the negative square root; moreover,

$$-3 = -\sqrt{1+C} + 1 \qquad \text{so} \qquad C = 15.$$

The solution to the initial value problem is then

$$y(t) = -\sqrt{t^2 + 15} + 1$$

24. $\left(\sqrt{y} + 1\right)y' = yte^{t^2}, \quad y(0) = 1$

SOLUTION First, we find the general solution of the differential equation. This is a separable equation that we rewrite as

$$\left(\frac{1}{\sqrt{y}} + \frac{1}{y}\right) dy = te^{t^2}\,dt.$$

Upon integrating both sides of this equation, we find

$$\int \left(\frac{1}{\sqrt{y}} + \frac{1}{y}\right) dy = \int te^{t^2}\,dt$$
$$2\sqrt{y} + \ln y = \frac{1}{2}e^{t^2} + C.$$

Note that we cannot solve explicitly for $y(t)$. The initial condition $y(0) = 1$ still allows us to determine the value of C:

$$2(1) + \ln 1 = \frac{1}{2} + C \qquad \text{so} \qquad C = \frac{3}{2}.$$

Hence, the general solution is given implicitly by the equation

$$2\sqrt{y} + \ln y = \frac{1}{2}e^{x^2} + \frac{3}{2}.$$

25. $\dfrac{dw}{dx} = k\dfrac{1+w^2}{x}, \quad w(1) = 1$

SOLUTION First, we find the general solution of the differential equation. This is a separable equation that we rewrite as

$$\frac{dw}{1+w^2} = \frac{k}{x}\,dx.$$

Upon integrating both sides of this equation, we find

$$\int \frac{dw}{1+w^2} = \int \frac{k}{x}\,dx$$
$$\tan^{-1} w = k\ln x + C$$
$$w(x) = \tan(k\ln x + C).$$

Because the initial condition is specified at $x = 1$, we are interested in the solution for $x > 0$; we can therefore omit the absolute value within the natural logarithm function. The initial condition $w(1) = 1$ allows us to determine the value of C:

$$1 = \tan(k\ln 1 + C) \qquad \text{so} \qquad C = \tan^{-1} 1 = \frac{\pi}{4}.$$

The solution to the initial value problem is then

$$w = \tan\left(k \ln x + \frac{\pi}{4}\right).$$

26. $y' + \dfrac{3y-1}{t} = t + 2$

SOLUTION We rewrite this first order linear equation in standard form:

$$y' + \frac{3}{t}y = t + 2 + \frac{1}{t}.$$

Thus, $A(t) = \frac{3}{t}$, $B(t) = t + 2 + \frac{1}{t}$, and the integrating factor is

$$\alpha(t) = e^{\int A(t)\,dt} = e^{3\ln t} = t^3.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(t^3 y)' = t^4 + 2t^3 + t^2.$$

Integration on both sides now yields

$$t^3 y = \frac{1}{5}t^5 + \frac{1}{2}t^4 + \frac{1}{3}t^3 + C;$$

hence,

$$y(t) = \frac{1}{5}t^2 + \frac{1}{2}t + \frac{1}{3} + \frac{C}{t^3}.$$

27. $y' + \dfrac{y}{x} = \sin x$

SOLUTION This is a first order linear equation with $A(x) = \frac{1}{x}$ and $B(x) = \sin x$. The integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{\ln x} = x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(xy)' = x\sin x.$$

Integration on both sides (integration by parts is needed for the integral on the right-hand side) now yields

$$xy = -x\cos x + \sin x + C;$$

hence,

$$y(x) = -\cos x + \frac{\sin x}{x} + \frac{C}{x}.$$

28. Find the solutions to $y' = 4(y - 12)$ satisfying $y(0) = 20$ and $y(0) = 0$, and sketch their graphs.

SOLUTION The general solution of the differential equation $y' = 4(y - 12)$ is

$$y(t) = 12 + Ce^{4t},$$

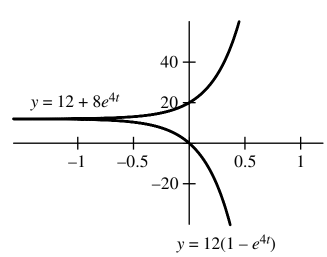
for some constant C. If $y(0) = 20$, then

$$20 = 12 + Ce^0 \quad \text{and} \quad C = 8.$$

Thus, $y(t) = 12 + 8e^{4t}$. If $y(0) = 0$, then

$$0 = 12 + Ce^0 \quad \text{and} \quad C = -12;$$

hence, $y(t) = 12(1 - e^{4t})$. The graphs of the two solutions are shown below.



29. Find the solutions to $y' = -2y + 8$ satisfying $y(0) = 3$ and $y(0) = 4$, and sketch their graphs.

SOLUTION First, rewrite the differential equation as $y' = -2(y - 4)$; from here we see that the general solution is

$$y(t) = 4 + Ce^{-2t},$$

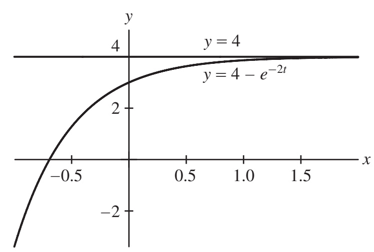
for some constant C. If $y(0) = 3$, then

$$3 = 4 + Ce^0 \quad \text{and} \quad C = -1.$$

Thus, $y(t) = 4 - e^{-2t}$. If $y(0) = 4$, then

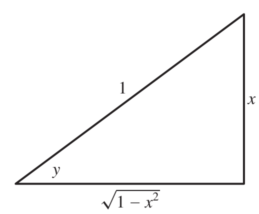
$$4 = 4 + Ce^0 \quad \text{and} \quad C = 0;$$

hence, $y(t) = 4$. The graphs of the two solutions are shown below.



30. Show that $y = \sin^{-1} x$ satisfies the differential equation $y' = \sec y$ with initial condition $y(0) = 0$.

SOLUTION Let $y = \sin^{-1} x$. Then $x = \sin y$ and we construct the right triangle shown below.



Thus,

$$\sec y = \frac{1}{\sqrt{1 - x^2}} = \frac{d}{dx} \sin^{-1} x = y'.$$

Moreover, $y(0) = \sin^{-1} 0 = 0$. Consequently, $y = \sin^{-1} x$ satisfies the differential equation $y' = \sec y$ with initial condition $y(0) = 0$.

31. What is the limit $\lim_{t\to\infty} y(t)$ if $y(t)$ is a solution of:

(a) $\dfrac{dy}{dt} = -4(y - 12)$?

(b) $\dfrac{dy}{dt} = 4(y - 12)$?

(c) $\dfrac{dy}{dt} = -4y - 12$?

SOLUTION

(a) The general solution of $\dfrac{dy}{dt} = -4(y - 12)$ is $y(t) = 12 + Ce^{-4t}$, where C is an arbitrary constant. Regardless of the value of C,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (12 + Ce^{-4t}) = 12.$$

(b) The general solution of $\dfrac{dy}{dt} = 4(y - 12)$ is $y(t) = 12 + Ce^{4t}$, where C is an arbitrary constant. Here, the limit depends on the value of C. Specifically,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (12 + Ce^{4t}) = \begin{cases} \infty, & C > 0 \\ 12, & C = 0 \\ -\infty, & C < 0 \end{cases}$$

(c) The general solution of $\frac{dy}{dt} = -4y - 12 = -4(y+3)$ is $y(t) = -3 + Ce^{-4t}$, where C is an arbitrary constant. Regardless of the value of C,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}(-3 + Ce^{-4t}) = -3.$$

In Exercises 32–35, let $P(t)$ denote the balance at time t (years) of an annuity that earns 5% interest continuously compounded and pays out \$20,000*/year continuously.*

32. Find the differential equation satisfied by $P(t)$.

SOLUTION Since money is withdrawn continuously at a rate of \$20,000 a year and the growth due to interest is $0.05P$, the rate of change of the balance is

$$P'(t) = 0.05P - 20{,}000.$$

Thus, the differential equation satisfied by $P(t)$ is

$$P'(t) = 0.05(P - 400{,}000).$$

33. Determine $P(5)$ if $P(0) = \$200{,}000$.

SOLUTION In the previous exercise we concluded that $P(t)$ satisfies the equation $P' = 0.05(P - 400{,}000)$. The general solution of this differential equation is

$$P(t) = 400{,}000 + Ce^{0.05t}.$$

Given $P(0) = 200{,}000$, it follows that

$$200{,}000 = 400{,}000 + Ce^{0.05\cdot 0} = 400{,}000 + C$$

or

$$C = -200{,}000.$$

Thus,

$$P(t) = 400{,}000 - 200{,}000e^{0.05t},$$

and

$$P(5) = 400{,}000 - 200{,}000e^{0.05(5)} \approx \$143{,}194.90.$$

34. When does the annuity run out of money if $P(0) = \$300{,}000$?

SOLUTION We found that

$$P(t) = 400{,}000 + Ce^{0.05t}.$$

If $P(0) = 300{,}000$, then

$$300{,}000 = 400{,}000 + Ce^{0.05\cdot 0} = 400{,}000 + C$$

or

$$C = -100{,}000.$$

Thus,

$$P(t) = 400{,}000 - 100{,}000e^{0.05t}.$$

The annuity runs out of money when $P(t) = 0$; that is, when

$$400{,}000 - 100{,}000e^{0.05t} = 0.$$

Solving for t yields

$$t = \frac{1}{0.05}\ln 4 = 20\ln 4 \approx 27.73.$$

The money runs out after roughly 27.73 years.

35. What is the minimum initial balance that will allow the annuity to make payments indefinitely?

SOLUTION In Exercise 33, we found that the balance at time t is

$$P(t) = 400{,}000 + Ce^{0.05t}.$$

If initial balance is P_0 then

$$P_0 = P(0) = 400{,}000 + Ce^{0.05\cdot 0} = 400{,}000 + C$$

or

$$C = P_0 - 400{,}000.$$

Thus,

$$P(t) = 400{,}000 + (P_0 - 400{,}000)\, e^{0.05t}.$$

If $P_0 \geq 400{,}000$, then $P(t)$ is always positive. Therefore, the minimum initial balance that allows the annuity to make payments indefinitely is $P_0 = \$400{,}000$.

36. State whether the differential equation can be solved using separation of variables, the method of integrating factors, both, or neither.

(a) $y' = y + x^2$

(b) $xy' = y + 1$

(c) $y' = y^2 + x^2$

(d) $xy' = y^2$

SOLUTION

(a) The equation $y' = y + x^2$ is a first order linear equation; hence, it can be solved by the method of integration factors. However, it cannot be written in the form $y' = f(x)g(y)$; therefore, separation of variables cannot be used.

(b) The equation $xy' = y + 1$ is a first order linear equation; hence, it can be solved using the method of integration factors. We can rewrite this equation as $y' = \frac{1}{x}(y+1)$; therefore, it can also be solved by separating the variables.

(c) The equation $y' = y^2 + x^2$ cannot be written in the form $y' = f(x)g(y)$; hence, separation of variables cannot be used. This equation is also not linear; hence, the method of integrating factors cannot be used.

(d) The equation $xy' = y^2$ can be rewritten as $y' = \frac{1}{x}y^2$; therefore, it can be solved by separating the variables. Since it is not a linear equation, the method of integrating factors cannot be used.

37. Let A and B be constants. Prove that if $A > 0$, then all solutions of $\frac{dy}{dt} + Ay = B$ approach the same limit as $t \to \infty$.

SOLUTION This is a linear first-order equation in standard form with integrating factor

$$\alpha(t) = e^{\int A\,dt} = e^{At}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^{At}y)' = Be^{At}.$$

Integration on both sides now yields

$$e^{At}y = \frac{B}{A}e^{At} + C;$$

hence,

$$y(t) = \frac{B}{A} + Ce^{-At}.$$

Because $A > 0$,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(\frac{B}{A} + Ce^{-At}\right) = \frac{B}{A}.$$

We conclude that if $A > 0$, all solutions approach the limit $\frac{B}{A}$ as $t \to \infty$.

38. At time $t = 0$, a tank of height 5 m in the shape of an inverted pyramid whose cross section at the top is a square of side 2 m is filled with water. Water flows through a hole at the bottom of area $0.002\ \text{m}^2$. Use Torricelli's Law to determine the time required for the tank to empty.

SOLUTION $y(t)$, the height of the water at time t, obeys the differential equation:

$$\frac{dy}{dt} = \frac{Bv(y)}{A(y)}$$

where $v(y)$ is the velocity of the water flowing through the hole when the height of the water is y, B is the area of the hole, and $A(y)$ is the cross-sectional area of the surface of the water when it is at height y. By Torricelli's Law, $v(y) = -\sqrt{19.6}\sqrt{y} = -14\sqrt{y}/\sqrt{10}$ m/s. The area of the hole is $B = 0.002$. To determine $A(y)$, note that the ratio of the length of a side of the square forming the surface of the water to the height of the water is $2/5$ (using similar triangles). Thus when the water is at height y, the area is $A(y) = \left(\frac{2}{5}y\right)^2 = \frac{4y^2}{25}$. Thus

$$\frac{dy}{dt} = \frac{-0.002 \cdot 14\sqrt{y} \cdot 25}{4y^2\sqrt{10}} = \frac{-0.175}{\sqrt{10}}y^{-3/2}$$

Separating variables gives

$$y^{3/2}\,dy = \frac{-0.175}{\sqrt{10}}\,dt$$

Integrating both sides gives

$$\frac{2}{5}y^{5/2} = \frac{-0.175}{\sqrt{10}}t + C \quad \text{so that} \quad y = \left(\frac{-0.4375}{\sqrt{10}}t + \frac{5}{2}C\right)^{2/5}$$

At $t = 0$, $y(t) = 5$, so that

$$5 = \left(\frac{5}{2}C\right)^{2/5} \quad \text{and} \quad C \approx 22.36$$

so that

$$y(t) \approx (-0.138t + 55.9)^{2/5}$$

The tank is empty when $y(t) = 0$, so when $t = 55.9/0.138 \approx 405.07$. The tank is empty after approximately 405 seconds, or 6 minutes 45 seconds.

39. The trough in Figure 3 (dimensions in centimeters) is filled with water. At time $t = 0$ (in seconds), water begins leaking through a hole at the bottom of area $4\ \text{cm}^2$. Let $y(t)$ be the water height at time t. Find a differential equation for $y(t)$ and solve it to determine when the water level decreases to 60 cm.

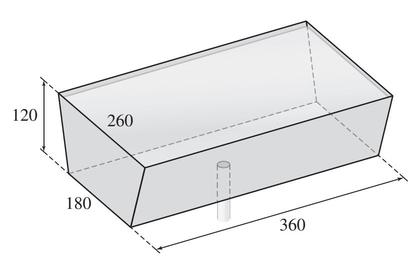


FIGURE 3

SOLUTION $y(t)$ obeys the differential equation:

$$\frac{dy}{dt} = \frac{Bv(y)}{A(y)},$$

where $v(y)$ denotes the velocity of the water flowing through the hole when the trough is filled to height y, B denotes the area of the hole and $A(y)$ denotes the area of the horizontal cross section of the trough at height y. Since measurements are all in centimeters, we will work in centimeters. We have

$$g = 9.8\ \text{m/s}^2 = 980\ \text{cm/s}^2$$

By Torricelli's Law, $v(y) = -\sqrt{2 \cdot 980}\sqrt{y} = -14\sqrt{10}\sqrt{y}$ m/s. The area of the hole is $B = 4$ cm^2. The horizontal cross section of the trough at height y is a rectangle of length 360 and width $w(y)$. As $w(y)$ varies linearly from 180 when $y = 0$ to 260 when $y = 120$, it follows that

$$w(y) = 180 + \frac{80y}{120} = 180 + \frac{2}{3}y$$

so that the area of the horizontal cross-section at height y is

$$A(y) = 360w(y) = 64800 + 240y = 240(y + 270)$$

The differential equation for $y(t)$ then becomes

$$\frac{dy}{dt} = \frac{Bv(y)}{A(y)} = \frac{-4 \cdot 14\sqrt{10}\sqrt{y}}{240(y+270)} = \frac{-7\sqrt{10}}{30} \cdot \frac{\sqrt{y}}{y+270}$$

This equation is separable, so

$$\frac{y+270}{\sqrt{y}}\,dy = \frac{-7\sqrt{10}}{30}\,dt$$

$$(y^{1/2} + 270y^{-1/2})\,dy = \frac{-7\sqrt{10}}{30}\,dt$$

$$\int (y^{1/2} + 270y^{-1/2})\,dy = \frac{-7\sqrt{10}}{30}\int 1\,dt$$

$$\frac{2}{3}y^{3/2} + 540y^{1/2} = -\frac{7\sqrt{10}}{30}t + C$$

$$y^{3/2} + 810y^{1/2} = -\frac{7\sqrt{10}}{20}t + C$$

The initial condition $y(0) = 120$ allows us to determine the value of C:

$$120^{3/2} + 810 \cdot 120^{1/2} = 0 + C \quad \text{so} \quad C = 930\sqrt{120} = 1860\sqrt{30}$$

Thus the height of the water is given implicitly by the equation

$$y^{3/2} + 810y^{1/2} = -\frac{7\sqrt{10}}{20}t + 1860\sqrt{30}$$

We want to find t such that $y(t) = 60$:

$$60^{3/2} + 810 \cdot 60^{1/2} = -\frac{7\sqrt{10}}{20}t + 1860\sqrt{30}$$

$$1740\sqrt{15} = -\frac{7\sqrt{10}}{20}t + 1860\sqrt{30}$$

$$t = \frac{120}{7}\sqrt{10}(31\sqrt{30} - 29\sqrt{15}) \approx 3115.88 \text{ s}$$

The height of the water in the tank is 60 cm after approximately 3116 seconds, or 51 minutes 56 seconds.

40. Find the solution of the logistic equation $\dot{y} = 0.4y(4 - y)$ satisfying $y(0) = 8$.

SOLUTION We can write the given equation as

$$\dot{y} = 1.6y\left(1 - \frac{y}{4}\right).$$

This is a logistic equation with $k = 1.6$ and $A = 4$. Therefore,

$$y(t) = \frac{A}{1 - e^{-kt}/C} = \frac{4}{1 - e^{-1.6t}/C}.$$

The initial condition $y(0) = 8$ allows us to determine the value of C:

$$8 = \frac{4}{1 - \frac{1}{C}}; \quad 1 - \frac{1}{C} = \frac{1}{2}; \quad \text{so} \quad C = 2.$$

Thus,

$$y(t) = \frac{4}{1 - e^{-1.6t}/2} = \frac{8}{2 - e^{-1.6t}}.$$

41. Let $y(t)$ be the solution of $\dot{y} = 0.3y(2-y)$ with $y(0) = 1$. Determine $\lim_{t\to\infty} y(t)$ without solving for y explicitly.

SOLUTION We write the given equation in the form

$$\dot{y} = 0.6y\left(1 - \frac{y}{2}\right).$$

This is a logistic equation with $A = 2$ and $k = 0.6$. Because the initial condition $y(0) = y_0 = 1$ satisfies $0 < y_0 < A$, the solution is increasing and approaches A as $t \to \infty$. That is, $\lim_{t\to\infty} y(t) = 2$.

42. Suppose that $y' = ky(1 - y/8)$ has a solution satisfying $y(0) = 12$ and $y(10) = 24$. Find k.

SOLUTION The given differential equation is a logistic equation with $A = 8$. Thus,

$$y(t) = \frac{8}{1 - e^{-kt}/C}.$$

The initial condition $y(0) = 12$ allows us to determine the value of C:

$$12 = \frac{8}{1 - \frac{1}{C}}; \quad 1 - \frac{1}{C} = \frac{2}{3}; \quad \text{so} \quad C = 3.$$

Hence,

$$y(t) = \frac{8}{1 - e^{-kt}/3} = \frac{24}{3 - e^{-kt}}.$$

Now, the condition $y(10) = 24$ allows us to determine the value of k:

$$24 = \frac{24}{3 - e^{-10k}}$$

$$3 - e^{-10k} = 1$$

$$k = -\frac{\ln 2}{10} \approx -0.0693.$$

43. A lake has a carrying capacity of 1000 fish. Assume that the fish population grows logistically with growth constant $k = 0.2\ \text{day}^{-1}$. How many days will it take for the population to reach 900 fish if the initial population is 20 fish?

SOLUTION Let $y(t)$ represent the fish population. Because the population grows logistically with $k = 0.2$ and $A = 1000$,

$$y(t) = \frac{1000}{1 - e^{-0.2t}/C}.$$

The initial condition $y(0) = 20$ allows us to determine the value of C:

$$20 = \frac{1000}{1 - \frac{1}{C}}; \quad 1 - \frac{1}{C} = 50; \quad \text{so} \quad C = -\frac{1}{49}.$$

Hence,

$$y(t) = \frac{1000}{1 + 49e^{-0.2t}}.$$

The population will reach 900 fish when

$$\frac{1000}{1 + 49e^{-0.2t}} = 900.$$

Solving for t, we find

$$t = 5 \ln 441 \approx 30.44 \text{ days}.$$

44. A rabbit population on an island increases exponentially with growth rate $k = 0.12\ \text{months}^{-1}$. When the population reaches 300 rabbits (say, at time $t = 0$), wolves begin eating the rabbits at a rate of r rabbits per month.

(a) Find a differential equation satisfied by the rabbit population $P(t)$.

(b) How large can r be without the rabbit population becoming extinct?

SOLUTION

(a) The rabbit population $P(t)$ obeys the differential equation

$$\frac{dP}{dt} = 0.12P - r,$$

where the term $0.12P$ accounts for the exponential growth of the population and the term $-r$ accounts for the rate of decline in the rabbit population due to their being food for wolves.

(b) Rewrite the linear differential equation from part (a) as

$$\frac{dP}{dt} - 0.12P = -r,$$

which is in standard form with $A = -0.12$ and $B = -r$. The integrating factor is

$$\alpha(t) = e^{\int A\,dt} = e^{\int -0.12\,dt} = e^{-0.12t}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(e^{-0.12t}P)' = -re^{-0.12t}.$$

Integration on both sides now yields

$$e^{-0.12t}P = \frac{r}{0.12}e^{-0.12t} + C;$$

hence,

$$P(t) = \frac{r}{0.12} + Ce^{0.12t}.$$

The initial condition $P(0) = 300$ allows us to determine the value of C:

$$300 = \frac{r}{0.12} + C \qquad \text{so} \qquad C = 300 - \frac{r}{0.12}.$$

The solution to the initial value problem is then

$$P(t) = \left(300 - \frac{r}{0.12}\right)e^{0.12t} + \frac{r}{0.12}.$$

Now, if $300 - \frac{r}{0.12} < 0$, then $\lim_{t\to\infty} P(t) = -\infty$, and the population becomes extinct. Therefore, in order for the population to survive, we must have

$$300 - \frac{r}{0.12} \ge 0 \qquad \text{or} \qquad r \le 36.$$

We conclude that the maximum rate at which the wolves can eat the rabbits without driving the rabbits to extinction is $r = 36$ rabbits per month.

45. Show that $y = \sin(\tan^{-1} x + C)$ is the general solution of $y' = \sqrt{1-y^2}/\left(1+x^2\right)$. Then use the addition formula for the sine function to show that the general solution may be written

$$y = \frac{(\cos C)x + \sin C}{\sqrt{1+x^2}}$$

SOLUTION Rewrite

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1+x^2} \qquad \text{as} \qquad \frac{dy}{\sqrt{1-y^2}} = \frac{dx}{1+x^2}.$$

Upon integrating both sides of this equation, we find

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{1+x^2}$$
$$\sin^{-1} y = \tan^{-1} x + C$$

Thus,

$$y(x) = \sin\left(\tan^{-1} x + C\right).$$

To express the solution in the required form, we use the addition formula

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

This yields

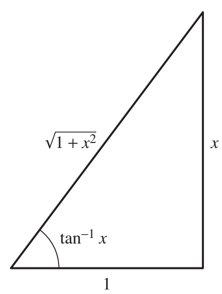
$$y(x) = \sin\left(\tan^{-1}x\right)\cos C + \sin C\cos\left(\tan^{-1}x\right).$$

Using the figure below, we see that

$$\sin\left(\tan^{-1}x\right) = \frac{x}{\sqrt{1+x^2}};\ \text{and}$$
$$\cos\left(\tan^{-1}x\right) = \frac{1}{\sqrt{1+x^2}}.$$

Finally,

$$y = \frac{x\cos C}{\sqrt{1+x^2}} + \frac{\sin C}{\sqrt{1+x^2}} = \frac{(\cos C)x + \sin C}{\sqrt{1+x^2}}.$$



46. A tank is filled with 300 liters of contaminated water containing 3 kg of toxin. Pure water is pumped in at a rate of 40 L/min, mixes instantaneously, and is then pumped out at the same rate. Let $y(t)$ be the quantity of toxin present in the tank at time t.

(a) Find a differential equation satisfied by $y(t)$.

(b) Solve for $y(t)$.

(c) Find the time at which there is 0.01 kg of toxin present.

SOLUTION

(a) The net flow of toxin into or out of the tank at time t is

$$\frac{dy}{dt} = \text{toxin rate in} - \text{toxin rate out} = \left(40\frac{\text{L}}{\text{min}}\right)\left(0\frac{\text{kg}}{\text{L}}\right) - \left(40\frac{\text{L}}{\text{min}}\right)\left(\frac{y(t)}{300}\frac{\text{kg}}{\text{L}}\right)$$
$$= -\frac{2}{15}y(t)$$

(b) This is a linear differential equation. Putting it in standard form gives

$$\frac{dy}{dt} + \frac{2}{15}y = 0$$

The integrating factor is

$$\alpha(t) = e^{\int (2/15)\,dt} = e^{2t/15}$$

When multiplied by the integrating factor, the differential equation becomes

$$(e^{2t/15}y)' = 0$$

Integrate both sides and multiply through by $e^{-2t/15}$ to get

$$y = Ce^{-2t/15}$$

Since there are initially 3 kg of toxin present, $y(0) = 3$ so that $C = 3$. Finally, we have

$$y = 3e^{-2t/15}$$

(c) We solve for t:

$$0.01 = 3e^{-2t/15} \quad \Rightarrow \quad t = -\frac{5}{2}\ln 0.01 \approx 11.51$$

There is 0.01 kg of toxin in the tank after about 11 and a half minutes.

47. At $t = 0$, a tank of volume 300 L is filled with 100 L of water containing salt at a concentration of 8 g/L. Fresh water flows in at a rate of 40 L/min, mixes instantaneously, and exits at the same rate. Let $c_1(t)$ be the salt concentration at time t.

(a) Find a differential equation satisfied by $c_1(t)$ *Hint:* Find the differential equation for the quantity of salt $y(t)$, and observe that $c_1(t) = y(t)/100$.

(b) Find the salt concentration $c_1(t)$ in the tank as a function of time.

SOLUTION

(a) Let $y(t)$ be the amount of salt in the tank at time t; then $c_1(t) = y(t)/100$. The rate of change of the amount of salt in the tank is

$$\begin{aligned}\frac{dy}{dt} &= \text{salt rate in} - \text{salt rate out} = \left(40\frac{\text{L}}{\text{min}}\right)\left(0\frac{\text{g}}{\text{L}}\right) - \left(40\frac{\text{L}}{\text{min}}\right)\left(\frac{y}{100}\cdot\frac{\text{g}}{\text{L}}\right)\\ &= -\frac{2}{5}y\end{aligned}$$

Now, $c_1'(t) = y'(t)/100$ and $c(t) = y(t)/100$, so that c_1 satisfies the same differential equation:

$$\frac{dc_1}{dt} = -\frac{2}{5}c_1$$

(b) This is a linear differential equation. Putting it in standard form gives

$$\frac{dc_1}{dt} + \frac{2}{5}c_1 = 0$$

The integrating factor is $e^{2t/5}$; multiplying both sides by the integrating factor gives

$$(e^{2t/5}c_1)' = 0$$

Integrate and multiply through by $e^{-2t/5}$ to get

$$c_1(t) = Ce^{-2t/5}$$

The initial condition tells us that $y(0) = Ce^{-2\cdot 0/5} = C = 8$, so that finally,

$$c_1(t) = 8e^{-2t/5}$$

48. The outflow of the tank in Exercise 47 is directed into a second tank containing V liters of fresh water where it mixes instantaneously and exits at the same rate of 40 L/min. Determine the salt concentration $c_2(t)$ in the second tank as a function of time in the following two cases:

(a) $V = 200$

(b) $V = 300$

In each case, determine the maximum concentration.

SOLUTION Let $y_2(t)$ be the amount of salt in the second tank at time t; then $y_2(t) = c_2(t)V$ and $y_2'(t) = c_2'(t)V$. The rate of change in the amount of salt in the second tank is

$$\begin{aligned}\frac{dy_2}{dt} &= \text{salt rate in} - \text{salt rate out} = \left(40\frac{\text{L}}{\text{min}}\right)\left(c_1\frac{\text{g}}{\text{L}}\right) - \left(40\frac{\text{L}}{\text{min}}\right)\left(\frac{y_2}{V}\cdot\frac{\text{g}}{\text{L}}\right)\\ &= 40c_1 - \frac{40}{V}y_2\end{aligned}$$

Substituting for $y_2(t)$ and $y_2'(t)$ gives

$$Vc_2'(t) = 40c_1(t) - \frac{40}{V}Vc_2(t) \quad \text{so} \quad c_2'(t) = \frac{40}{V}(c_1(t) - c_2(t))$$

This is a linear differential equation; in standard form, it is

$$c_2'(t) + \frac{40}{V}c_2(t) = \frac{40}{V}c_1(t)$$

From the previous problem, we know that $c_1(t) = 8e^{-2t/5}$; substituting gives

$$c_2'(t) + \frac{40}{V}c_2(t) = \frac{320}{V}e^{-2t/5}$$

The integrating factor is $e^{40t/V}$; multiplying through by this factor gives

$$(e^{40t/V}c_2)' = \frac{320}{V}e^{(40t/V)-(2t/5)} = \frac{320}{V}e^{(200-2V)t/5V}$$

Integrate both sides to get

$$e^{40t/V}c_2 = \frac{320}{V} \cdot \frac{5V}{200-2V}e^{(200-2V)t/5V} + C = \frac{800}{100-V}e^{(200-2V)t/5V} + C$$

Multiply through by $e^{-40t/V}$ to get

$$c_2(t) = \frac{800}{100-V}e^{-2t/5} + Ce^{-40t/V}$$

Since tank 2 initially contains fresh water, $c_2(0) = 0$, so that $C = -\frac{800}{100-V}$ and

$$c_2(t) = \frac{800}{100-V}(e^{-2t/5} - e^{-40t/V})$$

(a) If $V = 200$, we have

$$c_2(t) = -8(e^{-2t/5} - e^{-t/5}) = 8(e^{-t/5} - e^{-2t/5})$$

The concentration of salt is at a maximum when $c_2'(t) = 0$:

$$\begin{aligned} 0 = c_2'(t) &= \frac{16}{5}e^{-2t/5} - \frac{8}{5}e^{-t/5} \\ e^{-t/5} &= 2e^{-2t/5} \\ -\frac{t}{5} &= -\frac{2t}{5} + \ln 2 \\ t &= 5\ln 2 \approx 3.47 \end{aligned}$$

so that the concentration of salt is at a maximum after about 3 and a half minutes.
(b) If $V = 300$, we have

$$c_2(t) = -4(e^{-2t/5} - e^{-2t/15}) = 4(e^{-2t/15} - e^{-2t/5})$$

The concentration of salt is at a maximum when $c_2'(t) = 0$:

$$\begin{aligned} 0 = c_2'(t) &= \frac{8}{5}e^{-2t/5} - \frac{8}{15}e^{-2t/15} \\ e^{-2t/15} &= 3e^{-2t/5} \\ -\frac{2}{15}t &= -\frac{2}{5}t + \ln 3 \\ t &= \frac{15}{4}\ln 3 \approx 4.12 \end{aligned}$$

so that the concentration of salt is at a maximum after about 4 minutes 7 seconds.