8 FURTHER APPLICATIONS OF THE INTEGRAL AND TAYLOR POLYNOMIALS

8.1 Arc Length and Surface Area

Preliminary Questions

1. Which integral represents the length of the curve $y = \cos x$ between 0 and π?

$$\int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx, \qquad \int_0^{\pi} \sqrt{1 + \sin^2 x}\, dx$$

SOLUTION Let $y = \cos x$. Then $y' = -\sin x$, and $1 + (y')^2 = 1 + \sin^2 x$. Thus, the length of the curve $y = \cos x$ between 0 and π is

$$\int_0^{\pi} \sqrt{1 + \sin^2 x}\, dx.$$

2. Use the formula for arc length to show that for any constant C, the graphs $y = f(x)$ and $y = f(x) + C$ have the same length over every interval $[a, b]$. Explain geometrically.

SOLUTION The graph of $y = f(x) + C$ is a vertical translation of the graph of $y = f(x)$; hence, the two graphs should have the same arc length. We can explicitly establish this as follows:

$$\text{length of } y = f(x) + C = \int_a^b \sqrt{1 + \left[\frac{d}{dx}(f(x) + C)\right]^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \text{length of } y = f(x).$$

3. Use the formula for arc length to show that the length of a graph over $[1, 4]$ cannot be less than 3.

SOLUTION Note that $f'(x)^2 \geq 0$, so that $\sqrt{1 + [f'(x)]^2} \geq \sqrt{1} = 1$. Then the arc length of the graph of $f(x)$ on $[1, 4]$ is

$$\int_1^4 \sqrt{1 + [f'(x)]^2}\, dx \geq \int_1^4 1\, dx = 3$$

Exercises

1. Express the arc length of the curve $y = x^4$ between $x = 2$ and $x = 6$ as an integral (but do not evaluate).

SOLUTION Let $y = x^4$. Then $y' = 4x^3$ and

$$s = \int_2^6 \sqrt{1 + (4x^3)^2}\, dx = \int_2^6 \sqrt{1 + 16x^6}\, dx.$$

2. Express the arc length of the curve $y = \tan x$ for $0 \leq x \leq \frac{\pi}{4}$ as an integral (but do not evaluate).

SOLUTION Let $y = \tan x$. Then $y' = \sec^2 x$, and

$$s = \int_0^{\pi/4} \sqrt{1 + (\sec^2 x)^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \sec^4 x}\, dx.$$

3. Find the arc length of $y = \frac{1}{12}x^3 + x^{-1}$ for $1 \leq x \leq 2$. *Hint:* Show that $1 + (y')^2 = \left(\frac{1}{4}x^2 + x^{-2}\right)^2$.

SOLUTION Let $y = \dfrac{1}{12}x^3 + x^{-1}$. Then $y' = \dfrac{x^2}{4}x^{-2}$, and

$$(y')^2 + 1 = \left(\frac{x^2}{4} - x^{-2}\right)^2 + 1 = \frac{x^4}{16} - \frac{1}{2} + x^{-4} + 1 = \frac{x^4}{16} + \frac{1}{2} + x^{-4} = \left(\frac{x^2}{4} + x^{-2}\right)^2.$$

Thus,

$$s = \int_1^2 \sqrt{1+(y')^2}\,dx = \int_1^2 \sqrt{\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2}\,dx = \int_1^2 \left|\frac{x^2}{4}+\frac{1}{x^2}\right|\,dx$$
$$= \int_1^2 \left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx \quad \text{since} \quad \frac{x^2}{4}+\frac{1}{x^2} > 0$$
$$= \left(\frac{x^3}{12}-\frac{1}{x}\right)\Big|_1^2 = \frac{13}{12}.$$

4. Find the arc length of $y = \left(\frac{x}{2}\right)^4 + \frac{1}{2x^2}$ over [1, 4]. *Hint:* Show that $1+(y')^2$ is a perfect square.

SOLUTION Let $y = \left(\frac{x}{2}\right)^4 + \frac{1}{2x^2}$. Then

$$y' = 4\left(\frac{x}{2}\right)^3\left(\frac{1}{2}\right) - \frac{1}{x^3} = \frac{x^3}{4} - \frac{1}{x^3}$$

and

$$(y')^2 + 1 = \left(\frac{x^3}{4}-\frac{1}{x^3}\right)^2 + 1 = \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6} + 1 = \frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6} = \left(\frac{x^3}{4}+\frac{1}{x^3}\right)^2.$$

Hence,

$$s = \int_1^4 \sqrt{1+y'^2}\,dx = \int_1^4 \sqrt{\left(\frac{x^3}{4}+\frac{1}{x^3}\right)^2}\,dx = \int_1^4 \left|\frac{x^3}{4}+\frac{1}{x^3}\right|\,dx$$
$$= \int_1^4 \left(\frac{x^3}{4}+\frac{1}{x^3}\right)dx \quad \text{since} \quad \frac{x^3}{4}+\frac{1}{x^3} > 0 \text{ on } [1,4]$$
$$= \left(\frac{x^4}{16}+\frac{x^{-2}}{-2}\right)\Big|_1^4 = \frac{525}{32}.$$

In Exercises 5–10, calculate the arc length over the given interval.

5. $y = 3x + 1$, $[0, 3]$

SOLUTION Let $y = 3x+1$. Then $y' = 3$, and $s = \int_0^3 \sqrt{1+9}\,dx = 3\sqrt{10}$.

6. $y = 9 - 3x$, $[1, 3]$

SOLUTION Let $y = 9 - 3x$. Then $y' = -3$, and $s = \int_1^3 \sqrt{1+9}\,dx = 3\sqrt{10} - \sqrt{10} = 2\sqrt{10}$.

7. $y = x^{3/2}$, $[1, 2]$

SOLUTION Let $y = x^{3/2}$. Then $y' = \frac{3}{2}x^{1/2}$, and

$$s = \int_1^2 \sqrt{1+\frac{9}{4}x}\,dx = \frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\Big|_1^2 = \frac{8}{27}\left(\left(\frac{11}{2}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right) = \frac{1}{27}\left(22\sqrt{22} - 13\sqrt{13}\right).$$

8. $y = \frac{1}{3}x^{3/2} - x^{1/2}$, $[2, 8]$

SOLUTION Let $y = \frac{1}{3}x^{3/2} - x^{1/2}$. Then

$$y' = \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2},$$

and

$$1 + (y')^2 = 1 + \left(\frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2}\right)^2 = \frac{1}{4}x + \frac{1}{2} + \frac{1}{4}x^{-1} = \left(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}\right)^2.$$

Hence,

$$\begin{aligned} s &= \int_2^8 \sqrt{1+(y')^2}\,dx = \int_2^8 \sqrt{\left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right)^2}\,dx = \int_2^8 \left|\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right|\,dx \\ &= \int_2^8 \left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right)dx \quad \text{since} \quad \frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2} > 0 \\ &= \left(\frac{1}{3}x^{3/2}+x^{1/2}\right)\Big|_2^8 = \frac{17\sqrt{2}}{3}. \end{aligned}$$

9. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x, \quad [1, 2e]$

SOLUTION Let $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$. Then

$$y' = \frac{x}{2} - \frac{1}{2x},$$

and

$$1+(y')^2 = 1 + \left(\frac{x}{2}-\frac{1}{2x}\right)^2 = \frac{x^2}{4}+\frac{1}{2}+\frac{1}{4x^2} = \left(\frac{x}{2}+\frac{1}{2x}\right)^2.$$

Hence,

$$\begin{aligned} s &= \int_1^{2e} \sqrt{1+(y')^2}\,dx = \int_1^{2e} \sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\,dx = \int_1^{2e} \left|\frac{x}{2}+\frac{1}{2x}\right|\,dx \\ &= \int_1^{2e} \left(\frac{x}{2}+\frac{1}{2x}\right)dx \quad \text{since} \quad \frac{x}{2}+\frac{1}{2x} > 0 \text{ on } [1, 2e] \\ &= \left(\frac{x^2}{4}+\frac{1}{2}\ln x\right)\Big|_1^{2e} = e^2 + \frac{\ln 2}{2} + \frac{1}{4}. \end{aligned}$$

10. $y = \ln(\cos x), \quad \left[0, \frac{\pi}{4}\right]$

SOLUTION Let $y = \ln(\cos x)$. Then $y' = -\tan x$ and $1+(y')^2 = 1+\tan^2 x = \sec^2 x$. Hence,

$$\begin{aligned} s &= \int_0^{\pi/4} \sqrt{1+(y')^2}\,dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/4} |\sec x|\,dx \\ &= \int_0^{\pi/4} \sec x\,dx \quad \text{since} \quad \sec x > 0 \text{ on } \left[0, \frac{\pi}{4}\right] \\ &= \ln|\sec x + \tan x|\Big|_0^{\pi/4} = \ln(\sqrt{2}+1). \end{aligned}$$

In Exercises 11–14, approximate the arc length of the curve over the interval using the Trapezoidal Rule T_N, the Midpoint Rule M_N, or Simpson's Rule S_N as indicated.

11. $y = \frac{1}{4}x^4, \quad [1, 2], \quad T_5$

SOLUTION Let $y = \frac{1}{4}x^4$. Then

$$1+(y')^2 = 1+(x^3)^2 = 1+x^6.$$

Therefore, the arc length over [1, 2] is

$$\int_1^2 \sqrt{1+x^6}\,dx.$$

Now, let $f(x) = \sqrt{1+x^6}$. With $n = 5$,

$$\Delta x = \frac{2-1}{5} = \frac{1}{5} \quad \text{and} \quad \{x_i\}_{i=0}^5 = \left\{1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2\right\}.$$

Using the Trapezoidal Rule,

$$\int_1^2 \sqrt{1+x^6}\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2\sum_{i=1}^{4} f(x_i) + f(x_5)\right] = 3.957736.$$

The arc length is approximately 3.957736 units.

12. $y = \sin x$, $\left[0, \frac{\pi}{2}\right]$, M_8

SOLUTION Let $y = \sin x$. Then

$$1 + y'^2 = 1 + \cos^2 x.$$

Therefore, the arc length over $[0, \pi/2]$ is

$$\int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx.$$

Now, let $f(x) = \sqrt{1+\cos^2 x}$. With $n = 8$, we have:

$$\Delta x = \frac{\pi/2}{8} = \frac{\pi}{16} \quad \text{and} \quad \{x_i^*\}_{i=1}^{8} = \left\{\frac{\pi}{32}, \frac{3\pi}{32}, \frac{5\pi}{32}, \frac{7\pi}{32}, \frac{9\pi}{32}, \frac{11\pi}{32}, \frac{13\pi}{32}, \frac{15\pi}{32}\right\}.$$

Using the Midpoint Rule,

$$\int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx \approx \Delta x \sum_{i=1}^{8} f(x_i^*) = 1.910099.$$

The arc length is approximately 1.910099 units.

13. $y = x^{-1}$, $[1, 2]$, S_8

SOLUTION Let $y = x^{-1}$. Then $y' = -x^{-2}$ and

$$1 + (y')^2 = 1 + \frac{1}{x^4}.$$

Therefore, the arc length over $[1, 2]$ is

$$\int_1^2 \sqrt{1+\frac{1}{x^4}}\,dx.$$

Now, let $f(x) = \sqrt{1+\frac{1}{x^4}}$. With $n = 8$,

$$\Delta x = \frac{2-1}{8} = \frac{1}{8} \quad \text{and} \quad \{x_i\}_{i=0}^{8} = \left\{1, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}, \frac{3}{2}, \frac{13}{8}, \frac{7}{4}, \frac{15}{8}, 2\right\}.$$

Using Simpson's Rule,

$$\int_1^2 \sqrt{1+\frac{1}{x^4}}\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4\sum_{i=1}^{4} f(x_{2i-1}) + 2\sum_{i=1}^{3} f(x_{2i}) + f(x_8)\right] = 1.132123.$$

The arc length is approximately 1.132123 units.

14. $y = e^{-x^2}$, $[0, 2]$, S_8

SOLUTION Let $y = e^{-x^2}$. Then

$$1 + (y')^2 = 1 + 4x^2 e^{-2x^2}.$$

Therefore, the arc length over $[0, 2]$ is

$$\int_0^2 \sqrt{1+4x^2 e^{-2x^2}}\,dx.$$

Now, let $f(x) = \sqrt{1+4x^2 e^{-2x^2}}$. With $n = 8$,

$$\Delta x = \frac{2-0}{8} = \frac{1}{4} \quad \text{and} \quad \{x_i\}_{i=0}^{8} = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\right\}.$$

Using Simpson's Rule,

$$\int_0^2 \sqrt{1+4x^2e^{-2x^2}}\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4\sum_{i=1}^{4} f(x_{2i-1})+2\sum_{i=1}^{3} f(x_{2i})+f(x_8)\right] = 2.280718.$$

The arc length is approximately 2.280718 units.

15. Calculate the length of the astroid $x^{2/3}+y^{2/3}=1$ (Figure 11).

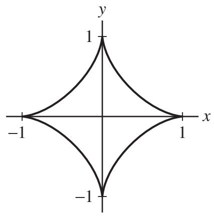


FIGURE 11 Graph of $x^{2/3}+y^{2/3}=1$.

SOLUTION We will calculate the arc length of the portion of the asteroid in the first quadrant and then multiply by 4. By implicit differentiation

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0,$$

so

$$y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}.$$

Thus

$$1+(y')^2 = 1+\frac{y^{2/3}}{x^{2/3}} = \frac{x^{2/3}+y^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}},$$

and

$$s=\int_0^1 \frac{1}{x^{1/3}}\,dx = \frac{3}{2}.$$

The total arc length is therefore $4\cdot\frac{3}{2}=6$.

16. Show that the arc length of the asteroid $x^{2/3}+y^{2/3}=a^{2/3}$ (for $a>0$) is proportional to a.

SOLUTION We will calculate the arc length of the portion of the asteroid in the first quadrant and then multiply by 4. By implicit differentiation

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0,$$

so

$$y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}.$$

Thus

$$1+(y')^2 = 1+\frac{y^{2/3}}{x^{2/3}} = \frac{x^{2/3}+y^{2/3}}{x^{2/3}} = \frac{a^{2/3}}{x^{2/3}},$$

and

$$s=\int_0^a \frac{a^{1/3}}{x^{1/3}}\,dx = a^{1/3}\left(\frac{3}{2}a^{2/3}\right) = \frac{3}{2}a.$$

The total arc length is therefore $4\cdot\frac{3}{2}a=6a$, which is proportional to a.

17. Let $a, r>0$. Show that the arc length of the curve $x^r+y^r=a^r$ for $0\le x\le a$ is proportional to a.

SOLUTION Using implicit differentiation, we find $y'=-(x/y)^{r-1}$ and

$$1+(y')^2 = 1+(x/y)^{2r-2} = \frac{x^{2r-1}+y^{2r-2}}{y^{2r-2}} = \frac{x^{2r-2}+(a^r-x^r)^{2-2/r}}{(a^r-x^r)^{2-2/r}}.$$

The arc length is then

$$s = \int_0^a \sqrt{\frac{x^{2r-2} + (a^r - x^r)^{2-2/r}}{(a^r - x^r)^{2-2/r}}}\, dx.$$

Using the substitution $x = au$, we obtain

$$s = a\int_0^1 \sqrt{\frac{u^{2r-2} + (1 - u^r)^{2-2/r}}{(1 - u^r)^{2-2/r}}}\, du,$$

where the integral is independent of a.

18. Find the arc length of the curve shown in Figure 12.

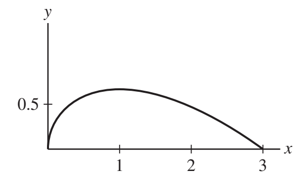


FIGURE 12 Graph of $9y^2 = x(x-3)^2$.

SOLUTION Using implicit differentiation,

$$18yy' = x(2)(x-3) + (x-3)^2 = 3(x-3)(x-1)$$

Hence,

$$(y')^2 = \frac{(x-3)^2(x-1)^2}{36y^2} = \frac{(x-3)^2(x-1)^2}{4(9y^2)} = \frac{(x-3)^2(x-1)^2}{4x(x-3)^2} = \frac{(x-1)^2}{4x}$$

and

$$1 + (y')^2 = \frac{(x-1)^2 + 4x}{4x} = \frac{(x+1)^2}{4x}.$$

Finally,

$$s = \int_0^3 \sqrt{\frac{(x+1)^2}{4x}}\, dx = \int_0^3 \frac{|x+1|}{2\sqrt{x}}\, dx$$

$$= \int_0^3 \frac{x+1}{2\sqrt{x}}\, dx \quad \text{since} \quad x+1 > 0 \text{ on } [0,3]$$

$$= \int_0^3 \left(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}\right) dx = \left(\frac{1}{3}x^{3/2} + x^{1/2}\right)\Big|_0^3 = 2\sqrt{3}.$$

19. Find the value of a such that the arc length of the *catenary* $y = \cosh x$ for $-a \le x \le a$ equals 10.

SOLUTION Let $y = \cosh x$. Then $y' = \sinh x$ and

$$1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x.$$

Thus,

$$s = \int_{-a}^{a} \cosh x\, dx = \sinh(a) - \sinh(-a) = 2\sinh a.$$

Setting this expression equal to 10 and solving for a yields $a = \sinh^{-1}(5) = \ln(5 + \sqrt{26})$.

20. Calculate the arc length of the graph of $f(x) = mx + r$ over $[a, b]$ in two ways: using the Pythagorean theorem (Figure 13) and using the arc length integral.

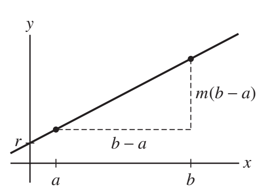


FIGURE 13

SOLUTION Let h denote the length of the hypotenuse. Then, by Pythagoras' Theorem,

$$h^2 = (b-a)^2 + m^2(b-a)^2 = (b-a)^2(1+m^2),$$

or

$$h = (b-a)\sqrt{1+m^2}$$

since $b > a$. Moreover, $(f'(x))^2 = m^2$, so

$$s = \int_a^b \sqrt{1+m^2}\,dx = (b-a)\sqrt{1+m^2} = h.$$

21. Show that the circumference of the unit circle is equal to

$$2\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \quad \text{(an improper integral)}$$

Evaluate, thus verifying that the circumference is 2π.

SOLUTION Note the circumference of the unit circle is twice the arc length of the upper half of the curve defined by $x^2 + y^2 = 1$. Thus, let $y = \sqrt{1-x^2}$. Then

$$y' = -\frac{x}{\sqrt{1-x^2}} \quad \text{and} \quad 1 + (y')^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}.$$

Finally, the circumference of the unit circle is

$$2\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 2\sin^{-1} x\Big|_{-1}^{1} = \pi - (-\pi) = 2\pi.$$

22. Generalize the result of Exercise 21 to show that the circumference of the circle of radius r is $2\pi r$.

SOLUTION Let $y = \sqrt{r^2 - x^2}$ denote the upper half of a circle of radius r centered at the origin. Then

$$1 + (y')^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2} = \frac{1}{1 - \frac{x^2}{r^2}},$$

and the circumference of the circle is given by

$$C = 2\int_{-r}^{r} \frac{dx}{\sqrt{1 - x^2/r^2}}.$$

Using the substitution $u = x/r$, $du = dx/r$, we find

$$C = 2r\int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = 2r\sin^{-1} u\Big|_{-1}^{1}$$
$$= 2r\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 2\pi r$$

23. Calculate the arc length of $y = x^2$ over $[0, a]$. *Hint:* Use trigonometric substitution. Evaluate for $a = 1$.

SOLUTION Let $y = x^2$. Then $y' = 2x$ and

$$s = \int_0^a \sqrt{1+4x^2}\,dx.$$

Using the substitution $2x = \tan\theta$, $2\,dx = \sec^2\theta\,d\theta$, we find

$$s = \frac{1}{2}\int_{x=0}^{x=a} \sec^3\theta\,d\theta.$$

Next, using a reduction formula for the integral of $\sec^3\theta$, we see that

$$s = \left(\frac{1}{4}\sec\theta\tan\theta + \frac{1}{4}\ln|\sec\theta + \tan\theta|\right)\Big|_{x=0}^{x=a} = \left(\frac{1}{2}x\sqrt{1+4x^2} + \frac{1}{4}\ln|\sqrt{1+4x^2} + 2x|\right)\Big|_0^a$$
$$= \frac{a}{2}\sqrt{1+4a^2} + \frac{1}{4}\ln|\sqrt{1+4a^2} + 2a|$$

Thus, when $a = 1$,

$$s = \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(\sqrt{5}+2) \approx 1.478943.$$

24. Express the arc length of $g(x) = \sqrt{x}$ over [0, 1] as a definite integral. Then use the substitution $u = \sqrt{x}$ to show that this arc length is equal to the arc length of x^2 over [0, 1] (but do not evaluate the integrals). Explain this result graphically.

SOLUTION Let $g(x) = \sqrt{x}$. Then

$$1 + g'(x)^2 = \frac{1+4x}{4x} \quad \text{and} \quad s = \int_0^1 \sqrt{\frac{1+4x}{4x}}\,dx = \int_0^1 \frac{\sqrt{1+4x}}{2\sqrt{x}}\,dx.$$

With the substitution $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$, this becomes

$$s = \int_0^1 \sqrt{1+4u^2}\,du.$$

Now, let $f(x) = x^2$. Then $1 + f'(x)^2 = 1 + 4x^2$, and

$$s = \int_0^1 \sqrt{1+4x^2}\,dx.$$

Thus, the two arc lengths are equal. This is explained graphically by the fact that for $x \geq 0$, x^2 and $\sqrt{x}$ are inverses of each other. This means that the two graphs are symmetric with respect to the line $y = x$. Moreover, the graphs of x^2 and $\sqrt{x}$ intersect at $x = 0$ and at $x = 1$. Thus, it is clear that the arc length of the two graphs on [0, 1] are equal.

25. Find the arc length of $y = e^x$ over $[0, a]$. *Hint:* Try the substitution $u = \sqrt{1+e^{2x}}$ followed by partial fractions.

SOLUTION Let $y = e^x$. Then $1 + (y')^2 = 1 + e^{2x}$, and the arc length over $[0, a]$ is

$$\int_0^a \sqrt{1+e^{2x}}\,dx.$$

Now, let $u = \sqrt{1+e^{2x}}$. Then

$$du = \frac{1}{2}\cdot\frac{2e^{2x}}{\sqrt{1+e^{2x}}}\,dx = \frac{u^2-1}{u}\,dx$$

and the arc length is

$$\begin{aligned}
\int_0^a \sqrt{1+e^{2x}}\,dx &= \int_{x=0}^{x=a} u\cdot\frac{u}{u^2-1}\,du = \int_{x=0}^{x=a}\frac{u^2}{u^2-1}\,du = \int_{x=0}^{x=a}\left(1+\frac{1}{u^2-1}\right)du \\
&= \int_{x=0}^{x=a}\left(1+\frac{1}{2}\frac{1}{u-1}-\frac{1}{2}\frac{1}{u+1}\right)du = \left(u+\frac{1}{2}\ln(u-1)-\frac{1}{2}\ln(u+1)\right)\Bigg|_{x=0}^{x=a} \\
&= \left[\sqrt{1+e^{2x}}+\frac{1}{2}\ln\left(\frac{\sqrt{1+e^{2x}}-1}{\sqrt{1+e^{2x}}+1}\right)\right]\Bigg|_0^a \\
&= \sqrt{1+e^{2a}}+\frac{1}{2}\ln\frac{\sqrt{1+e^{2a}}-1}{\sqrt{1+e^{2a}}+1}-\sqrt{2}+\frac{1}{2}\ln\frac{1+\sqrt{2}}{\sqrt{2}-1} \\
&= \sqrt{1+e^{2a}}+\frac{1}{2}\ln\frac{\sqrt{1+e^{2a}}-1}{\sqrt{1+e^{2a}}+1}-\sqrt{2}+\ln(1+\sqrt{2}).
\end{aligned}$$

26. Show that the arc length of $y = \ln(f(x))$ for $a \leq x \leq b$ is

$$\int_a^b \frac{\sqrt{f(x)^2+f'(x)^2}}{f(x)}\,dx$$ 4

SOLUTION Let $y = \ln(f(x))$. Then

$$y' = \frac{f'(x)}{f(x)} \qquad \text{and} \qquad 1+(y')^2 = \frac{f(x)^2+f'(x)^2}{f(x)^2}.$$

Therefore,

$$s = \int_a^b \frac{\sqrt{f(x)^2 + f'(x)^2}}{f(x)}\,dx$$

since $f(x) > 0$ in order for $y = \ln(f(x))$ to be defined on $[a, b]$.

27. Use Eq. (4) to compute the arc length of $y = \ln(\sin x)$ for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$.

SOLUTION With $f(x) = \sin x$, Eq. (4) yields

$$s = \int_{\pi/4}^{\pi/2} \frac{\sqrt{\sin^2 x + \cos^2 x}}{\sin x}\,dx = \int_{\pi/4}^{\pi/2} \csc x\,dx = \ln(\csc x - \cot x)\Big|_{\pi/4}^{\pi/2}$$

$$= \ln 1 - \ln(\sqrt{2} - 1) = \ln \frac{1}{\sqrt{2} - 1} = \ln(\sqrt{2} + 1).$$

28. Use Eq. (4) to compute the arc length of $y = \ln\left(\frac{e^x + 1}{e^x - 1}\right)$ over $[1, 3]$.

SOLUTION With $f(x) = \frac{e^x + 1}{e^x - 1}$,

$$f'(x) = \frac{(e^x - 1)e^x - (e^x + 1)e^x}{(e^x - 1)^2} = -\frac{2e^x}{(e^x - 1)^2}$$

and

$$f(x)^2 + f'(x)^2 = \left(\frac{e^x + 1}{e^x - 1}\right)^2 + \frac{4e^{2x}}{(e^x - 1)^4} = \frac{(e^{2x} - 1)^2 + 4e^{2x}}{(e^x - 1)^4} = \frac{(e^{2x} + 1)^2}{(e^x - 1)^4}.$$

Thus, by Eq. (4),

$$s = \int_1^3 \frac{e^{2x} + 1}{(e^x - 1)^2} \cdot \frac{e^x - 1}{e^x + 1}\,dx = \int_1^3 \frac{e^{2x} + 1}{e^{2x} - 1}\,dx.$$

Observe that

$$\frac{e^{2x} + 1}{e^{2x} - 1} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{(e^x + e^{-x})/2}{(e^x - e^{-x})/2} = \frac{\cosh x}{\sinh x}.$$

Therefore,

$$s = \int_1^3 \frac{\cosh x}{\sinh x}\,dx = \ln(\sinh x)\Big|_1^3 = \ln(\sinh 3) - \ln(\sinh 1).$$

29. Show that if $0 \le f'(x) \le 1$ for all x, then the arc length of $y = f(x)$ over $[a, b]$ is at most $\sqrt{2}(b - a)$. Show that for $f(x) = x$, the arc length equals $\sqrt{2}(b - a)$.

SOLUTION If $0 \le f'(x) \le 1$ for all x, then

$$s = \int_a^b \sqrt{1 + f'(x)^2}\,dx \le \int_a^b \sqrt{1 + 1}\,dx = \sqrt{2}(b - a).$$

If $f(x) = x$, then $f'(x) = 1$ and

$$s = \int_a^b \sqrt{1 + 1}\,dx = \sqrt{2}(b - a).$$

30. Use the Comparison Theorem (Section 5.2) to prove that the arc length of $y = x^{4/3}$ over $[1, 2]$ is not less than $\frac{5}{3}$.

SOLUTION Note that $f'(x) = \frac{4}{3}x^{1/3}$; for $x \in [1, 2]$, we have $x^{1/3} \ge 1$ so that $f'(x) \ge \frac{4}{3}$. Then

$$\sqrt{1 + f'(x)^2} \ge \sqrt{1 + \left(\frac{4}{3}\right)^2} = \sqrt{\frac{25}{9}} = \frac{5}{3}$$

and then the arc length is

$$\int_1^2 \sqrt{1 + f'(x)^2}\,dx \ge \int_1^2 \frac{5}{3}\,dx = \frac{5}{3}$$

31. Approximate the arc length of one-quarter of the unit circle (which we know is $\frac{\pi}{2}$) by computing the length of the polygonal approximation with $N = 4$ segments (Figure 14).

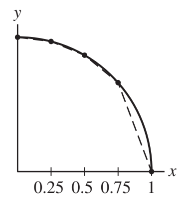


FIGURE 14 One-quarter of the unit circle

SOLUTION With $y = \sqrt{1 - x^2}$, the five points along the curve are

$$P_0(0, 1),\ P_1(1/4, \sqrt{15}/4),\ P_2(1/2, \sqrt{3}/2),\ P_3(3/4, \sqrt{7}/4),\ P_4(1, 0)$$

Then

$$\overline{P_0P_1} = \sqrt{\frac{1}{16} + \left(\frac{4 - \sqrt{15}}{4}\right)^2} \approx 0.252009$$

$$\overline{P_1P_2} = \sqrt{\frac{1}{16} + \left(\frac{2\sqrt{3} - \sqrt{15}}{4}\right)^2} \approx 0.270091$$

$$\overline{P_2P_3} = \sqrt{\frac{1}{16} + \left(\frac{2\sqrt{3} - \sqrt{7}}{4}\right)^2} \approx 0.323042$$

$$\overline{P_3P_4} = \sqrt{\frac{1}{16} + \frac{7}{16}} \approx 0.707108$$

and the total approximate distance is 1.552250 whereas $\pi/2 \approx 1.570796$.

32. CAS A merchant intends to produce specialty carpets in the shape of the region in Figure 15, bounded by the axes and graph of $y = 1 - x^n$ (units in yards). Assume that material costs \$50/yd^2 and that it costs $50L$ dollars to cut the carpet, where L is the length of the curved side of the carpet. The carpet can be sold for $150A$ dollars, where A is the carpet's area. Using numerical integration with a computer algebra system, find the whole number n for which the merchant's profits are maximal.

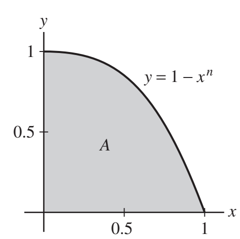


FIGURE 15

SOLUTION The area of the carpet is

$$A = \int_0^1 (1 - x^n)dx = \left(x - \frac{x^{n+1}}{n+1}\right)\Big|_0^1 = 1 - \frac{1}{n+1} = \frac{n}{n+1},$$

while the length of the curved side of the carpet is

$$L = \int_0^1 \sqrt{1 + (nx^{n-1})^2}\,dx = \int_0^1 \sqrt{1 + n^2x^{2n-2}}\,dx.$$

Using these formulas, we find that the merchant's profit is given by

$$150A - (50A + 50L) = 100A - 50L = \frac{100n}{n+1} - 50\int_0^1 \sqrt{1 + n^2x^{2n-2}}\,dx.$$

Using a CAS, we find that the merchant's profit is maximized (approximately \$3.31 per carpet) when $n = 13$. The table below lists the profit for $1 \le n \le 15$.

n	Profit	n	Profit
1	−20.71067810	9	3.06855532
2	−7.28047621	10	3.18862208
3	−2.39328273	11	3.25953632
4	−0.01147138	12	3.29668137
5	1.30534545	13	3.31024566
6	2.08684099	14	3.30715476
7	2.57017349	15	3.29222024
8	2.87535925		

In Exercises 33–40, compute the surface area of revolution about the x-axis over the interval.

33. $y = x$, $[0, 4]$

SOLUTION $1 + (y')^2 = 2$ so that

$$SA = 2\pi \int_0^4 x\sqrt{2}\,dx = 2\pi\sqrt{2}\frac{1}{2}x^2\Big|_0^4 = 16\pi\sqrt{2}$$

34. $y = 4x + 3$, $[0, 1]$

SOLUTION Let $y = 4x + 3$. Then $1 + (y')^2 = 17$ and

$$SA = 2\pi \int_0^1 (4x+3)\sqrt{17}\,dx = 2\pi\sqrt{17}\left(2x^2 + 3x\right)\Big|_0^1 = 10\pi\sqrt{17}.$$

35. $y = x^3$, $[0, 2]$

SOLUTION $1 + (y')^2 = 1 + 9x^4$, so that

$$SA = 2\pi \int_0^2 x^3\sqrt{1+9x^4}\,dx = \frac{2\pi}{36}\int_0^2 36x^3\sqrt{1+9x^4}\,dx = \frac{\pi}{18}(1+9x^4)^{3/2}\Big|_0^2 = \frac{\pi}{18}\left(145^{3/2} - 1\right)$$

36. $y = x^2$, $[0, 4]$

SOLUTION Let $y = x^2$. Then $y' = 2x$ and

$$SA = 2\pi \int_0^4 x^2\sqrt{1+4x^2}\,dx.$$

Using the substitution $2x = \tan\theta$, $2\,dx = \sec^2\theta\,d\theta$, we find that

$$\begin{aligned}\int x^2\sqrt{1+4x^2}\,dx &= \frac{1}{8}\int \sec^3\theta\tan^2\theta\,d\theta = \frac{1}{8}\int\left(\sec^5\theta - \sec^3\theta\right)d\theta\\ &= \frac{1}{8}\left(\frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}\sec\theta\tan\theta + \frac{3}{8}\ln|\sec\theta+\tan\theta| - \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta|\right) + C\\ &= \frac{x}{16}(1+4x^2)^{3/2} - \frac{x}{32}\sqrt{1+4x^2} - \frac{1}{64}\ln|\sqrt{1+4x^2}+2x| + C.\end{aligned}$$

Finally,

$$\begin{aligned}SA &= 2\pi\left(\frac{x}{16}(1+4x^2)^{3/2} - \frac{x}{32}\sqrt{1+4x^2} - \frac{1}{64}\ln|\sqrt{1+4x^2}+2x|\right)\Big|_0^4\\ &= 2\pi\left(\frac{1}{4}65^{3/2} - \frac{\sqrt{65}}{8} - \frac{1}{64}\ln(8+\sqrt{65})\right) = \frac{129\sqrt{65}}{4}\pi - \frac{\pi}{32}\ln(8+\sqrt{65}).\end{aligned}$$

37. $y = (4 - x^{2/3})^{3/2}$, $[0, 8]$

SOLUTION Let $y = (4 - x^{2/3})^{3/2}$. Then

$$y' = -x^{-1/3}(4 - x^{2/3})^{1/2},$$

and

$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}.$$

Therefore,

$$SA = 2\pi \int_0^8 (4 - x^{2/3})^{3/2} \left(\frac{2}{x^{1/3}}\right) dx.$$

Using the substitution $u = 4 - x^{2/3}$, $du = -\frac{2}{3}x^{-1/3}\,dx$, we find

$$SA = 2\pi \int_4^0 u^{3/2}(-3)\,du = 6\pi \int_0^4 u^{3/2}\,du = \frac{12}{5}\pi u^{5/2}\Big|_0^4 = \frac{384\pi}{5}.$$

38. $y = e^{-x}$, $[0, 1]$

SOLUTION Let $y = e^{-x}$. Then $y' = -e^{-x}$ and

$$SA = 2\pi \int_0^1 e^{-x}\sqrt{1 + e^{-2x}}\,dx.$$

Using the substitution $e^{-x} = \tan\theta$, $-e^{-x}\,dx = \sec^2\theta\,d\theta$, we find that

$$\begin{aligned}\int e^{-x}\sqrt{1+e^{-2x}}\,dx &= -\int \sec^3\theta\,d\theta = -\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C\\ &= -\frac{1}{2}e^{-x}\sqrt{1+e^{-2x}} - \frac{1}{2}\ln|\sqrt{1+e^{-2x}} + e^{-x}| + C.\end{aligned}$$

Finally,

$$\begin{aligned}SA &= \left(-\pi e^{-x}\sqrt{1+e^{-2x}} - \pi\ln|\sqrt{1+e^{-2x}} + e^{-x}|\right)\Big|_0^1\\ &= -\pi e^{-1}\sqrt{1+e^{-2}} - \pi\ln(\sqrt{1+e^{-2}} + e^{-1}) + \pi\sqrt{2} + \pi\ln(\sqrt{2}+1)\\ &= \pi\sqrt{2} - \pi e^{-1}\sqrt{1+e^{-2}} + \pi\ln\left(\frac{\sqrt{2}+1}{\sqrt{1+e^{-2}} + e^{-1}}\right).\end{aligned}$$

39. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $[1, e]$

SOLUTION We have $y' = \frac{x}{2} - \frac{1}{2x}$, and

$$1 + (y')^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2.$$

Thus,

$$\begin{aligned}SA &= 2\pi \int_1^e \left(\frac{x^2}{4} - \frac{\ln x}{2}\right)\left(\frac{x}{2} + \frac{1}{2x}\right) dx = 2\pi \int_1^e \frac{x^3}{8} + \frac{x}{8} - \frac{x\ln x}{4} - \frac{\ln x}{4x}\,dx\\ &= 2\pi\left(\frac{x^4}{32} + \frac{x^2}{16} - \frac{x^2\ln x}{8} + \frac{x^2}{16} - \frac{(\ln x)^2}{8}\right)\Big|_1^e\\ &= 2\pi\left(\frac{e^4}{32} + \frac{e^2}{16} - \frac{e^2}{8} + \frac{e^2}{16} - \frac{1}{8} - \left(\frac{1}{32} + \frac{1}{16} + 0 + \frac{1}{16} - 0\right)\right)\\ &= 2\pi\left(\frac{e^4}{32} - \frac{1}{8} - \frac{1}{32} - \frac{1}{16} - \frac{1}{16}\right)\\ &= \frac{\pi}{16}(e^4 - 9)\end{aligned}$$

40. $y = \sin x$, $[0, \pi]$

SOLUTION Let $y = \sin x$. Then $y' = \cos x$, and

$$SA = 2\pi \int_0^\pi \sin x\sqrt{1 + \cos^2 x}\,dx.$$

Using the substitution $\cos x = \tan\theta$, $-\sin x\,dx = \sec^2\theta\,d\theta$, we find that

$$\int \sin x\sqrt{1+\cos^2 x}\,dx = -\int \sec^3\theta\,d\theta = -\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= -\frac{1}{2}\cos x\sqrt{1+\cos^2 x} - \frac{1}{2}\ln|\sqrt{1+\cos^2 x}+\cos x| + C.$$

Finally,

$$SA = 2\pi\left(-\frac{1}{2}\cos x\sqrt{1+\cos^2 x} - \frac{1}{2}\ln|\sqrt{1+\cos^2 x}+\cos x|\right)\Big|_0^{\pi}$$

$$= 2\pi\left(\frac{1}{2}\sqrt{2} - \frac{1}{2}\ln(\sqrt{2}-1) + \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(\sqrt{2}+1)\right) = 2\pi\left(\sqrt{2}+\ln(\sqrt{2}+1)\right).$$

CAS *In Exercises 41–44, use a computer algebra system to find the approximate surface area of the solid generated by rotating the curve about the x-axis.*

41. $y = x^{-1}$, $[1, 3]$

SOLUTION

$$SA = 2\pi\int_1^3 \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx = 2\pi\int_1^3 \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \approx 7.603062807$$

using Maple.

42. $y = x^4$, $[0, 1]$

SOLUTION

$$SA = 2\pi\int_0^1 x^4\sqrt{1+(4x^3)^2}\,dx = 2\pi\int_0^1 x^4\sqrt{1+16x^6}\,dx \approx 3.436526697$$

using Maple.

43. $y = e^{-x^2/2}$, $[0, 2]$

SOLUTION

$$SA = 2\pi\int_0^2 e^{-x^2/2}\sqrt{1+(-xe^{-x^2/2})^2}\,dx = 2\pi\int_0^2 e^{-x^2/2}\sqrt{1+x^2e^{-x^2}}\,dx \approx 8.222695606$$

using Maple.

44. $y = \tan x$, $\left[0, \frac{\pi}{4}\right]$

SOLUTION Let $y = \tan x$. Then $y' = \sec^2 x$, $1+(y')^2 = 1+\sec^4 x$, and

$$SA = 2\pi\int_0^{\pi/4} \tan x\sqrt{1+\sec^4 x}\,dx.$$

Using a computer algebra system to approximate the value of the definite integral, we find

$$SA \approx 3.83908.$$

45. Find the area of the surface obtained by rotating $y = \cosh x$ over $[-\ln 2, \ln 2]$ around the x-axis.

SOLUTION Let $y = \cosh x$. Then $y' = \sinh x$, and

$$\sqrt{1+(y')^2} = \sqrt{1+\sinh^2 x} = \sqrt{\cosh^2 x} = \cosh x.$$

Therefore,

$$SA = 2\pi\int_{-\ln 2}^{\ln 2}\cosh^2 x\,dx = \pi\int_{-\ln 2}^{\ln 2}(1+\cosh 2x)\,dx = \pi\left(x+\frac{1}{2}\sinh 2x\right)\Big|_{-\ln 2}^{\ln 2}$$

$$= \pi\left(\ln 2 + \frac{1}{2}\sinh(2\ln 2) + \ln 2 - \frac{1}{2}\sinh(-2\ln 2)\right) = 2\pi\ln 2 + \pi\sinh(2\ln 2).$$

We can simplify this answer by recognizing that

$$\sinh(2\ln 2) = \frac{e^{2\ln 2} - e^{-2\ln 2}}{2} = \frac{4-\frac{1}{4}}{2} = \frac{15}{8}.$$

Thus,

$$SA = 2\pi\ln 2 + \frac{15\pi}{8}.$$

46. Show that the surface area of a spherical cap of height h and radius R (Figure 16) has surface area $2\pi Rh$.

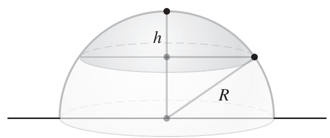


FIGURE 16

SOLUTION To determine the surface area of the cap, we will rotate a portion of a circle of radius R, centered at the origin, about the y-axis. Since the equation of the right half of the circle is $x = \sqrt{R^2 - y^2}$,

$$1 + (x')^2 = 1 + \frac{y^2}{R^2 - y^2} = \frac{R^2}{R^2 - y^2},$$

and

$$SA = 2\pi \int_{R-h}^{R} \sqrt{R^2 - y^2} \left(\frac{R}{\sqrt{R^2 - y^2}} \right) dy = 2\pi R\,(R - (R - h)) = 2\pi Rh.$$

47. Find the surface area of the torus obtained by rotating the circle $x^2 + (y - b)^2 = a^2$ about the x-axis (Figure 17).

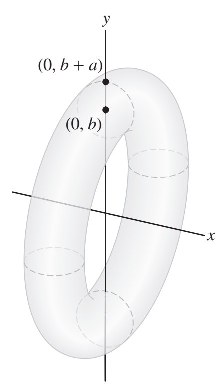


FIGURE 17 Torus obtained by rotating a circle about the x-axis.

SOLUTION $y = b + \sqrt{a^2 - x^2}$ gives the top half of the circle and $y = b - \sqrt{a^2 - x^2}$ gives the bottom half. Note that in each case,

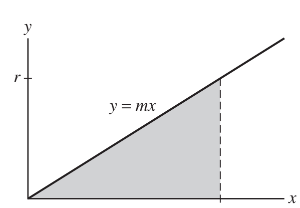
$$1 + (y')^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}.$$

Rotating the two halves of the circle around the x-axis then yields

$$\begin{aligned} SA &= 2\pi \int_{-a}^{a} (b + \sqrt{a^2 - x^2}) \frac{a}{\sqrt{a^2 - x^2}}\,dx + 2\pi \int_{-a}^{a} (b - \sqrt{a^2 - x^2}) \frac{a}{\sqrt{a^2 - x^2}}\,dx \\ &= 2\pi \int_{-a}^{a} 2b \frac{a}{\sqrt{a^2 - x^2}}\,dx = 4\pi ba \int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}}\,dx \\ &= 4\pi ba \cdot \sin^{-1}\left(\frac{x}{a}\right)\Big|_{-a}^{a} = 4\pi ba \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = 4\pi^2 ba. \end{aligned}$$

48. Show that the surface area of a right circular cone of radius r and height h is $\pi r \sqrt{r^2 + h^2}$. *Hint:* Rotate a line $y = mx$ about the x-axis for $0 \le x \le h$, where m is determined suitably by the radius r.

SOLUTION



From the figure, we see that $m = \frac{r}{h}$, so $y = \frac{rx}{h}$. Thus

$$SA = 2\pi \int_0^h \frac{rx}{h}\sqrt{1 + \frac{r^2}{h^2}}\,dx = \frac{2\pi r}{h}\sqrt{1 + \frac{r^2}{h^2}}\int_0^h x\,dx = \pi r\sqrt{h^2 + r^2}.$$

Further Insights and Challenges

49. Find the surface area of the ellipsoid obtained by rotating the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ about the x-axis.

SOLUTION Taking advantage of symmetry, we can find the surface area of the ellipsoid by doubling the surface area obtained by rotating the portion of the ellipse in the first quadrant about the x-axis. The equation for the portion of the ellipse in the first quadrant is

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Thus,

$$1 + (y')^2 = 1 + \frac{b^2x^2}{a^2(a^2 - x^2)} = \frac{a^4 + (b^2 - a^2)x^2}{a^2(a^2 - x^2)},$$

and

$$SA = 4\pi \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,\frac{\sqrt{a^4 + (b^2 - a^2)x^2}}{a\sqrt{a^2 - x^2}}\,dx = 4\pi b \int_0^a \sqrt{1 + \left(\frac{b^2 - a^2}{a^4}\right)x^2}\,dx.$$

We now consider two cases. If $b^2 > a^2$, then we make the substitution

$$\frac{\sqrt{b^2 - a^2}}{a^2}x = \tan\theta, \quad dx = \frac{a^2}{\sqrt{b^2 - a^2}}\sec^2\theta\,d\theta,$$

and find that

$$\begin{aligned} SA &= 4\pi b\frac{a^2}{\sqrt{b^2 - a^2}}\int_{x=0}^{x=a}\sec^3\theta\,d\theta = 2\pi b\frac{a^2}{\sqrt{b^2 - a^2}}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_{x=0}^{x=a} \\ &= \left(2\pi bx\sqrt{1 + \left(\frac{b^2 - a^2}{a^4}\right)x^2} + 2\pi b\frac{a^2}{\sqrt{b^2 - a^2}}\ln\left|\sqrt{1 + \left(\frac{b^2 - a^2}{a^4}\right)x^2} + \frac{\sqrt{b^2 - a^2}}{a^2}x\right|\right)\Bigg|_0^a \\ &= 2\pi b^2 + 2\pi b\frac{a^2}{\sqrt{b^2 - a^2}}\ln\left(\frac{b}{a} + \frac{\sqrt{b^2 - a^2}}{a}\right). \end{aligned}$$

On the other hand, if $a^2 > b^2$, then we make the substitution

$$\frac{\sqrt{a^2 - b^2}}{a^2}x = \sin\theta, \quad dx = \frac{a^2}{\sqrt{a^2 - b^2}}\cos\theta\,d\theta,$$

and find that

$$\begin{aligned} SA &= 4\pi b\frac{a^2}{\sqrt{a^2 - b^2}}\int_{x=0}^{x=a}\cos^2\theta\,d\theta = 2\pi b\frac{a^2}{\sqrt{a^2 - b^2}}\left(\theta + \sin\theta\cos\theta\right)\Big|_{x=0}^{x=a} \\ &= \left[2\pi bx\sqrt{1 - \left(\frac{a^2 - b^2}{a^4}\right)x^2} + 2\pi b\frac{a^2}{\sqrt{a^2 - b^2}}\sin^{-1}\left(\frac{\sqrt{a^2 - b^2}}{a^2}x\right)\right]\Bigg|_0^a \\ &= 2\pi b^2 + 2\pi b\frac{a^2}{\sqrt{a^2 - b^2}}\sin^{-1}\left(\frac{\sqrt{a^2 - b^2}}{a}\right). \end{aligned}$$

Observe that in both cases, as a approaches b, the value of the surface area of the ellipsoid approaches $4\pi b^2$, the surface area of a sphere of radius b.

50. Show that if the arc length of $f(x)$ over $[0, a]$ is proportional to a, then $f(x)$ must be a linear function.

SOLUTION

$$s = \int_0^a \sqrt{1 + f'(x)^2}\, dx$$

For s to be proportional to a, $\sqrt{1 + f'(x)^2}$ must be a constant, which implies $f'(x)$ is a constant. This, in turn, requires $f(x)$ be linear.

51. CAS Let L be the arc length of the upper half of the ellipse with equation

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

(Figure 18) and let $\eta = \sqrt{1 - (b^2/a^2)}$. Use substitution to show that

$$L = a \int_{-\pi/2}^{\pi/2} \sqrt{1 - \eta^2 \sin^2\theta}\, d\theta$$

Use a computer algebra system to approximate L for $a = 2$, $b = 1$.

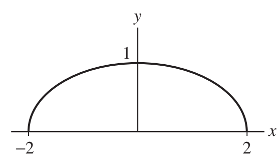


FIGURE 18 Graph of the ellipse $y = \frac{1}{2}\sqrt{4 - x^2}$.

SOLUTION Let $y = \frac{b}{a}\sqrt{a^2 - x^2}$. Then

$$1 + (y')^2 = \frac{b^2x^2 + a^2(a^2 - x^2)}{a^2(a^2 - x^2)}$$

and

$$s = \int_{-a}^{a} \sqrt{\frac{b^2x^2 + a^2(a^2 - x^2)}{a^2(a^2 - x^2)}}\, dx.$$

With the substitution $x = a\sin t$, $dx = a\cos t\, dt$, $a^2 - x^2 = a^2\cos^2 t$ and

$$s = a\int_{-\pi/2}^{\pi/2} \cos t \sqrt{\frac{a^2b^2\sin^2 t + a^2a^2\cos^2 t}{a^2(a^2\cos^2 t)}}\, dt = a\int_{\pi/2}^{\pi/2} \sqrt{\frac{b^2\sin^2 t}{a^2}} + \cos^2 t\, dt$$

Because

$$\eta = \sqrt{1 - \frac{b^2}{a^2}},\ \eta^2 = 1 - \frac{b^2}{a^2}$$

we then have

$$1 - \eta^2\sin^2 t = 1 - \left(1 - \frac{b^2}{a^2}\right)\sin^2 t = 1 - \sin^2 t + \frac{b^2}{a^2}\sin^2 t = \cos^2 t + \frac{b^2}{a^2}\sin^2 t$$

which is the same as the expression under the square root above. Substituting, we get

$$s = a\int_{-\pi/2}^{\pi/2} \sqrt{1 - \eta^2\sin^2 t}\, dt$$

When $a = 2$ and $b = 1$, $\eta^2 = \frac{3}{4}$. Using a computer algebra system to approximate the value of the definite integral, we find $s \approx 4.84422$.

52. Prove that the portion of a sphere of radius R seen by an observer located at a distance d above the North Pole has area $A = 2\pi dR^2/(d+R)$. *Hint:* According to Exercise 46, the cap has surface area is $2\pi Rh$. Show that $h = dR/(d+R)$ by applying the Pythagorean Theorem to the three right triangles in Figure 19.

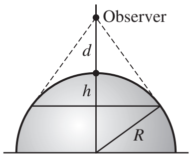
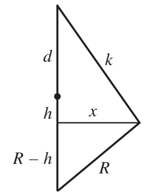


FIGURE 19 Spherical cap observed from a distance d above the North Pole.

SOLUTION Label distances as shown in the figure below.



By repeated application of the Pythagorean Theorem, we find

$$(d+R)^2 = R^2 + k^2 = R^2 + (d+h)^2 + x^2 = R^2 + (d+h)^2 + R^2 - (R-h)^2.$$

Solving for h yields

$$\begin{aligned} d^2 + 2dR + R^2 &= R^2 + d^2 + 2dh + h^2 + R^2 - R^2 + 2Rh - h^2 \\ 2dR &= 2dh + 2Rh \\ dR &= (d+R)h \\ h &= \frac{dR}{d+R} \end{aligned}$$

and thus

$$SA = 2\pi R\left(\frac{dR}{d+R}\right).$$

53. Suppose that the observer in Exercise 52 moves off to infinity—that is, $d \to \infty$. What do you expect the limiting value of the observed area to be? Check your guess by calculating the limit using the formula for the area in the previous exercise.

SOLUTION We would assume the observed surface area would approach $2\pi R^2$ which is the surface area of a hemisphere of radius R. To verify this, observe:

$$\lim_{d\to\infty} SA = \lim_{d\to\infty} \frac{2\pi R^2 d}{R+d} = \lim_{d\to\infty} \frac{2\pi R^2}{1} = 2\pi R^2.$$

54. Let M be the total mass of a metal rod in the shape of the curve $y = f(x)$ over $[a, b]$ whose mass density $\rho(x)$ varies as a function of x. Use Riemann sums to justify the formula

$$M = \int_a^b \rho(x)\sqrt{1 + f'(x)^2}\, dx$$

SOLUTION Divide the interval $[a, b]$ into n subintervals, which we shall denote by $[x_{j-1}, x_j]$ for $j = 1, 2, 3, \ldots, n$. On each subinterval, we will assume that the mass density of the rod is constant; hence, the mass of the corresponding segment of the rod will be approximately equal to the product of the mass density of the segment and the length of the segment. Specifically, let c_j be any point in the jth subinterval and approximate the mass of the segment by

$$\rho(c_j)\sqrt{1 + f'(c_j)^2}\,\Delta x,$$

where $\sqrt{1 + f'(c_j)^2}\,\Delta x$ is the approximate length of the segment. Thus,

$$M \approx \sum_{j=1}^{n} \rho(c_j)\sqrt{1 + f'(c_j)^2}\,\Delta x.$$

As $n \to \infty$, this Riemann sum approaches a definite integral, and we have

$$M = \int_a^b \rho(x)\sqrt{1 + f'(x)^2}\,dx.$$

55. Let $f(x)$ be an increasing function on $[a, b]$ and let $g(x)$ be its inverse. Argue on the basis of arc length that the following equality holds:

$$\int_a^b \sqrt{1 + f'(x)^2}\,dx = \int_{f(a)}^{f(b)} \sqrt{1 + g'(y)^2}\,dy \qquad \boxed{5}$$

Then use the substitution $u = f(x)$ to prove Eq. (5).

SOLUTION Since the graphs of $f(x)$ and $g(x)$ are symmetric with respect to the line $y = x$, the arc length of the curves will be equal on the respective domains. Since the domain of g is the range of f, on $f(a)$ to $f(b)$, $g(x)$ will have the same arc length as $f(x)$ on a to b. If $g(x) = f^{-1}(x)$ and $u = f(x)$, then $x = g(u)$ and $du = f'(x)\,dx$. But

$$g'(u) = \frac{1}{f'(g(u))} = \frac{1}{f'(x)} \Rightarrow f'(x) = \frac{1}{g'(u)}$$

Now substituting $u = f(x)$,

$$s = \int_a^b \sqrt{1 + f'(x)^2}\,dx = \int_{f(a)}^{f(b)} \sqrt{1 + \left(\frac{1}{g'(u)}\right)^2}\,g'(u)\,du = \int_{f(a)}^{f(b)} \sqrt{g'(u)^2 + 1}\,du$$

8.2 Fluid Pressure and Force

Preliminary Questions

1. How is pressure defined?

SOLUTION Pressure is defined as force per unit area.

2. Fluid pressure is proportional to depth. What is the factor of proportionality?

SOLUTION The factor of proportionality is the weight density of the fluid, $w = \rho g$, where ρ is the mass density of the fluid.

3. When fluid force acts on the side of a submerged object, in which direction does it act?

SOLUTION Fluid force acts in the direction perpendicular to the side of the submerged object.

4. Why is fluid pressure on a surface calculated using thin horizontal strips rather than thin vertical strips?

SOLUTION Pressure depends only on depth and does not change horizontally at a given depth.

5. If a thin plate is submerged horizontally, then the fluid force on one side of the plate is equal to pressure times area. Is this true if the plate is submerged vertically?

SOLUTION When a plate is submerged vertically, the pressure is not constant along the plate, so the fluid force is not equal to the pressure times the area.

Exercises

1. A box of height 6 m and square base of side 3 m is submerged in a pool of water. The top of the box is 2 m below the surface of the water.

(a) Calculate the fluid force on the top and bottom of the box.

(b) Write a Riemann sum that approximates the fluid force on a side of the box by dividing the side into N horizontal strips of thickness $\Delta y = 6/N$.

(c) To which integral does the Riemann sum converge?

(d) Compute the fluid force on a side of the box.

SOLUTION

(a) At a depth of 2 m, the pressure on the top of the box is $\rho g h = 10^3 \cdot 9.8 \cdot 2 = 19{,}600$ Pa. The top has area 9 m^2, and the pressure is constant, so the force on the top of the box is $19{,}600 \cdot 9 = 176{,}400N$. At a depth of 8 m, the pressure on the bottom of the box is $\rho g h = 10^3 \cdot 9.8 \cdot 8 = 78{,}400$ Pa, so the force on the bottom of the box is $78{,}400 \cdot 9 = 705{,}600N$.

(b) Let y_j denote the depth of the j^{th} strip, for $j = 1, 2, 3, \ldots, N$; the pressure at this depth is $10^3 \cdot 9.8 \cdot y_j = 9800y_j$ Pa. The strip has thickness Δy m and length 3 m, so has area $3\Delta y$ m^2. Thus the force on the strip is $29{,}400 y_j \Delta y$ N. Sum over all the strips to conclude that the force on one side of the box is approximately

$$F \approx \sum_{j=1}^{N} 29{,}400 y_j \Delta y.$$

(c) As $N \to \infty$, the Riemann sum in part (b) converges to the definite integral $29{,}400 \int_2^8 y\,dy$.

(d) Using the result from part (c), the fluid force on one side of the box is

$$29{,}400 \int_2^8 y\,dy = 14{,}700y^2 \Big|_2^8 = 882{,}000\ N$$

2. A plate in the shape of an isosceles triangle with base 1 m and height 2 m is submerged vertically in a tank of water so that its vertex touches the surface of the water (Figure 7).

(a) Show that the width of the triangle at depth y is $f(y) = \frac{1}{2}y$.

(b) Consider a thin strip of thickness Δy at depth y. Explain why the fluid force on a side of this strip is approximately equal to $\rho g \frac{1}{2} y^2 \Delta y$.

(c) Write an approximation for the total fluid force F on a side of the plate as a Riemann sum and indicate the integral to which it converges.

(d) Calculate F.

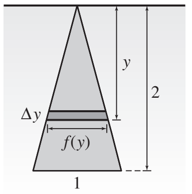


FIGURE 7

SOLUTION

(a) By similar triangles, $\dfrac{y}{2} = \dfrac{f(y)}{1}$ so $f(y) = \dfrac{y}{2}$.

(b) The pressure at a depth of y feet is $\rho g y$ Pa, and the area of the strip is approximately $f(y)\,\Delta y = \frac{1}{2} y \Delta y$ m^2. Therefore, the fluid force on this strip is approximately

$$\rho g y \left(\frac{1}{2} y \Delta y\right) = \frac{1}{2}\rho g y^2 \Delta y.$$

(c) $F \approx \displaystyle\sum_{j=1}^{N} \rho g \frac{y_j^2}{2} \Delta y$. As $N \to \infty$, the Riemann sum converges to the definite integral

$$\frac{\rho g}{2} \int_0^2 y^2\,dy.$$

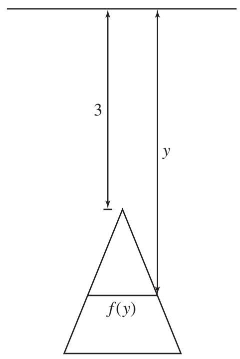
(d) Using the result of part (c),

$$F = \frac{\rho g}{2} \int_0^2 y^2\,dy = \frac{\rho g}{2} \left(\frac{y^3}{3}\right)\Bigg|_0^2 = \frac{9800}{2} \cdot \frac{8}{3} = \frac{39200}{3}\ \text{N}.$$

3. Repeat Exercise 2, but assume that the top of the triangle is located 3 m below the surface of the water.

SOLUTION

(a) Examine the figure below. By similar triangles, $\dfrac{y-3}{2} = \dfrac{f(y)}{1}$ so $f(y) = \dfrac{y-3}{2}$.



(b) The pressure at a depth of y feet is $\rho g y$ lb/ Pa, and the area of the strip is approximately $f(y)\,\Delta y = \frac{1}{2}(y-3)\Delta y$ m^2. Therefore, the fluid force on this strip is approximately

$$\rho g y\left(\frac{1}{2}(y-3)\Delta y\right) = \frac{1}{2}\rho g y(y-3)\Delta y \text{ N}.$$

(c) $F \approx \sum_{j=1}^{N} \rho g \frac{y_j^2 - 3y_j}{2}\Delta y$. As $N \to \infty$, the Riemann sum converges to the definite integral

$$\frac{\rho g}{2}\int_3^5 (y^2 - 3y)\,dy.$$

(d) Using the result of part (c),

$$F = \frac{\rho g}{2}\int_3^5 (y^2-3y)\,dy = \frac{\rho g}{2}\left(\frac{y^3}{3} - \frac{3y^2}{2}\right)\Bigg|_3^5 = \frac{9800}{2}\left[\left(\frac{125}{3} - \frac{75}{2}\right) - \left(9 - \frac{27}{2}\right)\right] = \frac{127{,}400}{3} \text{ N}.$$

4. The plate R in Figure 8, bounded by the parabola $y = x^2$ and $y = 1$, is submerged vertically in water (distance in meters).

(a) Show that the width of R at height y is $f(y) = 2\sqrt{y}$ and the fluid force on a side of a horizontal strip of thickness Δy at height y is approximately $(\rho g)2y^{1/2}(1-y)\Delta y$.

(b) Write a Riemann sum that approximates the fluid force F on a side of R and use it to explain why

$$F = \rho g\int_0^1 2y^{1/2}(1-y)\,dy$$

(c) Calculate F.

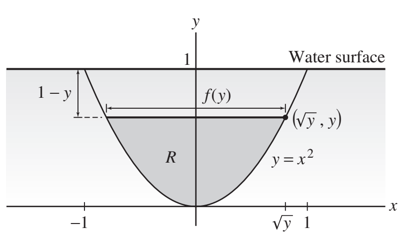


FIGURE 8

SOLUTION

(a) At height y, the thin plate R extends from the point $(-\sqrt{y}, y)$ on the left to the point $(\sqrt{y}, y)$ on the right; thus, the width of the plate is $f(y) = \sqrt{y} - (-\sqrt{y}) = 2\sqrt{y}$. Moreover, the area of a horizontal strip of thickness Δy at height y is $f(y)\,\Delta y = 2\sqrt{y}\,\Delta y$. Because the water surface is at height $y = 1$, the horizontal strip at height y is at a depth of $1 - y$. Consequently, the fluid force on the strip is approximately

$$\rho g(1-y) \times 2\sqrt{y}\Delta y = 2\rho g y^{1/2}(1-y)\Delta y.$$

(b) If the plate is divided into N strips with y_j being the representative height of the jth strip (for $j = 1, 2, 3, \ldots, N$), then the total fluid force exerted on the plate is

$$F \approx 2\rho g\sum_{j=1}^{N}(1-y_j)\sqrt{y_j}\Delta y.$$

As $N \to \infty$, the Riemann sum converges to the definite integral

$$2\rho g \int_0^1 (1-y)\sqrt{y}\,dy.$$

(c) Using the result from part (b),

$$F = 2\rho g \int_0^1 (1-y)\sqrt{y}\,dy = 2\rho g \left(\frac{2}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right)\Bigg|_0^1 = \frac{8}{15}\rho g.$$

Now, $\rho g = 9800\ \mathrm{N/m^3}$ so that $F = \frac{15680}{3}$ N.

5. Let F be the fluid force on a side of a semicircular plate of radius r meters, submerged vertically in water so that its diameter is level with the water's surface (Figure 9).

(a) Show that the width of the plate at depth y is $2\sqrt{r^2-y^2}$.

(b) Calculate F as a function of r using Eq. (2).

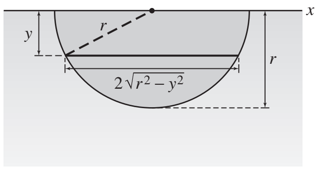


FIGURE 9

SOLUTION

(a) Place the origin at the center of the semicircle and point the positive y-axis downward. The equation for the edge of the semicircular plate is then $x^2+y^2=r^2$. At a depth of y, the plate extends from the point $(-\sqrt{r^2-y^2}, y)$ on the left to the point $(\sqrt{r^2-y^2}, y)$ on the right. The width of the plate at depth y is then

$$\sqrt{r^2-y^2} - \left(-\sqrt{r^2-y^2}\right) = 2\sqrt{r^2-y^2}.$$

(b) With $w = 9800\ \mathrm{N/m^3}$,

$$F = 2w\int_0^r y\sqrt{r^2-y^2}\,dy = -\frac{19{,}600}{3}(r^2-y^2)^{3/2}\Bigg|_0^r = \frac{19{,}600r^3}{3}\ \mathrm{N}.$$

6. Calculate the force on one side of a circular plate with radius 2 m, submerged vertically in a tank of water so that the top of the circle is tangent to the water surface.

SOLUTION Place the origin at the point where the top of the circle is tangent to the water surface and orient the positive y-axis pointing downward. The equation of the circle is then $x^2+(y-2)^2=4$, and the width at any depth y is $2\sqrt{4-(y-2)^2}$. Thus,

$$F = 2\rho g\int_0^4 y\sqrt{4-(y-2)^2}\,dy,$$

Using the substitution $y-2 = 2\sin\theta$, $dy = 2\cos\theta\,d\theta$, the limits of integration become $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so we find

$$\begin{aligned} F &= 2\rho g\int_0^4 y\sqrt{4-(y-2)^2}\,dy \\ &= 2\rho g\int_{-\pi/2}^{\pi/2}(2+2\sin\theta)(2\cos\theta)(2\cos\theta\,d\theta) = 16\rho g\int_{-\pi/2}^{\pi/2}\cos^2\theta + \sin\theta\cos^2\theta\,d\theta \\ &= 16\rho g\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta - \frac{1}{3}\cos^3\theta\right)\Bigg|_{-\pi/2}^{\pi/2} \\ &= 16\rho g\left(\frac{\pi}{4} + 0 - 0 - (-\frac{\pi}{4} + 0 - 0)\right) = 8\rho g\pi = 78{,}400\pi\ \mathrm{N}. \end{aligned}$$

7. A semicircular plate of radius r meters, oriented as in Figure 9, is submerged in water so that its diameter is located at a depth of m meters. Calculate the fluid force on one side of the plate in terms of m and r.

SOLUTION Place the origin at the center of the semicircular plate with the positive y-axis pointing downward. The water surface is then at $y=-m$. Moreover, at location y, the width of the plate is $2\sqrt{r^2-y^2}$ and the depth is $y+m$. Thus,

$$F = 2\rho g\int_0^r (y+m)\sqrt{r^2-y^2}\,dy.$$

Now,

$$\int_0^r y\sqrt{r^2 - y^2}\,dy = -\frac{1}{3}(r^2 - y^2)^{3/2}\Big|_0^r = \frac{1}{3}r^3.$$

Geometrically,

$$\int_0^r \sqrt{r^2 - y^2}\,dy$$

represents the area of one quarter of a circle of radius r, and thus has the value $\frac{\pi r^2}{4}$. Bringing these results together, we find that

$$F = 2\rho g\left(\frac{1}{3}r^3 + \frac{\pi}{4}r^2\right) = \frac{19{,}600}{3}r^3 + 4900mr^2 \text{ N}.$$

8. A plate extending from depth $y = 2$ m to $y = 5$ m is submerged in a fluid of density $\rho = 850$ kg/m^3. The horizontal width of the plate at depth y is $f(y) = 2(1 + y^2)^{-1}$. Calculate the fluid force on one side of the plate.

SOLUTION The fluid force on one side of the plate is given by

$$F = \rho g \int_2^5 y f(y)\,dy = \rho g \int_2^5 2y(1 + y^2)^{-1}\,dy = \rho g \ln(1 + y^2)\Big|_2^5 = \rho g(\ln 26 - \ln 5)$$
$$= 8330 \ln \frac{26}{5} \approx 13733.32 \text{ N}.$$

9. Figure 10 shows the wall of a dam on a water reservoir. Use the Trapezoidal Rule and the width and depth measurements in the figure to estimate the fluid force on the wall.

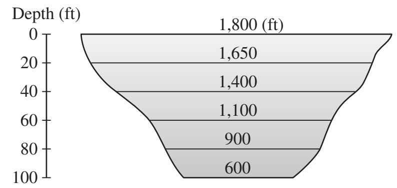


FIGURE 10

SOLUTION Let $f(y)$ denote the width of the dam wall at depth y feet. Then the force on the dam wall is

$$F = w\int_0^{100} y f(y)\,dy.$$

Using the Trapezoidal Rule and the width and depth measurements in the figure,

$$F \approx w\frac{20}{2}[0 \cdot f(0) + 2 \cdot 20 \cdot f(20) + 2 \cdot 40 \cdot f(40) + 2 \cdot 60 \cdot f(60) + 2 \cdot 80 \cdot f(80) + 100 \cdot f(100)]$$
$$= 10w(0 + 66{,}000 + 112{,}000 + 132{,}000 + 144{,}000 + 60{,}000) = 321{,}250{,}000 \text{ lb}.$$

10. Calculate the fluid force on a side of the plate in Figure 11(A), submerged in water.

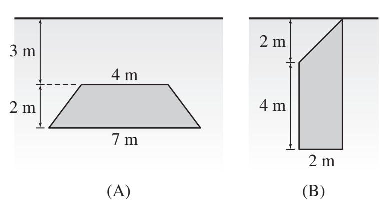


FIGURE 11

SOLUTION The width of the plate varies linearly from 4 meters at a depth of 3 meters to 7 meters at a depth of 5 meters. Thus, at depth y, the width of the plate is

$$4 + \frac{3}{2}(y - 3) = \frac{3}{2}y - \frac{1}{2}.$$

Finally, the force on a side of the plate is

$$F = w\int_3^5 y\left(\frac{3}{2}y - \frac{1}{2}\right)dy = w\left(\frac{1}{2}y^3 - \frac{1}{4}y^2\right)\Big|_3^5 = 45w = 441{,}000 \text{ N}.$$

11. Calculate the fluid force on a side of the plate in Figure 11(B), submerged in a fluid of mass density $\rho = 800\ \text{kg/m}^3$.

SOLUTION Because the fluid has a mass density of $\rho = 800\ \text{kg/m}^3$,

$$w = (800)(9.8) = 7840\ \text{N/m}^3.$$

For depths up to 2 meters, the width of the plate at depth y is y; for depths from 2 meters to 6 meters, the width of the plate is a constant 2 meters. Thus,

$$F = w\int_0^2 y(y)\,dy + w\int_2^6 2y\,dy = w\frac{y^3}{3}\Big|_0^2 + wy^2\Big|_2^6 = \frac{8w}{3} + 32w = \frac{104w}{3} = \frac{815{,}360}{3}\ \text{N}.$$

12. Find the fluid force on the side of the plate in Figure 12, submerged in a fluid of density $\rho = 1200\ \text{kg/m}^3$. The top of the place is level with the fluid surface. The edges of the plate are the curves $y = x^{1/3}$ and $y = -x^{1/3}$.

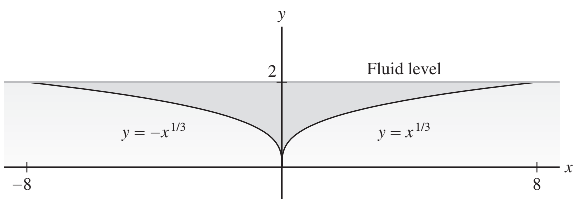


FIGURE 12

SOLUTION At height y, the plate extends from the point $(-y^3, y)$ on the left to the point (y^3, y) on the right; thus, the width of the plate is $f(y) = y^3 - (-y^3) = 2y^3$. Because the water surface is at height $y = 2$, the horizontal strip at height y is at a depth of $2 - y$. Consequently,

$$F = \rho g\int_0^2 (2-y)(2y^3)\,dy = 2\rho g\left(\frac{1}{2}y^4 - \frac{1}{5}y^5\right)\Big|_0^2 = \frac{16\rho g}{5} = \frac{16\cdot 1200\cdot 9.8}{5} = 37{,}632\ \text{N}.$$

13. Let R be the plate in the shape of the region under $y = \sin x$ for $0 \le x \le \frac{\pi}{2}$ in Figure 13(A). Find the fluid force on a side of R if it is rotated counterclockwise by 90° and submerged in a fluid of density $1100\ \text{kg/m}^3$ with its top edge level with the surface of the fluid as in (B).

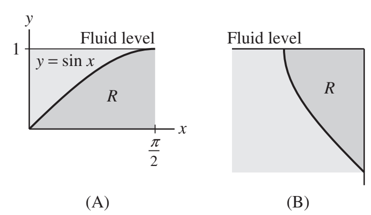


FIGURE 13

SOLUTION Place the origin at the bottom corner of the plate with the positive y-axis pointing upward. The fluid surface is then at height $y = \frac{\pi}{2}$, and the horizontal strip of the plate at height y is at a depth of $\frac{\pi}{2} - y$ and has a width of $\sin y$. Now, using integration by parts we find

$$F = \rho g\int_0^{\pi/2}\left(\frac{\pi}{2} - y\right)\sin y\,dy = \rho g\left[-\left(\frac{\pi}{2} - y\right)\cos y - \sin y\right]\Big|_0^{\pi/2} = \rho g\left(\frac{\pi}{2} - 1\right)$$
$$= 1100\cdot 9.8\left(\frac{\pi}{2} - 1\right) \approx 6153.184\ \text{N}.$$

14. In the notation of Exercise 13, calculate the fluid force on a side of the plate R if it is oriented as in Figure 13(A). You may need to use Integration by Parts and trigonometric substitution.

SOLUTION Place the origin at the lower left corner of the plate. Because the fluid surface is at height $y = 1$, the horizontal strip at height y is at a depth of $1 - y$. Moreover, this strip has a width of

$$\frac{\pi}{2} - \sin^{-1} y = \cos^{-1} y.$$

Thus,

$$F = \rho g\int_0^1 (1-y)\cos^{-1} y\,dy.$$

Starting with integration by parts, we find

$$\int_0^1 (1-y)\cos^{-1} y\,dy = \left(y - \frac{1}{2}y^2\right)\cos^{-1} y\Big|_0^1 + \int_0^1 \frac{y - \frac{1}{2}y^2}{\sqrt{1-y^2}}\,dy$$

$$= \frac{1}{2}\cos^{-1} 1 + \int_0^1 \frac{y - \frac{1}{2}y^2}{\sqrt{1-y^2}}\,dy = \int_0^1 \frac{y}{\sqrt{1-y^2}}\,dy - \frac{1}{2}\int_0^1 \frac{y^2}{\sqrt{1-y^2}}\,dy.$$

Now,

$$\int_0^1 \frac{y}{\sqrt{1-y^2}}\,dy = -\sqrt{1-y^2}\Big|_0^1 = 1.$$

For the remaining integral, we use the trigonometric substitution $y = \sin\theta$, $dy = \cos\theta\,d\theta$ and find

$$\frac{1}{2}\int_0^1 \frac{y^2}{\sqrt{1-y^2}}\,dy = \frac{1}{2}\int_{y=0}^{y=1} \sin^2\theta\,d\theta = \frac{1}{4}(\theta - \sin\theta\cos\theta)\Big|_{y=0}^{y=1}$$

$$= \frac{1}{4}\left(\sin^{-1} y - y\sqrt{1-y^2}\right)\Big|_0^1 = \frac{\pi}{8}.$$

Finally,

$$F = \rho g\left(1 - \frac{\pi}{8}\right) = 1100 \cdot 9.8\left(1 - \frac{\pi}{8}\right) \approx 6546.70\text{ N}.$$

15. Calculate the fluid force on one side of a plate in the shape of region A shown Figure 14. The water surface is at $y = 1$, and the fluid has density $\rho = 900\text{ kg/m}^3$.

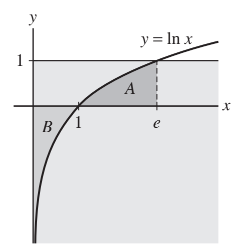


FIGURE 14

SOLUTION Because the fluid surface is at height $y = 1$, the horizontal strip at height y is at a depth of $1 - y$. Moreover, this strip has a width of $e - e^y$. Thus,

$$F = \rho g\int_0^1 (1-y)(e - e^y)\,dy = e\rho g\int_0^1 (1-y)\,dy - \rho g\int_0^1 (1-y)e^y\,dy.$$

Now,

$$\int_0^1 (1-y)\,dy = \left(y - \frac{1}{2}y^2\right)\Big|_0^1 = \frac{1}{2},$$

and using integration by parts

$$\int_0^1 (1-y)e^y\,dy = \left((1-y)e^y + e^y\right)\Big|_0^1 = e - 2.$$

Combining these results, we find that

$$F = \rho g\left(\frac{1}{2}e - (e-2)\right) = \rho g\left(2 - \frac{1}{2}e\right) = 900 \cdot 9.8\left(2 - \frac{1}{2}e\right) \approx 5652.37\text{ N}.$$

16. Calculate the fluid force on one side of the "infinite" plate B in Figure 14, assuming the fluid has density $\rho = 900\text{ kg/m}^3$.

SOLUTION Because the fluid surface is at height $y = 1$, the horizontal strip at height y is at a depth of $1 - y$. Moreover, this strip has a width of e^y. Thus,

$$F = \rho g\int_{-\infty}^0 (1-y)e^y\,dy.$$

Using integration by parts, we find

$$\int_{-\infty}^{0} (1-y)e^{y}\,dy = \left[(1-y)e^{y} + e^{y}\right]\Big|_{-\infty}^{0} = 2.$$

Thus, $F = 2\rho g = 2 \cdot 900 \cdot 9.8 = 17{,}640$ N.

17. Figure 15(A) shows a ramp inclined at 30° leading into a swimming pool. Calculate the fluid force on the ramp.

SOLUTION A horizontal strip at depth y has length 6 and width

$$\frac{\Delta y}{\sin 30^\circ} = 2\Delta y.$$

Thus,

$$F = 2\rho g \int_{0}^{4} 6y\,dy = 96\rho g.$$

If distances are in feet, then $\rho g = w = 62.5$ lb/ft^3 and $F = 6000$ lb; if distances are in meters, then $\rho g = 9800$ N/m^3 and $F = 940{,}800$ N.

18. Calculate the fluid force on one side of the plate (an isosceles triangle) shown in Figure 15(B).

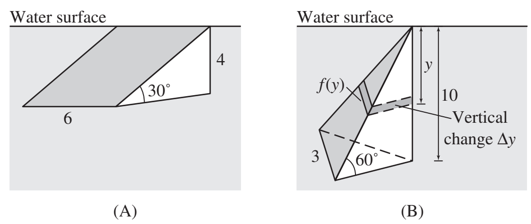


FIGURE 15

SOLUTION A horizontal strip at depth y has length $f(y) = \frac{3}{10}y$ and width

$$\frac{\Delta y}{\sin 60^\circ} = \frac{2}{\sqrt{3}}\Delta y.$$

Thus,

$$F = \frac{\sqrt{3}}{5} w \int_{0}^{10} y^2\,dy = \frac{200\sqrt{3}}{3} w.$$

If distances are in feet, then $w = 62.5$ lb/ft^3 and $F \approx 7216.88$ lb; if distances are in meters, then $w = 9800$ N/m^3 and $F \approx 1{,}131{,}606.5$ N.

19. The massive Three Gorges Dam on China's Yangtze River has height 185 m (Figure 16). Calculate the force on the dam, assuming that the dam is a trapezoid of base 2000 m and upper edge 3000 m, inclined at an angle of 55° to the horizontal (Figure 17).

FIGURE 16 Three Gorges Dam on the Yangtze River

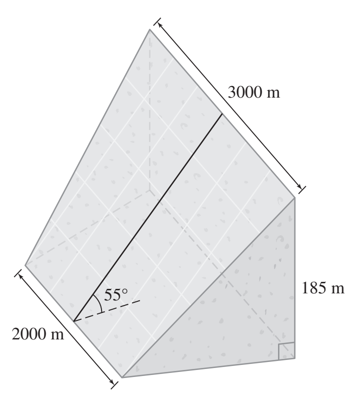


FIGURE 17

SOLUTION Let $y = 0$ be at the bottom of the dam, so that the top of the dam is at $y = 185$. Then the width of the dam at height y is $2000 + \frac{1000y}{185}$. The dam is inclined at an angle of 55° to the horizontal, so the height of a horizontal strip is

$$\frac{\Delta y}{\sin 55^\circ} \approx 1.221\Delta y$$

so that the area of such a strip is

$$1.221\left(2000+\frac{1000y}{185}\right)\Delta y$$

Then

$$F=\rho g\int_0^{185}1.221y\left(2000+\frac{1000y}{185}\right)dy=\rho g\int_0^{185}2442y+6.6y^2\,dy=\rho g(1221y^2+2.2y^3)\Big|_0^{185}$$

$$=55{,}718{,}300\rho g=55{,}718{,}300\cdot 9800=5.460393400\times 10^{11}\text{ N.}$$

20. A square plate of side 3 m is submerged in water at an incline of 30° with the horizontal. Calculate the fluid force on one side of the plate if the top edge of the plate lies at a depth of 6 m.

SOLUTION Because the plate is 3 meters on a side, is submerged at a horizontal angle of 30°, and has its top edge located at a depth of 6 meters, the bottom edge of the plate is located at a depth of $6+3\sin 30^\circ=\frac{15}{2}$ meters. Let y denote the depth at any point of the plate. The width of each horizontal strip of the plate is then

$$\frac{\Delta y}{\sin 30^\circ}=2\Delta y,$$

and

$$F=\rho g\int_6^{15/2}(2)3y\,dy=(\rho g)\frac{243}{4}=595{,}350\text{ N.}$$

21. The trough in Figure 18 is filled with corn syrup, whose weight density is 90 lb/ft^3. Calculate the force on the front side of the trough.

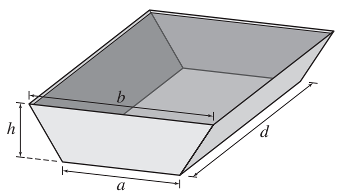


FIGURE 18

SOLUTION Place the origin along the top edge of the trough with the positive y-axis pointing downward. The width of the front side of the trough varies linearly from b when $y=0$ to a when $y=h$; thus, the width of the front side of the trough at depth y feet is given by

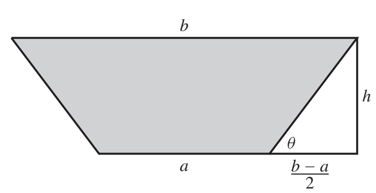
$$b+\frac{a-b}{h}y.$$

Now,

$$F=w\int_0^h y\left(b+\frac{a-b}{h}y\right)dy=w\left(\frac{1}{2}by^2+\frac{a-b}{3h}y^3\right)\Big|_0^h=w\left(\frac{b}{6}+\frac{a}{3}\right)h^2=(15b+30a)h^2\text{ lb.}$$

22. Calculate the fluid pressure on one of the slanted sides of the trough in Figure 18 when it is filled with corn syrup as in Exercise 21.

SOLUTION



The diagram above displays a side view of the trough. From this diagram, we see that

$$\sin\theta=\frac{h}{\sqrt{\left(\frac{b-a}{2}\right)^2+h^2}}.$$

Thus,

$$F=\frac{w}{\sin\theta}\int_0^h d\cdot y\,dy=\frac{90\sqrt{\left(\frac{b-a}{2}\right)^2+h^2}}{h}\frac{dh^2}{2}=45dh\sqrt{\left(\frac{b-a}{2}\right)^2+h^2}.$$

Further Insights and Challenges

23. The end of the trough in Figure 19 is an equilateral triangle of side 3. Assume that the trough is filled with water to height H. Calculate the fluid force on each side of the trough as a function of H and the length l of the trough.

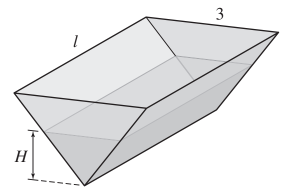


FIGURE 19

SOLUTION Place the origin at the lower vertex of the trough and orient the positive y-axis pointing upward. First, consider the faces at the front and back ends of the trough. A horizontal strip at height y has a length of $\frac{2y}{\sqrt{3}}$ and is at a depth of $H - y$. Thus,

$$F = w\int_0^H (H-y)\frac{2y}{\sqrt{3}}\,dy = w\left(\frac{H}{\sqrt{3}}y^2 - \frac{2}{3\sqrt{3}}y^3\right)\Big|_0^H = \frac{\sqrt{3}}{9}wH^3.$$

For the slanted sides, we note that each side makes an angle of 60° with the horizontal. If we let ℓ denote the length of the trough, then

$$F = \frac{2w\ell}{\sqrt{3}}\int_0^H (H-y)\,dy = \frac{\sqrt{3}}{3}\ell w H^2.$$

24. A rectangular plate of side ℓ is submerged vertically in a fluid of density w, with its top edge at depth h. Show that if the depth is increased by an amount Δh, then the force on a side of the plate increases by $wA\Delta h$, where A is the area of the plate.

SOLUTION Let F_1 be the force on a side of the plate when its top edge is at depth h and F_2 be the force on a side of the plate when its top edge is at depth $h + \Delta h$. Further, let b denote the width of the rectangular plate. Then

$$F_1 = w\int_h^{h+\ell} yb\,dy = bw\left(\frac{y^2}{2}\right)\Big|_h^{h+\ell} = bw\left(\frac{\ell^2 + 2\ell h}{2}\right)$$

$$F_2 = w\int_{h+\Delta h}^{h+\ell+\Delta h} yb\,dy = bw\left(\frac{y^2}{2}\right)\Big|_{h+\Delta h}^{h+\ell+\Delta h} = bw\frac{\ell^2 + 2\ell h + 2\ell\Delta h}{2}$$

and $F_2 - F_1 = bw\ell\Delta h = wA\Delta h$.

25. Prove that the force on the side of a rectangular plate of area A submerged vertically in a fluid is equal to p_0A, where p_0 is the fluid pressure at the center point of the rectangle.

SOLUTION Let ℓ denote the length of the vertical side of the rectangle, x denote the length of the horizontal side of the rectangle, and suppose the top edge of the rectangle is at depth $y = m$. The pressure at the center of the rectangle is then

$$p_0 = w\left(m + \frac{\ell}{2}\right),$$

and the force on the side of the rectangular plate is

$$F = \int_m^{\ell+m} wxy\,dy = \frac{wx}{2}\left[(\ell+m)^2 - m^2\right] = \frac{wx\ell}{2}(\ell + 2m) = Aw\left(\frac{\ell}{2} + m\right) = Ap_0.$$

26. If the density of a fluid varies with depth, then the pressure at depth y is a function $p(y)$ (which need not equal wy as in the case of constant density). Use Riemann sums to argue that the total force F on the flat side of a submerged object submerged vertically is $F = \int_a^b f(y)p(y)\,dy$, where $f(y)$ is the width of the side at depth y.

SOLUTION Suppose the object extends from a depth of $y = a$ to a depth of $y = b$. Divide the object into N horizontal strips, each of width Δy. Let $p(y)$ denote the pressure within the fluid at depth y and $f(y)$ denote the width of the flat side of the submerged object at depth y. The approximate force on the jth strip ($j = 1, 2, 3, \ldots, N$) is

$$p(y_j)f(y_j)\Delta y,$$

where y_j is a depth associated with the jth strip. Summing over all of the strips,

$$F \approx \sum_{j=1}^{N} p(y_j) f(y_j) \Delta y.$$

As $N \to \infty$, this Riemann sum converges to a definite integral, and

$$F = \int_a^b p(y) f(y)\, dy.$$

8.3 Center of Mass

Preliminary Questions

1. What are the x- and y-moments of a lamina whose center of mass is located at the origin?

SOLUTION Because the center of mass is located at the origin, it follows that $M_x = M_y = 0$.

2. A thin plate has mass 3. What is the x-moment of the plate if its center of mass has coordinates $(2, 7)$?

SOLUTION The x-moment of the plate is the product of the mass of the plate and the y-coordinate of the center of mass. Thus, $M_x = 3(7) = 21$.

3. The center of mass of a lamina of total mass 5 has coordinates $(2, 1)$. What are the lamina's x- and y-moments?

SOLUTION The x-moment of the plate is the product of the mass of the plate and the y-coordinate of the center of mass, whereas the y-moment is the product of the mass of the plate and the x-coordinate of the center of mass. Thus, $M_x = 5(1) = 5$, and $M_y = 5(2) = 10$.

4. Explain how the Symmetry Principle is used to conclude that the centroid of a rectangle is the center of the rectangle.

SOLUTION Because a rectangle is symmetric with respect to both the vertical line and the horizontal line through the center of the rectangle, the Symmetry Principle guarantees that the centroid of the rectangle must lie along both of these lines. The only point in common to both lines of symmetry is the center of the rectangle, so the centroid of the rectangle must be the center of the rectangle.

Exercises

1. Four particles are located at points $(1, 1)$, $(1, 2)$, $(4, 0)$, $(3, 1)$.

(a) Find the moments M_x and M_y and the center of mass of the system, assuming that the particles have equal mass m.

(b) Find the center of mass of the system, assuming the particles have masses 3, 2, 5, and 7, respectively.

SOLUTION

(a) Because each particle has mass m,

$$M_x = m(1) + m(2) + m(0) + m(1) = 4m;$$
$$M_y = m(1) + m(1) + m(4) + m(3) = 9m;$$

and the total mass of the system is $4m$. Thus, the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{9m}{4m}, \frac{4m}{4m}\right) = \left(\frac{9}{4}, 1\right).$$

(b) With the indicated masses of the particles,

$$M_x = 3(1) + 2(2) + 5(0) + 7(1) = 14;$$
$$M_y = 3(1) + 2(1) + 5(4) + 7(3) = 46;$$

and the total mass of the system is 17. Thus, the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{46}{17}, \frac{14}{17}\right).$$

2. Find the center of mass for the system of particles of masses 4, 2, 5, 1 located at $(1, 2)$, $(-3, 2)$, $(2, -1)$, $(4, 0)$.

SOLUTION With the indicated masses and locations of the particles

$$M_x = 4(2) + 2(2) + 5(-1) + 1(0) = 7;$$

$$M_y = 4(1) + 2(-3) + 5(2) + 1(4) = 12;$$

and the total mass of the system is 12. Thus, the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(1, \frac{7}{12}\right).$$

3. Point masses of equal size are placed at the vertices of the triangle with coordinates $(a, 0)$, $(b, 0)$, and $(0, c)$. Show that the center of mass of the system of masses has coordinates $\left(\frac{1}{3}(a+b), \frac{1}{3}c\right)$.

SOLUTION Let each particle have mass m. The total mass of the system is then $3m$. and the moments are

$$M_x = 0(m) + 0(m) + c(m) = cm; \text{ and}$$

$$M_y = a(m) + b(m) + 0(m) = (a+b)m.$$

Thus, the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{(a+b)m}{3m}, \frac{cm}{3m}\right) = \left(\frac{a+b}{3}, \frac{c}{3}\right).$$

4. Point masses of mass m_1, m_2, and m_3 are placed at the points $(-1, 0)$, $(3, 0)$, and $(0, 4)$.

(a) Suppose that $m_1 = 6$. Find m_2 such that the center of mass lies on the y-axis.

(b) Suppose that $m_1 = 6$ and $m_2 = 4$. Find the value of m_3 such that $y_{CM} = 2$.

SOLUTION With the given masses and locations, we find

$$M_x = m_1(0) + m_2(0) + m_3(4) = 4m_3;$$

$$M_y = m_1(-1) + m_2(3) + m_3(0) = 3m_2 - m_1;$$

and the total mass of the system is $m_1 + m_2 + m_3$. Thus, the coordinates of the center of mass are

$$\left(\frac{3m_2 - m_1}{m_1 + m_2 + m_3}, \frac{4m_3}{m_1 + m_2 + m_3}\right).$$

(a) For the center of mass to lie on the y-axis, we must have $3m_2 - m_1 = 0$, or $m_2 = \frac{1}{3}m_1$. Given $m_1 = 6$, it follows that $m_2 = 2$.

(b) To have $y_{CM} = 2$ requires

$$\frac{4m_3}{m_1 + m_2 + m_3} = 2 \quad \text{or} \quad m_3 = m_1 + m_2.$$

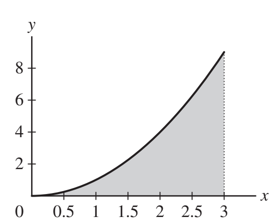
Given $m_1 = 6$ and $m_2 = 4$, it follows that $m_3 = 10$.

5. Sketch the lamina S of constant density $\rho = 3$ g/cm^2 occupying the region beneath the graph of $y = x^2$ for $0 \le x \le 3$.

(a) Use Eqs. (1) and (2) to compute M_x and M_y.

(b) Find the area and the center of mass of S.

SOLUTION A sketch of the lamina is shown below



(a) Using Eq. (2),

$$M_x = 3\int_0^9 y(3 - \sqrt{y})\,dy = \left(\frac{9y^2}{2} - \frac{6}{5}y^{5/2}\right)\Big|_0^9 = \frac{729}{10}.$$

Using Eq. (1),

$$M_y = 3\int_0^3 x(x^2)\,dx = \frac{3x^4}{4}\Big|_0^3 = \frac{243}{4}.$$

(b) The area of the lamina is

$$A = \int_0^3 x^2\,dx = \left.\frac{x^3}{3}\right|_0^3 = 9\text{ cm}^2.$$

With a constant density of $\rho = 3\text{ g/cm}^2$, the mass of the lamina is $M = 27$ grams, and the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{243/4}{27}, \frac{729/10}{27}\right) = \left(\frac{9}{4}, \frac{27}{10}\right).$$

6. Use Eqs. (1) and (3) to find the moments and center of mass of the lamina S of constant density $\rho = 2\text{ g/cm}^2$ occupying the region between $y = x^2$ and $y = 9x$ over $[0, 3]$. Sketch S, indicating the location of the center of mass.

SOLUTION With $\rho = 2\text{ g/cm}^2$,

$$M_x = \frac{1}{2}(2)\int_0^3 \left((9x)^2 - (x^2)^2\right) dx = \frac{3402}{5},$$

and

$$M_y = 2\int_0^3 x(9x - x^2)\,dx = \frac{243}{2}.$$

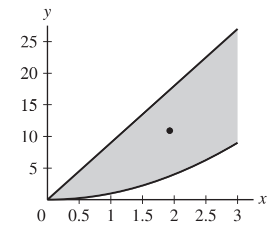
The mass of the lamina is

$$M = 2\int_0^3 (9x - x^2)\,dx = 63\text{ g},$$

so the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{243}{126}, \frac{3402}{315}\right).$$

A sketch of the lamina, with the location of the center of mass indicated, is shown below.



7. Find the moments and center of mass of the lamina of uniform density ρ occupying the region underneath $y = x^3$ for $0 \le x \le 2$.

SOLUTION With uniform density ρ,

$$M_x = \frac{1}{2}\rho\int_0^2 (x^3)^2\,dx = \frac{64\rho}{7} \quad\text{and}\quad M_y = \rho\int_0^2 x(x^3)\,dx = \frac{32\rho}{5}.$$

The mass of the lamina is

$$M = \rho\int_0^2 x^3\,dx = 4\rho,$$

so the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{8}{5}, \frac{16}{7}\right).$$

8. Calculate M_x (assuming $\rho = 1$) for the region underneath the graph of $y = 1 - x^2$ for $0 \le x \le 1$ in two ways, first using Eq. (2) and then using Eq. (3).

SOLUTION By Eq. (2),

$$M_x = \int_0^1 y\sqrt{1-y}\,dy.$$

Using the substitution $u = 1 - y$, $du = -dy$, we find

$$M_x = \int_0^1 (1-u)\sqrt{u}\,du = \left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_0^1 = \frac{4}{15}.$$

By Eq. (3),

$$M_x = \frac{1}{2}\int_0^1 (1-x^2)^2\,dx = \frac{1}{2}\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 = \frac{4}{15}.$$

9. Let T be the triangular lamina in Figure 17.
(a) Show that the horizontal cut at height y has length $4 - \frac{2}{3}y$ and use Eq. (2) to compute M_x (with $\rho = 1$).
(b) Use the Symmetry Principle to show that $M_y = 0$ and find the center of mass.

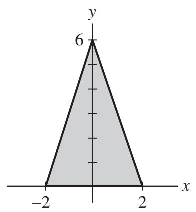


FIGURE 17 Isosceles triangle.

SOLUTION
(a) The equation of the line from (2, 0) to (0, 6) is $y = -3x + 6$, so

$$x = 2 - \frac{1}{3}y.$$

The length of the horizontal cut at height y is then

$$2\left(2 - \frac{1}{3}y\right) = 4 - \frac{2}{3}y,$$

and

$$M_x = \int_0^6 y\left(4 - \frac{2}{3}y\right)dy = 24.$$

(b) Because the triangular lamina is symmetric with respect to the y-axis, $x_{cm} = 0$, which implies that $M_y = 0$. The total mass of the lamina is

$$M = 2\int_0^2 (-3x+6)\,dx = 12,$$

so $y_{cm} = 24/12$. Finally, the coordinates of the center of mass are (0, 2).

In Exercises 10–17, find the centroid of the region lying underneath the graph of the function over the given interval.

10. $f(x) = 6 - 2x$, $[0, 3]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^3 (6-2x)^2\,dx = 18 \quad \text{and} \quad M_y = \int_0^3 x(6-2x)\,dx = 9.$$

The area of the region is

$$A = \int_0^3 (6-2x)\,dx = 9,$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = (1, 2).$$

11. $f(x) = \sqrt{x}$, $[1, 4]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_1^4 x\,dx = \frac{15}{4} \quad \text{and} \quad M_y = \int_1^4 x\sqrt{x}\,dx = \frac{62}{5}.$$

The area of the region is

$$A = \int_1^4 \sqrt{x}\,dx = \frac{14}{3},$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{93}{35}, \frac{45}{56}\right).$$

12. $f(x) = x^3, \quad [0, 1]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^1 x^6\,dx = \frac{1}{14} \quad \text{and} \quad M_y = \int_0^1 x^4\,dx = \frac{1}{5}.$$

The area of the region is

$$A = \int_0^1 x^3\,dx = \frac{1}{4},$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{4}{5}, \frac{2}{7}\right).$$

13. $f(x) = 9 - x^2, \quad [0, 3]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^3 (9 - x^2)^2\,dx = \frac{324}{5} \quad \text{and} \quad M_y = \int_0^3 x(9 - x^2)\,dx = \frac{81}{4}.$$

The area of the region is

$$A = \int_0^3 (9 - x^2)\,dx = 18,$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{9}{8}, \frac{18}{5}\right).$$

14. $f(x) = (1 + x^2)^{-1/2}, \quad [0, 3]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^3 \frac{1}{1 + x^2}\,dx = \left.\frac{\tan^{-1} x}{2}\right|_0^3 = \frac{1}{2}\tan^{-1} 3 \quad \text{and} \quad M_y = \int_0^3 \frac{x}{\sqrt{1 + x^2}}\,dx = \sqrt{10} - 1.$$

The area of the region is

$$A = \int_0^3 \frac{1}{\sqrt{1 + x^2}}\,dx = \ln|x + \sqrt{1 + x^2}|\Big|_0^3 = \ln(3 + \sqrt{10}),$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{\sqrt{10} - 1}{\ln(3 + \sqrt{10})}, \frac{\tan^{-1} 3}{2\ln(3 + \sqrt{10})}\right).$$

15. $f(x) = e^{-x}, \quad [0, 4]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^4 e^{-2x}\,dx = \frac{1}{4}\left(1 - e^{-8}\right) \quad \text{and} \quad M_y = \int_0^4 xe^{-x}\,dx = -e^{-x}(x + 1)\Big|_0^4 = 1 - 5e^{-4}.$$

The area of the region is

$$A = \int_0^4 e^{-x}\,dx = 1 - e^{-4},$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{1 - 5e^{-4}}{1 - e^{-4}}, \frac{1 - e^{-8}}{4(1 - e^{-4})}\right).$$

16. $f(x) = \ln x, \quad [1, 2]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_1^2 (\ln x)^2\,dx = \frac{1}{2}(x(\ln x)^2 - 2x\ln x + 2x)\Big|_1^2 = (\ln 2)^2 - 2\ln 2 + 1; \text{ and}$$

$$M_y = \int_1^2 x\ln x\,dx = \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_1^2 = 2\ln 2 - \frac{3}{4}.$$

The area of the region is

$$A = \int_1^2 \ln x\,dx = (x\ln x - x)\Big|_1^2 = 2\ln 2 - 1,$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{2\ln 2 - \frac{3}{4}}{2\ln 2 - 1}, \frac{(\ln 2)^2 - 2\ln 2 + 1}{2\ln 2 - 1}\right).$$

17. $f(x) = \sin x, \quad [0, \pi]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^\pi \sin^2 x\,dx = \frac{1}{4}(x - \sin x\cos x)\Big|_0^\pi = \frac{\pi}{4}; \text{ and}$$

$$M_y = \int_0^\pi x\sin x\,dx = (-x\cos x + \sin x)\Big|_0^\pi = \pi.$$

The area of the region is

$$A = \int_0^\pi \sin x\,dx = 2,$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{\pi}{2}, \frac{\pi}{8}\right).$$

18. Calculate the moments and center of mass of the lamina occupying the region between the curves $y = x$ and $y = x^2$ for $0 \le x \le 1$.

SOLUTION The moments of the lamina are

$$M_x = \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx = \frac{1}{15} \quad \text{and} \quad M_y = \int_0^1 x(x - x^2)\,dx = \frac{1}{12}.$$

The area of the lamina is

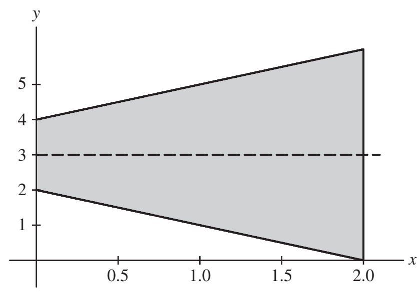
$$A = \int_0^1 (x - x^2)\,dx = \frac{1}{6},$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{1}{2}, \frac{2}{5}\right).$$

19. Sketch the region between $y = x + 4$ and $y = 2 - x$ for $0 \le x \le 2$. Using symmetry, explain why the centroid of the region lies on the line $y = 3$. Verify this by computing the moments and the centroid.

SOLUTION A sketch of the region is shown below.



The region is clearly symmetric about the line $y = 3$, so we expect the centroid of the region to lie along this line. We find

$$M_x = \frac{1}{2}\int_0^2 \left((x+4)^2 - (2-x)^2\right) dx = 24;$$

$$M_y = \int_0^2 x\,((x+4) - (2-x))\,dx = \frac{28}{3}; \text{ and}$$

$$A = \int_0^2 ((x+4) - (2-x))\,dx = 8.$$

Thus, the coordinates of the centroid are $\left(\frac{7}{6}, 3\right)$.

In Exercises 20–25, find the centroid of the region lying between the graphs of the functions over the given interval.

20. $y = x, \quad y = \sqrt{x}, \quad [0, 1]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^1 (x - x^2)\,dx = \frac{1}{12} \quad \text{and} \quad M_y = \int_0^1 x(\sqrt{x} - x)\,dx = \frac{1}{15}.$$

The area of the region is

$$A = \int_0^1 (\sqrt{x} - x)\,dx = \frac{1}{6},$$

so the coordinates of the centroid are

$$\left(\frac{6}{15}, \frac{1}{2}\right).$$

21. $y = x^2, \quad y = \sqrt{x}, \quad [0, 1]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^1 (x - x^4)\,dx = \frac{3}{20} \quad \text{and} \quad M_y = \int_0^1 x(\sqrt{x} - x^2)\,dx = \frac{3}{20}.$$

The area of the region is

$$A = \int_0^1 (\sqrt{x} - x^2)\,dx = \frac{1}{3},$$

so the coordinates of the centroid are

$$\left(\frac{9}{20}, \frac{9}{20}\right).$$

Note: This makes sense, since the functions are inverses of each other. This makes the region symmetric with respect to the line $y = x$. Thus, by the symmetry principle, the center of mass must lie on that line.

22. $y = x^{-1}, \quad y = 2 - x, \quad [1, 2]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_1^2 \left[\left(\frac{1}{x}\right)^2 - (2-x)^2\right] dx = \frac{1}{12} \quad \text{and} \quad M_y = \int_1^2 x\left(\frac{1}{x} - (2-x)\right) dx = \frac{1}{3}.$$

The area of the region is

$$A = \int_1^2 \left(\frac{1}{x} - (2-x)\right) dx = \ln 2 - \frac{1}{2},$$

so the coordinates of the centroid are

$$\left(\frac{2}{6\ln 2 - 3}, \frac{1}{12\ln 2 - 6}\right).$$

23. $y = e^x, \quad y = 1, \quad [0, 1]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^1 (e^{2x} - 1)\, dx = \frac{e^2 - 3}{4} \quad \text{and} \quad M_y = \int_0^1 x(e^x - 1)\, dx = \left(xe^x - e^x - \frac{1}{2}x^2\right)\Big|_0^1 = \frac{1}{2}.$$

The area of the region is

$$A = \int_0^1 (e^x - 1)\, dx = e - 2,$$

so the coordinates of the centroid are

$$\left(\frac{1}{2(e-2)}, \frac{e^2 - 3}{4(e-2)}\right).$$

24. $y = \ln x, \quad y = x - 1, \quad [1, 3]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_1^3 \left[(x-1)^2 - (\ln x)^2\right] dx = \left(\frac{1}{3}x^3 - x^2 - x - x(\ln x)^2 + 2x\ln x\right)\Big|_1^3 = 3\ln 3 - \frac{3}{2}(\ln 3)^2 - \frac{2}{3}; \text{ and}$$

$$M_y = \int_1^3 x((x-1) - \ln x)\, dx = \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_1^3 = \frac{20}{3} - \frac{9}{2}\ln 3.$$

The area of the region is

$$A = \int_1^3 (x - 1 - \ln x)\, dx = \left(\frac{1}{2}x^2 - x\ln x\right)\Big|_1^3 = 4 - 3\ln 3,$$

so the coordinates of the centroid are

$$\left(\frac{40 - 27\ln 3}{24 - 18\ln 3}, \frac{18\ln 3 - 9(\ln 3)^2 - 4}{24 - 18\ln 3}\right).$$

25. $y = \sin x, \quad y = \cos x, \quad [0, \pi/4]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^{\pi/4} (\cos^2 x - \sin^2 x)\, dx = \frac{1}{2}\int_0^{\pi/4} \cos 2x\, dx = \frac{1}{4}; \text{ and}$$

$$M_y = \int_0^{\pi/4} x(\cos x - \sin x)\, dx = [(x-1)\sin x + (x+1)\cos x]\Big|_0^{\pi/4} = \frac{\pi\sqrt{2}}{4} - 1.$$

The area of the region is

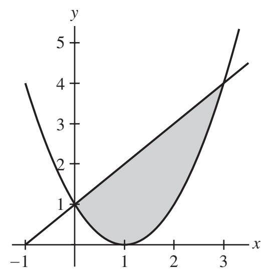
$$A = \int_0^{\pi/4} (\cos x - \sin x)\, dx = \sqrt{2} - 1,$$

so the coordinates of the centroid are

$$\left(\frac{\pi\sqrt{2} - 4}{4(\sqrt{2} - 1)}, \frac{1}{4(\sqrt{2} - 1)}\right).$$

26. Sketch the region enclosed by $y = x + 1$, and $y = (x-1)^2$, and find its centroid.

SOLUTION A sketch of the region is shown below.



The moments of the region are

$$M_x = \frac{1}{2}\int_0^3 (x+1)^2 - (x-1)^4\,dx = \frac{1}{2}\left(\frac{1}{3}(x+1)^3 - \frac{1}{5}(x-1)^5\right)\Big|_0^3 = \frac{1}{2}\left(\frac{64}{3} - \frac{32}{5} - \frac{1}{3} - \frac{1}{5}\right) = \frac{36}{5}$$

$$M_y = \int_0^3 x((x+1) - (x-1)^2)\,dx = \int_0^3 3x^2 - x^3\,dx = \left(x^3 - \frac{1}{4}x^4\right)\Big|_0^3 = \frac{27}{4}$$

The area of the region is

$$A = \int_0^3 (x+1) - (x-1)^2\,dx = \int_0^3 -x^2 + 3x\,dx = \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2\right)\Big|_0^3 = \frac{9}{2}$$

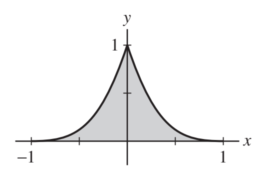
so that the coordinates of the centroid are

$$\left(\frac{27}{4}\cdot\frac{2}{9}, \frac{36}{5}\cdot\frac{2}{9}\right) = \left(\frac{3}{2}, \frac{8}{5}\right)$$

s

27. Sketch the region enclosed by $y = 0$, $y = (x+1)^3$, and $y = (1-x)^3$, and find its centroid.

SOLUTION A sketch of the region is shown below.



The moments of the region are

$$M_x = \frac{1}{2}\left(\int_{-1}^0 (x+1)^6\,dx + \int_0^1 (1-x)^6\,dx\right) = \frac{1}{7};\ \text{and}$$

$M_y = 0$ by the Symmetry Principle.

The area of the region is

$$A = \int_{-1}^0 (x+1)^3\,dx + \int_0^1 (1-x)^3\,dx = \frac{1}{2},$$

so the coordinates of the centroid are $\left(0, \frac{2}{7}\right)$.

In Exercises 28–32, find the centroid of the region.

28. Top half of the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$

SOLUTION The equation of the top half of the ellipse is $y = \sqrt{16 - 4x^2}$. Thus,

$$M_x = \frac{1}{2}\int_{-2}^2 \left(\sqrt{16-4x^2}\right)^2\,dx = \frac{64}{3}.$$

By the Symmetry Principle, $M_y = 0$. The area of the region is one-half the area of an ellipse with major axis 4 and minor axis 2; i.e., $\frac{1}{2}\pi(4)(2) = 4\pi$. Finally, the coordinates of the centroid are

$$\left(0, \frac{16}{3\pi}\right).$$

29. Top half of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ for arbitrary $a, b > 0$

SOLUTION The equation of the top half of the ellipse is

$$y = \sqrt{b^2 - \frac{b^2x^2}{a^2}}$$

Thus,

$$M_x = \frac{1}{2}\int_{-a}^{a}\left(\sqrt{b^2 - \frac{b^2x^2}{a^2}}\right)^2 dx = \frac{2ab^2}{3}.$$

By the Symmetry Principle, $M_y = 0$. The area of the region is one-half the area of an ellipse with axes of length a and b; i.e., $\frac{1}{2}\pi ab$. Finally, the coordinates of the centroid are

$$\left(0, \frac{4b}{3\pi}\right).$$

30. Semicircle of radius r with center at the origin

SOLUTION The equation of the top half of the circle is $y = \sqrt{r^2 - x^2}$. Thus,

$$M_x = \frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx = \frac{2r^3}{3}.$$

By the Symmetry Principle, $M_y = 0$. The area of the region is one-half the area of a circle of radius r; i.e., $\frac{1}{2}\pi r^2$. Finally, the coordinates of the centroid are

$$\left(0, \frac{4r}{3\pi}\right).$$

31. Quarter of the unit circle lying in the first quadrant

SOLUTION By the Symmetry Principle, the center of mass must lie on the line $y = x$ in the first quadrant. Therefore, we need only calculate one of the moments of the region. With $y = \sqrt{1 - x^2}$, we find

$$M_y = \int_0^1 x\sqrt{1 - x^2}\, dx = \frac{1}{3}.$$

The area of the region is one-quarter of the area of a unit circle; i.e., $\frac{1}{4}\pi$. Thus, the coordinates of the centroid are

$$\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right).$$

32. Triangular plate with vertices $(-c, 0)$, $(0, c)$, (a, b), where $a, b, c > 0$, and $b < c$

SOLUTION By symmetry, the center of mass must lie on the line connecting $(-c, 0)$ and the midpoint $(a/2, (b + c)/2)$ of the opposite side:

$$\ell_1 : y = \frac{b + c}{a + 2c}(x + c)$$

Also by symmetry, the center of mass must lie on the line connecting $(0, c)$ and the midpoint $((a - c)/2, b/2)$ of the opposite side:

$$\ell_2 : y = \frac{b - 2c}{a - c}x + c$$

These lines intersect at one point (x_{cm}, y_{cm}). Equating the formulas for the two lines and solving for x yields

$$x = \frac{a - c}{3}.$$

Substituting this value for x into the equation for ℓ_2 gives

$$y = \frac{b - 2c}{a - c}\frac{a - c}{3} + c = \frac{b + c}{3}.$$

Hence, the coordinates of the centroid are

$$\left(\frac{a - c}{3}, \frac{b + c}{3}\right).$$

33. Find the centroid for the shaded region of the semicircle of radius r in Figure 18. What is the centroid when $r = 1$ and $h = \frac{1}{2}$? *Hint:* Use geometry rather than integration to show that the *area* of the region is $r^2 \sin^{-1}(\sqrt{1 - h^2/r^2}) - h\sqrt{r^2 - h^2}$).

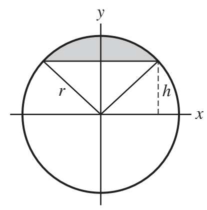


FIGURE 18

SOLUTION From the symmetry of the region, it is obvious that the centroid lies along the y-axis. To determine the y-coordinate of the centroid, we must calculate the moment about the x-axis and the area of the region. Now, the length of the horizontal cut of the semicircle at height y is

$$\sqrt{r^2 - y^2} - \left(-\sqrt{r^2 - y^2}\right) = 2\sqrt{r^2 - y^2}.$$

Therefore, taking $\rho = 1$, we find

$$M_x = 2\int_h^r y\sqrt{r^2 - y^2}\,dy = \frac{2}{3}(r^2 - h^2)^{3/2}.$$

Observe that the region is comprised of a sector of the circle with the triangle between the two radii removed. The angle of the sector is 2θ, where $\theta = \sin^{-1}\sqrt{1 - h^2/r^2}$, so the area of the sector is $\frac{1}{2}r^2(2\theta) = r^2 \sin^{-1}\sqrt{1 - h^2/r^2}$. The triangle has base $2\sqrt{r^2 - h^2}$ and height h, so the area is $h\sqrt{r^2 - h^2}$. Therefore,

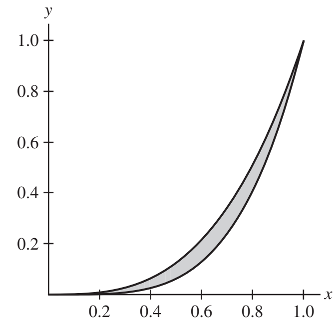
$$Y_{CM} = \frac{M_x}{A} = \frac{\frac{2}{3}(r^2 - h^2)^{3/2}}{r^2 \sin^{-1}\sqrt{1 - h^2/r^2} - h\sqrt{r^2 - h^2}}.$$

When $r = 1$ and $h = 1/2$, we find

$$Y_{CM} = \frac{\frac{2}{3}(3/4)^{3/2}}{\sin^{-1}\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}} = \frac{3\sqrt{3}}{4\pi - 3\sqrt{3}}.$$

34. Sketch the region between $y = x^n$ and $y = x^m$ for $0 \le x \le 1$, where $m > n \ge 0$ and find the COM of the region. Find a pair (n, m) such that the COM lies outside the region.

SOLUTION A sketch of the region for x^3 and x^4 is below.



Since $m > n \ge 0$, the graph of x^n lies above that of x^m for x between 0 and 1. Thus the moments are

$$M_x = \frac{1}{2}\int_0^1 x^{2n} - x^{2m}\,dx = \frac{1}{2}\left(\frac{1}{2n+1}x^{2n+1} - \frac{1}{2m+1}x^{2m+1}\right)\Bigg|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{2n+1} - \frac{1}{2m+1}\right) = \frac{m-n}{(2n+1)(2m+1)}$$

$$M_y = \int_0^1 x(x^n - x^m)\,dx = \int_0^1 x^{n+1} - x^{m+1}\,dx = \left(\frac{1}{n+2}x^{n+2} - \frac{1}{m+2}x^{m+2}\right)\Bigg|_0^1$$

$$= \frac{1}{n+2} - \frac{1}{m+2} = \frac{m-n}{(n+2)(m+2)}$$

The area of the region is

$$A = \int_0^1 x^n - x^m \, dx = \frac{1}{n+1} - \frac{1}{m+1} = \frac{m-n}{(n+1)(m+1)}$$

Thus the center of mass has coordinates

$$\left(\frac{(n+1)(m+1)}{(n+2)(m+2)}, \frac{(n+1)(m+1)}{(2n+1)(2m+1)} \right)$$

In the case graphed above, for $n = 3$, $m = 4$, the center of mass is

$$\left(\frac{20}{30}, \frac{20}{63} \right) = \left(\frac{2}{3}, \frac{20}{63} \right)$$

and

$$\left(\frac{2}{3} \right)^3 = \frac{8}{27} < \frac{20}{63}$$

Thus the point $\left(\frac{2}{3}, \frac{8}{27} \right)$ lies on $y = x^3$ and then the curve $y = x^3$ lies below the center of mass of the region.

In Exercises 35–37, use the additivity of moments to find the COM of the region.

35. Isosceles triangle of height 2 on top of a rectangle of base 4 and height 3 (Figure 19)

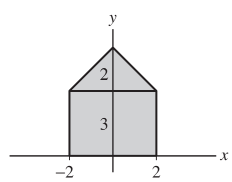


FIGURE 19

SOLUTION The region is symmetric with respect to the y-axis, so $M_y = 0$ by the Symmetry Principle. The moment about the x-axis for the rectangle is

$$M_x^{\text{rect}} = \frac{1}{2} \int_{-2}^{2} 3^2 \, dx = 18,$$

whereas the moment about the x-axis for the triangle is

$$M_x^{\text{triangle}} = \int_3^5 y(10 - 2y) \, dy = \frac{44}{3}.$$

The total moment about the x-axis is then

$$M_x = M_x^{\text{rect}} + M_x^{\text{triangle}} = 18 + \frac{44}{3} = \frac{98}{3}.$$

Because the area of the region is $12 + 4 = 16$, the coordinates of the center of mass are

$$\left(0, \frac{49}{24} \right).$$

36. An ice cream cone consisting of a semicircle on top of an equilateral triangle of side 6 (Figure 20)

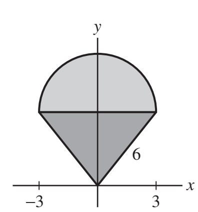


FIGURE 20

SOLUTION The region is symmetric with respect to the y-axis, so $M_y = 0$ by the Symmetry Principle. The moment about the x-axis for the triangle is

$$M_x^{\text{triangle}} = \frac{2}{\sqrt{3}} \int_0^{3\sqrt{3}} y^2\, dy = 54.$$

For the semicircle, first note that the center is $(0, 3\sqrt{3})$, so the equation is $x^2 + (y - 3\sqrt{3})^2 = 9$, and

$$M_x^{\text{semi}} = 2 \int_{3\sqrt{3}}^{3+3\sqrt{3}} y\sqrt{9 - (y - 3\sqrt{3})^2}\, dy.$$

Using the substitution $w = y - 3\sqrt{3}$, $dw = dy$, we find

$$\begin{aligned} M_x^{\text{semi}} &= 2 \int_0^3 (w + 3\sqrt{3})\sqrt{9 - w^2}\, dw \\ &= 2 \int_0^3 w\sqrt{9 - w^2}\, dw + 6\sqrt{3} \int_0^3 \sqrt{9 - w^2}\, dw = 18 + \frac{27\pi\sqrt{3}}{2}, \end{aligned}$$

where we have used the fact that $\int_0^3 \sqrt{9 - w^2}\, dw$ represents the area of one-quarter of a circle of radius 3. The total moment about the x-axis is then

$$M_x = M_x^{\text{triangle}} + M_x^{\text{semi}} = 72 + \frac{27\pi\sqrt{3}}{2}.$$

Because the area of the region is $9\sqrt{3} + \frac{9\pi}{2}$, the coordinates of the center of mass are

$$\left(0, \frac{16 + 3\pi\sqrt{3}}{\pi + 2\sqrt{3}}\right).$$

37. Three-quarters of the unit circle (remove the part in the fourth quadrant)

SOLUTION By the Symmetry Principle, the center of mass must lie on the line $y = -x$. Let region 1 be the semicircle above the x-axis and region 2 be the quarter circle in the third quadrant. Because region 1 is symmetric with respect to the y-axis, $M_y^1 = 0$ by the Symmetry Principle. Furthermore

$$M_y^2 = \int_{-1}^{0} x\sqrt{1 - x^2}\, dx = -\frac{1}{3}.$$

Thus, $M_y = M_y^1 + M_y^2 = 0 + (-\frac{1}{3}) = -\frac{1}{3}$. The area of the region is $3\pi/4$, so the coordinates of the centroid are

$$\left(-\frac{4}{9\pi}, \frac{4}{9\pi}\right).$$

38. Let S be the lamina of mass density $\rho = 1$ obtained by removing a circle of radius r from the circle of radius $2r$ shown in Figure 21. Let M_x^S and M_y^S denote the moments of S. Similarly, let M_y^{big} and M_y^{small} be the y-moments of the larger and smaller circles.

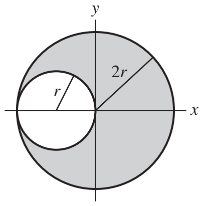


FIGURE 21

(a) Use the Symmetry Principle to show that $M_x^S = 0$.

(b) Show that $M_y^S = M_y^{\text{big}} - M_y^{\text{small}}$ using the additivity of moments.

(c) Find M_y^{big} and M_y^{small} using the fact that the COM of a circle is its center. Then compute M_y^S using (b).

(d) Determine the COM of S.

SOLUTION

(a) Because S is symmetric with respect to the x-axis, $M_x^S = 0$.

(b) Because the small circle together with the region S comprise the big circle, by the additivity of moments,

$$M_y^S + M_y^{\text{small}} = M_y^{\text{big}}.$$

Thus $M_y^S = M_y^{\text{big}} - M_y^{\text{small}}$.

(c) The center of the big circle is the origin, so $x_{\text{cm}}^{\text{big}} = 0$; consequently, $M_y^{\text{big}} = 0$. On the other hand, the center of the small circle is $(-r, 0)$, so $x_{\text{cm}}^{\text{small}} = -r$; consequently

$$M_y^{\text{small}} = x_{\text{cm}}^{\text{small}} \cdot A^{\text{small}} = -r \cdot \pi r^2 = -\pi r^3.$$

By the result of part (b), it follows that $M_y^S = 0 - (-\pi r^3) = \pi r^3$.

(d) The area of the region S is $4\pi r^2 - \pi r^2 = 3\pi r^2$. The coordinates of the center of mass of the region S are then

$$\left(\frac{\pi r^3}{3\pi r^2}, 0\right) = \left(\frac{r}{3}, 0\right).$$

39. Find the COM of the laminas in Figure 22 obtained by removing squares of side 2 from a square of side 8.

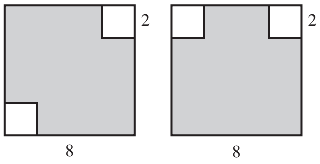


FIGURE 22

SOLUTION Start with the square on the left. Place the square so that the bottom left corner is at $(0, 0)$. By the Symmetry Principle, the center of mass must lie on the lines $y = x$ and $y = 8 - x$. The only point in common to these two lines is $(4, 4)$, so the center of mass is $(4, 4)$.

Now consider the square on the right. Place the square as above. By the symmetry principle, $x_{cm} = 4$. Now, let $s1$ denote the square in the upper left, $s2$ denote the square in the upper right, and B denote the entire square. Then

$$M_x^{s1} = \frac{1}{2}\int_0^2 (8^2 - 6^2)\,dx = 28;$$

$$M_x^{s2} = \frac{1}{2}\int_6^8 (8^2 - 6^2)\,dx = 28; \text{ and}$$

$$M_x^B = \frac{1}{2}\int_0^8 8^2\,dx = 256.$$

By the additivity of moments, $M_x = 256 - 28 - 28 = 200$. Finally, the area of the region is $A = 64 - 4 - 4 = 56$, so the coordinates of the center of mass are

$$\left(4, \frac{200}{56}\right) = \left(4, \frac{25}{7}\right).$$

Further Insights and Challenges

40. A **median** of a triangle is a segment joining a vertex to the midpoint of the opposite side. Show that the centroid of a triangle lies on each of its medians, at a distance two-thirds down from the vertex. Then use this fact to prove that the three medians intersect at a single point. *Hint:* Simplify the calculation by assuming that one vertex lies at the origin and another on the x-axis.

SOLUTION Orient the triangle by placing one vertex at $(0, 0)$ and the long side of the triangle along the x-axis. Label the vertices $(0, 0)$, $(a, 0)$, (b, c). Thus, the equations of the short sides are $y = \frac{cx}{b}$ and $y = \frac{cx}{b-a} - \frac{ac}{b-a}$. Now,

$$M_x = \frac{1}{2}\int_0^b (cx/b)^2\,dx + \frac{1}{2}\int_b^a \left(\frac{cx - ac}{b - a}\right)^2 dx = \frac{ac^2}{6};$$

$$M_y = \int_0^b x(cx/b)\,dx + \int_b^a x\left(\frac{cx - ac}{b - a}\right) dx = \frac{ac(a + b)}{6}; \text{ and}$$

$$M = \frac{ac}{2}.$$

so the center of mass is $\left(\frac{a+b}{3}, \frac{c}{3}\right)$. To show that the centroid lies on each median, let y_1 be the median from (b, c), y_2 the median from $(0, 0)$ and y_3 the median from $(a, 0)$. We find

$$y_1(x) = \frac{2c}{2b-a}(x - a/2), \qquad \text{so} \qquad y_1\left(\frac{a+b}{3}\right) = \frac{c}{3};$$

$$y_2(x) = \frac{c}{a+b}x, \qquad \text{so} \qquad y_2\left(\frac{a+b}{3}\right) = \frac{c}{3};$$

$$y_3(x) = \frac{c}{b-2a}(x - a), \qquad \text{so} \qquad y_3\left(\frac{a+b}{3}\right) = \frac{c}{3}.$$

This shows that the center of mass lies on each median. We now show that the center of mass is $\frac{2}{3}$ of the way from each vertex. For y_1, note that $x = b$ gives the vertex and $x = \frac{a}{2}$ gives the midpoint of the opposite side, so two-thirds of this distance is

$$x = b + \frac{2}{3}\left(\frac{a}{2} - b\right) = \frac{a+b}{3},$$

the x-coordinate of the center of mass. Likewise, for y_2, two-thirds of the distance from $x = 0$ to $x = \frac{a+b}{2}$ is $\frac{a+b}{3}$, and for y_3, the two-thirds point is

$$x = a + \frac{2}{3}\left(\frac{b}{2} - a\right) = \frac{a+b}{3}.$$

A similar method shows that the y-coordinate is also two-thirds of the way along the median. Thus, since the centroid lies on all three medians, we can conclude that all three medians meet at a single point, namely the centroid.

41. Let P be the COM of a system of two weights with masses m_1 and m_2 separated by a distance d. Prove Archimedes' Law of the (weightless) Lever: P is the point on a line between the two weights such that $m_1L_1 = m_2L_2$, where L_j is the distance from mass j to P.

SOLUTION Place the lever along the x-axis with mass m_1 at the origin. Then $M_y = m_2 d$ and the x-coordinate of the center of mass, P, is

$$\frac{m_2 d}{m_1 + m_2}.$$

Thus,

$$L_1 = \frac{m_2 d}{m_1 + m_2}, \quad L_2 = d - \frac{m_2 d}{m_1 + m_2} = \frac{m_1 d}{m_1 + m_2},$$

and

$$L_1 m_1 = m_1 \frac{m_2 d}{m_1 + m_2} = m_2 \frac{m_1 d}{m_1 + m_2} = L_2 m_2.$$

42. Find the COM of a system of two weights of masses m_1 and m_2 connected by a lever of length d whose mass density ρ is uniform. *Hint:* The moment of the system is the sum of the moments of the weights and the lever.

SOLUTION Let A be the cross-sectional area of the rod. Place the rod with m_1 at the origin and rod lying on the positive x-axis. The y-moment of the rod is $M_y = \frac{1}{2}\rho A d^2$, the y-moment of the mass m_2 is $M_y = m_2 d$, and the total mass of the system is $M = m_1 + m_2 + \rho A d$. Therefore, the x-coordinate of the center of mass is

$$\frac{m_2 d + \frac{1}{2}\rho A d^2}{m_1 + m_2 + \rho A d}.$$

43. **Symmetry Principle** Let $\mathcal{R}$ be the region under the graph of $f(x)$ over the interval $[-a, a]$, where $f(x) \geq 0$. Assume that $\mathcal{R}$ is symmetric with respect to the y-axis.

(a) Explain why $f(x)$ is even—that is, why $f(x) = f(-x)$.

(b) Show that $xf(x)$ is an *odd* function.

(c) Use (b) to prove that $M_y = 0$.

(d) Prove that the COM of $\mathcal{R}$ lies on the y-axis (a similar argument applies to symmetry with respect to the x-axis).

SOLUTION

(a) By the definition of symmetry with respect to the y-axis, $f(x) = f(-x)$, so f is even.

(b) Let $g(x) = xf(x)$ where f is even. Then

$$g(-x) = -xf(-x) = -xf(x) = -g(x),$$

and thus g is odd.

(c) $M_y = \rho \int_{-a}^{a} x f(x)\,dx = 0$ since $xf(x)$ is an odd function.

(d) By part (c), $x_{cm} = \dfrac{M_y}{M} = \dfrac{0}{M} = 0$ so the center of mass lies along the y-axis.

44. Prove directly that Eqs. (2) and (3) are equivalent in the following situation. Let $f(x)$ be a positive decreasing function on $[0, b]$ such that $f(b) = 0$. Set $d = f(0)$ and $g(y) = f^{-1}(y)$. Show that

$$\frac{1}{2}\int_0^b f(x)^2\,dx = \int_0^d y g(y)\,dy$$

Hint: First apply the substitution $y = f(x)$ to the integral on the left and observe that $dx = g'(y)\,dy$. Then apply Integration by Parts.

SOLUTION $f(x) \geq 0$ and $f'(x) < 0$ shows that f has an inverse g on $[a, b]$. Because $f(b) = 0$, $f(0) = d$, and $f^{-1}(x) = g(x)$, it follows that $g(d) = 0$ and $g(0) = b$. If we let $x = g(y)$, then $dx = g'(y)\,dy$. Thus, with $y = f(x)$,

$$\frac{1}{2}\int_0^b f(x)^2\,dx = \frac{1}{2}\int_0^b y^2\,dx = \frac{1}{2}\int_d^0 y^2 g'(y)\,dy.$$

Using Integration by Parts with $u = y^2$ and $v' = g'(y)\,dy$, we find

$$\frac{1}{2}\int_d^0 y^2 g'(y)\,dy = \frac{1}{2}\left[y^2 g(y)\Big|_d^0 - 2\int_d^0 y g(y)\,dy\right] = \frac{1}{2}\left[0 - d^2 g(d)\right] - \int_d^0 y g(y)\,dy = \int_0^d y g(y)\,dy.$$

45. Let R be a lamina of uniform density submerged in a fluid of density w (Figure 23). Prove the following law: The fluid force on one side of R is equal to the area of R times the fluid pressure on the centroid. *Hint:* Let $g(y)$ be the horizontal width of R at depth y. Express both the fluid pressure [Eq. (2) in Section 8.2] and y-coordinate of the centroid in terms of $g(y)$.

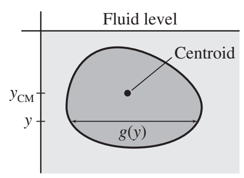


FIGURE 23

SOLUTION Let ρ denote the uniform density of the submerged lamina. Then

$$M_x = \rho \int_a^b y g(y)\,dy,$$

and the mass of the lamina is

$$M = \rho \int_a^b g(y)\,dy = \rho A,$$

where A is the area of the lamina. Thus, the y-coordinate of the centroid is

$$y_{\text{cm}} = \frac{\rho \int_a^b y g(y)\,dy}{\rho A} = \frac{\int_a^b y g(y)\,dy}{A}.$$

Now, the fluid force on the lamina is

$$F = w\int_a^b y g(y)\,dy = w\frac{\int_a^b y g(y)\,dy}{A} A = w y_{\text{cm}} A.$$

In other words, the fluid force on the lamina is equal to the fluid pressure at the centroid of the lamina times the area of the lamina.

8.4 Taylor Polynomials

Preliminary Questions

1. What is $T_3(x)$ centered at $a = 3$ for a function $f(x)$ such that $f(3) = 9$, $f'(3) = 8$, $f''(3) = 4$, and $f'''(3) = 12$?

SOLUTION In general, with $a = 3$,

$$T_3(x) = f(3) + f'(3)(x - 3) + \frac{f''(3)}{2}(x - 3)^2 + \frac{f'''(3)}{6}(x - 3)^3.$$

Using the information provided, we find

$$T_3(x) = 9 + 8(x-3) + 2(x-3)^2 + 2(x-3)^3.$$

2. The dashed graphs in Figure 9 are Taylor polynomials for a function $f(x)$. Which of the two is a Maclaurin polynomial?

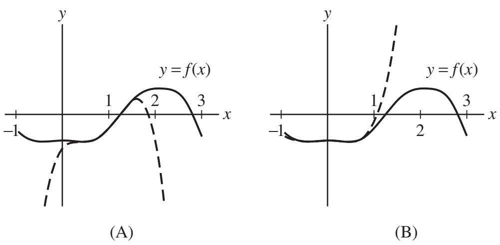


FIGURE 9

SOLUTION A Maclaurin polynomial always gives the value of $f(0)$ exactly. This is true for the Taylor polynomial sketched in (B); thus, this is the Maclaurin polynomial.

3. For which value of x does the Maclaurin polynomial $T_n(x)$ satisfy $T_n(x) = f(x)$, no matter what $f(x)$ is?

SOLUTION A Maclaurin polynomial always gives the value of $f(0)$ exactly.

4. Let $T_n(x)$ be the Maclaurin polynomial of a function $f(x)$ satisfying $|f^{(4)}(x)| \le 1$ for all x. Which of the following statements follow from the error bound?

(a) $|T_4(2) - f(2)| \le \frac{2}{3}$

(b) $|T_3(2) - f(2)| \le \frac{2}{3}$

(c) $|T_3(2) - f(2)| \le \frac{1}{3}$

SOLUTION For a function $f(x)$ satisfying $|f^{(4)}(x)| \le 1$ for all x,

$$|T_3(2) - f(2)| \le \frac{1}{24}|f^{(4)}(x)|2^4 \le \frac{16}{24} < \frac{2}{3}.$$

Thus, **(b)** is the correct answer.

Exercises

In Exercises 1–14, calculate the Taylor polynomials $T_2(x)$ and $T_3(x)$ centered at $x = a$ for the given function and value of a.

1. $f(x) = \sin x, \quad a = 0$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{array}{ll} f(x) = \sin x & f(a) = 0 \\ f'(x) = \cos x & f'(a) = 1 \\ f''(x) = -\sin x & f''(a) = 0 \\ f'''(x) = -\cos x & f'''(a) = -1 \end{array}$$

Now,

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 0 + 1(x-0) + \frac{0}{2}(x-0)^2 = x; \text{ and}$$

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$$

$$= 0 + 1(x-0) + \frac{0}{2}(x-0)^2 + \frac{-1}{6}(x-0)^3 = x - \frac{1}{6}x^3.$$

2. $f(x) = \sin x, \quad a = \frac{\pi}{2}$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{array}{ll} f(x) = \sin x & f(a) = 1 \\ f'(x) = \cos x & f'(a) = 0 \\ f''(x) = -\sin x & f''(a) = -1 \\ f'''(x) = -\cos x & f'''(a) = 0 \end{array}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\ &= 1 + 0\left(x - \frac{\pi}{2}\right) + \frac{-1}{2}\left(x - \frac{\pi}{2}\right)^2 = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= 1 + 0\left(x - \frac{\pi}{2}\right) + \frac{-1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{0}{6}\left(x - \frac{\pi}{2}\right)^3 = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2. \end{aligned}$$

3. $f(x) = \frac{1}{1+x}, \quad a = 2$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{array}{ll} f(x) = \frac{1}{1+x} & f(a) = \frac{1}{3} \\ f'(x) = \frac{-1}{(1+x)^2} & f'(a) = -\frac{1}{9} \\ f''(x) = \frac{2}{(1+x)^3} & f''(a) = \frac{2}{27} \\ f'''(x) = \frac{-6}{(1+x)^4} & f'''(a) = -\frac{2}{27} \end{array}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{2/27}{2!}(x-2)^2 \\ &= \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &= \frac{1}{3} - \frac{1}{9}(x-2) + \frac{2/27}{2!}(x-2)^2 - \frac{2/27}{3!}(x-2)^3 = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3 \end{aligned}$$

4. $f(x) = \frac{1}{1+x^2}, \quad a = -1$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{array}{ll} f(x) = \frac{1}{1+x^2} & f(a) = 1/2 \\ f'(x) = \frac{-2x}{(x^2+1)^2} & f'(a) = 1/2 \\ f''(x) = \frac{2(3x^2-1)}{(x^2+1)^3} & f''(a) = 1/2 \\ f'''(x) = \frac{-24x(x^2-1)}{(x^2+1)^4} & f'''(a) = 0 \end{array}$$

Now,

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$
$$= \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1/2}{2}(x+1)^2 = \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1}{4}(x+1)^2; \text{ and}$$
$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$$
$$= \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1/2}{2}(x+1)^2 + \frac{0}{6}(x+1)^3 = \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1}{4}(x+1)^2.$$

5. $f(x) = x^4 - 2x, \quad a = 3$

SOLUTION First calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= x^4 - 2x & f(a) &= 75 \\ f'(x) &= 4x^3 - 2 & f'(a) &= 106 \\ f''(x) &= 12x^2 & f''(a) &= 108 \\ f'''(x) &= 24x & f'''(a) &= 72 \end{aligned}$$

Now,

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 75 + 106(x-3) + \frac{108}{2}(x-3)^2$$
$$= 75 + 106(x-3) + 54(x-3)^2$$
$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$
$$= 75 + 106(x-3) + \frac{108}{2}(x-3)^2 + \frac{72}{3!}(x-3)^3$$
$$= 75 + 106(x-3) + 54(x-3)^2 + 12(x-3)^3$$

6. $f(x) = \dfrac{x^2+1}{x+1}, \quad a = -2$

SOLUTION First calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \frac{x^2+1}{x+1} & f(a) &= -5 \\ f'(x) &= \frac{x^2+2x-1}{(x+1)^2} & f'(a) &= -1 \\ f''(x) &= \frac{4}{(x+1)^3} & f''(a) &= -4 \\ f'''(x) &= \frac{-12}{(x+1)^4} & f'''(a) &= -12 \end{aligned}$$

Now,

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = -5 - (x+2) + \frac{-4}{2}(x+2)^2$$
$$= -5 - (x+2) - 2(x+2)^2$$
$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$
$$= -5 - (x+2) + \frac{-4}{2}(x+2)^2 + \frac{-12}{3!}(x+2)^3$$
$$= -5 - (x+2) - 2(x+2)^2 - 2(x+2)^3$$

7. $f(x) = \tan x, \quad a = 0$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \tan x & f(a) &= 0 \\ f'(x) &= \sec^2 x & f'(a) &= 1 \\ f''(x) &= 2\sec^2 x \tan x & f''(a) &= 0 \\ f'''(x) &= 2\sec^4 x + 4\sec^2 x \tan^2 x & f'''(a) &= 2 \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 0 + 1(x-0) + \frac{0}{2}(x-0)^2 = x; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= 0 + 1(x-0) + \frac{0}{2}(x-0)^2 + \frac{2}{6}(x-0)^3 = x + \frac{1}{3}x^3. \end{aligned}$$

8. $f(x) = \tan x, \quad a = \frac{\pi}{4}$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \tan x & f(a) &= 1 \\ f'(x) &= \sec^2 x & f'(a) &= 2 \\ f''(x) &= 2\sec^2 x \tan x & f''(a) &= 4 \\ f'''(x) &= 2\sec^4 x + 4\sec^2 x \tan^2 x & f'''(a) &= 16 \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2}\left(x - \frac{\pi}{4}\right)^2 \\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{6}\left(x - \frac{\pi}{4}\right)^3 = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3. \end{aligned}$$

9. $f(x) = e^{-x} + e^{-2x}, \quad a = 0$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= e^{-x} + e^{-2x} & f(a) &= 2 \\ f'(x) &= -e^{-x} - 2e^{-2x} & f'(a) &= -3 \\ f''(x) &= e^{-x} + 4e^{-2x} & f''(a) &= 5 \\ f'''(x) &= -e^{-x} - 8e^{-2x} & f'''(a) &= -9 \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\ &= 2 + (-3)(x-0) + \frac{5}{2}(x-0)^2 = 2 - 3x + \frac{5}{2}x^2; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= 2 + (-3)(x-0) + \frac{5}{2}(x-0)^2 + \frac{-9}{6}(x-0)^3 = 2 - 3x + \frac{5}{2}x^2 - \frac{3}{2}x^3. \end{aligned}$$

10. $f(x) = e^{2x}, \quad a = \ln 2$

SOLUTION First calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= e^{2x} & f(a) &= 4 \\ f'(x) &= 2e^{2x} & f'(a) &= 8 \\ f''(x) &= 4e^{2x} & f''(a) &= 16 \\ f'''(x) &= 8e^{2x} & f'''(a) &= 32 \end{aligned}$$

Now

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = 4 + 8(x - \ln 2) + \frac{16}{2!}(x - \ln 2)^2 \\ &= 4 + 8(x - \ln 2) + 8(x - \ln 2)^2 \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &= 4 + 8(x - \ln 2) + \frac{16}{2!}(x - \ln 2)^2 + \frac{32}{6}(x - \ln 2)^3 \\ &= 4 + 8(x - \ln 2) + 8(x - \ln 2)^2 + \frac{16}{3}(x - \ln 2)^3 \end{aligned}$$

11. $f(x) = x^2 e^{-x}, \quad a = 1$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= x^2 e^{-x} & f(a) &= 1/e \\ f'(x) &= (2x - x^2)e^{-x} & f'(a) &= 1/e \\ f''(x) &= (x^2 - 4x + 2)e^{-x} & f''(a) &= -1/e \\ f'''(x) &= (-x^2 + 6x - 6)e^{-x} & f'''(a) &= -1/e \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\ &= \frac{1}{e} + \frac{1}{e}(x-1) + \frac{-1/e}{2}(x-1)^2 = \frac{1}{e} + \frac{1}{e}(x-1) - \frac{1}{2e}(x-1)^2; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= \frac{1}{e} + \frac{1}{e}(x-1) + \frac{-1/e}{2}(x-1)^2 + \left(\frac{-1/e}{6}\right)(x-1)^3 \\ &= \frac{1}{e} + \frac{1}{e}(x-1) - \frac{1}{2e}(x-1)^2 - \frac{1}{6e}(x-1)^3. \end{aligned}$$

12. $f(x) = \cosh 2x, \quad a = 0$

SOLUTION First calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \cosh 2x & f(a) &= 1 \\ f'(x) &= 2\sinh 2x & f'(a) &= 0 \\ f''(x) &= 4\cosh 2x & f''(a) &= 4 \\ f'''(x) &= 8\sinh 2x & f'''(a) &= 0 \end{aligned}$$

so that

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = 1 + 0(x-0) + \frac{4}{2!}(x-0)^2 \\ &= 1 + 2x^2 \end{aligned}$$

$$\begin{aligned}T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &= 1 + 0(x-0) + 2(x-0)^2 + \frac{0}{3!}(x-0)^3 \\ &= 1 + 2x^2\end{aligned}$$

13. $f(x) = \dfrac{\ln x}{x}, \quad a = 1$

SOLUTION First calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \frac{\ln x}{x} & f(a) &= 0 \\ f'(x) &= \frac{1-\ln x}{x^2} & f(a) &= 1 \\ f''(x) &= \frac{-3+2\ln x}{x^3} & f(a) &= -3 \\ f'''(x) &= \frac{11-6\ln x}{x^4} & f(a) &= 11 \end{aligned}$$

so that

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = 0 + 1(x-1) + \frac{-3}{2!}(x-1)^2 \\ &= (x-1) - \frac{3}{2}(x-1)^2 \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &= 0 + 1(x-1) + \frac{-3}{2!}(x-1)^2 + \frac{11}{3!}(x-1)^3 \\ &= (x-1) - \frac{3}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 \end{aligned}$$

14. $f(x) = \ln(x+1), \quad a = 0$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \ln(x+1) & f(a) &= 0 \\ f'(x) &= \frac{1}{x+1} & f'(a) &= 1 \\ f''(x) &= \frac{-1}{(x+1)^2} & f''(a) &= -1 \\ f'''(x) &= \frac{2}{(x+1)^3} & f'''(a) &= 2 \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 0 + 1(x-0) + \frac{-1}{2}(x-0)^2 = x - \frac{1}{2}x^2; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= 0 + 1(x-0) + \frac{-1}{2}(x-0)^2 + \frac{2}{6}(x-0)^3 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3. \end{aligned}$$

15. Show that the nth Maclaurin polynomial for e^x is

$$T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

SOLUTION With $f(x) = e^x$, it follows that $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for all n. Thus,

$$T_n(x) = 1 + 1(x-0) + \frac{1}{2}(x-0)^2 + \cdots + \frac{1}{n!}(x-0)^n = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.$$

16. Show that the nth Taylor polynomial for $\frac{1}{x+1}$ at $a = 1$ is

$$T_n(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8} + \cdots + (-1)^n \frac{(x-1)^n}{2^{n+1}}$$

SOLUTION Let $f(x) = \frac{1}{x+1}$. Then

$$\begin{aligned}
f(x) &= \frac{1}{x+1} & f(1) &= \frac{1}{2} = \frac{(-1)^0 0!}{2^{0+1}} \\
f'(x) &= \frac{-1}{(x+1)^2} & f'(1) &= -\frac{1}{4} = \frac{(-1)^1 1!}{2^{1+1}} \\
f''(x) &= \frac{2}{(x+1)^3} & f''(1) &= \frac{1}{4} = \frac{(-1)^2 2!}{2^{2+1}} \\
&\vdots & &\vdots \\
f^{(n)}(x) &= \frac{(-1)^n n!}{(x+1)^{n+1}} & f^{(n)}(1) &= \frac{(-1)^n n!}{2^{n+1}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
T_n(x) &= \frac{1}{2} + \left(-\frac{1}{4}\right)(x-1) + \frac{1}{4}\frac{(x-1)^2}{2!} + \cdots + \frac{(-1)^n n!}{2^{n+1}}\frac{(x-1)^n}{n!} \\
&= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{(x-1)^2}{8} + \cdots + (-1)^n \frac{(x-1)^n}{2^{n+1}}.
\end{aligned}$$

17. Show that the Maclaurin polynomials for $\sin x$ are

$$T_{2n+1}(x) = T_{2n+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

SOLUTION Let $f(x) = \sin x$. Then

$$\begin{aligned}
f(x) &= \sin x & f(0) &= 0 \\
f'(x) &= \cos x & f'(0) &= 1 \\
f''(x) &= -\sin x & f''(0) &= 0 \\
f'''(x) &= -\cos x & f'''(0) &= -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \\
&\vdots & &\vdots
\end{aligned}$$

Consequently,

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

and

$$T_{2n+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + 0 = T_{2n+1}(x).$$

18. Show that the Maclaurin polynomials for $\ln(1+x)$ are

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n}$$

SOLUTION Let $f(x) = \ln(1+x)$. Then

$$\begin{aligned}
f(x) &= \ln(1+x) & f(0) &= 0 \\
f'(x) &= (1+x)^{-1} & f'(0) &= 1
\end{aligned}$$

$$\begin{aligned} f''(x) &= -(1+x)^{-2} & f''(0) &= -1 \\ f'''(x) &= 2(1+x)^{-3} & f'''(0) &= 2 \\ f^{(4)}(x) &= -3!(1+x)^{-4} & f^{(4)}(0) &= -6 \\ f^{(5)}(x) &= 4!(1+x)^{-5} & f^{(5)}(0) &= 24 \end{aligned}$$

so that in general

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n} \qquad f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

Thus

$$T_n(x) = x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \cdots + \frac{(-1)^{n-1}(n-1)!}{n!}x^n = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n}$$

In Exercises 19–24, find $T_n(x)$ at $x = a$ for all n.

19. $f(x) = \dfrac{1}{1+x}, \quad a = 0$

SOLUTION We have

$$\frac{1}{1+x} = (\ln(1+x))'$$

so that from Exercise 18, letting $g(x) = \ln(1+x)$,

$$f^{(n)}(x) = g^{(n+1)}(x) = (-1)^n n!(x+1)^{-1-n} \quad \text{and} \quad f^{(n)}(0) = (-1)^n n!$$

Then

$$\begin{aligned} T_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= 1 - x + \frac{2!}{2!}x^2 - \frac{3!}{3!}x^3 + \cdots + (-1)^n\frac{n!}{n!}x^n \\ &= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n \end{aligned}$$

20. $f(x) = \dfrac{1}{x-1}, \quad a = 4$

SOLUTION Let $f(x) = \frac{1}{x-1}$. Then

$$\begin{aligned} f(x) &= \frac{1}{x-1} & f(4) &= \frac{1}{3} = \frac{(-1)^0 0!}{3^{0+1}} \\ f'(x) &= \frac{-1}{(x-1)^2} & f'(4) &= -\frac{1}{9} = \frac{(-1)^1 1!}{3^{1+1}} \\ f''(x) &= \frac{2}{(x-1)^3} & f''(4) &= \frac{2}{27} = \frac{(-1)^2 2!}{3^{2+1}} \\ &\vdots & &\vdots \\ f^{(n)}(x) &= \frac{(-1)^n n!}{(x-1)^{n+1}} & f^{(n)}(4) &= \frac{(-1)^n n!}{3^{n+1}} \end{aligned}$$

Therefore,

$$\begin{aligned} T_n(x) &= \frac{1}{3} + \left(-\frac{1}{9}\right)(x-4) + \frac{2/27}{2}(x-4)^2 + \cdots + \frac{(-1)^n n!}{3^{n+1}}\frac{(x-4)^n}{n!} \\ &= \frac{1}{3} - \frac{1}{9}(x-4) + \frac{1}{27}(x-4)^2 + \cdots + \frac{(-1)^n}{3^{n+1}}(x-4)^n. \end{aligned}$$

21. $f(x) = e^x, \quad a = 1$

SOLUTION Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ and $f^{(n)}(1) = e$ for all n. Therefore,

$$T_n(x) = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \cdots + \frac{e}{n!}(x-1)^n.$$

22. $f(x) = x^{-2}, \quad a = 2$

SOLUTION We have

$$\begin{aligned} f(x) &= x^{-2} & f(2) &= \frac{1}{4} \\ f'(x) &= -2x^{-3} & f'(2) &= -\frac{1}{4} \\ f''(x) &= 6x^{-4} & f''(2) &= \frac{3}{8} \\ f'''(x) &= -24x^{-5} & f'''(2) &= -\frac{3}{4} \\ &\vdots & &\vdots \\ f^{(n)}(x) &= (-1)^n (n+1)! x^{-n-2} & f^{(n)}(2) &= (-1)^n \frac{(n+1)!}{2^{n+2}} \end{aligned}$$

so that

$$\begin{aligned} T_n(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 + \cdots + (-1)^n \frac{n+1}{2^{n+2}}(x-2)^n \end{aligned}$$

23. $f(x) = \cos x, \quad a = \dfrac{\pi}{4}$

SOLUTION Let $f(x) = \cos x$. Then

$$\begin{aligned} f(x) &= \cos x & f(\pi/4) &= \frac{1}{\sqrt{2}} \\ f'(x) &= -\sin x & f'(\pi/4) &= -\frac{1}{\sqrt{2}} \\ f''(x) &= -\cos x & f''(\pi/4) &= -\frac{1}{\sqrt{2}} \\ f'''(x) &= \sin x & f'''(\pi/4) &= \frac{1}{\sqrt{2}} \end{aligned}$$

This pattern of four values repeats indefinitely. Thus,

$$f^{(n)}(\pi/4) = \begin{cases} (-1)^{(n+1)/2} \dfrac{1}{\sqrt{2}}, & n \text{ odd} \\ (-1)^{n/2} \dfrac{1}{\sqrt{2}}, & n \text{ even} \end{cases}$$

and

$$T_n(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 \cdots.$$

In general, the coefficient of $(x - \pi/4)^n$ is

$$\pm \frac{1}{(\sqrt{2})n!}$$

with the pattern of signs $+, -, -, +, +, -, -, \ldots$.

24. $f(\theta) = \sin 3\theta, \quad a = 0$

SOLUTION We have

$$\begin{aligned} f(\theta) &= \sin 3\theta & f(0) &= 0 \\ f'(\theta) &= 3\cos 3\theta & f'(0) &= 3 \\ f''(\theta) &= -9\sin 3\theta & f''(0) &= 0 \\ f'''(\theta) &= -27\cos 3\theta & f'''(0) &= -27 \\ f^{(4)}(\theta) &= 81\sin 3\theta & f^{(4)}(0) &= 0 \end{aligned}$$

and in general

$$f^{(2n)}(\theta) = (-1)^n 3^{2n} \sin 3\theta \qquad f^{(2n)}(0) = 0$$
$$f^{(2n+1)}(\theta) = (-1)^n 3^{2n+1} \cos 3\theta \qquad f^{(2n+1)}(0) = (-1)^n 3^{2n+1}$$

Thus

$$T_n(x) = 3\theta - \frac{27}{3!}\theta^3 + \frac{243}{5!}\theta^5 - \dots$$

where the coefficient of θ^{2n+1} is $(-1)^n \frac{3^{2n+1}}{(2n+1)!}$.

In Exercises 25–28, find $T_2(x)$ and use a calculator to compute the error $|f(x) - T_2(x)|$ for the given values of a and x.

25. $y = e^x, \quad a = 0, \quad x = -0.5$

SOLUTION Let $f(x) = e^x$. Then $f'(x) = e^x$, $f''(x) = e^x$, $f(a) = 1$, $f'(a) = 1$ and $f''(a) = 1$. Therefore

$$T_2(x) = 1 + 1(x-0) + \frac{1}{2}(x-0)^2 = 1 + x + \frac{1}{2}x^2,$$

and

$$T_2(-0.5) = 1 + (-0.5) + \frac{1}{2}(-0.5)^2 = 0.625.$$

Using a calculator, we find

$$f(-0.5) = \frac{1}{\sqrt{e}} = 0.606531,$$

so

$$|T_2(-0.5) - f(-0.5)| = 0.0185.$$

26. $y = \cos x, \quad a = 0, \quad x = \frac{\pi}{12}$

SOLUTION Let $f(x) = \cos x$. Then $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f(a) = 1$, $f'(a) = 0$, and $f''(a) = -1$. Therefore

$$T_2(x) = 1 + 0(x-0) + \frac{-1}{2}(x-0)^2 = 1 - \frac{1}{2}x^2,$$

and

$$T_2\left(\frac{\pi}{12}\right) = 1 - \frac{1}{2}\left(\frac{\pi}{12}\right)^2 \approx 0.965731.$$

Using a calculator, we find

$$f\left(\frac{\pi}{12}\right) = 0.965926,$$

so

$$\left|T_2\left(\frac{\pi}{12}\right) - f\left(\frac{\pi}{12}\right)\right| = 0.000195.$$

27. $y = x^{-2/3}, \quad a = 1, \quad x = 1.2$

SOLUTION Let $f(x) = x^{-2/3}$. Then $f'(x) = -\frac{2}{3}x^{-5/3}$, $f''(x) = \frac{10}{9}x^{-8/3}$, $f(1) = 1$, $f'(1) = -\frac{2}{3}$, and $f''(1) = \frac{10}{9}$. Thus

$$T_2(x) = 1 - \frac{2}{3}(x-1) + \frac{10}{2 \cdot 9}(x-1)^2 = 1 - \frac{2}{3}(x-1) + \frac{5}{9}(x-1)^2$$

and

$$T_2(1.2) = 1 - \frac{2}{3}(0.2) + \frac{5}{9}(0.2)^2 = \frac{8}{9} \approx 0.88889$$

Using a calculator, $f(1.2) = (1.2)^{-2/3} \approx 0.88555$ so that

$$|T_2(1.2) - f(1.2)| \approx 0.00334$$

28. $y = e^{\sin x}, \quad a = \frac{\pi}{2}, \quad x = 1.5$

SOLUTION Let $f(x) = e^{\sin x}$. Then $f'(x) = \cos x e^{\sin x}$, $f''(x) = \cos^2 x e^{\sin x} - \sin x e^{\sin x}$, $f(a) = e$, $f'(a) = 0$ and $f''(a) = -e$. Therefore

$$T_2(x) = e + 0\left(x - \frac{\pi}{2}\right) + \frac{-e}{2}\left(x - \frac{\pi}{2}\right)^2 = e - \frac{e}{2}\left(x - \frac{\pi}{2}\right)^2,$$

and

$$T_2(1.5) = e - \frac{e}{2}\left(1.5 - \frac{\pi}{2}\right)^2 \approx 2.711469651.$$

Using a calculator, we find $f(1.5) = 2.711481018$, so

$$|T_2(1.5) - f(1.5)| = 1.14 \times 10^{-5}.$$

29. GU Compute $T_3(x)$ for $f(x) = \sqrt{x}$ centered at $a = 1$. Then use a plot of the error $|f(x) - T_3(x)|$ to find a value $c > 1$ such that the error on the interval $[1, c]$ is at most 0.25.

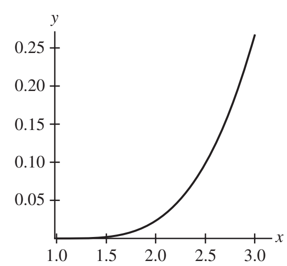
SOLUTION We have

$$\begin{aligned} f(x) &= x^{1/2} & f(1) &= 1 \\ f'(x) &= \frac{1}{2}x^{-1/2} & f'(1) &= \frac{1}{2} \\ f''(x) &= -\frac{1}{4}x^{-3/2} & f''(1) &= -\frac{1}{4} \\ f'''(x) &= \frac{3}{8}x^{-5/2} & f'''(1) &= \frac{3}{8} \end{aligned}$$

Therefore

$$T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4 \cdot 2!}(x-1)^2 + \frac{3}{8 \cdot 3!}(x-1)^3 = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$$

A plot of $|f(x) - T_3(x)|$ is below.



It appears that for $x \in [1, 2.9]$ that the error does not exceed 0.25. The error at $x = 3$ appears to just exceed 0.25.

30. CAS Plot $f(x) = 1/(1+x)$ together with the Taylor polynomials $T_n(x)$ at $a = 1$ for $1 \le n \le 4$ on the interval $[-2, 8]$ (be sure to limit the upper plot range).

(a) Over which interval does $T_4(x)$ appear to approximate $f(x)$ closely?

(b) What happens for $x < -1$?

(c) Use your computer algebra system to produce and plot T_{30} together with $f(x)$ on $[-2, 8]$. Over which interval does T_{30} appear to give a close approximation?

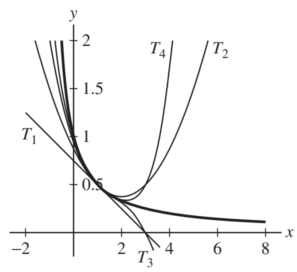
SOLUTION Let $f(x) = \frac{1}{1+x}$. Then

$$\begin{aligned} f(x) &= \frac{1}{1+x} & f(1) &= \frac{1}{2} \\ f'(x) &= -\frac{1}{(1+x)^2} & f'(1) &= -\frac{1}{4} \\ f''(x) &= \frac{2}{(1+x)^3} & f''(1) &= \frac{1}{4} \\ f'''(x) &= -\frac{6}{(1+x)^4} & f'''(1) &= -\frac{3}{8} \\ f^{(4)}(x) &= \frac{24}{(1+x)^5} & f^{(4)}(1) &= \frac{3}{4} \end{aligned}$$

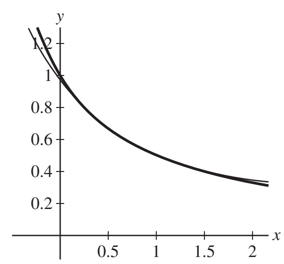
and

$$T_1(x) = \frac{1}{2} - \frac{1}{4}(x-1);$$
$$T_2(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2;$$
$$T_3(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3; \text{ and}$$
$$T_4(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-1)^4.$$

A plot of $f(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$ and $T_4(x)$ is shown below.

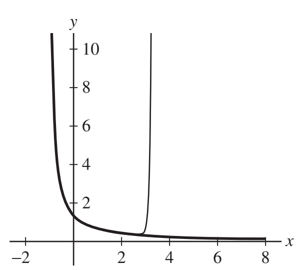


(a) The graph below displays $f(x)$ and $T_4(x)$ over the interval $[-0.5, 2.5]$. It appears that $T_4(x)$ gives a close approximation to $f(x)$ over the interval $(0.1, 2)$.



(b) For $x < -1$, $f(x)$ is negative, but the Taylor polynomials are positive; thus, the Taylor polynomials are poor approximations for $x < -1$.

(c) The graph below displays $f(x)$ and $T_{30}(x)$ over the interval $[-2, 8]$. It appears that $T_{30}(x)$ gives a close approximation to $f(x)$ over the interval $(-1, 3)$.



31. Let $T_3(x)$ be the Maclaurin polynomial of $f(x) = e^x$. Use the error bound to find the maximum possible value of $|f(1.1) - T_3(1.1)|$. Show that we can take $K = e^{1.1}$.

SOLUTION Since $f(x) = e^x$, we have $f^{(n)}(x) = e^x$ for all n; since e^x is increasing, the maximum value of e^x on the interval $[0, 1.1]$ is $K = e^{1.1}$. Then by the error bound,

$$\left| e^{1.1} - T_3(1.1) \right| \le K \frac{(1.1-0)^4}{4!} = \frac{e^{1.1} 1.1^4}{24} \approx 0.183$$

32. Let $T_2(x)$ be the Taylor polynomial of $f(x) = \sqrt{x}$ at $a = 4$. Apply the error bound to find the maximum possible value of the error $|f(3.9) - T_2(3.9)|$.

SOLUTION We have $f(x) = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, and $f'''(x) = \frac{3}{8}x^{-5/2}$. This is a decreasing function of x, so its maximum value on $[3.9, 4]$ is achieved at $x = 3.9$; that value is $\frac{3}{8\cdot 3.9^{5/2}} \approx 0.0125$, so we can take $K = 0.0125$. Then

$$|f(x) - T_2(x)| \le K\frac{|3.9-4|^3}{3!} = 0.0125\frac{0.001}{6} \approx 2.08 \times 10^{-6}$$

In Exercises 33–36, compute the Taylor polynomial indicated and use the error bound to find the maximum possible size of the error. Verify your result with a calculator.

33. $f(x) = \cos x, \quad a = 0; \quad |\cos 0.25 - T_5(0.25)|$

SOLUTION The Maclaurin series for $\cos x$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

so that

$$T_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$
$$T_5(0.25) \approx 0.9689127604$$

In addition, $f^{(6)}(x) = -\cos x$ so that $|f^{(6)}(x)| \le 1$ and we may take $K = 1$ in the error bound formula. Then

$$|\cos 0.25 - T_5(0.25)| \le K\frac{0.25^6}{6!} = \frac{1}{2^{12}\cdot 6!} \approx 3.390842014 \cdot 10^{-7}$$

(The true value is $\cos 0.25 \approx 0.9689124217$ and the difference is in fact $\approx 3.387 \cdot 10^{-7}$.)

34. $f(x) = x^{11/2}, \quad a = 1; \quad |f(1.2) - T_4(1.2)|$

SOLUTION Let $f(x) = x^{11/2}$. Then

$$\begin{aligned} f(x) &= x^{11/2} & f(1) &= 1 \\ f'(x) &= \frac{11}{2}x^{9/2} & f'(1) &= \frac{11}{2} \\ f''(x) &= \frac{99}{4}x^{7/2} & f''(1) &= \frac{99}{4} \\ f'''(x) &= \frac{693}{8}x^{5/2} & f'''(1) &= \frac{693}{8} \\ f^{(4)}(x) &= \frac{3465}{16}x^{3/2} & f^{(4)}(1) &= \frac{3465}{16} \end{aligned}$$

and

$$T_4(x) = 1 + \frac{11}{2}(x-1) + \frac{99}{8}(x-1)^2 + \frac{231}{16}(x-1)^3 + \frac{1155}{128}(x-1)^4.$$

Using the Error Bound,

$$|f(1.2) - T_4(1.2)| \le \frac{K|1.2-1|^5}{5!} = \frac{K}{375{,}000},$$

where K is a number such that $|f^{(5)}(x)| \le K$ for x between 1 and 1.2. Now,

$$f^{(5)}(x) = \frac{10{,}395}{32}x^{1/2},$$

which is increasing for $x > 1$. Consequently, on the interval $[1, 1.2]$, $f^{(5)}(x)$ is maximized at $x = 1.2$. We can therefore take $K = \frac{10{,}395}{32}\sqrt{1.2}$, and then

$$|f(1.2) - T_4(1.2)| \le \frac{10{,}395}{(32)(375{,}000)}\sqrt{1.2} \approx 9.489 \times 10^{-4}.$$

35. $f(x) = x^{-1/2}, \quad a = 4; \quad |f(4.3) - T_3(4.3)|$

SOLUTION We have

$$\begin{aligned} f(x) &= x^{-1/2} & f(4) &= \frac{1}{2} \\ f'(x) &= -\frac{1}{2}x^{-3/2} & f'(4) &= -\frac{1}{16} \\ f''(x) &= \frac{3}{4}x^{-5/2} & f''(4) &= \frac{3}{128} \\ f'''(x) &= -\frac{15}{8}x^{-7/2} & f'''(4) &= -\frac{15}{1024} \\ f^{(4)}(x) &= \frac{105}{16}x^{-9/2} \end{aligned}$$

so that

$$T_3(x) = \frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{256}(x-4)^2 - \frac{5}{2048}(x-4)^3$$

Using the error bound formula,

$$|f(4.3) - T_3(4.3)| \le K\frac{|4.3-4|^4}{4!} = \frac{27K}{80{,}000}$$

where K is a number such that $|f^{(4)}(x)| \le K$ for x between 4 and 4.3. Now, $f^{(4)}(x)$ is a decreasing function for $x > 1$, so it takes its maximum value on $[4, 4.3]$ at $x = 4$; there, its value is

$$K = \frac{105}{16}4^{-9/2} = \frac{105}{8192}$$

so that

$$|f(4.3) - T_3(4.3)| \le \frac{27\frac{105}{8192}}{80{,}000} = \frac{27 \cdot 105}{8192 \cdot 80{,}000} \approx 4.3258667 \cdot 10^{-6}$$

36. $f(x) = \sqrt{1+x}, \quad a = 8; \quad |\sqrt{9.02} - T_3(8.02)|$

SOLUTION Let $f(x) = \sqrt{1+x}$. Then

$$\begin{aligned} f(x) &= \sqrt{1+x} & f(8) &= 3 \\ f'(x) &= \frac{1}{2}(x+1)^{-1/2} & f'(8) &= \frac{1}{6} \\ f''(x) &= \frac{-1}{4}(x+1)^{-3/2} & f''(8) &= \frac{-1}{108} \\ f'''(x) &= \frac{3}{8}(x+1)^{-5/2} & f'''(8) &= \frac{1}{648} \end{aligned}$$

and

$$T_3(x) = 3 + \frac{1}{6}(x-8) - \frac{1}{108 \cdot 2!}(x-8)^2 + \frac{1}{648 \cdot 3!}(x-8)^3 = 3 + \frac{1}{6}(x-8) - \frac{1}{216}(x-8)^2 + \frac{1}{3888}(x-8)^3.$$

Therefore

$$T_3(8.02) = 3 + \frac{1}{6}(0.02) - \frac{1}{216}(0.02)^2 + \frac{1}{3888}(0.02)^3 = 3.003331484.$$

Using the Error Bound, we have

$$|\sqrt{9.02} - T_3(8.02)| \le K\frac{|8.02-8|^4}{4!} = \frac{K}{150{,}000{,}000},$$

where K is a number such that $|f^{(4)}(x)| \le K$ for x between 8 and 8.02. Now

$$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2},$$

which is a decreasing function for $8 \le x \le 8.02$, so we may take

$$K = \frac{15}{16}9^{-7/2} = \frac{15}{34992}.$$

Thus,

$$|\sqrt{9.02} - T_3(8.02)| \le \frac{15/34992}{150{,}000{,}000} \approx 2.858 \times 10^{-12}.$$

37. Calculate the Maclaurin polynomial $T_3(x)$ for $f(x) = \tan^{-1} x$. Compute $T_3\left(\frac{1}{2}\right)$ and use the error bound to find a bound for the error $\left|\tan^{-1}\frac{1}{2} - T_3\left(\frac{1}{2}\right)\right|$. Refer to the graph in Figure 10 to find an acceptable value of K. Verify your result by computing $\left|\tan^{-1}\frac{1}{2} - T_3\left(\frac{1}{2}\right)\right|$ using a calculator.

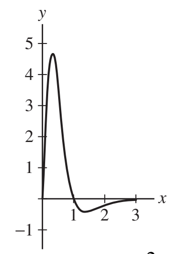


FIGURE 10 Graph of $f^{(4)}(x) = \dfrac{-24x(x^2-1)}{(x^2+1)^4}$, where $f(x) = \tan^{-1} x$.

SOLUTION Let $f(x) = \tan^{-1} x$. Then

$$f(x) = \tan^{-1} x \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1+x^2} \qquad f'(0) = 1$$

$$f''(x) = \frac{-2x}{(1+x^2)^2} \qquad f''(0) = 0$$

$$f'''(x) = \frac{(1+x^2)^2(-2) - (-2x)(2)(1+x^2)(2x)}{(1+x^2)^4} \qquad f'''(0) = -2$$

and

$$T_3(x) = 0 + 1(x-0) + \frac{0}{2}(x-0)^2 + \frac{-2}{6}(x-0)^3 = x - \frac{x^3}{3}.$$

Since $f^{(4)}(x) \le 5$ for $x \ge 0$, we may take $K = 5$ in the error bound; then,

$$\left|\tan^{-1}\left(\frac{1}{2}\right) - T_3\left(\frac{1}{2}\right)\right| \le \frac{5(1/2)^4}{4!} = \frac{5}{384}.$$

38. Let $f(x) = \ln(x^3 - x + 1)$. The third Taylor polynomial at $a = 1$ is

$$T_3(x) = 2(x-1) + (x-1)^2 - \frac{7}{3}(x-1)^3$$

Find the maximum possible value of $|f(1.1) - T_3(1.1)|$, using the graph in Figure 11 to find an acceptable value of K. Verify your result by computing $|f(1.1) - T_3(1.1)|$ using a calculator.

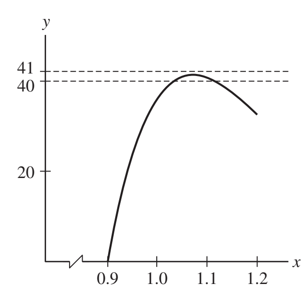


FIGURE 11 Graph of $f^{(4)}(x)$, where $f(x) = \ln(x^3 - x + 1)$.

SOLUTION The maximum value of $f^{(4)}(x)$ on $[1.0, 1.1]$ is less than 41, so we may take $K = 41$. Then

$$|f(1.1) - T_3(1.1)| \le K\frac{|1.1-1|^4}{4!} = \frac{41}{24 \cdot 10{,}000} \approx 0.00017083$$

In fact, we have

$$f(1.1) = \ln(1.1^3 - 1.1 + 1) = \ln(1.231) \approx 0.2078268472$$

$$T_3(1.1) = 2(1.1 - 1) + (1.1 - 1)^2 - \frac{7}{3}(1.1 - 1)^3 \approx 0.2076666667$$

$$|f(1.1) - T_3(1.1)| \approx 0.2078268472 - 0.2076666667 = 0.0001601805$$

which is in accordance with the error bound above.

39. GU Calculate the $T_3(x)$ at $a = 0.5$ for $f(x) = \cos(x^2)$, and use the error bound to find the maximum possible value of $|f(0.6) - T_3(0.6)|$. Plot $f^{(4)}(x)$ to find an acceptable value of K.

SOLUTION We have

$$f(x) = \cos(x^2) \qquad f(0.5) = \cos(0.25) \approx 0.9689$$

$$f'(x) = -2x\sin(x^2) \qquad f'(0.5) = -\sin(0.25) \approx -0.2474039593$$

$$f''(x) = -4x^2\cos(x^2) - 2\sin(x^2) \qquad f''(0.5) = -\cos(0.25) - 2\sin(0.25) \approx -1.463720340$$

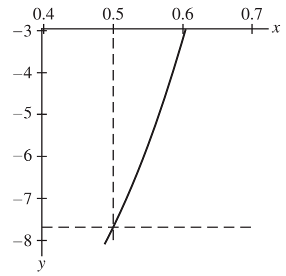
$$f'''(x) = 8x^3\sin(x^2) - 12x\cos(x^2) \qquad f'''(0.5) = \sin(0.25) - 6\cos(0.25) \approx -5.566070571$$

$$f^{(4)}(x) = 16x^4\cos(x^2) + 48x^2\sin(x^2) - 12\cos(x^2)$$

so that

$$T_3(x) = 0.9689 - 0.2472039593(x - 0.5) - 0.73186017(x - 0.5)^2 - 0.92767843(x - 0.5)^3$$

and $T_3(0.6) \approx 0.9359257453$. A graph of $f^{(4)}(x)$ for x near 0.5 is below.



Clearly the maximum value of $|f^{(4)}(x)|$ on $[0.5, 0.6]$ is bounded by 8 (near $x = 0.5$), so we may take $K = 8$; then

$$|f(0.6) - T_3(0.6)| \le K\frac{|0.6 - 0.5|^4}{4!} = \frac{8}{240{,}000} \approx 0.000033333$$

40. GU Calculate the Maclaurin polynomial $T_2(x)$ for $f(x) = \operatorname{sech} x$ and use the error bound to find the maximum possible value of $\left|f\left(\frac{1}{2}\right) - T_2\left(\frac{1}{2}\right)\right|$. Plot $f'''(x)$ to find an acceptable value of K.

SOLUTION To compute $T_2(x)$ for $f(x) = \operatorname{sech} x$, we take the first two derivatives:

$$f(x) = \operatorname{sech} x \qquad f(0) = 1$$

$$f'(x) = -\operatorname{sech} x \tanh x \qquad f'(0) = 0$$

$$f''(x) = \operatorname{sech} x \tanh^2 x - \operatorname{sech}^3 x \qquad f''(0) = -1$$

From this,

$$T_2(x) = 1 - \frac{1}{2}x^2,$$

and

$$T_2\left(\frac{1}{2}\right) = 1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 = 1 - \frac{1}{8} = \frac{7}{8}.$$

Using the Error Bound, we have

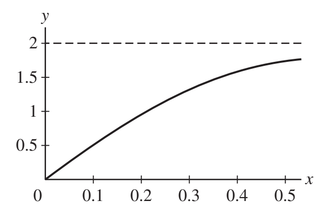
$$\left|f\left(\frac{1}{2}\right) - T_2\left(\frac{1}{2}\right)\right| \le K\frac{|1/2|^3}{6} = \frac{K}{48},$$

where K is a number such that $|f'''(x)| \le K|$ for x between 0 and $\frac{1}{2}$. Here,

$$f'''(x) = -\operatorname{sech} x \tanh^3 x + 2\operatorname{sech}^3 x \tanh x + 3\operatorname{sech}^2 x(\operatorname{sech} x \tanh x)$$
$$= 5\operatorname{sech}^2 x \tanh x - \operatorname{sech} x \tanh^3 x.$$

A plot of $f'''(x)$ is given below. From the plot, we see that $|f'''(x)| \le 2$ for all x between 0 and 1/2. Thus,

$$\left| f\left(\frac{1}{2}\right) - T_2\left(\frac{1}{2}\right) \right| \le \frac{2}{48} = \frac{1}{24}.$$



In Exercises 41–44, use the error bound to find a value of n for which the given inequality is satisfied. Then verify your result using a calculator.

41. $|\cos 0.1 - T_n(0.1)| \le 10^{-7}, \quad a = 0$

SOLUTION Using the error bound with $K = 1$ (every derivative of $f(x) = \cos x$ is $\pm\sin x$ or $\pm\cos x$, so $|f^{(n)}(x)| \le 1$ for all n), we have

$$|T_n(0.1) - \cos 0.1| \le \frac{(0.1)^{n+1}}{(n+1)!}.$$

With $n = 3$,

$$\frac{(0.1)^4}{4!} \approx 4.17 \times 10^{-6} > 10^{-7},$$

but with $n = 4$,

$$\frac{(0.1)^5}{5!} \approx 8.33 \times 10^{-8} < 10^{-7},$$

so we choose $n = 4$. Now,

$$T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4,$$

so

$$T_4(0.1) = 1 - \frac{1}{2}(0.1)^2 + \frac{1}{24}(0.1)^4 = 0.995004166.$$

Using a calculator, $\cos 0.1 = 0.995004165$, so

$$|T_4(0.1) - \cos 0.1| = 1.387 \times 10^{-8} < 10^{-7}.$$

42. $|\ln 1.3 - T_n(1.3)| \le 10^{-4}, \quad a = 1$

SOLUTION Let $f(x) = \ln x$. Then $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -6x^{-4}$, etc. In general,

$$f^{(n)}(x) = (-1)^{n+1}(n-1)!x^{-n}.$$

Now, $|f^{(n+1)}(x)|$ is decreasing on the interval $[1, 1.3]$, so $|f^{(n+1)}(x)| \le |f^{(n+1)}(1)| = n!$ for all $x \in [1, 1.3]$. We can therefore take $K = n!$ in the error bound, and

$$|\ln 1.3 - T_n(1.3)| \le n!\frac{|1.3 - 1|^{n+1}}{(n+1)!} = \frac{(0.3)^{n+1}}{n+1}.$$

With $n = 5$,

$$\frac{(0.3)^6}{6} = 1.215 \times 10^{-4} > 10^{-4},$$

but with $n = 6$,

$$\frac{(0.3)^7}{7} = 3.124 \times 10^{-5} < 10^{-4}.$$

Therefore, the error is guaranteed to be below 10^{-4} for $n = 6$. Now,

$$T_6(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6$$

and $T_6(1.3) \approx 0.2623395000$. Using a calculator, $\ln(1.3) \approx 0.2623642645$; the difference is

$$\ln(1.3) - T_6(1.3) \approx 0.0000247645 < 10^{-4}$$

43. $|\sqrt{1.3} - T_n(1.3)| \le 10^{-6}, \quad a = 1$

SOLUTION Using the Error Bound, we have

$$|\sqrt{1.3} - T_n(1.3)| \le K\frac{|1.3-1|^{n+1}}{(n+1)!} = K\frac{|0.3|^{n+1}}{(n+1)!},$$

where K is a number such that $|f^{(n+1)}(x)| \le K$ for x between 1 and 1.3. For $f(x) = \sqrt{x}$, $|f^{(n)}(x)|$ is decreasing for $x > 1$, hence the maximum value of $|f^{(n+1)}(x)|$ occurs at $x = 1$. We may therefore take

$$K = |f^{(n+1)}(1)| = \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2^{n+1}}$$
$$= \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2^{n+1}} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n+2)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} = \frac{(2n+2)!}{(n+1)!2^{2n+2}}.$$

Then

$$|\sqrt{1.3} - T_n(1.3)| \le \frac{(2n+2)!}{(n+1)!2^{2n+2}} \cdot \frac{|0.3|^{n+1}}{(n+1)!} = \frac{(2n+2)!}{[(n+1)!]^2}(0.075)^{n+1}.$$

With $n = 9$

$$\frac{(20)!}{[(10)!]^2}(0.075)^{10} = 1.040 \times 10^{-6} > 10^{-6},$$

but with $n = 10$

$$\frac{(22)!}{[(11)!]^2}(0.075)^{11} = 2.979 \times 10^{-7} < 10^{-6}.$$

Hence, $n = 10$ will guarantee the desired accuracy. Using technology to compute and evaluate $T_{10}(1.3)$ gives

$$T_{10}(1.3) \approx 1.140175414, \qquad \sqrt{1.3} \approx 1.140175425$$

and

$$|\sqrt{1.3} - T_{10}(1.3)| \approx 1.1 \times 10^{-8} < 10^{-6}$$

44. $|e^{-0.1} - T_n(-0.1)| \le 10^{-6}, \quad a = 0$

SOLUTION Using the Error Bound, we have

$$|e^{-0.1} - T_n(-0.1)| \le K\frac{|-0.1-0|^{n+1}}{(n+1)!} = K\frac{1}{10^{n+1}(n+1)!}$$

where K is a number such that $|f^{(n+1)}(x)| \le K$ for x between -0.1 and 0. Since $f(x) = e^x$, $f^{(n)}(x) = e^x$ for all n; this is an increasing function, so it takes its maximum value at $x = 0$; this value is 1. So we may take $K = 1$ and then

$$|e^{-0.1} - T_n(-0.1)| \le \frac{1}{10^{n+1}(n+1)!}$$

With $n = 3$

$$\frac{1}{10^4 \cdot 24} = \frac{1}{240{,}000} \approx 4.166666667 \times 10^{-6} > 10^{-6}$$

but with $n = 4$

$$\frac{1}{10^5 \cdot 120} = \frac{1}{12{,}000{,}000} \approx 8.333333333 \times 10^{-8} < 10^{-6}$$

Thus $n = 4$ will guarantee the desired accuracy. Using technology to compute $T_4(x)$ and evalute,

$$T_4(-0.1) \approx 0.9048375000, \qquad e^{-0.1} \approx 0.9048374180$$

and

$$|e^{-0.1} - T_4(-0.1)| \approx 8.2 \times 10^{-8} < 10^{-6}$$

45. Let $f(x) = e^{-x}$ and $T_3(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$. Use the error bound to show that for all $x \geq 0$,

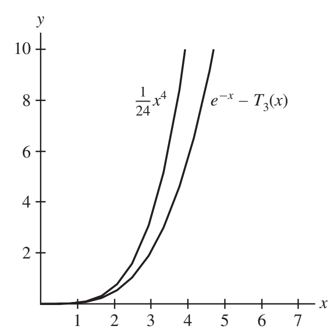
$$|f(x) - T_3(x)| \leq \frac{x^4}{24}$$

If you have a GU, illustrate this inequality by plotting $f(x) - T_3(x)$ and $x^4/24$ together over [0, 1].

SOLUTION Note that $f^{(n)}(x) = \pm e^{-x}$, so that $|f^{(n)}(x)| = f(x)$. Now, $f(x)$ is a decreasing function for $x \geq 0$, so that for any $c > 0$, $|f^{(n)}(x)|$ takes its maximum value at $x = 0$; this value is $e^0 = 1$. Thus we may take $K = 1$ in the error bound equation. Thus for any x,

$$|f(x) - T_3(x)| \leq K\frac{|x-0|^4}{4!} = \frac{x^4}{24}$$

A plot of $f(x) - T_3(x)$ and $\frac{x^4}{24}$ is shown below.



46. Use the error bound with $n = 4$ to show that

$$\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \leq \frac{|x|^5}{120} \quad \text{(for all } x\text{)}$$

SOLUTION Note that all derivatives of $\sin x$ are either $\pm\cos x$ or $\pm\sin x$ so are bounded in absolute value by 1. Thus we may take $K = 1$ in the Error Bound. Now,

$$T_4(x) = x - \frac{x^3}{3!}$$

so that

$$|\sin x - T_4(x)| = \left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \leq K\frac{|x-0|^5}{5!} = \frac{|x|^5}{120}$$

47. Let $T_n(x)$ be the Taylor polynomial for $f(x) = \ln x$ at $a = 1$, and let $c > 1$. Show that

$$|\ln c - T_n(c)| \leq \frac{|c-1|^{n+1}}{n+1}$$

Then find a value of n such that $|\ln 1.5 - T_n(1.5)| \leq 10^{-2}$.

SOLUTION With $f(x) = \ln x$, we have

$$f'(x) = x^{-1},\ f''(x) = -x^{-2},\ f'''(x) = 2x^{-3},\ f^{(4)}(x) = -6x^{-4},$$

and, in general,

$$f^{(k+1)}(x) = (-1)^k k!\, x^{-k-1}.$$

Notice that $|f^{(k+1)}(x)| = k!|x|^{-k-1}$ is a decreasing function for $x > 0$. Therefore, the maximum value of $|f^{(k+1)}(x)|$ on $[1, c]$ is $|f^{(k+1)}(1)|$. Using the Error Bound, we have

$$|\ln c - T_n(c)| \le K\frac{|c-1|^{n+1}}{(n+1)!},$$

where K is a number such that $|f^{(n+1)}(x)| \le K$ for x between 1 and c. From part (a), we know that we may take $K = |f^{(n+1)}(1)| = n!$. Then

$$|\ln c - T_n(c)| \le n!\frac{|c-1|^{n+1}}{(n+1)!} = \frac{|c-1|^{n+1}}{n+1}.$$

Evaluating at $c = 1.5$ gives

$$|\ln 1.5 - T_n(1.5)| \le \frac{|1.5-1|^{n+1}}{n+1} = \frac{(0.5)^{n+1}}{n+1}.$$

With $n = 3$,

$$\frac{(0.5)^4}{4} = 0.015625 > 10^{-2}.$$

but with $n = 4$,

$$\frac{(0.5)^5}{5} = 0.00625 < 10^{-2}.$$

Hence, $n = 4$ will guarantee the desired accuracy.

48. Let $n \ge 1$. Show that if $|x|$ is small, then

$$(x+1)^{1/n} \approx 1 + \frac{x}{n} + \frac{1-n}{2n^2}x^2$$

Use this approximation with $n = 6$ to estimate $1.5^{1/6}$.

SOLUTION Let $f(x) = (x+1)^{1/n}$. Then

$$\begin{aligned} f(x) &= (x+1)^{1/n} & f(0) &= 1 \\ f'(x) &= \frac{1}{n}(x+1)^{1/n-1} & f'(0) &= \frac{1}{n} \\ f''(x) &= \frac{1}{n}\left(\frac{1}{n}-1\right)(x+1)^{1/n-2} & f''(0) &= \frac{1}{n}\left(\frac{1}{n}-1\right) \end{aligned}$$

and

$$T_2(x) = 1 + \frac{1}{n}(x) + \left(\frac{1}{n^2} - \frac{1}{n}\right)\frac{x^2}{2} = 1 + \frac{x}{n} + \left(\frac{1-n}{2n^2}\right)x^2.$$

With $n = 6$ and $x = 0.5$,

$$1.5^{1/6} \approx T_2(0.5) = \frac{307}{288} \approx 1.065972.$$

49. Verify that the third Maclaurin polynomial for $f(x) = e^x \sin x$ is equal to the product of the third Maclaurin polynomials of e^x and $\sin x$ (after discarding terms of degree greater than 3 in the product).

SOLUTION Let $f(x) = e^x \sin x$. Then

$$\begin{aligned} f(x) &= e^x \sin x & f(0) &= 0 \\ f'(x) &= e^x(\cos x + \sin x) & f'(0) &= 1 \\ f''(x) &= 2e^x \cos x & f''(0) &= 2 \\ f'''(x) &= 2e^x(\cos x - \sin x) & f'''(0) &= 2 \end{aligned}$$

and

$$T_3(x) = 0 + (1)x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 = x + x^2 + \frac{x^3}{3}.$$

Now, the third Maclaurin polynomial for e^x is $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$, and the third Maclaurin polynomial for $\sin x$ is $x - \frac{x^3}{6}$. Multiplying these two polynomials, and then discarding terms of degree greater than 3, yields

$$e^x \sin x \approx x + x^2 + \frac{x^3}{3},$$

which agrees with the Maclaurin polynomial obtained from the definition.

50. Find the fourth Maclaurin polynomial for $f(x) = \sin x \cos x$ by multiplying the fourth Maclaurin polynomials for $f(x) = \sin x$ and $f(x) = \cos x$.

SOLUTION The fourth Maclaurin polynomial for $\sin x$ is $x - \frac{x^3}{6}$, and the fourth Maclaurin polynomial for $\cos x$ is $1 - \frac{x^2}{2} + \frac{x^4}{24}$. Multiplying these two polynomials, and then discarding terms of degree greater than 4, we find that the fourth Maclaurin polynomial for $f(x) = \sin x \cos x$ is

$$T_4(x) = x - \frac{2x^3}{3}.$$

51. Find the Maclaurin polynomials $T_n(x)$ for $f(x) = \cos(x^2)$. You may use the fact that $T_n(x)$ is equal to the sum of the terms up to degree n obtained by substituting x^2 for x in the nth Maclaurin polynomial of $\cos x$.

SOLUTION The Maclaurin polynomials for $\cos x$ are of the form

$$T_{2n}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Accordingly, the Maclaurin polynomials for $\cos(x^2)$ are of the form

$$T_{4n}(x) = 1 - \frac{x^4}{2} + \frac{x^8}{4!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!}.$$

52. Find the Maclaurin polynomials of $1/(1 + x^2)$ by substituting $-x^2$ for x in the Maclaurin polynomials of $1/(1 - x)$.

SOLUTION The Maclaurin polynomials for $\frac{1}{1-x}$ are of the form

$$T_n(x) = 1 + x + x^2 + \cdots + x^n.$$

Accordingly, the Maclaurin polynomials for $\frac{1}{1+x^2}$ are of the form

$$T_{2n}(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-x^2)^n.$$

53. Let $f(x) = 3x^3 + 2x^2 - x - 4$. Calculate $T_j(x)$ for $j = 1, 2, 3, 4, 5$ at both $a = 0$ and $a = 1$. Show that $T_3(x) = f(x)$ in both cases.

SOLUTION Let $f(x) = 3x^3 + 2x^2 - x - 4$. Then

$f(x) = 3x^3 + 2x^2 - x - 4$	$f(0) = -4$	$f(1) = 0$
$f'(x) = 9x^2 + 4x - 1$	$f'(0) = -1$	$f'(1) = 12$
$f''(x) = 18x + 4$	$f''(0) = 4$	$f''(1) = 22$
$f'''(x) = 18$	$f'''(0) = 18$	$f'''(1) = 18$
$f^{(4)}(x) = 0$	$f^{(4)}(0) = 0$	$f^{(4)}(1) = 0$
$f^{(5)}(x) = 0$	$f^{(5)}(0) = 0$	$f^{(5)}(1) = 0$

At $a = 0$,

$$T_1(x) = -4 - x;$$
$$T_2(x) = -4 - x + 2x^2;$$
$$T_3(x) = -4 - x + 2x^2 + 3x^3 = f(x);$$
$$T_4(x) = T_3(x);\ \text{and}$$
$$T_5(x) = T_3(x).$$

At $a = 1$,

$$T_1(x) = 12(x-1);$$
$$T_2(x) = 12(x-1) + 11(x-1)^2;$$
$$T_3(x) = 12(x-1) + 11(x-1)^2 + 3(x-1)^3 = -4 - x + 2x^2 + 3x^3 = f(x);$$
$$T_4(x) = T_3(x); \text{ and}$$
$$T_5(x) = T_3(x).$$

54. Let $T_n(x)$ be the nth Taylor polynomial at $x = a$ for a polynomial $f(x)$ of degree n. Based on the result of Exercise 53, guess the value of $|f(x) - T_n(x)|$. Prove that your guess is correct using the error bound.

SOLUTION Based on Exercise 53, we expect $|f(x) - T_n(x)| = 0$. From the Error Bound,

$$|f(x) - T_n(x)| \le K \frac{|x-a|^{n+1}}{(n+1)!},$$

where K is a number such that $|f^{(n+1)}(u)| \le K$ for u between a and x. Since $f^{(n+1)}(x) = 0$ for an nth degree polynomial, we may take $K = 0$; the Error Bound then becomes $|f(x) - T_n(x)| = 0$.

55. Let $s(t)$ be the distance of a truck to an intersection. At time $t = 0$, the truck is 60 meters from the intersection, is traveling at a velocity of 24 m/s, and begins to slow down with an acceleration of $a = -3$ m/s^2. Determine the second Maclaurin polynomial of $s(t)$, and use it to estimate the truck's distance from the intersection after 4 s.

SOLUTION Place the origin at the intersection, so that $s(0) = 60$ (the truck is traveling away from the intersection). The second Maclaurin polynomial of $s(t)$ is

$$T_2(t) = s(0) + s'(0)t + \frac{s''(0)}{2}t^2$$

The conditions of the problem tell us that $s(0) = 60$, $s'(0) = 24$, and $s''(0) = -3$. Thus

$$T_2(t) = 60 + 24t - \frac{3}{2}t^2$$

so that after 4 seconds,

$$T_2(4) = 60 + 24 \cdot 4 - \frac{3}{2} \cdot 4^2 = 132 \text{ m}$$

The truck is 132 m past the intersection.

56. A bank owns a portfolio of bonds whose value $P(r)$ depends on the interest rate r (measured in percent; for example, $r = 5$ means a 5% interest rate). The bank's quantitative analyst determines that

$$P(5) = 100{,}000, \qquad \left.\frac{dP}{dr}\right|_{r=5} = -40{,}000, \qquad \left.\frac{d^2P}{dr^2}\right|_{r=5} = 50{,}000$$

In finance, this second derivative is called **bond convexity**. Find the second Taylor polynomial of $P(r)$ centered at $r = 5$ and use it to estimate the value of the portfolio if the interest rate moves to $r = 5.5\%$.

SOLUTION The second Taylor polynomial of $P(r)$ at $r = 5$ is

$$T_2(r) = P(5) + P'(5)(r-5) + \frac{P''(5)}{2}(r-5)^2$$

From the conditions of the problem, $P(5) = 100{,}000$, $P'(5) = -40{,}000$, and $P''(5) = 50{,}000$, so that

$$T_2(r) = 100{,}000 - 40{,}000(r-5) + 25{,}000(r-5)^2$$

If the interest rate moves to 5.5%, then the value of the portfolio can be estimated by

$$T_2(5.5) = 100{,}000 - 40{,}000(0.5) + 25{,}000(0.5)^2 = 86{,}250$$

57. A narrow, negatively charged ring of radius R exerts a force on a positively charged particle P located at distance x above the center of the ring of magnitude

$$F(x) = -\frac{kx}{(x^2 + R^2)^{3/2}}$$

where $k > 0$ is a constant (Figure 12).

(a) Compute the third-degree Maclaurin polynomial for $F(x)$.

(b) Show that $F \approx -(k/R^3)x$ to second order. This shows that when x is small, $F(x)$ behaves like a restoring force similar to the force exerted by a spring.

(c) Show that $F(x) \approx -k/x^2$ when x is large by showing that

$$\lim_{x\to\infty} \frac{F(x)}{-k/x^2} = 1$$

Thus, $F(x)$ behaves like an inverse square law, and the charged ring looks like a point charge from far away.

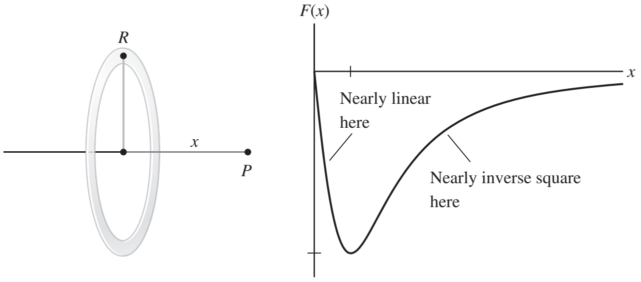


FIGURE 12

SOLUTION

(a) Start by computing and evaluating the necessary derivatives:

$$F(x) = -\frac{kx}{(x^2+R^2)^{3/2}} \qquad F(0) = 0$$

$$F'(x) = \frac{k(2x^2-R^2)}{(x^2+R^2)^{5/2}} \qquad F'(0) = -\frac{k}{R^3}$$

$$F''(x) = \frac{3kx(3R^2-2x^2)}{(x^2+R^2)^{7/2}} \qquad F''(0) = 0$$

$$F'''(x) = \frac{3k(8x^4-24x^2R^2+3R^4)}{(x^2+R^2)^{9/2}} \qquad F'''(0) = \frac{9k}{R^5}$$

so that

$$T_3(x) = F(0) + F'(0)x + \frac{F''(0)}{2!}x^2 + \frac{F'''(0)}{3!}x^3 = -\frac{k}{R^3}x + \frac{3k}{2R^5}x^3$$

(b) To degree 2, $F(x) \approx T_3(x) \approx -\frac{k}{R^3}x$ as we may ignore the x^3 term of $T_3(x)$.

(c) We have

$$\lim_{x\to\infty}\frac{F(x)}{-k/x^2} = \lim_{x\to\infty}\left(-\frac{x^2}{k}\cdot\frac{-kx}{(x^2+R^2)^{3/2}}\right) = \lim_{x\to\infty}\frac{x^3}{(x^2+R^2)^{3/2}}$$

$$= \lim_{x\to\infty}\frac{1}{x^{-3}(x^2+R^2)^{3/2}} = \lim_{x\to\infty}\frac{1}{(1+R^2/x^2)^{3/2}}$$

$$= 1$$

Thus as x grows large, $F(x)$ looks like an inverse square function.

58. A light wave of wavelength λ travels from A to B by passing through an aperture (circular region) located in a plane that is perpendicular to $\overline{AB}$ (see Figure 13 for the notation). Let $f(r) = d' + h'$; that is, $f(r)$ is the distance $AC + CB$ as a function of r.

(a) Show that $f(r) = \sqrt{d^2+r^2} + \sqrt{h^2+r^2}$, and use the Maclaurin polynomial of order 2 to show that

$$f(r) \approx d + h + \frac{1}{2}\left(\frac{1}{d}+\frac{1}{h}\right)r^2$$

(b) The **Fresnel zones**, used to determine the optical disturbance at B, are the concentric bands bounded by the circles of radius R_n such that $f(R_n) = d + h + n\lambda/2$. Show that $R_n \approx \sqrt{n\lambda L}$, where $L = (d^{-1}+h^{-1})^{-1}$.

(c) Estimate the radii R_1 and R_{100} for blue light ($\lambda = 475 \times 10^{-7}$ cm) if $d = h = 100$ cm.

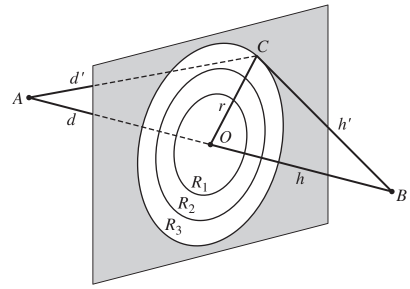


FIGURE 13 The Fresnel zones are the regions between the circles of radius R_n.

SOLUTION

(a) From the diagram, we see that $\overline{AC} = \sqrt{d^2 + r^2}$ and $\overline{CB} = \sqrt{h^2 + r^2}$. Therefore, $f(r) = \sqrt{d^2 + r^2} + \sqrt{h^2 + r^2}$. Moreover,

$$f'(r) = \frac{r}{\sqrt{d^2 + r^2}} + \frac{r}{\sqrt{h^2 + r^2}}, \qquad f''(r) = \frac{d^2}{(d^2 + r^2)^{3/2}} + \frac{h^2}{(h^2 + r^2)^{3/2}},$$

$f(0) = d + h$, $f'(0) = 0$ and $f''(0) = d^{-1} + h^{-1}$. Thus,

$$f(r) \approx T_2(r) = d + h + \frac{1}{2}\left(\frac{1}{d} + \frac{1}{h}\right) r^2.$$

(b) Solving

$$f(R_n) \approx d + h + \frac{1}{2}\left(\frac{1}{d} + \frac{1}{h}\right) R_n^2 = d + h + \frac{n\lambda}{2}$$

yields

$$R_n = \sqrt{n\lambda(d^{-1} + h^{-1})^{-1}} = \sqrt{n\lambda L},$$

where $L = (d^{-1} + h^{-1})^{-1}$.

(c) With $d = h = 100$ cm, $L = 50$ cm. Taking $\lambda = 475 \times 10^{-7}$ cm, it follows that

$$R_1 \approx \sqrt{\lambda L} = 0.04873 \text{ cm; and}$$
$$R_{100} \approx \sqrt{100\lambda L} = 0.4873 \text{ cm.}$$

59. Referring to Figure 14, let a be the length of the chord $\overline{AC}$ of angle θ of the unit circle. Derive the following approximation for the excess of the arc over the chord.

$$\theta - a \approx \frac{\theta^3}{24}$$

Hint: Show that $\theta - a = \theta - 2\sin(\theta/2)$ and use the third Maclaurin polynomial as an approximation.

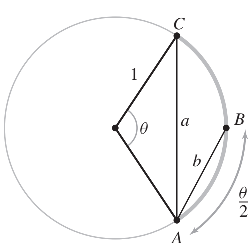


FIGURE 14 Unit circle.

SOLUTION Draw a line from the center O of the circle to B, and label the point of intersection of this line with AC as D. Then $CD = \frac{a}{2}$, and the angle COB is $\frac{\theta}{2}$. Since $CO = 1$, we have

$$\sin\frac{\theta}{2} = \frac{a}{2}$$

so that $a = 2\sin(\theta/2)$. Thus $\theta - a = \theta - 2\sin(\theta/2)$. Now, the third Maclaurin polynomial for $f(\theta) = \sin(\theta/2)$ can be computed as follows: $f(0) = 0$, $f'(x) = \frac{1}{2}\cos(\theta/2)$ so that $f'(0) = \frac{1}{2}$. $f''(x) = -\frac{1}{4}\sin(\theta/2)$ and $f''(0) = 0$. Finally, $f'''(x) = -\frac{1}{8}\cos(\theta/2)$ and $f'''(0) = -\frac{1}{8}$. Thus

$$T_3(\theta) = f(0) + f'(0)\theta + \frac{f''(0)}{2!}\theta^2 + \frac{f'''(0)}{3!}\theta^3 = \frac{1}{2}\theta - \frac{1}{48}\theta^3$$

Finally,

$$\theta - a = \theta - 2\sin\frac{\theta}{2} \approx \theta - 2T_3(\theta) = \theta - \left(\theta - \frac{1}{24}\theta^3\right) = \frac{\theta^3}{24}$$

60. To estimate the length θ of a circular arc of the unit circle, the seventeenth-century Dutch scientist Christian Huygens used the approximation $\theta \approx (8b - a)/3$, where a is the length of the chord $\overline{AC}$ of angle θ and b is length of the chord $\overline{AB}$ of angle $\theta/2$ (Figure 14).

(a) Prove that $a = 2\sin(\theta/2)$ and $b = 2\sin(\theta/4)$, and show that the Huygens approximation amounts to the approximation

$$\theta \approx \frac{16}{3}\sin\frac{\theta}{4} - \frac{2}{3}\sin\frac{\theta}{2}$$

(b) Compute the fifth Maclaurin polynomial of the function on the right.

(c) Use the error bound to show that the error in the Huygens approximation is less than $0.00022|\theta|^5$.

SOLUTION

(a) By the Law of Cosines and the identity $\sin^2(\theta/2) = (1 - \cos\theta)/2$:

$$a^2 = 1^2 + 1^2 - 2\cos\theta = 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2}$$

and so $a = 2\sin(\theta/2)$. Similarly, $b = 2\sin(\theta/4)$. Substituting these expressions for a and b into the Huygens approximation yields

$$\theta \approx \frac{8}{3}\cdot 2\sin\frac{\theta}{4} - \frac{1}{3}\cdot 2\sin\frac{\theta}{2} = \frac{16}{3}\sin\frac{\theta}{4} - \frac{2}{3}\sin\frac{\theta}{2}.$$

(b) The fifth Maclaurin polynomial for $\sin x$ is $x - \frac{x^3}{6} + \frac{x^5}{120}$; therefore, the fifth Maclaurin polynomial for $\sin(\theta/2)$ is

$$\frac{\theta}{2} - \frac{(\theta/2)^3}{6} + \frac{(\theta/2)^5}{120} = \frac{\theta}{2} - \frac{\theta^3}{48} + \frac{\theta^5}{3840},$$

and the fifth Maclaurin polynomial for $\sin(\theta/4)$ is

$$\frac{\theta}{4} - \frac{(\theta/4)^3}{6} + \frac{(\theta/4)^5}{120} = \frac{\theta}{4} - \frac{\theta^3}{384} + \frac{\theta^5}{122{,}880}.$$

Thus, the fifth Maclaurin polynomial for $f(\theta) = \frac{16}{3}\sin\frac{\theta}{4} - \frac{2}{3}\sin\frac{\theta}{2}$ is

$$\theta - \frac{1}{7680}\theta^5.$$

(c) Based on the result from part (b), the Huygens approximation for θ is equal to the fourth Maclaurin polynomial $T_4(\theta)$ for $f(\theta)$, and the error is at most $K|\theta|^5/5!$, where K is the maximum value of the absolute value of the fifth derivative $f^{(5)}(\theta)$. Because

$$f^{(5)}(\theta) = \frac{1}{192}\cos\frac{\theta}{4} - \frac{1}{48}\cos\frac{\theta}{2},$$

we may take $K = 1/48 + 1/192 = 0.0260417$, so the error is at most $|\theta|^5$ times the constant

$$\frac{0.0261}{5!} = 0.00022.$$

Further Insights and Challenges

61. Show that the nth Maclaurin polynomial of $f(x) = \arcsin x$ for n odd is

$$T_n(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots + \frac{1\cdot 3\cdot 5\cdots(n-2)}{2\cdot 4\cdot 6\cdots(n-1)}\frac{x^n}{n}$$

SOLUTION Let $f(x) = \sin^{-1} x$. Then

$$\begin{aligned}
f(x) &= \sin^{-1} x & f(0) &= 0\\
f'(x) &= \frac{1}{\sqrt{1-x^2}} & f'(0) &= 1\\
f''(x) &= -\frac{1}{2}(1-x^2)^{-3/2}(-2x) & f''(0) &= 0\\
f'''(x) &= \frac{2x^2+1}{(1-x^2)^{5/2}} & f'''(0) &= 1\\
f^{(4)}(x) &= \frac{-3x(2x^2+3)}{(1-x^2)^{7/2}} & f^{(4)}(0) &= 0\\
f^{(5)}(x) &= \frac{24x^4+72x^2+9}{(1-x^2)^{9/2}} & f^{(5)}(0) &= 9\\
&\vdots & &\vdots\\
& & f^{(7)}(0) &= 225
\end{aligned}$$

and

$$T_7(x) = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \frac{225x^7}{7!} = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7}.$$

Thus, we can infer that

$$T_n(x) = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \cdots + \frac{1}{2}\frac{3}{4}\cdots\frac{n-2}{n-1}\frac{x^n}{n}.$$

62. Let $x \ge 0$ and assume that $f^{(n+1)}(t) \ge 0$ for $0 \le t \le x$. Use Taylor's Theorem to show that the nth Maclaurin polynomial $T_n(x)$ satisfies

$$T_n(x) \le f(x) \quad \text{for all } x \ge 0$$

SOLUTION From Taylor's Theorem,

$$R_n(x) = f(x) - T_n(x) = \frac{1}{n!}\int_0^x (x-u)^n f^{(n+1)}(u)\,du.$$

If $f^{(n+1)}(t) \ge 0$ for all t then

$$\frac{1}{n!}\int_0^x (x-u)^n f^{(n+1)}(u)\,du \ge 0$$

since $(x-u)^n \ge 0$ for $0 \le u \le x$. Thus, $f(x) - T_n(x) \ge 0$, or $f(x) \ge T_n(x)$.

63. Use Exercise 62 to show that for $x \ge 0$ and all n,

$$e^x \ge 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

Sketch the graphs of e^x, $T_1(x)$, and $T_2(x)$ on the same coordinate axes. Does this inequality remain true for $x < 0$?

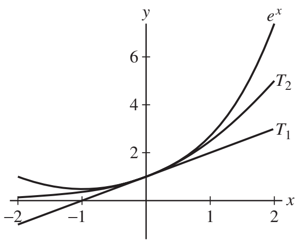
SOLUTION Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ for all n. Because $e^x > 0$ for all x, it follows from Exercise 62 that $f(x) \ge T_n(x)$ for all $x \ge 0$ and for all n. For $f(x) = e^x$,

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!},$$

thus,

$$e^x \ge 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

From the figure below, we see that the inequality does not remain true for $x < 0$, as $T_2(x) \ge e^x$ for $x < 0$.



64. This exercise is intended to reinforce the proof of Taylor's Theorem.

(a) Show that $f(x) = T_0(x) + \int_a^x f'(u)\,du$.

(b) Use Integration by Parts to prove the formula

$$\int_a^x (x-u) f^{(2)}(u)\,du = -f'(a)(x-a) + \int_a^x f'(u)\,du$$

(c) Prove the case $n = 2$ of Taylor's Theorem:

$$f(x) = T_1(x) + \int_a^x (x-u) f^{(2)}(u)\,du.$$

SOLUTION

(a)

$$T_0(x) + \int_a^x f'(u)\,du = T_0(x) + f(x) - f(a) \quad \text{(from FTC2)}$$
$$= f(a) + f(x) - f(a) = f(x).$$

(b) Using Integration by Parts with $w = x - u$ and $v' = f''(u)\,du$,

$$\int_a^x (x-u) f''(u)\,du = f'(u)(x-u)\Big|_a^x + \int_a^x f'(u)\,du$$
$$= f'(x)(x-x) - f'(a)(x-a) + \int_a^x f'(u)\,du$$
$$= -f'(a)(x-a) + \int_a^x f'(u)\,du.$$

(c)

$$T_1(x) + \int_a^x (x-u) f''(u)\,du = f(a) + f'(a)(x-a) + \big(-f'(a)(x-a)\big) + \int_a^x f'(u)\,du$$
$$= f(a) + f(x) - f(a) = f(x).$$

In Exercises 65–69, we estimate integrals using Taylor polynomials. Exercise 66 is used to estimate the error.

65. Find the fourth Maclaurin polynomial $T_4(x)$ for $f(x) = e^{-x^2}$, and calculate $I = \int_0^{1/2} T_4(x)\,dx$ as an estimate $\int_0^{1/2} e^{-x^2}\,dx$. A CAS yields the value $I \approx 0.4794255$. How large is the error in your approximation? *Hint:* $T_4(x)$ is obtained by substituting $-x^2$ in the second Maclaurin polynomial for e^x.

SOLUTION Following the hint, since the second Maclaurin polynomial for e^x is

$$1 + x + \frac{x^2}{2}$$

we substitute $-x^2$ for x to get the fourth Maclaurin polynomial for e^{x^2}:

$$T_4(x) = 1 - x^2 + \frac{x^4}{2}$$

Then

$$\int_0^{1/2} e^{-x^2}\,dx \approx \int_0^{1/2} T_4(x)\,dx = \left(x - \frac{1}{3}x^3 + \frac{1}{10}x^5\right)\Big|_0^{1/2} = \frac{443}{960} \approx 0.4614583333$$

Using a CAS, we have $\int_0^{1/2} e^{-x^2}\,dx \approx 0.4612810064$, so the error is about 1.77×10^{-4}.

66. Approximating Integrals Let $L > 0$. Show that if two functions $f(x)$ and $g(x)$ satisfy $|f(x) - g(x)| < L$ for all $x \in [a, b]$, then

$$\left| \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| < L(b-a)$$

SOLUTION Because $f(x) - g(x) \le |f(x) - g(x)|$, it follows that

$$\left| \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| = \left| \int_a^b (f(x) - g(x))\,dx \right| \le \int_a^b |f(x) - g(x)|\,dx$$
$$< \int_a^b L\,dx = L(b-a).$$

67. Let $T_4(x)$ be the fourth Maclaurin polynomial for $\cos x$.

(a) Show that $|\cos x - T_4(x)| \le \left(\frac{1}{2}\right)^6/6!$ for all $x \in \left[0, \frac{1}{2}\right]$. *Hint:* $T_4(x) = T_5(x)$.

(b) Evaluate $\int_0^{1/2} T_4(x)\,dx$ as an approximation to $\int_0^{1/2} \cos x\,dx$. Use Exercise 66 to find a bound for the size of the error.

SOLUTION

(a) Let $f(x) = \cos x$. Then

$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Moreover, with $a = 0$, $T_4(x) = T_5(x)$ and

$$|\cos x - T_4(x)| \le K\frac{|x|^6}{6!},$$

where K is a number such that $|f^{(6)}(u)| \le K$ for u between 0 and x. Now $|f^{(6)}(u)| = |\cos u| \le 1$, so we may take $K = 1$. Finally, with the restriction $x \in [0, \frac{1}{2}]$,

$$|\cos x - T_4(x)| \le \frac{(1/2)^6}{6!} \approx 0.000022.$$

(b)

$$\int_0^{1/2} \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) dx = \frac{1841}{3840} \approx 0.479427.$$

By (a) and Exercise 66, the error associated with this approximation is less than or equal to

$$\frac{(1/2)^6}{6!}\left(\frac{1}{2} - 0\right) = \frac{1}{92{,}160} \approx 1.1 \times 10^{-5}.$$

Note that $\displaystyle\int_0^{1/2} \cos x\,dx \approx 0.4794255$, so the actual error is roughly 1.5×10^{-6}.

68. Let $Q(x) = 1 - x^2/6$. Use the error bound for $\sin x$ to show that

$$\left| \frac{\sin x}{x} - Q(x) \right| \le \frac{|x|^4}{5!}$$

Then calculate $\int_0^1 Q(x)\,dx$ as an approximation to $\int_0^1 (\sin x/x)\,dx$ and find a bound for the error.

SOLUTION The third Maclaurin polynomial for $\sin x$ is

$$T_3(x) = x - \frac{1}{3!}x^3 = x - \frac{1}{6}x^3 = xQ(x)$$

Additionally, this is also $T_4(x)$ since $(\sin x)^{(4)}(0) = 0$. All derivatives of $\sin x$ are either $\pm \sin x$ or $\pm \cos x$, which are bounded in absolute value by 1. Thus we may take $K = 1$ in the Error Bound, so

$$|\sin x - xQ(x)| = |\sin x - T_3(x)| = |\sin x - T_4(x)| \le K\frac{|x|^5}{5!} = \frac{|x|^5}{5!}$$

Divide both sides of this inequality by $|x|$ to get

$$\left|\frac{\sin x}{x} - Q(x)\right| \le \frac{|x|^4}{5!}$$

We can thus estimate $\int_0^1 (\sin x/x)\,dx$ by

$$\int_0^1 Q(x)\,dx = \int_0^1 1 - \frac{x^2}{6}\,dx = \left(x - \frac{x^3}{18}\right)\Big|_0^1 = \frac{17}{18} \approx 0.9444444444$$

The error in this approximation is at most

$$\frac{|1|^4}{5!} = \frac{1}{120} \approx 0.008333333333$$

The true value of the integral is approximately 0.9460830704, which is consistent with the error bound.

69. (a) Compute the sixth Maclaurin polynomial $T_6(x)$ for $\sin(x^2)$ by substituting x^2 in $P(x) = x - x^3/6$, the third Maclaurin polynomial for $\sin x$.

(b) Show that $|\sin(x^2) - T_6(x)| \le \dfrac{|x|^{10}}{5!}$.

Hint: Substitute x^2 for x in the error bound for $|\sin x - P(x)|$, noting that $P(x)$ is also the fourth Maclaurin polynomial for $\sin x$.

(c) Use $T_6(x)$ to approximate $\displaystyle\int_0^{1/2} \sin(x^2)\,dx$ and find a bound for the error.

SOLUTION Let $s(x) = \sin x$ and $f(x) = \sin(x^2)$. Then

(a) The third Maclaurin polynomial for $\sin x$ is

$$S_3(x) = x - \frac{x^3}{6}$$

so, substituting x^2 for x, we see that the sixth Maclaurin polynomial for $\sin(x^2)$ is

$$T_6(x) = x^2 - \frac{x^6}{6}$$

(b) Since all derivatives of $s(x)$ are either $\pm\cos x$ or $\pm\sin x$, they are bounded in magnitude by 1, so we may take $K = 1$ in the Error Bound for $\sin x$. Since the third Maclaurin polynomial $S_3(x)$ for $\sin x$ is also the fourth Maclaurin polynomial $S_4(x)$, we have

$$|\sin x - S_3(x)| = |\sin x - S_4(x)| \le K\frac{|x|^5}{5!} = \frac{|x|^5}{5!}$$

Now substitute x^2 for x in the above inequality and note from part (a) that $S_3(x^2) = T_6(x)$ to get

$$|\sin(x^2) - S_3(x^2)| = |\sin(x^2) - T_6(x)| \le \frac{|x^2|^5}{5!} = \frac{|x|^{10}}{5!}$$

(c)

$$\int_0^{1/2} \sin(x^2)\,dx \approx \int_0^{1/2} T_6(x)\,dx = \left(\frac{1}{3}x^3 - \frac{1}{42}x^7\right)\Big|_0^{1/2} \approx 0.04148065476$$

From part (b), the error is bounded by

$$\frac{x^{10}}{5!} = \frac{(1/2)^{10}}{120} = \frac{1}{1024 \cdot 120} \approx 8.138020833 \times 10^{-6}$$

The true value of the integral is approximately 0.04148102420, which is consistent with the computed error bound.

70. Prove by induction that for all k,

$$\frac{d^j}{dx^j}\left(\frac{(x-a)^k}{k!}\right) = \frac{k(k-1)\cdots(k-j+1)(x-a)^{k-j}}{k!}$$

$$\frac{d^j}{dx^j}\left(\frac{(x-a)^k}{k!}\right)\Big|_{x=a} = \begin{cases} 1 & \text{for } k = j \\ 0 & \text{for } k \ne j \end{cases}$$

Use this to prove that $T_n(x)$ agrees with $f(x)$ at $x = a$ to order n.

SOLUTION The first formula is clearly true for $j = 0$. Suppose the formula is true for an arbitrary j. Then

$$\frac{d^{j+1}}{dx^{j+1}}\left(\frac{(x-a)^k}{k!}\right) = \frac{d}{dx}\frac{d^j}{dx^j}\left(\frac{(x-a)^k}{k!}\right) = \frac{d}{dx}\left(\frac{k(k-1)\cdots(k-j+1)(x-a)^{k-j}}{k!}\right)$$
$$= \frac{k(k-1)\cdots(k-j+1)(k-(j+1)+1)(x-a)^{k-(j+1)}}{k!}$$

as desired. Note that if $k = j$, then the numerator is $k!$, the denominator is $k!$ and the value of the derivative is 1; otherwise, the value of the derivative is 0 at $x = a$. In other words,

$$\frac{d^j}{dx^j}\left(\frac{(x-a)^k}{k!}\right)\bigg|_{x=a} = \begin{cases} 1 & \text{for } k = j \\ 0 & \text{for } k \neq j \end{cases}$$

Applying this latter formula, it follows that

$$\frac{d^j}{dx^j}T_n(a)\bigg|_{x=a} = \sum_{k=0}^{n}\frac{d^j}{dx^j}\left(\frac{f^{(k)}(a)}{k!}(x-a)^k\right)\bigg|_{x=a} = f^{(j)}(a)$$

as required.

71. Let a be any number and let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0$$

be a polynomial of degree n or less.

(a) Show that if $P^{(j)}(a) = 0$ for $j = 0, 1, \ldots, n$, then $P(x) = 0$, that is, $a_j = 0$ for all j. *Hint:* Use induction, noting that if the statement is true for degree $n-1$, then $P'(x) = 0$.

(b) Prove that $T_n(x)$ is the only polynomial of degree n or less that agrees with $f(x)$ at $x = a$ to order n. *Hint:* If $Q(x)$ is another such polynomial, apply (a) to $P(x) = T_n(x) - Q(x)$.

SOLUTION

(a) Note first that if $n = 0$, i.e. if $P(x) = a_0$ is a constant, then the statement holds: if $P^{(0)}(a) = P(a) = 0$, then $a_0 = 0$ so that $P(x) = 0$. Next, assume the statement holds for all polynomials of degree $n-1$ or less, and let $P(x)$ be a polynomial of degree at most n with $P^{(j)}(a) = 0$ for $j = 0, 1, \ldots, n$. If $P(x)$ has degree less than n, then we know $P(x) = 0$ by induction, so assume the degree of $P(x)$ is exactly n. Then

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$; also,

$$P'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$$

Note that $P^{(j+1)}(a) = (P')^{(j)}(a)$ for $j = 0, 1, \ldots, n-1$. But then

$$0 = P^{(j+1)}(a) = (P')^{(j)}(a) \quad \text{for all } j = 0, 1, \ldots, n-1$$

Since $P'(x)$ has degree at most $n-1$, it follows by induction that $P'(x) = 0$. Thus $a_n = a_{n-1} = \cdots = a_1 = 0$ so that $P(x) = a_0$. But $P(a) = 0$ so that $a_0 = 0$ as well and thus $P(x) = 0$.

(b) Suppose $Q(x)$ is a polynomial of degree at most n that agrees with $f(x)$ at $x = a$ up to order n. Let $P(x) = T_n(x) - Q(x)$. Note that $P(x)$ is a polynomial of degree at most n since both $T_n(x)$ and $Q(x)$ are. Since both $T_n(x)$ and $Q(x)$ agree with $f(x)$ at $x = a$ to order n, we have

$$T_n^{(j)}(a) = f^{(j)}(a) = Q^{(j)}(a), \quad j = 0, 1, 2, \ldots, n$$

Thus

$$P^{(j)}(a) = T_n^{(j)}(a) - Q^{(j)}(a) = 0 \quad \text{for } j = 0, 1, 2, \ldots, n$$

But then by part (a), $P(x) = 0$ so that $T_n(x) = Q(x)$.

CHAPTER REVIEW EXERCISES

In Exercises 1–4, calculate the arc length over the given interval.

1. $y = \dfrac{x^5}{10} + \dfrac{x^{-3}}{6}$, $[1, 2]$

SOLUTION Let $y = \dfrac{x^5}{10} + \dfrac{x^{-3}}{6}$. Then

$$1 + (y')^2 = 1 + \left(\frac{x^4}{2} - \frac{x^{-4}}{2}\right)^2 = 1 + \frac{x^8}{4} - \frac{1}{2} + \frac{x^{-8}}{4}$$
$$= \frac{x^8}{4} + \frac{1}{2} + \frac{x^{-8}}{4} = \left(\frac{x^4}{2} + \frac{x^{-4}}{2}\right)^2.$$

Because $\frac{1}{2}(x^4 + x^{-4}) > 0$ on $[1, 2]$, the arc length is

$$s = \int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \left(\frac{x^4}{2} + \frac{x^{-4}}{2}\right) dx = \left(\frac{x^5}{10} - \frac{x^{-3}}{6}\right)\Bigg|_1^2 = \frac{779}{240}.$$

2. $y = e^{x/2} + e^{-x/2}$, $[0, 2]$

SOLUTION Let $y = e^{x/2} + e^{-x/2} = 2\cosh\frac{x}{2}$. Then, $y' = \sinh\frac{x}{2}$ and

$$\sqrt{1 + (y')^2} = \sqrt{1 + \sinh^2\frac{x}{2}} = \sqrt{\cosh^2\left(\frac{x}{2}\right)} = \cosh\frac{x}{2}.$$

Thus,

$$s = \int_0^2 \cosh\left(\frac{x}{2}\right) dx = 2\sinh\left(\frac{x}{2}\right)\Bigg|_0^2 = 2\left(\sinh\left(\frac{2}{2}\right) - \sinh(0)\right) = 2\sinh(1).$$

Alternately, $y' = \frac{1}{2}(e^{x/2} - e^{-x/2})$, so

$$1 + (y')^2 = \frac{1}{4}(e^x - 2 + e^{-x}) + 1 = \frac{1}{4}(e^x + 2 + e^{-x}) = \left[\frac{1}{2}(e^{x/2} + e^{-x/2})\right]^2.$$

Because $\frac{1}{2}(e^{x/2} + e^{-x/2}) > 0$ on $[0, 2]$,

$$s = \int_0^2 \frac{1}{2}(e^{x/2} + e^{-x/2})\, dx = (e^{x/2} - e^{-x/2})\Big|_0^2 = e - e^{-1} = 2\sinh(1).$$

3. $y = 4x - 2$, $[-2, 2]$

SOLUTION Let $y = 4x - 2$. Then

$$\sqrt{1 + (y')^2} = \sqrt{1 + 4^2} = \sqrt{17}.$$

Hence,

$$s = \int_{-2}^2 \sqrt{17}\, dx = 4\sqrt{17}.$$

4. $y = x^{2/3}$, $[1, 8]$

SOLUTION Let $y = x^{2/3}$. Then $y' = \frac{2}{3}x^{-1/3}$, and

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{4}{9}x^{-2/3}} = \sqrt{\frac{4}{9}x^{-2/3}\left(\frac{9}{4}x^{2/3} + 1\right)} = \frac{2}{3}x^{-1/3}\sqrt{1 + \frac{9}{4}x^{2/3}}.$$

The arc length is

$$s = \int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \frac{2}{3}x^{-1/3}\sqrt{1 + \frac{9}{4}x^{2/3}}\, dx.$$

Now, we make the substitution $u = 1 + \frac{9}{4}x^{2/3}$, $du = \frac{3}{2}x^{-1/3}\,dx$. Then

$$s = \int_{13/4}^{10} \sqrt{u} \cdot \frac{4}{9}\,du = \frac{8}{27}u^{3/2}\Big|_{13/4}^{10} = \frac{8}{27}\left[10^{3/2} - \left(\frac{\sqrt{13}}{2}\right)^3\right]$$

$$= \frac{8}{27}\left(10\sqrt{10} - \frac{13\sqrt{13}}{8}\right) \approx 7.633705415.$$

5. Show that the arc length of $y = 2\sqrt{x}$ over $[0, a]$ is equal to $\sqrt{a(a+1)} + \ln(\sqrt{a} + \sqrt{a+1})$. *Hint:* Apply the substitution $x = \tan^2\theta$ to the arc length integral.

SOLUTION Let $y = 2\sqrt{x}$. Then $y' = \frac{1}{\sqrt{x}}$, and

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{1}{\sqrt{x}}\sqrt{x+1}.$$

Thus,

$$s = \int_0^a \frac{1}{\sqrt{x}}\sqrt{1+x}\,dx.$$

We make the substitution $x = \tan^2\theta$, $dx = 2\tan\theta\sec^2\theta\,d\theta$. Then

$$s = \int_{x=0}^{x=a} \frac{1}{\tan\theta}\sec\theta \cdot 2\tan\theta\sec^2\theta\,d\theta = 2\int_{x=0}^{x=a}\sec^3\theta\,d\theta.$$

We use a reduction formula to obtain

$$s = 2\left(\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right)\Big|_{x=0}^{x=a} = (\sqrt{x}\sqrt{1+x} + \ln|\sqrt{1+x} + \sqrt{x}|)\Big|_0^a$$

$$= \sqrt{a}\sqrt{1+a} + \ln|\sqrt{1+a} + \sqrt{a}| = \sqrt{a(a+1)} + \ln\left(\sqrt{a} + \sqrt{a+1}\right).$$

6. CAS Compute the trapezoidal approximation T_5 to the arc length s of $y = \tan x$ over $\left[0, \frac{\pi}{4}\right]$.

SOLUTION Let $y = \tan x$. With $N = 5$, the subintervals are $\left[(i-1)\frac{\pi}{20}, i\frac{\pi}{20}\right]$, $i = 1, 2, 3, 4, 5$. Now,

$$1 + (y')^2 = 1 + (\sec^2 x)^2 = 1 + \sec^4 x$$

so the arc length is approximately

$$s = \int_1^{\pi/4}\sqrt{1 + \sec^4 x}\,dx$$

$$\approx \frac{\pi}{40}\left(\sqrt{1+\sec^4 0} + 2\sqrt{1+\sec^4\frac{\pi}{20}} + 2\sqrt{1+\sec^4\frac{\pi}{10}} + 2\sqrt{1+\sec^4\frac{3\pi}{20}} + 2\sqrt{1+\sec^4\frac{\pi}{5}}\right.$$
$$\left. + \sqrt{1+\sec^4\frac{\pi}{4}}\right)$$

$$\approx \frac{\pi}{40}(1.41421356 + 2\cdot 1.43206164 + 2\cdot 1.49073513 + 2\cdot 1.60830125 + 2\cdot 1.82602534 + 2.23606797)$$

$$\approx 1.285267058$$

In Exercises 7–10, calculate the surface area of the solid obtained by rotating the curve over the given interval about the x-axis.

7. $y = x + 1$, $[0, 4]$

SOLUTION Let $y = x + 1$. Then $y' = 1$, and

$$y\sqrt{1 + y'^2} = (x+1)\sqrt{1+1} = \sqrt{2}(x+1).$$

Thus,

$$SA = 2\pi\int_0^4 \sqrt{2}(x+1)\,dx = 2\sqrt{2}\pi\left(\frac{x^2}{2} + x\right)\Big|_0^4 = 24\sqrt{2}\pi.$$

8. $y = \frac{2}{3}x^{3/4} - \frac{2}{5}x^{5/4}$, $[0, 1]$

SOLUTION Let $y = \frac{2}{3}x^{3/4} - \frac{2}{5}x^{5/4}$. Then

$$y' = \frac{x^{-1/4}}{2} - \frac{x^{1/4}}{2},$$

and

$$1 + (y')^2 = 1 + \left(\frac{x^{-1/4}}{2} - \frac{x^{1/4}}{2}\right)^2 = \frac{x^{-1/2}}{4} + \frac{1}{2} + \frac{x^{1/2}}{4} = \left(\frac{x^{-1/4}}{2} + \frac{x^{1/4}}{2}\right)^2.$$

Because $\frac{1}{2}(x^{-1/4} + x^{1/4}) \geq 0$, the surface area is

$$\begin{aligned} 2\pi \int_0^1 y\sqrt{1 + (y')^2}\,dy &= 2\pi \int_0^1 \left(\frac{2x^{3/4}}{3} - \frac{2x^{5/4}}{5}\right)\left(\frac{x^{1/4}}{2} + \frac{x^{-1/4}}{2}\right) dx \\ &= 2\pi \int_0^1 \left(-\frac{x^{3/2}}{5} - \frac{x}{5} + \frac{x}{3} + \frac{\sqrt{x}}{3}\right) dx \\ &= 2\pi \left(-\frac{2x^{5/2}}{25} + \frac{x^2}{15} + \frac{2x^{3/2}}{9}\right)\Bigg|_0^1 = \frac{94}{225}\pi. \end{aligned}$$

9. $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$, $[1, 2]$

SOLUTION Let $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$. Then

$$y' = \sqrt{x} - \frac{1}{4\sqrt{x}},$$

and

$$1 + \left(y'\right)^2 = 1 + \left(\sqrt{x} - \frac{1}{4\sqrt{x}}\right)^2 = 1 + \left(x - \frac{1}{2} + \frac{1}{16x}\right) = x + \frac{1}{2} + \frac{1}{16x} = \left(\sqrt{x} + \frac{1}{4\sqrt{x}}\right)^2.$$

Because $\sqrt{x} + \frac{1}{\sqrt{x}} \geq 0$, the surface area is

$$\begin{aligned} 2\pi \int_a^b y\sqrt{1 + \left(y'\right)^2}\,dx &= 2\pi \int_1^2 \left(\frac{2}{3}x^{3/2} - \frac{\sqrt{x}}{2}\right)\left(\sqrt{x} + \frac{1}{4\sqrt{x}}\right) dx \\ &= 2\pi \int_1^2 \left(\frac{2}{3}x^2 + \frac{1}{6}x - \frac{1}{2}x - \frac{1}{8}\right) dx = 2\pi \left(\frac{2x^3}{9} - \frac{x^2}{6} - \frac{1}{8}x\right)\Bigg|_1^2 = \frac{67}{36}\pi. \end{aligned}$$

10. $y = \frac{1}{2}x^2$, $[0, 2]$

SOLUTION Let $y = \frac{1}{2}x^2$. Then $y' = x$ and

$$SA = 2\pi \int_0^2 \frac{1}{2}x^2\sqrt{1 + x^2}\,dx = \pi \int_0^2 x^2\sqrt{1 + x^2}\,dx.$$

Using the substitution $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, we find that

$$\begin{aligned} \int x^2\sqrt{1 + x^2}\,dx &= \int \sec^3\theta\tan^2\theta\,d\theta = \int \left(\sec^5\theta - \sec^3\theta\right) d\theta \\ &= \left(\frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}\sec\theta\tan\theta + \frac{3}{8}\ln|\sec\theta + \tan\theta| - \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\right) + C \\ &= \frac{x}{4}(1 + x^2)^{3/2} - \frac{x}{8}\sqrt{1 + x^2} - \frac{1}{8}\ln|\sqrt{1 + x^2} + x| + C. \end{aligned}$$

Finally,

$$\begin{aligned} SA &= \pi\left(\frac{x}{4}(1 + x^2)^{3/2} - \frac{x}{8}\sqrt{1 + x^2} - \frac{1}{8}\ln|\sqrt{1 + x^2} + x|\right)\Bigg|_0^2 \\ &= \pi\left(\frac{5\sqrt{5}}{2} - \frac{\sqrt{5}}{4} - \frac{1}{8}\ln(2 + \sqrt{5})\right) = \frac{9\sqrt{5}}{4}\pi - \frac{\pi}{8}\ln(2 + \sqrt{5}). \end{aligned}$$

11. Compute the total surface area of the coin obtained by rotating the region in Figure 1 about the x-axis. The top and bottom parts of the region are semicircles with a radius of 1 mm.

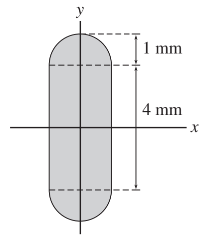


FIGURE 1

SOLUTION The generating half circle of the edge is $y = 2 + \sqrt{1 - x^2}$. Then,

$$y' = \frac{-2x}{2\sqrt{1 - x^2}} = \frac{-x}{\sqrt{1 - x^2}},$$

and

$$1 + (y')^2 = 1 + \frac{x^2}{1 - x^2} = \frac{1}{1 - x^2}.$$

The surface area of the edge of the coin is

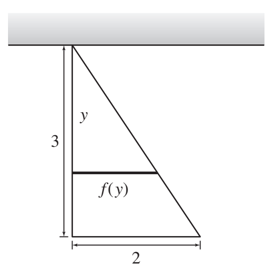
$$\begin{aligned} 2\pi \int_{-1}^{1} y\sqrt{1 + (y')^2}\,dx &= 2\pi \int_{-1}^{1} \left(2 + \sqrt{1 - x^2}\right) \frac{1}{\sqrt{1 - x^2}}\,dx \\ &= 2\pi \left(2 \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} + \int_{-1}^{1} \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}}\,dx\right) \\ &= 2\pi \left(2 \arcsin x\Big|_{-1}^{1} + \int_{-1}^{1} dx\right) \\ &= 2\pi(2\pi + 2) = 4\pi^2 + 4\pi. \end{aligned}$$

We now add the surface area of the two sides of the disk, which are circles of radius 2. Hence the surface area of the coin is:

$$\left(4\pi^2 + 4\pi\right) + 2\pi \cdot 2^2 = 4\pi^2 + 12\pi.$$

12. Calculate the fluid force on the side of a right triangle of height 3 m and base 2 m submerged in water vertically, with its upper vertex at the surface of the water.

SOLUTION To find the fluid force, we must find an expression for the horizontal width $f(y)$ of the triangle at depth y.



By similar triangles we have:

$$\frac{y}{f(y)} = \frac{3}{2} \qquad \text{so} \qquad f(y) = \frac{2y}{3}.$$

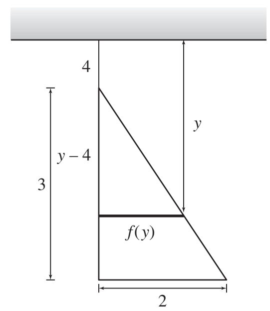
Therefore, the fluid force on the side of the triangle is

$$F = \rho g \int_0^3 y f(y)\,dy = \rho g \int_0^3 \frac{2y^2}{3}\,dy = \rho g \cdot \frac{2y^3}{9}\Big|_0^3 = 6\rho g.$$

For water, $\rho = 10^3$; $g = 9.8$, so $F = 6 \cdot 9800 = 58{,}800$ N.

13. Calculate the fluid force on the side of a right triangle of height 3 m and base 2 m submerged in water vertically, with its upper vertex located at a depth of 4 m.

SOLUTION We need to find an expression for the horizontal width $f(y)$ at depth y.



By similar triangles we have:

$$\frac{f(y)}{y-4} = \frac{2}{3} \qquad \text{so} \qquad f(y) = \frac{2(y-4)}{3}.$$

Hence, the force on the side of the triangle is

$$F = \rho g \int_4^7 y f(y)\,dy = \frac{2\rho g}{3}\int_4^7 \left(y^2 - 4y\right)\,dy = \frac{2\rho g}{3}\left(\frac{y^3}{3} - 2y^2\right)\bigg|_4^7 = 18\rho g.$$

For water, $\rho = 10^3$; $g = 9.8$, so $F = 18 \cdot 9800 = 176{,}400$ N.

14. A plate in the shape of the shaded region in Figure 2 is submerged in water. Calculate the fluid force on a side of the plate if the water surface is $y = 1$.

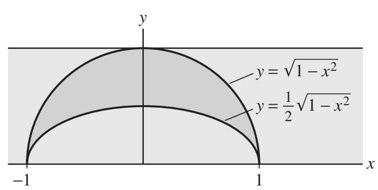


FIGURE 2

SOLUTION Here, we can proceed as follows: Calculate the force that would be exerted on the entire semicircle and then subtract the force that would be exerted on the "missing" portion of the ellipse. The force on the semicircle is

$$2w\int_0^1 (1-y)\sqrt{1-y^2}\,dy = 2w\int_0^1 \sqrt{1-y^2}\,dy - 2w\int_0^1 y\sqrt{1-y^2}\,dy.$$

The first integral can be interpreted as the area of one-quarter of a circle of radius 1. Hence,

$$\int_0^1 \sqrt{1-y^2}\,dy = \frac{\pi}{4}.$$

On the other hand,

$$\int_0^1 y\sqrt{1-y^2}\,dy = -\frac{1}{3}(1-y^2)^{3/2}\bigg|_0^1 = \frac{1}{3}.$$

Thus, the force on the semicircle is

$$2w\left(\frac{\pi}{4} - \frac{1}{3}\right).$$

Now for the ellipse. The force that would be exerted on the upper half of the ellipse is

$$2w\int_0^{1/2}(1-y)\sqrt{1-4y^2}\,dy = 2w\int_0^{1/2}\sqrt{1-4y^2}\,dy - 2w\int_0^{1/2} y\sqrt{1-4y^2}\,dy.$$

Using the substitution $w = 2y$, $dw = 2\,dy$, it follows that

$$\int_0^{1/2}\sqrt{1-4y^2}\,dy = \frac{1}{2}\int_0^1 \sqrt{1-w^2}\,dw = \frac{\pi}{8},$$

and

$$\int_0^{1/2} y\sqrt{1-4y^2}\,dy = \frac{1}{4}\int_0^1 w\sqrt{1-w^2}\,dw = \frac{1}{12}.$$

Thus, the force on the "missing" ellipse is

$$2w\left(\frac{\pi}{8} - \frac{1}{12}\right).$$

Finally, the force exerted on the plate shown in Figure 2 is

$$F = 2w\left(\frac{\pi}{4} - \frac{1}{3}\right) - 2w\left(\frac{\pi}{8} - \frac{1}{12}\right) = \frac{\pi-2}{4}w.$$

15. Figure 3 shows an object whose face is an equilateral triangle with 5-m sides. The object is 2 m thick and is submerged in water with its vertex 3 m below the water surface. Calculate the fluid force on both a triangular face and a slanted rectangular edge of the object.

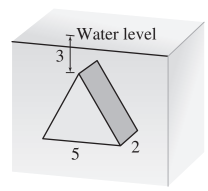


FIGURE 3

SOLUTION Start with each triangular face of the object. Place the origin at the upper vertex of the triangle, with the positive y-axis pointing downward. Note that because the equilateral triangle has sides of length 5 feet, the height of the triangle is $\frac{5\sqrt{3}}{2}$ feet. Moreover, the width of the triangle at location y is $\frac{2y}{\sqrt{3}}$. Thus,

$$F = \frac{2\rho g}{\sqrt{3}}\int_0^{5\sqrt{3}/2}(y+3)y\,dy = \frac{2\rho g}{\sqrt{3}}\left(\frac{1}{3}y^3 + \frac{3}{2}y^2\right)\Bigg|_0^{5\sqrt{3}/2} = \frac{\rho g}{4}(125 + 75\sqrt{3}) \approx 624{,}514 \text{ N}.$$

Now, consider the slanted rectangular edges of the object. Each edge is a constant 2 feet wide and makes an angle of 60° with the horizontal. Therefore,

$$F = \frac{\rho g}{\sin 60^\circ}\int_0^{5\sqrt{3}/2} 2(y+3)\,dy = \frac{2\rho g}{\sqrt{3}}\left(y^2 + 6y\right)\Bigg|_0^{5\sqrt{3}/2} = \rho g\left(\frac{25\sqrt{3}}{2} + 30\right) \approx 506{,}176 \text{ N}.$$

The force on the bottom face can be computed without calculus:

$$F = \left(3 + \frac{5\sqrt{3}}{2}\right)(2)(5)\rho g \approx 718{,}352 \text{ N}.$$

16. The end of a horizontal oil tank is an ellipse (Figure 4) with equation $(x/4)^2 + (y/3)^2 = 1$ (length in meters). Assume that the tank is filled with oil of density 900 kg/m^3.

(a) Calculate the total force F on the end of the tank when the tank is full.

(b) Would you expect the total force on the lower half of the tank to be greater than, less than, or equal to $\frac{1}{2}F$? Explain. Then compute the force on the lower half exactly and confirm (or refute) your expectation.

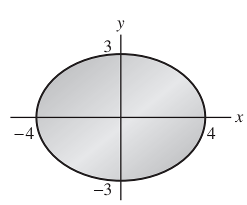


FIGURE 4

SOLUTION

(a) Solving the equation of the ellipse for x yields

$$x = \frac{4}{3}\sqrt{9 - y^2}.$$

Therefore, a horizontal strip of the ellipse at height y has width $\frac{8}{3}\sqrt{9-y^2}$. This strip is at a depth of $3-y$, so the total force on the end of the tank is

$$F = \rho g \int_{-3}^{3} (3-y) \cdot \frac{8}{3}\sqrt{9-y^2}\,dy = 8\rho g \int_{-3}^{3} \sqrt{9-y^2}\,dy - \frac{8}{3}\rho g \int_{-3}^{3} y\sqrt{9-y^2}\,dy.$$

The first integral can be interpreted as the area of one-half of a circle of radius 3, so the value of this integral is $\frac{9\pi}{2}$. The second integral is zero, since the integrand is an odd function and the interval of integration is symmetric about zero. Hence,

$$F = 8\rho g \frac{9\pi}{2} - \frac{8}{3}\rho g(0) = 8 \cdot 900 \cdot 9.8 \cdot \frac{9\pi}{2} \approx 997{,}518 \text{ N}.$$

(b) The oil in the lower half of the tank is at a greater depth than the oil in the upper half, therefore we expect the total force F_l on the lower half of the tank to be greater than the total force F_u on the upper half. We compute the two forces to verify our expectation. Now,

$$F_l = \rho g \int_{-3}^{0} (3-y) \cdot \frac{8}{3}\sqrt{9-y^2}\,dy = 8\rho g \int_{-3}^{0} \sqrt{9-y^2}\,dy - \frac{8}{3}\rho g \int_{-3}^{0} y\sqrt{9-y^2}\,dy.$$

Similarly,

$$F_u = 8\rho g \int_{0}^{3} \sqrt{9-y^2}\,dy - \frac{8}{3}\rho g \int_{0}^{3} y\sqrt{9-y^2}\,dy.$$

The first integral in each expression,

$$\int_{-3}^{0} \sqrt{9-y^2}\,dy \qquad \text{and} \qquad \int_{0}^{3} \sqrt{9-y^2}\,dy,$$

can be interpreted as the area of one-quarter of a circle of radius 3, so both integrals have the value $\frac{9\pi}{4}$. Using the substitution $u = 9 - y^2$, $du = -2y\,dy$ we find

$$\int_{-3}^{0} y\sqrt{9-y^2}\,dy = \int_{0}^{9} \sqrt{u}\left(-\frac{1}{2}\right)du = -\frac{1}{3}u^{3/2}\Big|_0^9 = -9.$$

Moreover, since the integrand is an odd function, we have

$$\int_{0}^{3} y\sqrt{9-y^2}\,dy = -\int_{-3}^{0} y\sqrt{9-y^2}\,dy = 9.$$

Thus,

$$F_l = 8\rho g \frac{9\pi}{4} - \frac{8}{3}\rho g(-9) = (18\pi + 24)\rho g; \text{ and}$$
$$F_u = 8\rho g \frac{9\pi}{4} - \frac{8}{3}\rho g(9) = (18\pi - 24)\rho g.$$

We see that $F_l > F_u$. That is, the total force on the lower half of the tank is greater than the total force on the upper half, as expected.

17. Calculate the moments and COM of the lamina occupying the region under $y = x(4-x)$ for $0 \le x \le 4$, assuming a density of $\rho = 1200$ kg/m^3.

SOLUTION Because the lamina is symmetric with respect to the vertical line $x = 2$, by the symmetry principle, we know that $x_{\text{cm}} = 2$. Now,

$$M_x = \frac{\rho}{2}\int_0^4 f(x)^2\,dx = \frac{1200}{2}\int_0^4 x^2(4-x)^2\,dx = \frac{1200}{2}\left(\frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5\right)\Big|_0^4 = 20{,}480.$$

Moreover, the mass of the lamina is

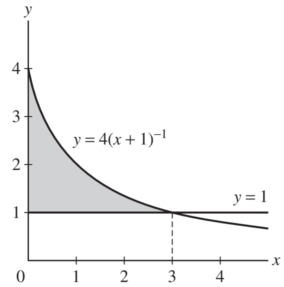
$$M = \rho\int_0^4 f(x)\,dx = 1200\int_0^4 x(4-x)\,dx = 1200\left(2x^2 - \frac{1}{3}x^3\right)\Big|_0^4 = 12{,}800.$$

Thus, the coordinates of the center of mass are

$$\left(2, \frac{20{,}480}{12{,}800}\right) = \left(2, \frac{8}{5}\right).$$

18. Sketch the region between $y = 4(x+1)^{-1}$ and $y = 1$ for $0 \le x \le 3$, and find its centroid.

SOLUTION



First, we calculate the moments:

$$M_x = \frac{1}{2}\int_0^3 \left(\frac{16}{(x+1)^2} - 1\right) dx = \frac{1}{2}\left(-\frac{16}{x+1} - x\right)\Bigg|_0^3 = \frac{9}{2},$$

and

$$\begin{aligned} M_y &= \int_0^3 x\left(4(x+1)^{-1} - 1\right) dx = \int_0^3 \left(\frac{4x}{x+1} - x\right) dx \\ &= \int_0^3 \left(\frac{4(x+1)-4}{x+1} - x\right) dx = \int_0^3 \left(4 - \frac{4}{x+1} - x\right) dx \\ &= \left(4x - \frac{x^2}{2} - 4\ln(x+1)\right)\Bigg|_0^3 = \frac{15}{2} - 4\ln 4. \end{aligned}$$

The area of the region is

$$A = \int_0^3 \left(\frac{4}{x+1} - 1\right) dx = \left(4\ln(x+1) - x\right)\Big|_0^3 = 4\ln 4 - 3,$$

so the coordinates of the centroid are:

$$\left(\frac{15 - 8\ln 4}{8\ln 4 - 6}, \frac{9}{8\ln 4 - 6}\right).$$

19. Find the centroid of the region between the semicircle $y = \sqrt{1-x^2}$ and the top half of the ellipse $y = \frac{1}{2}\sqrt{1-x^2}$ (Figure 2).

SOLUTION Since the region is symmetric with respect to the y-axis, the centroid lies on the y-axis. To find $y_{\rm cm}$ we calculate

$$\begin{aligned} M_x &= \frac{1}{2}\int_{-1}^1 \left[\left(\sqrt{1-x^2}\right)^2 - \left(\frac{\sqrt{1-x^2}}{2}\right)^2\right] dx \\ &= \frac{1}{2}\int_{-1}^1 \frac{3}{4}\left(1-x^2\right) dx = \frac{3}{8}\left(x - \frac{1}{3}x^3\right)\Bigg|_{-1}^1 = \frac{1}{2}. \end{aligned}$$

The area of the lamina is $\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$, so the coordinates of the centroid are

$$\left(0, \frac{1/2}{\pi/4}\right) = \left(0, \frac{2}{\pi}\right).$$

20. Find the centroid of the shaded region in Figure 5 bounded on the left by $x = 2y^2 - 2$ and on the right by a semicircle of radius 1. *Hint:* Use symmetry and additivity of moments.

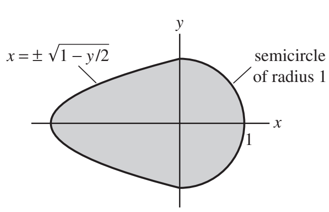


FIGURE 5

SOLUTION The region is symmetric with respect to the x-axis, hence the centroid lies on the x-axis; that is, $y_{\text{cm}} = 0$. To compute the area and the moment with respect to the y-axis, we treat the left side and the right side of the region separately. Starting with the left side, we find

$$M_y^{\text{left}} = 2\int_{-2}^{0} x\sqrt{\frac{x}{2}+1}\,dx \qquad \text{and} \qquad A^{\text{left}} = 2\int_{-2}^{0}\sqrt{\frac{x}{2}+1}\,dx.$$

In each integral we make the substitution $u = \frac{x}{2}+1$, $du = \frac{1}{2}\,dx$, and find

$$M_y^{\text{left}} = 8\int_0^1 (u-1)u^{1/2}\,du = 8\int_0^1 \left(u^{3/2}-u^{1/2}\right)du = 8\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)\Big|_0^1 = -\frac{32}{15}$$

and

$$A^{\text{left}} = 4\int_0^1 u^{1/2}\,du = \frac{8}{3}u^{3/2}\Big|_0^1 = \frac{8}{3}.$$

On the right side of the region

$$M_y^{\text{right}} = 2\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{2}{3}(1-x^2)^{3/2}\Big|_0^1 = \frac{2}{3},$$

and $A^{\text{right}} = \frac{\pi}{2}$ (because the right side of the region is one-half of a circle of radius 1). Thus,

$$M_y = M_y^{\text{left}} + M_y^{\text{right}} = -\frac{32}{15}+\frac{2}{3} = -\frac{22}{15};$$
$$A = A^{\text{left}} + A^{\text{right}} = \frac{8}{3}+\frac{\pi}{2} = \frac{16+3\pi}{6};$$

and the coordinates of the centroid are

$$\left(\frac{-22/15}{(16+3\pi)/6}, 0\right) = \left(-\frac{44}{80+15\pi}, 0\right).$$

In Exercises 21–26, find the Taylor polynomial at $x = a$ for the given function.

21. $f(x) = x^3, \quad T_3(x), \quad a = 1$

SOLUTION We start by computing the first three derivatives of $f(x) = x^3$:

$$f'(x) = 3x^2$$
$$f''(x) = 6x$$
$$f'''(x) = 6$$

Evaluating the function and its derivatives at $x = 1$, we find

$$f(1) = 1,\ f'(1) = 3,\ f''(1) = 6,\ f'''(1) = 6.$$

Therefore,

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-2)^2 + \frac{f'''(1)}{3!}(x-1)^3$$
$$= 1 + 3(x-1) + \frac{6}{2!}(x-2)^2 + \frac{6}{3!}(x-1)^3$$
$$= 1 + 3(x-1) + 3(x-2)^2 + (x-1)^3.$$

22. $f(x) = 3(x+2)^3 - 5(x+2), \quad T_3(x), \quad a = -2$

SOLUTION $T_3(x)$ is the Taylor polynomial of f consisting of powers of $(x+2)$ up to three. Since $f(x)$ is already in this form we conclude that $T_3(x) = f(x)$.

23. $f(x) = x\ln(x), \quad T_4(x), \quad a = 1$

SOLUTION We start by computing the first four derivatives of $f(x) = x\ln x$:

$$f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$
$$f''(x) = \frac{1}{x}$$
$$f'''(x) = -\frac{1}{x^2}$$
$$f^{(4)}(x) = \frac{2}{x^3}$$

Evaluating the function and its derivatives at $x = 1$, we find

$$f(1) = 0,\ f'(1) = 1,\ f''(1) = 1,\ f'''(1) = -1,\ f^{(4)}(1) = 2.$$

Therefore,

$$\begin{aligned}T_4(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4\\ &= 0 + 1(x-1) + \frac{1}{2!}(x-1)^2 - \frac{1}{3!}(x-1)^3 + \frac{2}{4!}(x-1)^4\\ &= (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4.\end{aligned}$$

24. $f(x) = (3x+2)^{1/3}, \quad T_3(x), \quad a = 2$

SOLUTION We start by computing the first three derivatives of $f(x) = (3x+2)^{1/3}$:

$$f'(x) = \frac{1}{3}(3x+2)^{-2/3} \cdot 3 = (3x+2)^{-2/3}$$
$$f''(x) = -\frac{2}{3}(3x+2)^{-5/3} \cdot 3 = -2(3x+2)^{-5/3}$$
$$f'''(x) = \frac{10}{3}(3x+2)^{-8/3} \cdot 3 = 10(3x+2)^{-8/3}$$

Evaluating the function and its derivatives at $x = 2$, we find

$$f(2) = 2,\ f'(2) = \frac{1}{4},\ f''(2) = -\frac{1}{16},\ f'''(2) = \frac{5}{128}.$$

Therefore,

$$\begin{aligned}T_3(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3\\ &= 2 + \frac{1}{4}(x-2) + \frac{-1/16}{2!}(x-2)^2 + \frac{5/128}{3!}(x-2)^3\\ &= 2 + \frac{1}{4}(x-2) - \frac{1}{32}(x-2)^2 - \frac{5}{768}(x-2)^3.\end{aligned}$$

25. $f(x) = xe^{-x^2}, \quad T_4(x), \quad a = 0$

SOLUTION We start by computing the first four derivatives of $f(x) = xe^{-x^2}$:

$$\begin{aligned}f'(x) &= e^{-x^2} + x \cdot (-2x)e^{-x^2} = (1-2x^2)e^{-x^2}\\ f''(x) &= -4xe^{-x^2} + (1-2x^2)\cdot(-2x)e^{-x^2} = (4x^3-6x)e^{-x^2}\\ f'''(x) &= (12x^2-6)e^{-x^2} + (4x^3-6x)\cdot(-2x)e^{-x^2} = (-8x^4+24x^2-6)e^{-x^2}\\ f^{(4)}(x) &= (-32x^3+48x)e^{-x^2} + (-8x^4+24x^2-6)\cdot(-2x)e^{-x^2} = (16x^5-80x^3+60x)e^{-x^2}\end{aligned}$$

Evaluating the function and its derivatives at $x = 0$, we find

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -6,\ f^{(4)}(0) = 0.$$

Therefore,

$$T_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$= 0 + x + 0 \cdot x^2 - \frac{6}{3!}x^3 + 0 \cdot x^4 = x - x^3.$$

26. $f(x) = \ln(\cos x), \quad T_3(x), \quad a = 0$

SOLUTION We start by computing the first three derivatives of $f(x) = \ln(\cos x)$:

$$f'(x) = -\frac{\sin x}{\cos x} = -\tan x$$
$$f''(x) = -\sec^2 x$$
$$f'''(x) = -2\sec^2 x \tan x$$

Evaluating the function and its derivatives at $x = 0$, we find

$$f(0) = 0,\ f'(0) = 0,\ f''(0) = -1,\ f'''(0) = 0.$$

Therefore,

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 0 + \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 = -\frac{x^2}{2}.$$

27. Find the nth Maclaurin polynomial for $f(x) = e^{3x}$.

SOLUTION We differentiate the function $f(x) = e^{3x}$ repeatedly, looking for a pattern:

$$f'(x) = 3e^{3x} = 3^1 e^{3x}$$
$$f''(x) = 3 \cdot 3e^{3x} = 3^2 e^{3x}$$
$$f'''(x) = 3 \cdot 3^2 e^{3x} = 3^3 e^{3x}$$

Thus, for general n, $f^{(n)}(x) = 3^n e^{3x}$ and $f^{(n)}(0) = 3^n$. Substituting into the formula for the nth Taylor polynomial, we obtain:

$$T_n(x) = 1 + \frac{3x}{1!} + \frac{3^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{3^4x^4}{4!} + \cdots + \frac{3^nx^n}{n!}$$
$$= 1 + 3x + \frac{1}{2!}(3x)^2 + \frac{1}{3!}(3x)^3 + \cdots + \frac{1}{n!}(3x)^n.$$

28. Use the fifth Maclaurin polynomial of $f(x) = e^x$ to approximate $\sqrt{e}$. Use a calculator to determine the error.

SOLUTION Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for all n. Hence,

$$T_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.$$

For $x = \frac{1}{2}$ we have

$$T_5\left(\frac{1}{2}\right) = 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \frac{\left(\frac{1}{2}\right)^4}{4!} + \frac{\left(\frac{1}{2}\right)^5}{5!}$$
$$= 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \frac{1}{3840} = 1.648697917$$

Using a calculator, we find that $\sqrt{e} = 1.648721271$. The error in the Taylor polynomial approximation is

$$|1.648697917 - 1.648721271| = 2.335 \times 10^{-5}.$$

29. Use the third Taylor polynomial of $f(x) = \tan^{-1} x$ at $a = 1$ to approximate $f(1.1)$. Use a calculator to determine the error.

SOLUTION We start by computing the first three derivatives of $f(x) = \tan^{-1} x$:

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = -\frac{2x}{\left(1+x^2\right)^2}$$

$$f'''(x) = \frac{-2\left(1+x^2\right)^2 + 2x \cdot 2\left(1+x^2\right) \cdot 2x}{\left(1+x^2\right)^4} = \frac{2\left(3x^2-1\right)}{\left(1+x^2\right)^3}$$

Evaluating the function and its derivatives at $x = 1$, we find

$$f(1) = \frac{\pi}{4},\; f'(1) = \frac{1}{2},\; f''(1) = -\frac{1}{2},\; f'''(1) = \frac{1}{2}.$$

Therefore,

$$\begin{aligned} T_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3. \end{aligned}$$

Setting $x = 1.1$ yields:

$$T_3(1.1) = \frac{\pi}{4} + \frac{1}{2}(0.1) - \frac{1}{4}(0.1)^2 + \frac{1}{12}(0.1)^3 = 0.832981496.$$

Using a calculator, we find $\tan^{-1} 1.1 = 0.832981266$. The error in the Taylor polynomial approximation is

$$\left|T_3(1.1) - \tan^{-1} 1.1\right| = |0.832981496 - 0.832981266| = 2.301 \times 10^{-7}.$$

30. Let $T_4(x)$ be the Taylor polynomial for $f(x) = \sqrt{x}$ at $a = 16$. Use the error bound to find the maximum possible size of $|f(17) - T_4(17)|$.

SOLUTION Using the Error Bound, we have

$$|f(17) - T_4(17)| \le K\frac{(17-16)^5}{5!} = \frac{K}{5!},$$

where K is a number such that $\left|f^{(5)}(x)\right| \le K$ for all $16 \le x \le 17$. Starting from $f(x) = \sqrt{x}$ we find

$$f'(x) = \frac{1}{2}x^{-1/2},\; f''(x) = -\frac{1}{4}x^{-3/2},\; f'''(x) = \frac{3}{8}x^{-5/2},\; f^{(4)}(x) = -\frac{15}{16}x^{-7/2},$$

and

$$f^{(5)}(x) = \frac{105}{32}x^{-9/2}.$$

For $16 \le x \le 17$,

$$\left|f^{(5)}(x)\right| = \frac{105}{32x^{9/2}} \le \frac{105}{32 \cdot 16^{9/2}} = \frac{105}{8{,}388{,}608}.$$

Therefore, we may take

$$K = \frac{105}{8{,}388{,}608}.$$

Finally,

$$|f(17) - T_4(17)| \le \frac{105}{8{,}388{,}608} \cdot \frac{1}{5!} \approx 1.044 \cdot 10^{-7}.$$

31. Find n such that $|e - T_n(1)| < 10^{-8}$, where $T_n(x)$ is the nth Maclaurin polynomial for $f(x) = e^x$.

SOLUTION Using the Error Bound, we have

$$|e - T_n(1)| \le K\frac{|1-0|^{n+1}}{(n+1)!} = \frac{K}{(n+1)!}$$

where K is a number such that $\left|f^{(n+1)}(x)\right| = e^x \le K$ for all $0 \le x \le 1$. Since e^x is increasing, the maximum value on the interval $0 \le x \le 1$ is attained at the endpoint $x = 1$. Thus, for $0 \le u \le 1$, $e^u \le e^1 < 2.8$. Hence we may take $K = 2.8$ to obtain:

$$|e - T_n(1)| \le \frac{2.8}{(n+1)!}$$

We now choose n such that

$$\frac{2.8}{(n+1)!} < 10^{-8}$$
$$\frac{(n+1)!}{2.8} > 10^8$$
$$(n+1)! > 2.8 \times 10^8$$

For $n = 10$, $(n+1)! = 3.99 \times 10^7 < 2.8 \times 10^8$ and for $n = 11$, $(n+1)! = 4.79 \times 10^8 > 2.8 \times 10^8$. Hence, to make the error less than 10^{-8}, $n = 11$ is sufficient; that is,

$$|e - T_{11}(1)| < 10^{-8}.$$

32. Let $T_4(x)$ be the Taylor polynomial for $f(x) = x \ln x$ at $a = 1$ computed in Exercise 23. Use the error bound to find a bound for $|f(1.2) - T_4(1.2)|$.

SOLUTION Using the Error Bound, we have

$$|f(1.2) - T_4(1.2)| \le K\frac{(1.2-1)^5}{5!} = \frac{(0.2)^5}{120}K,$$

where K is a number such that $\left|f^{(5)}x\right| \le K$ for all $1 \le x \le 1.2$. Starting from $f(x) = x \ln x$, we find

$$f'(x) = \ln x + x\frac{1}{x} = \ln x + 1,\ f''(x) = \frac{1}{x},\ f'''(x) = -\frac{1}{x^2},\ f^{(4)}(x) = \frac{2}{x^3},$$

and

$$f^{(5)}(x) = \frac{-6}{x^4}.$$

For $1 \le x \le 1.2$,

$$\left|f^{(5)}(x)\right| = \frac{6}{x^4} \le \frac{6}{1^4} = 6.$$

Hence we may take $K = 6$ to obtain:

$$|f(1.2) - T_4(1.2)| \le \frac{(0.2)^5}{120}6 = 1.6 \times 10^{-5}.$$

33. Verify that $T_n(x) = 1 + x + x^2 + \cdots + x^n$ is the nth Maclaurin polynomial of $f(x) = 1/(1-x)$. Show using substitution that the nth Maclaurin polynomial for $f(x) = 1/(1-x/4)$ is

$$T_n(x) = 1 + \frac{1}{4}x + \frac{1}{4^2}x^2 + \cdots + \frac{1}{4^n}x^n$$

What is the nth Maclaurin polynomial for $g(x) = \dfrac{1}{1+x}$?

SOLUTION Let $f(x) = (1-x)^{-1}$. Then, $f'(x) = (1-x)^{-2}$, $f''(x) = 2(1-x)^{-3}$, $f'''(x) = 3!(1-x)^{-4}$, and, in general, $f^{(n)}(x) = n!(1-x)^{-(n+1)}$. Therefore, $f^{(n)}(0) = n!$ and

$$T_n(x) = 1 + \frac{1!}{1!}x + \frac{2!}{2!}x^2 + \cdots + \frac{n!}{n!}x^n = 1 + x + x^2 + \cdots + x^n.$$

Upon substituting $x/4$ for x, we find that the nth Maclaurin polynomial for $f(x) = \dfrac{1}{1 - x/4}$ is

$$T_n(x) = 1 + \frac{1}{4}x + \frac{1}{4^2}x^2 + \cdots + \frac{1}{4^n}x^n.$$

Substituting $-x$ for x, the nth Maclaurin polynomial for $g(x) = \dfrac{1}{1+x}$ is

$$T_n(x) = 1 - x + x^2 - x^3 + - \cdots + (-x)^n.$$

34. Let $f(x) = \dfrac{5}{4 + 3x - x^2}$ and let a_k be the coefficient of x^k in the Maclaurin polynomial $T_n(x)$ of for $k \le n$.

(a) Show that $f(x) = \left(\dfrac{1/4}{1 - x/4} + \dfrac{1}{1+x}\right)$.

(b) Use Exercise 33 to show that $a_k = \dfrac{1}{4^{k+1}} + (-1)^k$.

(c) Compute $T_3(x)$.

SOLUTION

(a) Start by factoring the denominator and writing the form of the partial fraction decomposition:

$$f(x) = \frac{5}{4 + 3x - x^2} = \frac{5}{(x+1)(4-x)} = \frac{A}{x+1} + \frac{B}{4-x}.$$

Multiplying through by $(x+1)(4-x)$, we obtain:

$$5 = A(4 - x) + B(x + 1).$$

Substituting $x = 4$ yields $5 = A(0) + B(5)$, so $B = 1$; substituting $x = -1$ yields $5 = A(5) + B(0)$, so $A = 1$. Thus,

$$f(x) = \frac{1}{x+1} + \frac{1}{4-x} = \frac{1}{x+1} + \frac{\frac{1}{4}}{1 - \frac{x}{4}}.$$

(b) The nth Maclaurin polynomial for $f(x) = \frac{\frac{1}{4}}{1-\frac{x}{4}} + \frac{1}{x+1}$ is the sum of the nth Maclaurin polynomials for the functions $g(x) = \frac{1}{4} \cdot \frac{1}{1-\frac{x}{4}}$ and $h(x) = \frac{1}{1+x}$. In Exercise 33, we found that the nth Maclaurin polynomials $P_n(x)$ and $Q_n(x)$ for g and h are

$$P_n(x) = \frac{1}{4}\left(1 + \frac{1}{4}x + \frac{1}{4^2}x^2 + \cdots + \frac{1}{4^n}x^n\right) = \frac{1}{4} + \frac{1}{4^2}x + \frac{1}{4^3}x^2 + \cdots + \frac{1}{4^{n+1}}x^n = \sum_{k=0}^{n} \frac{x^k}{4^{k+1}}$$

$$Q_n(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{k=0}^{n} (-1)^k x^k$$

Therefore,

$$T_n(x) = P_n(x) + Q_n(x) = \sum_{k=0}^{n} \frac{x^k}{4^{k+1}} + \sum_{k=0}^{n} (-1)^k x^k = \sum_{k=0}^{n} \left[\frac{1}{4^{k+1}} + (-1)^k\right] x^k;$$

that is, the coefficient of x^k in T_n for $k \le n$ is

$$a_k = \frac{1}{4^{k+1}} + (-1)^k.$$

(c) From part (b),

$$a_0 = \frac{1}{4} + 1, \quad a_1 = \frac{1}{4^2} - 1, \quad a_2 = \frac{1}{4^3} + 1, \quad a_3 = \frac{1}{4^4} - 1$$

so that

$$T_3(x) = \frac{5}{4} - \frac{15}{16}x + \frac{65}{64}x^2 - \frac{255}{256}x^3$$

35. Let $T_n(x)$ be the nth Maclaurin polynomial for the function $f(x) = \sin x + \sinh x$.
(a) Show that $T_5(x) = T_6(x) = T_7(x) = T_8(x)$.
(b) Show that $|f^n(x)| \le 1 + \cosh x$ for all n. *Hint:* Note that $|\sinh x| \le |\cosh x|$ for all x.
(c) Show that $|T_8(x) - f(x)| \le \dfrac{2.6}{9!}|x|^9$ for $-1 \le x \le 1$.

SOLUTION
(a) Let $f(x) = \sin x + \sinh x$. Then

$$\begin{aligned} f'(x) &= \cos x + \cosh x \\ f''(x) &= -\sin x + \sinh x \\ f'''(x) &= -\cos x + \cosh x \\ f^{(4)}(x) &= \sin x + \sinh x. \end{aligned}$$

From this point onward, the pattern of derivatives repeats indefinitely. Thus

$$\begin{aligned} f(0) = f^{(4)}(0) = f^{(8)}(0) &= \sin 0 + \sinh 0 = 0 \\ f'(0) = f^{(5)}(0) &= \cos 0 + \cosh 0 = 2 \\ f''(0) = f^{(6)}(0) &= -\sin 0 + \sinh 0 = 0 \\ f'''(0) = f^{(7)}(0) &= -\cos 0 + \cosh 0 = 0. \end{aligned}$$

Consequently,

$$T_5(x) = f'(0)x + \frac{f^{(5)}(0)}{5!}x^5 = 2x + \frac{1}{60}x^5,$$

and, because $f^{(6)}(0) = f^{(7)}(0) = f^{(8)}(0) = 0$, it follows that

$$T_6(x) = T_7(x) = T_8(x) = T_5(x) = 2x + \frac{1}{60}x^5.$$

(b) First note that $|\sin x| \le 1$ and $|\cos x| \le 1$ for all x. Moreover,

$$|\sinh x| = \left|\frac{e^x - e^{-x}}{2}\right| \le \frac{e^x + e^{-x}}{2} = \cosh x.$$

Now, recall from part (a), that all derivatives of $f(x)$ contain two terms: the first is $\pm\sin x$ or $\pm\cos x$, while the second is either $\sinh x$ or $\cosh x$. In absolute value, the trigonometric term is always less than or equal to 1, while the hyperbolic term is always less than or equal to $\cosh x$. Thus, for all n,

$$f^{(n)}(x) \le 1 + \cosh x.$$

(c) Using the Error Bound, we have

$$|T_8(x) - f(x)| \le \frac{K|x-0|^9}{9!} = \frac{K|x|^9}{9!},$$

where K is a number such that $\left|f^{(9)}(u)\right| \le K$ for all u between 0 and x. By part (b), we know that

$$f^{(9)}(u) \le 1 + \cosh u.$$

Now, $\cosh u$ is an even function that is increasing on $(0, \infty)$. The maximum value for u between 0 and x is therefore $\cosh x$. Moreover, for $-1 \le x \le 1$, $\cosh x \le \cosh 1 \approx 1.543 < 1.6$. Hence, we may take $K = 1 + 1.6 = 2.6$, and

$$|T_8(x) - f(x)| \le \frac{2.6}{9!}|x|^9.$$