7 TECHNIQUES OF INTEGRATION

7.1 Integration by Parts

Preliminary Questions

1. Which derivative rule is used to derive the Integration by Parts formula?

SOLUTION The Integration by Parts formula is derived from the Product Rule.

2. For each of the following integrals, state whether substitution or Integration by Parts should be used:

$$\int x\cos(x^2)\,dx, \qquad \int x\cos x\,dx, \qquad \int x^2e^x\,dx, \qquad \int xe^{x^2}\,dx$$

SOLUTION

(a) $\int x\cos(x^2)\,dx$: use the substitution $u = x^2$.

(b) $\int x\cos x\,dx$: use Integration by Parts.

(c) $\int x^2e^x\,dx$; use Integration by Parts.

(d) $\int xe^{x^2}\,dx$; use the substitution $u = x^2$.

3. Why is $u = \cos x$, $v' = x$ a poor choice for evaluating $\int x\cos x\,dx$?

SOLUTION Transforming $v' = x$ into $v = \frac{1}{2}x^2$ increases the power of x and makes the new integral harder than the original.

Exercises

In Exercises 1–6, evaluate the integral using the Integration by Parts formula with the given choice of u and v′.

1. $\int x\sin x\,dx; \quad u = x,\ v' = \sin x$

SOLUTION Using the given choice of u and v' results in

$$\begin{aligned} u &= x & v &= -\cos x \\ u' &= 1 & v' &= \sin x \end{aligned}$$

Using Integration by Parts,

$$\int x\sin x\,dx = x(-\cos x) - \int (1)(-\cos x)\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

2. $\int xe^{2x}\,dx; \quad u = x,\ v' = e^{2x}$

SOLUTION Using $u = x$ and $v' = e^{2x}$ gives us

$$\begin{aligned} u &= x & v &= \tfrac{1}{2}e^{2x} \\ u' &= 1 & v' &= e^{2x} \end{aligned}$$

Integration by Parts gives us

$$\int xe^{2x}\,dx = x\left(\frac{1}{2}e^{2x}\right) - \int (1)\frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}\right)e^{2x} + C = \frac{1}{4}e^{2x}(2x-1) + C.$$

3. $\int (2x+9)e^x\,dx; \quad u = 2x+9,\ v' = e^x$

SOLUTION Using $u = 2x+9$ and $v' = e^x$ gives us

$$\begin{array}{ll} u = 2x+9 & v = e^x \\ u' = 2 & v' = e^x \end{array}$$

Integration by Parts gives us

$$\int (2x+9)e^x\,dx = (2x+9)e^x - \int 2e^x\,dx = (2x+9)e^x - 2e^x + C = e^x(2x+7) + C.$$

4. $\int x\cos 4x\,dx; \quad u = x,\ v' = \cos 4x$

SOLUTION Using $u = x$ and $v' = \cos 4x$ gives us

$$\begin{array}{ll} u = x & v = \frac{1}{4}\sin 4x \\ u' = 1 & v' = \cos 4x \end{array}$$

Integration by Parts gives us

$$\begin{aligned} \int x\cos 4x\,dx &= \frac{1}{4}x\sin 4x - \int (1)\frac{1}{4}\sin 4x\,dx = \frac{1}{4}x\sin 4x - \frac{1}{4}\left(-\frac{1}{4}\cos 4x\right) + C \\ &= \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C. \end{aligned}$$

5. $\int x^3\ln x\,dx; \quad u = \ln x,\ v' = x^3$

SOLUTION Using $u = \ln x$ and $v' = x^3$ gives us

$$\begin{array}{ll} u = \ln x & v = \frac{1}{4}x^4 \\ u' = \frac{1}{x} & v' = x^3 \end{array}$$

Integration by Parts gives us

$$\begin{aligned} \int x^3\ \ln x\,dx &= (\ln x)\left(\frac{1}{4}x^4\right) - \int \left(\frac{1}{x}\right)\left(\frac{1}{4}x^4\right)dx \\ &= \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C = \frac{x^4}{16}(4\ln x - 1) + C. \end{aligned}$$

6. $\int \tan^{-1}x\,dx; \quad u = \tan^{-1}x,\ v' = 1$

SOLUTION Using $u = \tan^{-1}x$ and $v' = 1$ gives us

$$\begin{array}{ll} u = \tan^{-1}x & v = x \\ u' = \dfrac{1}{x^2+1} & v' = 1 \end{array}$$

Integration by Parts gives us

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \left(\frac{1}{x^2+1}\right)x\,dx.$$

For the integral on the right we'll use the substitution $w = x^2+1$, $dw = 2x\,dx$. Then we have

$$\begin{aligned} \int \tan^{-1}x\,dx &= x\tan^{-1}x - \frac{1}{2}\int \left(\frac{1}{x^2+1}\right)2x\,dx = x\tan^{-1}x - \frac{1}{2}\int \frac{dw}{w} \\ &= x\tan^{-1}x - \frac{1}{2}\ln|w| + C = x\tan^{-1}x - \frac{1}{2}\ln|x^2+1| + C. \end{aligned}$$

In Exercises 7–36, evaluate using Integration by Parts.

7. $\int (4x-3)e^{-x}\,dx$

SOLUTION Let $u = 4x - 3$ and $v' = e^{-x}$. Then we have

$$\begin{aligned} u &= 4x-3 & v &= -e^{-x} \\ u' &= 4 & v' &= e^{-x} \end{aligned}$$

Using Integration by Parts, we get

$$\int (4x-3)e^{-x}\,dx = (4x-3)(-e^{-x}) - \int (4)(-e^{-x})\,dx$$
$$= -e^{-x}(4x-3) + 4\int e^{-x}\,dx = -e^{-x}(4x-3) - 4e^{-x} + C = -e^{-x}(4x+1) + C.$$

8. $\int (2x+1)e^{x}\,dx$

SOLUTION Let $u = 2x + 1$ and $v' = e^{-x}$. Then we have

$$\begin{aligned} u &= 2x+1 & v &= -e^{-x} \\ u' &= 2 & v' &= e^{-x} \end{aligned}$$

Using Integration by Parts, we get

$$\int (2x+1)\,e^{-x}\,dx = (2x+1)(-e^{-x}) - \int (2)(-e^{-x})\,dx$$
$$= -(2x+1)e^{-x} + 2\int e^{-x}\,dx = -(2x+1)e^{-x} - 2e^{-x} + C = -e^{-x}(2x+3) + C.$$

9. $\int x\,e^{5x+2}\,dx$

SOLUTION Let $u = x$ and $v' = e^{5x+2}$. Then we have

$$\begin{aligned} u &= x & v &= \frac{1}{5}e^{5x+2} \\ u' &= 1 & v' &= e^{5x+2} \end{aligned}$$

Using Integration by Parts, we get

$$\int xe^{5x+2}\,dx = x\left(\frac{1}{5}e^{5x+2}\right) - \int (1)\left(\frac{1}{5}e^{5x+2}\right)dx = \frac{1}{5}xe^{5x+2} - \frac{1}{5}\int e^{5x+2}\,dx$$
$$= \frac{1}{5}xe^{5x+2} - \frac{1}{25}e^{5x+2} + C = \left(\frac{x}{5} - \frac{1}{25}\right)e^{5x+2} + C$$

10. $\int x^2e^x\,dx$

SOLUTION Let $u = x^2$ and $v' = e^x$. Then we have

$$\begin{aligned} u &= x^2 & v &= e^x \\ u' &= 2x & v' &= e^x \end{aligned}$$

Using Integration by Parts, we get

$$\int x^2\,e^x\,dx = x^2e^x - 2\int xe^x\,dx.$$

We must apply Integration by Parts again to evaluate $\int xe^x\,dx$. Taking $u = x$ and $v' = e^x$, we get

$$\int xe^x\,dx = xe^x - \int (1)e^x\,dx = xe^x - e^x + C.$$

Plugging this into the original equation gives us

$$\int x^2\,e^x\,dx = x^2e^x - 2\left(xe^x - e^x\right) + C = e^x(x^2 - 2x + 2) + C.$$

11. $\int x \cos 2x\,dx$

SOLUTION Let $u = x$ and $v' = \cos 2x$. Then we have

$$u = x \quad v = \tfrac{1}{2}\sin 2x$$
$$u' = 1 \quad v' = \cos 2x$$

Using Integration by Parts, we get

$$\int x\ \cos 2x\,dx = x\left(\frac{1}{2}\sin 2x\right) - \int (1)\left(\frac{1}{2}\sin 2x\right)dx$$
$$= \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C.$$

12. $\int x\sin(3-x)\,dx$

SOLUTION Let $u = x$ and $v' = \sin(3-x)$. Then we have

$$u = x \quad v = \cos(3-x)$$
$$u' = 1 \quad v' = \sin(3-x)$$

Using Integration by Parts, we get

$$\int x\sin(3-x)\,dx = x\cos(3-x) - \int (1)\cos(3-x)\,dx = x\cos(3-x) + \sin(3-x) + C$$

13. $\int x^2 \sin x\,dx$

SOLUTION Let $u = x^2$ and $v' = \sin x$. Then we have

$$u = x^2 \quad v = -\cos x$$
$$u' = 2x \quad v' = \sin x$$

Using Integration by Parts, we get

$$\int x^2\sin x\,dx = x^2(-\cos x) - \int 2x(-\cos x)\,dx = -x^2\cos x + 2\int x\cos x\,dx.$$

We must apply Integration by Parts again to evaluate $\int x\cos x\,dx$. Taking $u = x$ and $v' = \cos x$, we get

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Plugging this into the original equation gives us

$$\int x^2\sin x\,dx = -x^2\cos x + 2(x\sin x + \cos x) + C = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

14. $\int x^2\cos 3x\,dx$

SOLUTION Let $u = x^2$ and $v' = \cos 3x$. Then we have

$$u = x^2 \quad v = \frac{1}{3}\sin 3x$$
$$u' = 2x \quad v' = \cos 3x$$

Using Integration by Parts, we get

$$\int x^2\cos 3x\,dx = \frac{1}{3}x^2\sin 3x - \int (2x)\frac{1}{3}\sin 3x\,dx = \frac{1}{3}x^2\sin 3x - \frac{2}{3}\int x\sin 3x\,dx$$

Use Integration by Parts again on this integral, with $u = x$ and $v' = \sin 3x$ to get

$$\int x^2\cos 3x\,dx = \frac{1}{3}x^2\sin 3x - \frac{2}{3}\left(-\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx\right)$$
$$= \frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C$$

15. $\int e^{-x} \sin x \, dx$

SOLUTION Let $u = e^{-x}$ and $v' = \sin x$. Then we have

$$u = e^{-x} \qquad v = -\cos x$$
$$u' = -e^{-x} \quad v' = \sin x$$

Using Integration by Parts, we get

$$\int e^{-x} \sin x \, dx = -e^{-x} \cos x - \int (-e^{-x})(-\cos x) \, dx = -e^{-x} \cos x - \int e^{-x} \cos x \, dx.$$

We must apply Integration by Parts again to evaluate $\int e^{-x} \cos x \, dx$. Using $u = e^{-x}$ and $v' = \cos x$, we get

$$\int e^{-x} \cos x \, dx = e^{-x} \sin x - \int (-e^{-x})(\sin x) \, dx = e^{-x} \sin x + \int e^{-x} \sin x \, dx.$$

Plugging this into the original equation, we get

$$\int e^{-x} \sin x \, dx = -e^{-x} \cos x - \left[e^{-x} \sin x + \int e^{-x} \sin x \, dx \right].$$

Solving this equation for $\int e^{-x} \sin x \, dx$ gives us

$$\int e^{-x} \sin x \, dx = -\frac{1}{2} e^{-x} (\sin x + \cos x) + C.$$

16. $\int e^{x} \sin 2x \, dx$

SOLUTION Let $u = \sin 2x$ and $v' = e^x$. Then we have

$$u = \sin 2x \qquad v = e^x$$
$$u' = 2\cos 2x \quad v' = e^x$$

Using Integration by Parts, we get

$$\int e^x \sin 2x \, dx = e^x \sin 2x - 2 \int e^x \cos 2x \, dx.$$

We must apply Integration by Parts again to evaluate $\int e^x \cos 2x \, dx$. Using $u = \cos 2x$ and $v' = e^x$, we get

$$\int e^x \cos 2x \, dx = e^x \cos 2x - \int (-2 \sin 2x) e^x \, dx = e^x \cos 2x + 2 \int e^x \sin 2x \, dx.$$

Plugging this into the original equation, we get

$$\int e^x \sin 2x \, dx = e^x \sin 2x - 2 \left[e^x \cos 2x + 2 \int e^x \sin 2x \, dx \right] = e^x \sin 2x - 2e^x \cos 2x - 4 \int e^x \sin 2x \, dx.$$

Solving this equation for $\int e^x \sin 2x \, dx$ gives us

$$\int e^x \sin 2x \, dx = \frac{1}{5} e^x (\sin 2x - 2\cos 2x) + C.$$

17. $\int e^{-5x} \sin x \, dx$

SOLUTION Let $u = \sin x$ and $v' = e^{-5x}$. Then we have

$$u = \sin x \qquad v = -\frac{1}{5} e^{-5x}$$
$$u' = \cos x \quad v' = e^{-5x}$$

Using Integration by Parts, we get

$$\int e^{-5x} \sin x \, dx = -\frac{1}{5} e^{-5x} \sin x - \int \cos x \left(-\frac{1}{5} e^{-5x} \right) dx = -\frac{1}{5} e^{-5x} \sin x + \frac{1}{5} \int e^{-5x} \cos x \, dx$$

Apply Integration by Parts again to this integral, with $u = \cos x$ and $v' = e^{-5x}$ to get

$$\int e^{-5x} \cos x \, dx = -\frac{1}{5} e^{-5x} \cos x - \frac{1}{5} \int e^{-5x} \sin x \, dx$$

Plugging this into the original equation, we get

$$\begin{aligned} \int e^{-5x} \sin x \, dx &= -\frac{1}{5} e^{-5x} \sin x + \frac{1}{5} \left(-\frac{1}{5} e^{-5x} \cos x - \frac{1}{5} \int e^{-5x} \sin x \, dx \right) \\ &= -\frac{1}{5} e^{-5x} \sin x - \frac{1}{25} e^{-5x} \cos x - \frac{1}{25} \int e^{-5x} \sin x \, dx \end{aligned}$$

Solving this equation for $\int e^{-5x} \sin x \, dx$ gives us

$$\int e^{-5x} \sin x \, dx = -\frac{5}{26} e^{-5x} \sin x - \frac{1}{26} e^{-5x} \cos x + C = -\frac{1}{26} e^{-5x} (5 \sin x + \cos x) + C$$

18. $\int e^{3x} \cos 4x \, dx$

SOLUTION Let $u = \cos 4x$ and $v' = e^{3x}$. Then we have

$$\begin{aligned} u &= \cos 4x & v &= \frac{1}{3} e^{3x} \\ u' &= -4 \sin 4x & v' &= e^{3x} \end{aligned}$$

Using Integration by Parts, we get

$$\int e^{3x} \cos 4x \, dx = \frac{1}{3} e^{3x} \cos 4x - \int \frac{1}{3} e^{3x} (-4 \sin 4x) \, dx = \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3} \int e^{3x} \sin 4x \, dx$$

Apply Integration by Parts again to this integral, with $u = \sin 4x$ and $v' = e^{3x}$, to get

$$\int e^{3x} \sin 4x \, dx = \frac{1}{3} e^{3x} \sin 4x - \int \frac{1}{3} e^{3x} \cdot 4 \cos 4x \, dx = \frac{1}{3} e^{3x} \sin 4x - \frac{4}{3} \int e^{3x} \cos 4x \, dx$$

Plugging this into the original equation, we get

$$\begin{aligned} \int e^{3x} \cos 4x \, dx &= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3} \left(\frac{1}{3} e^{3x} \sin 4x - \frac{4}{3} \int e^{3x} \cos 4x \, dx \right) \\ &= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{9} e^{3x} \sin 4x - \frac{16}{9} \int e^{3x} \cos 4x \, dx \end{aligned}$$

Solving this equation for $\int e^{3x} \cos 4x \, dx$ gives us

$$\int e^{3x} \cos 4x \, dx = \frac{3}{25} e^{3x} \cos 4x + \frac{4}{25} e^{3x} \sin 4x = \frac{1}{25} e^{3x} (3 \cos 4x + 4 \sin 4x) + C$$

19. $\int x \ln x \, dx$

SOLUTION Let $u = \ln x$ and $v' = x$. Then we have

$$\begin{aligned} u &= \ln x & v &= \tfrac{1}{2} x^2 \\ u' &= \tfrac{1}{x} & v' &= x \end{aligned}$$

Using Integration by Parts, we get

$$\begin{aligned} \int x \ln x \, dx &= \frac{1}{2} x^2 \ln x - \int \left(\frac{1}{x} \right) \left(\frac{1}{2} x^2 \right) dx \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx = \frac{1}{2} x^2 \ln x - \frac{1}{2} \left(\frac{x^2}{2} \right) + C = \frac{1}{4} x^2 (2 \ln x - 1) + C. \end{aligned}$$

20. $\int \frac{\ln x}{x^2}\,dx$

SOLUTION Let $u = \ln x$ and $v' = x^{-2}$. Then we have

$$u = \ln x \qquad v = -x^{-1}$$
$$u' = \tfrac{1}{x} \qquad v' = x^{-2}$$

Using Integration by Parts, we get

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x - \int \frac{1}{x}\left(\frac{-1}{x}\right)dx = -\frac{1}{x}\ln x + \int x^{-2}\,dx$$
$$= -\frac{1}{x}\ln x - \frac{1}{x} + C = -\frac{1}{x}(\ln x + 1) + C.$$

21. $\int x^2 \ln x\,dx$

SOLUTION Let $u = \ln x$ and $v' = x^2$. Then we have

$$u = \ln x \qquad v = \tfrac{1}{3}x^3$$
$$u' = \tfrac{1}{x} \qquad v' = x^2$$

Using Integration by Parts, we get

$$\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{x}\left(\frac{1}{3}x^3\right)dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\,dx$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{x^3}{3}\right) + C = \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C.$$

22. $\int x^{-5} \ln x\,dx$

SOLUTION Let $u = \ln x$ and $v' = x^{-5}$. Then we have

$$u = \ln x \qquad v = -\frac{1}{4}x^{-4}$$
$$u' = \frac{1}{x} \qquad v = x^{-5}$$

Using Integration by Parts, we get

$$\int x^{-5} \ln x\,dx = -\frac{1}{4}x^{-4}\ln x + \int \frac{1}{4}x^{-4}\frac{1}{x}\,dx = -\frac{1}{4}x^{-4}\ln x + \frac{1}{4}\int x^{-5}\,dx$$
$$= -\frac{1}{4}x^{-4}\ln x - \frac{1}{16}x^{-4} + C = -\frac{1}{4x^4}\left(\ln x + \frac{1}{4}\right) + C$$

23. $\int (\ln x)^2\,dx$

SOLUTION Let $u = (\ln x)^2$ and $v' = 1$. Then we have

$$u = (\ln x)^2 \qquad v = x$$
$$u' = \frac{2}{x}\ln x \qquad v' = 1$$

Using Integration by Parts, we get

$$\int (\ln x)^2\,dx = (\ln x)^2(x) - \int \left(\frac{2}{x}\ln x\right)x\,dx = x(\ln x)^2 - 2\int \ln x\,dx.$$

We must apply Integration by Parts again to evaluate $\int \ln x\,dx$. Using $u = \ln x$ and $v' = 1$, we have

$$\int \ln x\,dx = x\ln x - \int \frac{1}{x}\cdot x\,dx = x\ln x - \int dx = x\ln x - x + C.$$

Plugging this into the original equation, we get

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\,(x\ln x - x) + C = x\left[(\ln x)^2 - 2\ln x + 2\right] + C.$$

24. $\int x(\ln x)^2\,dx$

SOLUTION Let $u = (\ln x)^2$, $v' = x$. Then we have

$$u = (\ln x)^2 \qquad v = \frac{1}{2}x^2$$
$$u' = \frac{2\ln x}{x} \qquad v' = x$$

Using Integration by Parts, we get

$$\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \int x^2\frac{\ln x}{x}\,dx = \frac{1}{2}x^2(\ln x)^2 - \int x\ln x\,dx$$

Apply Integration by Parts again to this integral, with $u = \ln x$, $v' = x$, to get

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\frac{1}{x}\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

Plug this back into the first formula to get

$$\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right) + C = \frac{1}{2}x^2\left((\ln x)^2 - \ln x + \frac{1}{2}\right) + C$$

25. $\int x\sec^2 x\,dx$

SOLUTION Let $u = x$ and $v' = \sec^2 x$. Then we have

$$u = x \qquad v = \tan x$$
$$u' = 1 \qquad v' = \sec^2 x$$

Using Integration by Parts, we get

$$\int x\sec^2 x\,dx = x\tan x - \int (1)\tan x\,dx = x\tan x - \ln|\sec x| + C.$$

26. $\int x\tan x\sec x\,dx$

SOLUTION Let $u = x$ and $v' = \tan x\sec x$. Then we have

$$u = x \qquad v = \sec x$$
$$u' = 1 \qquad v' = \tan x\sec x$$

Using Integration by Parts, we get

$$\int x\tan x\sec x\,dx = x\sec x - \int \sec x\,dx = x\sec x - \ln|\sec x + \tan x| + C$$

27. $\int \cos^{-1} x\,dx$

SOLUTION Let $u = \cos^{-1} x$ and $v' = 1$. Then we have

$$u = \cos^{-1} x \qquad v = x$$
$$u' = \frac{-1}{\sqrt{1-x^2}} \qquad v' = 1$$

Using Integration by Parts, we get

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \int \frac{-x}{\sqrt{1-x^2}}\,dx.$$

We can evaluate $\int \frac{-x}{\sqrt{1-x^2}}\,dx$ by making the substitution $w = 1 - x^2$. Then $dw = -2x\,dx$, and we have

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}} = x\cos^{-1} x - \frac{1}{2}\int w^{-1/2}\,dw$$
$$= x\cos^{-1} x - \frac{1}{2}(2w^{1/2}) + C = x\cos^{-1} x - \sqrt{1-x^2} + C.$$

28. $\int \sin^{-1} x\, dx$

SOLUTION Let $u = \sin^{-1} x$ and $v' = 1$. Then we have

$$u = \sin^{-1} x \qquad v = x$$
$$u' = \frac{1}{\sqrt{1-x^2}} \qquad v' = 1$$

Using Integration by Parts, we get

$$\int \sin^{-1} x\, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx.$$

We can evaluate $\int \frac{x}{\sqrt{1-x^2}}\, dx$ by making the substitution $w = 1 - x^2$. Then $dw = -2x\, dx$, and we have

$$\int \sin^{-1} x\, dx = x \sin^{-1} x + \frac{1}{2} \int \frac{-2x\, dx}{\sqrt{1-x^2}} = x \sin^{-1} x + \frac{1}{2} \int w^{-1/2}\, dw$$
$$= x \sin^{-1} x + \frac{1}{2}(2w^{1/2}) + C = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

29. $\int \sec^{-1} x\, dx$

SOLUTION We are forced to choose $u = \sec^{-1} x$, $v' = 1$, so that $u' = \frac{1}{x\sqrt{x^2-1}}$ and $v = x$. Using Integration by parts, we get:

$$\int \sec^{-1} x\, dx = x \sec^{-1} x - \int \frac{x\, dx}{x\sqrt{x^2-1}} = x \sec^{-1} x - \int \frac{dx}{\sqrt{x^2-1}}.$$

Via the substitution $\sqrt{x^2-1} = \tan\theta$ (so that $x = \sec\theta$ and $dx = \sec\theta \tan\theta d\theta$), we get:

$$\int \sec^{-1} x\, dx = x \sec^{-1} x - \int \frac{\sec\theta \tan\theta d\theta}{\tan\theta} = x \sec^{-1} x - \int \sec\theta d\theta$$
$$= x \sec^{-1} x - \ln|\sec\theta + \tan\theta| + C = x \sec^{-1} x - \ln|x + \sqrt{x^2-1}| + C.$$

30. $\int x 5^x\, dx$

SOLUTION Let $u = x$ and $v' = 5^x$. Then we have

$$u = x \qquad v = \frac{5^x}{\ln 5}$$
$$u' = 1 \qquad v' = 5^x$$

Using Integration by Parts, we get

$$\int x\, 5^x\, dx = x\left(\frac{5^x}{\ln 5}\right) - \int (1) \frac{5^x}{\ln 5}\, dx = \frac{x\, 5^x}{\ln 5} - \frac{1}{\ln 5} \int 5^x\, dx$$
$$= \frac{x\, 5^x}{\ln 5} - \frac{1}{\ln 5}\left(\frac{5^x}{\ln 5}\right) + C = \frac{5^x}{\ln 5}\left(x - \frac{1}{\ln 5}\right) + C.$$

31. $\int 3^x \cos x\, dx$

SOLUTION Let $u = \cos x$ and $v' = 3^x$. Then we have

$$u = \cos x \qquad v = \frac{3^x}{\ln 3}$$
$$u' = -\sin x \qquad v' = 3^x$$

Using Integration by Parts, we get

$$\int 3^x \cos x\, dx = \frac{3^x}{\ln 3} \cos x + \frac{1}{\ln 3} \int 3^x \sin x\, dx$$

Apply Integration by Parts to the remaining integral, with $u = \sin x$ and $v' = 3^x$; then

$$\int 3^x \sin x\, dx = \frac{3^x}{\ln 3} \sin x - \frac{1}{\ln 3} \int 3^x \cos x\, dx$$

Plug this into the first equation to get

$$\begin{aligned}\int 3^x \cos x\, dx &= \frac{3^x}{\ln 3} \cos x + \frac{1}{\ln 3}\left(\frac{3^x}{\ln 3} \sin x - \frac{1}{\ln 3} \int 3^x \cos x\, dx\right)\\ &= \frac{3^x}{\ln 3} \cos x + \frac{3^x}{(\ln 3)^2} \sin x - \frac{1}{(\ln 3)^2} \int 3^x \cos x\, dx\end{aligned}$$

Solving for $\int 3^x \cos x\, dx$ gives

$$\int 3^x \cos x\, dx = \frac{3^x \ln 3 \cos x}{1 + (\ln 3)^2} + \frac{3^x \sin x}{1 + (\ln 3)^2} + C = \frac{3^x}{1 + (\ln 3)^2}(\ln 3 \cos x + \sin x) + C$$

32. $\int x \sinh x\, dx$

SOLUTION Let $u = x$, $v' = \sinh x$. Then

$$\begin{aligned} u &= x & v &= \cosh x\\ u' &= 1 & v' &= \sinh x\end{aligned}$$

Integration by Parts gives us

$$\int x \sinh x\, dx = x \cosh x - \int \cosh x\, dx = x \cosh x - \sinh x + C$$

33. $\int x^2 \cosh x\, dx$

SOLUTION Let $u = x^2$, $v' = \cosh x$. Then

$$\begin{aligned} u &= x^2 & v &= \sinh x\\ u' &= 2x & v' &= \cosh x\end{aligned}$$

Integration by Parts gives us (along with Exercise 32)

$$\int x^2 \cosh x\, dx = x^2 \sinh x - 2\int x \sinh x, dx = x^2 \sinh x - 2x \cosh x + 2 \sinh x + C$$

34. $\int \cos x \cosh x\, dx$

SOLUTION Let $u = \cos x$ and $v' = \cosh x$. Then

$$\begin{aligned} u &= \cos x & v &= \sinh x\\ u' &= -\sin x & v' &= \cosh x\end{aligned}$$

Integration by Parts gives us

$$\int \cos x \cosh x\, dx = \cos x \sinh x - \int (-\sin x) \sinh x\, dx = \cos x \sinh x + \int \sin x \sinh x\, dx.$$

We must apply Integration by Parts again to evaluate $\int \sin x \sinh x\, dx$. Using $u = \sin x$ and $v' = \sinh x$, we find

$$\int \sin x \sinh x\, dx = \sin x \cosh x - \int \cos x \cosh x\, dx.$$

Plugging this into the original equation, we have

$$\int \cos x \cosh x\, dx = \cos x \sinh x + \sin x \cosh x - \int \cos x \cosh x\, dx.$$

Solving this equation for $\int \cos x \cosh x\, dx$ yields

$$\int \cos x \cosh x\, dx = \frac{1}{2}(\cos x \sinh x + \sin x \cosh x) + C.$$

35. $\int \tanh^{-1} 4x\, dx$

SOLUTION Using $u = \tanh^{-1} 4x$ and $v' = 1$ gives us

$$\begin{aligned} u &= \tanh^{-1} 4x & v &= x \\ u' &= \frac{4}{1-16x^2} & v' &= 1 \end{aligned}$$

Integration by Parts gives us

$$\int \tanh^{-1} 4x\, dx = x \tanh^{-1} 4x - \int \left(\frac{4}{1-16x^2}\right) x\, dx.$$

For the integral on the right we'll use the substitution $w = 1 - 16x^2$, $dw = -32x\, dx$. Then we have

$$\begin{aligned} \int \tanh^{-1} 4x\, dx &= x \tanh^{-1} 4x + \frac{1}{8}\int \frac{dw}{w} = x \tanh^{-1} 4x + \frac{1}{8}\ln|w| + C \\ &= x \tanh^{-1} 4x + \frac{1}{8}\ln|1-16x^2| + C. \end{aligned}$$

36. $\int \sinh^{-1} x\, dx$

SOLUTION Using $u = \sinh^{-1} x$ and $v' = 1$ gives us

$$\begin{aligned} u &= \sinh^{-1} x & v &= x \\ u' &= \frac{1}{\sqrt{1+x^2}} & v' &= 1 \end{aligned}$$

Integration by Parts gives us

$$\int \sinh^{-1} x\, dx = x \sinh^{-1} x - \int \left(\frac{1}{\sqrt{1+x^2}}\right) x\, dx.$$

For the integral on the right we'll use the substitution $w = 1 + x^2$, $dw = 2x\, dx$. Then we have

$$\begin{aligned} \int \sinh^{-1} x\, dx &= x \sinh^{-1} x - \frac{1}{2}\int \frac{dw}{\sqrt{w}} = x \sinh^{-1} x - \sqrt{w} + C \\ &= x \sinh^{-1} x - \sqrt{1+x^2} + C. \end{aligned}$$

In Exercises 37 and 38, evaluate using substitution and then Integration by Parts.

37. $\int e^{\sqrt{x}}\, dx$ *Hint:* Let $u = x^{1/2}$

SOLUTION Let $w = x^{1/2}$. Then $dw = \frac{1}{2}x^{-1/2}dx$, or $dx = 2\,x^{1/2}\, dw = 2w\, dw$. Now,

$$\int e^{\sqrt{x}}\, dx = 2\int we^w\, dw.$$

Using Integration by Parts with $u = w$ and $v' = e^w$, we get

$$2\int we^w\, dw = 2(we^w - e^w) + C.$$

Substituting back, we find

$$\int e^{\sqrt{x}}\, dx = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C.$$

38. $\int x^3 e^{x^2}\,dx$

SOLUTION Let $w = x^2$. Then $dw = 2x\,dx$, and

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int we^w dw.$$

Using Integration by Parts, we let $u = w$ and $v' = e^w$. Then we have

$$\int we^w\,dw = we^w - \int (1)e^w\,dw = we^w - e^w + C.$$

Substituting back in terms of x, we get

$$\int x^3\,e^{x^2}\,dx = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C.$$

In Exercises 39–48, evaluate using Integration by Parts, substitution, or both if necessary.

39. $\int x\cos 4x\,dx$

SOLUTION Let $u = x$ and $v' = \cos 4x$. Then we have

$$u = x \qquad v = \tfrac{1}{4}\sin 4x$$
$$u' = 1 \qquad v' = \cos 4x$$

Using Integration by Parts, we get

$$\int x\cos 4x\,dx = \frac{1}{4}x\sin 4x - \int (1)\frac{1}{4}\sin 4x\,dx = \frac{1}{4}x\sin 4x - \frac{1}{4}\left(-\frac{1}{4}\cos 4x\right) + C$$
$$= \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C.$$

40. $\int \frac{\ln(\ln x)\,dx}{x}$

SOLUTION Let $w = \ln x$. Then $dw = dx/x$, and we have

$$\int \frac{\ln(\ln x)\,dx}{x} = \int \ln w\,dw$$

Now we can use Integration by Parts, letting $u = \ln w$ and $v' = 1$. Then $u' = 1/w$, $v = w$, and

$$\int \ln w\,dw = w\ln w - \int \frac{1}{w}(w)\,dw = w\ln w - w + C.$$

Substituting back in terms of x, we get

$$\int \frac{\ln(\ln x)\,dx}{x} = (\ln x)\ln(\ln x) - \ln x + C.$$

41. $\int \frac{x\,dx}{\sqrt{x+1}}$

SOLUTION Let $u = x + 1$. Then $du = dx$, $x = u - 1$, and

$$\int \frac{x\,dx}{\sqrt{x+1}} = \int \frac{(u-1)\,du}{\sqrt{u}} = \int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right)du = \int (u^{1/2} - u^{-1/2})\,du$$
$$= \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C.$$

42. $\int x^2(x^3+9)^{15}\,dx$

SOLUTION Note that $(x^3 + 0)' = 3x^2$, so use substitution with $u = x^3 + 9$, $du = 3x^2\,dx$. Then

$$\int x^2(x^3+9)^{15}\,dx = \frac{1}{3}\int u^{15}\,du = \frac{1}{48}u^{16} + C = \frac{1}{48}(x^3+9)^{16} + C$$

43. $\int \cos x \ln(\sin x)\,dx$

SOLUTION Let $w = \sin x$. Then $dw = \cos x\,dx$, and

$$\int \cos x\ \ln(\sin x)\,dx = \int \ln w\,dw.$$

Now use Integration by Parts with $u = \ln w$ and $v' = 1$. Then $u' = 1/w$ and $v = w$, which gives us

$$\int \cos x\ \ln(\sin x)\,dx = \int \ln w\,dw = w\ln w - w + C = \sin x \ln(\sin x) - \sin x + C.$$

44. $\int \sin\sqrt{x}\,dx$

SOLUTION First use substitution, with $w = \sqrt{x}$ and $dw = dx/(2\sqrt{x})$. This gives us

$$\int \sin\sqrt{x}\,dx = \int \frac{(2\sqrt{x})\sin\sqrt{x}\,dx}{(2\sqrt{x})} = 2\int w\sin w\,dw.$$

Now use Integration by Parts, with $u = w$ and $v' = \sin w$. Then we have

$$\int \sin\sqrt{x}\,dx = 2\int w\sin w\,dw = 2\left(-w\cos w - \int -\cos w\,dw\right)$$
$$= 2(-w\cos w + \sin w) + C = 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + C.$$

45. $\int \sqrt{x}e^{\sqrt{x}}\,dx$

SOLUTION Let $w = \sqrt{x}$. Then $dw = \frac{1}{2\sqrt{x}}\,dx$ and

$$\int \sqrt{x}e^{\sqrt{x}}\,dx = 2\int w^2e^w\,dw.$$

Now, use Integration by Parts with $u = w^2$ and $v' = e^w$. This gives

$$\int \sqrt{x}e^{\sqrt{x}}\,dx = 2\int w^2e^w\,dw = 2w^2e^w - 4\int we^w\,dw.$$

We need to use Integration by Parts again, this time with $u = w$ and $v' = e^w$. We find

$$\int we^w\,dw = we^w - \int e^w\,dw = we^w - e^w + C;$$

finally,

$$\int \sqrt{x}e^{\sqrt{x}}\,dx = 2w^2e^w - 4we^w + 4e^w + C = 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C.$$

46. $\int \frac{\tan\sqrt{x}\,dx}{\sqrt{x}}$

SOLUTION Let $u = \sqrt{x}$ and $du = \frac{1}{2}x^{-1/2}$. Then

$$\int \frac{\tan\sqrt{x}\,dx}{\sqrt{x}} = 2\int \tan u\,du = -2\ln|\cos u| + C = -2\ln|\cos\sqrt{x}| + C$$

47. $\int \frac{\ln(\ln x)\ln x\,dx}{x}$

SOLUTION Let $w = \ln x$. Then $dw = dx/x$, and

$$\int \frac{\ln(\ln x)\ln x\,dx}{x} = \int w\ln w\,dw.$$

Now use Integration by Parts, with $u = \ln w$ and $v' = w$. Then,

$$u = \ln w \qquad v = \frac{1}{2}w^2$$
$$u' = w^{-1} \qquad v' = w$$

and

$$\int \frac{\ln(\ln x)\ln x\,dx}{x} = \frac{1}{2}w^2\ln w - \frac{1}{2}\int w\,dw = \frac{1}{2}w^2\ln w - \frac{1}{2}\left(\frac{w^2}{2}\right) + C$$
$$= \frac{1}{2}(\ln x)^2\ln(\ln x) - \frac{1}{4}(\ln x)^2 + C = \frac{1}{4}(\ln x)^2[2\ln(\ln x) - 1] + C.$$

48. $\int \sin(\ln x)\,dx$

SOLUTION Let $u = \sin(\ln x)$ and $v' = 1$. Then we have

$$u = \sin(\ln x) \qquad v = x$$
$$u' = \frac{\cos(\ln x)}{x} \qquad v' = 1$$

Using Integration by Parts, we get

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int (x)\frac{\cos(\ln x)}{x}\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

We must use Integration by Parts again to evaluate $\int \cos(\ln x)\,dx$. Let $u = \cos(\ln x)$ and $v' = 1$. Then

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \left[x\cos(\ln x) - \int(-\sin(\ln x))\,dx\right]$$
$$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx.$$

Solving this equation for $\int \sin(\ln x)\,dx$, we get

$$\int \sin(\ln x)\,dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$$

In Exercises 49–54, compute the definite integral.

49. $\int_0^3 xe^{4x}\,dx$

SOLUTION Let $u = x$, $v' = e^{4x}$. Then $u' = 1$ and $v = \frac{1}{4}e^{4x}$. Using Integration by Parts,

$$\int_0^3 xe^{4x}\,dx = \left(\frac{1}{4}xe^{4x}\right)\Big|_0^3 - \frac{1}{4}\int_0^3 e^{4x}\,dx = \frac{3}{4}e^{12} - \frac{1}{16}e^{12} + \frac{1}{16} = \frac{11}{16}e^{12} + \frac{1}{16}$$

50. $\int_0^{\pi/4} x\sin 2x\,dx$

SOLUTION Let $u = x$ and $v' = \sin 2x$. Then $u' = 1$ and $v = -\frac{1}{2}\cos 2x$. Using Integration by Parts,

$$\int_0^{\pi/4} x\sin(2x)\,dx = -\frac{1}{2}x\cos 2x\Big|_0^{\pi/4} - \int_0^{\pi/4}\left(-\frac{1}{2}\cos 2x\right)\,dx = \left(-\frac{1}{2}x\cos 2x + \left(\frac{1}{2}\right)\frac{\sin 2x}{2}\right)\Big|_0^{\pi/4}$$
$$= \left(-\frac{1}{2}\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right) - (0+0) = \frac{1}{4}.$$

51. $\int_1^2 x\ln x\,dx$

SOLUTION Let $u = \ln x$ and $v' = x$. Then $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$. Using Integration by Parts gives

$$\int_1^2 x\ln x\,dx = \left(\frac{1}{2}x^2\ln x\right)\Big|_1^2 - \frac{1}{2}\int_1^2 x\,dx = 2\ln 2 - \frac{1}{4}x^2\Big|_1^2 = 2\ln 2 - \frac{3}{4}$$

52. $\int_1^e \frac{\ln x\,dx}{x^2}$

SOLUTION Let $u = \ln x$ and $v' = x^{-2}$. Then $u' = x^{-1}$ and $v = -x^{-1}$. Using Integration by Parts gives

$$\int_1^e \frac{\ln x\,dx}{x^2} = -\frac{\ln x}{x}\Big|_1^e + \int_1^e x^{-2}\,dx = -e^{-1} - x^{-1}\Big|_1^e = 1 - \frac{2}{e}$$

53. $\int_0^{\pi} e^x \sin x\,dx$

SOLUTION Let $u = \sin x$ and $v' = e^x$; then $u' = \cos x$ and $v = e^x$. Integration by Parts gives

$$\int_0^{\pi} e^x \sin x\,dx = e^x \sin x\Big|_0^{\pi} - \int_0^{\pi} e^x \cos x\,dx = -\int_0^{\pi} e^x \cos x\,dx$$

Apply integration by parts again to this integral, with $u = \cos x$ and $v' = e^x$; then $u' = -\sin x$ and $v = e^x$, so we get

$$\int_0^{\pi} e^x \sin x\,dx = -\left(\left(e^x \cos x\right)\Big|_0^{\pi} + \int_0^{\pi} e^x \sin x\,dx\right) = e^{\pi} + 1 - \int_0^{\pi} e^x \sin x\,dx$$

Solving for $\int_0^{\pi} e^x \sin x\,dx$ gives

$$\int_0^{\pi} e^x \sin x\,dx = \frac{e^{\pi} + 1}{2}$$

54. $\int_0^1 \tan^{-1} x\,dx$

SOLUTION Let $u = \tan^{-1} x$ and $v' = 1$. Then we have

$$u = \tan^{-1} x \qquad v = x$$
$$u' = \frac{1}{x^2 + 1} \qquad v' = 1$$

Integration by Parts gives us

$$\int \tan^{-1} x\,dx = x \tan^{-1} x - \int \left(\frac{1}{x^2 + 1}\right) x\,dx.$$

For the integral on the right we'll use the substitution $w = x^2 + 1$, $dw = 2x\,dx$. Then we have

$$\int \tan^{-1} x\,dx = x \tan^{-1} x - \frac{1}{2}\int \frac{dw}{w} = x \tan^{-1} x - \frac{1}{2}\ln|w| + C = x \tan^{-1} x - \frac{1}{2}\ln|x^2 + 1| + C.$$

Now we can compute the definite integral:

$$\int_0^1 \tan^{-1} x\,dx = \left(x \tan^{-1} x - \frac{1}{2}\ln|x^2 + 1|\right)\Big|_0^1 = \left((1)\tan^{-1}(1) - \frac{1}{2}\ln 2\right) - (0) = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

55. Use Eq. (5) to evaluate $\int x^4 e^x\,dx$.

SOLUTION

$$\begin{aligned}
\int x^4 e^x\,dx &= x^4 e^x - 4\int x^3 e^x\,dx = x^4 e^x - 4\left[x^3 e^x - 3\int x^2 e^x\,dx\right] \\
&= x^4 e^x - 4x^3 e^x + 12\int x^2 e^x\,dx = x^4 e^x - 4x^3 e^x + 12\left[x^2 e^x - 2\int x e^x\,dx\right] \\
&= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24\int x e^x\,dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24\left[x e^x - \int e^x\,dx\right] \\
&= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24\left[x e^x - e^x\right] + C.
\end{aligned}$$

Thus,

$$\int x^4 e^x\,dx = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C.$$

56. Use substitution and then Eq. (5) to evaluate $\int x^4 e^{7x}\,dx$.

SOLUTION Let $u = 7x$. Then $du = 7dx$, and

$$\int x^4 e^{7x}\,dx = \frac{1}{7^5}\int (7x)^4 e^{7x}(7dx) = \frac{1}{7^5}\int u^4 e^u\,du.$$

Now use the result from Exercise 55:

$$\begin{aligned}\int x^4 e^{7x}\,dx &= \frac{1}{7^5}e^u[u^4 - 4u^3 + 12u^2 - 24u + 24] + C \\ &= \frac{1}{7^5}e^{7x}[(7x)^4 - 4(7x)^3 + 12(7x)^2 - 24(7x) + 24] + C \\ &= \frac{1}{7^5}e^{7x}[2401x^4 - 1372x^3 + 588x^2 - 168x + 24] + C.\end{aligned}$$

57. Find a reduction formula for $\int x^n e^{-x}\,dx$ similar to Eq. (5).

SOLUTION Let $u = x^n$ and $v' = e^{-x}$. Then

$$\begin{aligned} u &= x^n & v &= -e^{-x} \\ u' &= nx^{n-1} & v' &= e^{-x}\end{aligned}$$

Using Integration by Parts, we get

$$\int x^n e^{-x}\,dx = -x^n e^{-x} - \int nx^{n-1}(-e^{-x})\,dx = -x^n e^{-x} + n\int x^{n-1}e^{-x}\,dx.$$

58. Evaluate $\int x^n \ln x\,dx$ for $n \neq -1$. Which method should be used to evaluate $\int x^{-1}\ln x\,dx$?

SOLUTION Let $u = \ln x$ and $v' = x^n$. Then we have

$$\begin{aligned} u &= \ln x & v &= \frac{x^{n+1}}{n+1} \\ u' &= \frac{1}{x} & v' &= x^n\end{aligned}$$

and

$$\begin{aligned}\int x^n \ln x\,dx &= \frac{x^{n+1}}{n+1}\ln x - \int \frac{1}{x}\cdot\frac{x^{n+1}}{n+1}\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n\,dx \\ &= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\cdot\frac{x^{n+1}}{n+1} = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C.\end{aligned}$$

For $n = -1$, $\int x^{-1}\ln x\,dx$, use the substitution $u = \ln x$, $du = dx/x$. Then

$$\int x^{-1}\ln x\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C.$$

In Exercises 59–66, indicate a good method for evaluating the integral (but do not evaluate). Your choices are algebraic manipulation, substitution (specify u and du), and Integration by Parts (specify u and v′). If it appears that the techniques you have learned thus far are not sufficient, state this.

59. $\int \sqrt{x}\ln x\,dx$

SOLUTION Use Integration by Parts, with $u = \ln x$ and $v' = \sqrt{x}$.

60. $\int \frac{x^2 - \sqrt{x}}{2x}\,dx$

SOLUTION Use algebraic manipulation:

$$\frac{x^2 - \sqrt{x}}{2x} = \frac{x}{2} - \frac{1}{2\sqrt{x}}.$$

61. $\displaystyle\int \frac{x^3\,dx}{\sqrt{4-x^2}}$

SOLUTION Use substitution, followed by algebraic manipulation: Let $u = 4 - x^2$. Then $du = -2x\,dx$, $x^2 = 4 - u$, and

$$\int \frac{x^3}{\sqrt{4-x^2}}dx = -\frac{1}{2}\int \frac{(x^2)(-2x\,dx)}{\sqrt{u}} = -\frac{1}{2}\int \frac{(4-u)(du)}{\sqrt{u}} = -\frac{1}{2}\int \left(\frac{4}{\sqrt{u}} - \frac{u}{\sqrt{u}}\right) du.$$

62. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

SOLUTION The techniques learned so far are insufficient. This problem requires the technique of trigonometric substitution.

63. $\displaystyle\int \frac{x+2}{x^2+4x+3}\,dx$

SOLUTION Use substitution. Let $u = x^2 + 4x + 3$; then $du = 2x + 4\,dx = 2(x+2)\,dx$, and

$$\int \frac{x+2}{x^2+4x+3}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$$

64. $\displaystyle\int \frac{dx}{(x+2)(x^2+4x+3)}$

SOLUTION The techniques learned so far are insufficient. This problem requires the technique of trigonometric substitution.

65. $\displaystyle\int x\sin(3x+4)\,dx$

SOLUTION Use Integration by Parts, with $u = x$ and $v' = \sin(3x+4)$.

66. $\displaystyle\int x\cos(9x^2)\,dx$

SOLUTION Use substitution, with $u = 9x^2$ and $du = 18x\,dx$.

67. Evaluate $\displaystyle\int (\sin^{-1}x)^2\,dx$. *Hint:* Use Integration by Parts first and then substitution.

SOLUTION First use integration by parts with $v' = 1$ to get

$$\int (\sin^{-1}x)^2\,dx = x(\sin^{-1}x)^2 - 2\int \frac{x\sin^{-1}x\,dx}{\sqrt{1-x^2}}.$$

Now use substitution on the integral on the right, with $u = \sin^{-1}x$. Then $du = dx/\sqrt{1-x^2}$ and $x = \sin u$, and we get (using Integration by Parts again)

$$\int \frac{x\sin^{-1}x\,dx}{\sqrt{1-x^2}} = \int u\sin u\,du = -u\cos u + \sin u + C = -\sqrt{1-x^2}\,\sin^{-1}x + x + C.$$

where $\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-x^2}$. So the final answer is

$$\int (\sin^{-1}x)^2\,dx = x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - 2x + C.$$

68. Evaluate $\displaystyle\int \frac{(\ln x)^2\,dx}{x^2}$. *Hint:* Use substitution first and then Integration by Parts.

SOLUTION Let $w = \ln x$. Then $dw = dx/x$, $e^w = x$, and

$$\int \frac{(\ln x)^2\,dx}{x^2} = \int \frac{w^2dw}{e^w}.$$

Now use Integration by Parts, with $u = w^2$ and $v' = e^{-w}$:

$$\int \frac{w^2dw}{e^w} = -w^2e^{-w} - \int 2w(-e^{-w})\,dw = -w^2e^{-w} + 2(-we^{-w} - e^{-w}) + C$$
$$= -e^{-w}(w^2 + 2w + 2) + C = -e^{-\ln x}((\ln x)^2 + 2\ln x + 2) + C.$$

The final answer is

$$\int \frac{(\ln x)^2\,dx}{x^2} = \frac{-[(\ln x)^2 + 2\ln x + 2]}{x} + C.$$

69. Evaluate $\int x^7 \cos(x^4)\,dx$.

SOLUTION First, let $w = x^4$. Then $dw = 4x^3\,dx$ and

$$\int x^7 \cos(x^4)\,dx = \frac{1}{4}\int w \cos x\,dw.$$

Now, use Integration by Parts with $u = w$ and $v' = \cos w$. Then

$$\int x^7 \cos(x^4)\,dx = \frac{1}{4}\left(w \sin w - \int \sin w\,dw\right) = \frac{1}{4} w \sin w + \frac{1}{4}\cos w + C = \frac{1}{4}x^4 \sin(x^4) + \frac{1}{4}\cos(x^4) + C.$$

70. Find $f(x)$, assuming that

$$\int f(x)e^x\,dx = f(x)e^x - \int x^{-1}e^x\,dx$$

SOLUTION We see that Integration by Parts was applied to $\int f(x)\,e^x\,dx$ with $u = f(x)$ and $v' = e^x$, and that therefore $f'(x) = u' = x^{-1}$. Thus $f(x) = \ln x + C$ for any constant C.

71. Find the volume of the solid obtained by revolving the region under $y = e^x$ for $0 \le x \le 2$ about the y-axis.

SOLUTION By the Method of Cylindrical Shells, the volume V of the solid is

$$V = \int_a^b (2\pi r)h\,dx = 2\pi \int_0^2 xe^x\,dx.$$

Using Integration by Parts with $u = x$ and $v' = e^x$, we find

$$V = 2\pi \;\; (xe^x - e^x)\Big|_0^2 = 2\pi\big[(2e^2 - e^2) - (0 - 1)\big] = 2\pi(e^2 + 1).$$

72. Find the area enclosed by $y = \ln x$ and $y = (\ln x)^2$.

SOLUTION The two graphs intersect at $x = 1$ and at $x = e$, and $\ln x$ is above $(\ln x)^2$, so the area is

$$\int_1^e \ln x - (\ln x)^2\,dx = \int_1^e \ln x\,dx - \int_1^e (\ln x)^2\,dx$$

Using integration by parts for the second integral, let $u = (\ln x)^2$, $v' = 1$; then $u' = \frac{2\ln x}{x}$ and $v = x$, so that

$$\int_1^e (\ln x)^2\,dx = \left(x(\ln x)^2\right)\Big|_1^e - 2\int_1^e \ln x = e - 2\int_1^e \ln x$$

Substituting this back into the original equation gives

$$\int_1^e \ln x - (\ln x)^2\,dx = 3\int_1^e \ln x\,dx - e$$

We use integration by parts to evaluate the remaining integral, with $u = \ln x$ and $v' = 1$; then $u' = \frac{1}{x}$ and $v = x$, so that

$$\int_1^e \ln x\,dx = x \ln x\Big|_1^e - \int_1^e 1\,dx = e - (e - 1) = 1$$

and thus, substituting back in, the value of the original integral is

$$\int_1^e \ln x - (\ln x)^2\,dx = 3\int_1^e \ln x\,dx - e = 3 - e$$

73. Recall that the *present value* (PV) of an investment that pays out income continuously at a rate $R(t)$ for T years is $\int_0^T R(t)e^{-rt}\,dt$, where r is the interest rate. Find the PV if $R(t) = 5000 + 100t$ \$/year, $r = 0.05$ and $T = 10$ years.

SOLUTION The present value is given by

$$PV = \int_0^T R(t)e^{-rt}\,dt = \int_0^{10} (5000 + 100t)e^{-rt}\,dt = 5000\int_0^{10} e^{-rt}\,dt + 100\int_0^{10} te^{-rt}\,dt.$$

Using Integration by Parts for the integral on the right, with $u = t$ and $v' = e^{-rt}$, we find

$$PV = 5000\left(-\frac{1}{r}e^{-rt}\right)\Big|_0^{10} + 100\left[\left(-\frac{t}{r}e^{-rt}\right)\Big|_0^{10} - \int_0^{10}\frac{-1}{r}e^{-rt}\,dt\right]$$
$$= -\frac{5000}{r}e^{-rt}\Big|_0^{10} - \frac{100}{r}\left(te^{-rt} + \frac{1}{r}e^{-rt}\right)\Big|_0^{10}$$
$$= -\frac{5000}{r}(e^{-10r} - 1) - \frac{100}{r}\left[\left(10e^{-10r} + \frac{1}{r}e^{-10r}\right) - \left(0 + \frac{1}{r}\right)\right]$$
$$= e^{-10r}\left[-\frac{5000}{r} - \frac{1000}{r} - \frac{100}{r^2}\right] + \frac{5000}{r} + \frac{100}{r^2}$$
$$= \frac{5000r + 100 - e^{-10r}(6000r + 100)}{r^2}.$$

74. Derive the reduction formula

$$\int (\ln x)^k\,dx = x(\ln x)^k - k\int (\ln x)^{k-1}\,dx \qquad \boxed{6}$$

SOLUTION Use Integration by Parts with $u = (\ln x)^k$ and $v' = 1$. Then $u' = k(\ln x)^{k-1}/x$, $v = x$, and we get

$$\int (\ln x)^k\,dx = x(\ln x)^k - k\int \frac{(\ln x)^{k-1}x\,dx}{x} = x(\ln x)^k - k\int (\ln x)^{k-1}\,dx.$$

75. Use Eq. (6) to calculate $\int (\ln x)^k\,dx$ for $k = 2, 3$.

SOLUTION

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx = x(\ln x)^2 - 2(x\ln x - x) + C = x(\ln x)^2 - 2x\ln x + 2x + C;$$
$$\int (\ln x)^3\,dx = x(\ln x)^3 - 3\int (\ln x)^2\,dx = x(\ln x)^3 - 3\left[x(\ln x)^2 - 2x\ln x + 2x\right] + C$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C.$$

76. Derive the reduction formulas

$$\int x^n\cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx$$
$$\int x^n\sin x\,dx = -x^n\cos x + n\int x^{n-1}\cos x\,dx$$

SOLUTION For $\int x^n\cos x\,dx$, let $u = x^n$ and $v' = \cos x$. Then we have

$$u = x^n \qquad v = \sin x$$
$$u' = nx^{n-1} \qquad v' = \cos x$$

Using Integration by Parts, we get

$$\int x^n\cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx.$$

For $\int x^n\sin x\,dx$, let $u = x^n$ and $v' = \sin x$. Then we have

$$u = x^n \qquad v = -\cos x$$
$$u' = nx^{n-1} \qquad v' = \sin x$$

Using Integration by Parts, we get

$$\int x^n\sin x\,dx = -x^n\cos x + n\int x^{n-1}\cos x\,dx.$$

77. Prove that $\int xb^x\,dx = b^x\left(\dfrac{x}{\ln b} - \dfrac{1}{\ln^2 b}\right) + C$.

SOLUTION Let $u = x$ and $v' = b^x$. Then $u' = 1$ and $v = b^x/\ln b$. Using Integration by Parts, we get

$$\int x\,b^x\,dx = \frac{xb^x}{\ln b} - \frac{1}{\ln b}\int b^x\,dx = \frac{xb^x}{\ln b} - \frac{1}{\ln b}\cdot\frac{b^x}{\ln b} + C = b^x\left(\frac{x}{\ln b} - \frac{1}{(\ln b)^2}\right) + C.$$

78. Define $P_n(x)$ by

$$\int x^n e^x\,dx = P_n(x)\,e^x + C$$

Use Eq. (5) to prove that $P_n(x) = x^n - nP_{n-1}(x)$. Use this recursion relation to find $P_n(x)$ for $n = 1, 2, 3, 4$. Note that $P_0(x) = 1$.

SOLUTION Use induction on n. Clearly for $n = 0$, we have

$$\int x^0 e^x\,dx = \int e^x\,dx = e^x + C = (1)e^x + C$$

so we may take $P_0(x) = 1 = x^0 - 0$. Now assume that

$$\int x^n e^x\,dx = P_n(x)e^x + C$$

where $P_n(x) = x^n - nP_{n-1}(x)$. Then using Eq. (5) with $n+1$ in place of n gives

$$\begin{aligned}\int x^{n+1}e^x\,dx &= x^{n+1}e^x - (n+1)\int x^n e^x\,dx = x^{n+1}e^x - (n+1)(P_n(x)e^x + C_1)\\ &= (x^{n+1} - (n+1)P_n(x))e^x + C\end{aligned}$$

Thus we may define $P_{n+1}(x) = x^{n+1} - (n+1)P_n(x)$ and we get

$$\int x^{n+1}e^x\,dx = P_{n+1}(x)e^x + C$$

as required.

Further Insights and Challenges

79. The Integration by Parts formula can be written

$$\int u(x)v(x)\,dx = u(x)V(x) - \int u'(x)V(x)\,dx \qquad \boxed{7}$$

where $V(x)$ satisfies $V'(x) = v(x)$.

(a) Show directly that the right-hand side of Eq. (7) does not change if $V(x)$ is replaced by $V(x) + C$, where C is a constant.

(b) Use $u = \tan^{-1} x$ and $v = x$ in Eq. (7) to calculate $\int x\tan^{-1} x\,dx$, but carry out the calculation twice: first with $V(x) = \frac{1}{2}x^2$ and then with $V(x) = \frac{1}{2}x^2 + \frac{1}{2}$. Which choice of $V(x)$ results in a simpler calculation?

SOLUTION

(a) Replacing $V(x)$ with $V(x) + C$ in the expression $u(x)V(x) - \int V(x)u'(x)\,dx$, we get

$$\begin{aligned}u(x)(V(x)+C) - \int (V(x)+C)u'(x)\,dx &= u(x)V(x) + u(x)C - \int V(x)u'(x)\,dx - C\int u'(x)\,dx\\ &= u(x)V(x) - \int V(x)u'(x)\,dx + C\left[u(x) - \int u'(x)\,dx\right]\\ &= u(x)V(x) - \int V(x)u'(x)\,dx + C\,[u(x) - u(x)]\\ &= u(x)V(x) - \int V(x)u'(x)\,dx.\end{aligned}$$

(b) If we evaluate $\int x \tan^{-1} x\, dx$ with $u = \tan^{-1} x$ and $v' = x$, and if we don't add a constant to v, Integration by Parts gives us

$$\int x \tan^{-1} x\, dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2 dx}{x^2 + 1}.$$

The integral on the right requires algebraic manipulation in order to evaluate. But if we take $V(x) = \frac{1}{2}x^2 + \frac{1}{2}$ instead of $V(x) = \frac{1}{2}x^2$, then

$$\int x \tan^{-1} x\, dx = \left(\frac{1}{2}x^2 + \frac{1}{2}\right) \tan^{-1} x - \frac{1}{2} \int \frac{x^2 + 1}{x^2 + 1}\, dx = \frac{1}{2}(x^2 + 1) \tan^{-1} x - \frac{1}{2}x + C$$
$$= \frac{1}{2}(x^2 \tan^{-1} x - x + \tan^{-1} x) + C.$$

80. Prove in two ways that

$$\int_0^a f(x)\, dx = af(a) - \int_0^a xf'(x)\, dx \qquad \boxed{8}$$

First use Integration by Parts. Then assume $f(x)$ is increasing. Use the substitution $u = f(x)$ to prove that $\int_0^a xf'(x)\, dx$ is equal to the area of the shaded region in Figure 1 and derive Eq. (8) a second time.

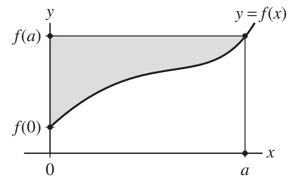


FIGURE 1

SOLUTION Let $u = f(x)$ and $v' = 1$. Then Integration by Parts gives

$$\int_0^a f(x)\, dx = xf(x)\Big|_0^a - \int_0^a xf'(x)\, dx = af(a) - \int_0^a xf'(x)\, dx.$$

Alternately, let $u = f(x)$. Then $du = f'(x)\, dx$, and if $f(x)$ is either increasing or decreasing, it has an inverse function, and $x = f^{-1}(u)$. Thus,

$$\int_{x=0}^{x=a} xf'(x)\, dx = \int_{f(0)}^{f(a)} f^{-1}(u)\, du$$

which is precisely the area of the shaded region in Figure 1 (integrating along the vertical axis). Since the area of the entire rectangle is $af(a)$, the difference between the areas of the two regions is $\int_0^a f(x)\, dx$.

81. Assume that $f(0) = f(1) = 0$ and that f'' exists. Prove

$$\int_0^1 f''(x) f(x)\, dx = -\int_0^1 f'(x)^2\, dx \qquad \boxed{9}$$

Use this to prove that if $f(0) = f(1) = 0$ and $f''(x) = \lambda f(x)$ for some constant λ, then $\lambda < 0$. Can you think of a function satisfying these conditions for some λ?

SOLUTION Let $u = f(x)$ and $v' = f''(x)$. Using Integration by Parts, we get

$$\int_0^1 f''(x) f(x)\, dx = f(x) f'(x)\Big|_0^1 - \int_0^1 f'(x)^2\, dx = f(1)f'(1) - f(0)f'(0) - \int_0^1 f'(x)^2\, dx = -\int_0^1 f'(x)^2\, dx.$$

Now assume that $f''(x) = \lambda f(x)$ for some constant λ. Then

$$\int_0^1 f''(x) f(x)\, dx = \lambda \int_0^1 [f(x)]^2\, dx = -\int_0^1 f'(x)^2\, dx < 0.$$

Since $\int_0^1 [f(x)]^2\, dx > 0$, we must have $\lambda < 0$. An example of a function satisfying these properties for some λ is $f(x) = \sin \pi x$.

82. Set $I(a,b) = \int_0^1 x^a(1-x)^b\,dx$, where a, b are whole numbers.

(a) Use substitution to show that $I(a,b) = I(b,a)$.

(b) Show that $I(a,0) = I(0,a) = \dfrac{1}{a+1}$.

(c) Prove that for $a \geq 1$ and $b \geq 0$,

$$I(a,b) = \frac{a}{b+1}\,I(a-1,b+1)$$

(d) Use (b) and (c) to calculate $I(1,1)$ and $I(3,2)$.

(e) Show that $I(a,b) = \dfrac{a!\,b!}{(a+b+1)!}$.

SOLUTION

(a) Let $u = 1-x$. Then $du = -dx$ and

$$I(a,b) = \int_{u=1}^{u=0} (1-u)^a u^b(-du) = \int_0^1 u^b(1-u)^a\,du = I(b,a).$$

(b) $I(a,0) = I(0,a)$ by part (a). Further,

$$I(a,0) = \int_0^1 x^a(1-x)^0\,dx = \int_0^1 x^a\,dx = \frac{1}{a+1}.$$

(c) Using Integration by Parts with $u = (1-x)^b$ and $v' = x^a$ gives

$$I(a,b) = \left.(1-x)^b\frac{x^{a+1}}{a+1}\right|_0^1 + \frac{b}{a+1}\int_0^1 x^{a+1}(1-x)^{b-1}\,dx = \frac{b}{a+1}I(a+1,b-1).$$

The other equality arises from Integration by Parts with $u = x^a$ and $v' = (1-x)^b$.

(d)

$$I(1,1) = \frac{1}{1+1}I(1-1,1+1) = \frac{1}{2}I(0,2) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$$
$$I(3,2) = \frac{1}{2}I(4,2) = \frac{1}{2}\cdot\frac{1}{5}I(5,0) = \frac{1}{10}\cdot\frac{1}{6} = \frac{1}{60}.$$

(e) We proceed as follows:

$$\begin{aligned}
I(a,b) &= \frac{a}{b+1}I(a-1,b+1) = \frac{a}{b+1}\cdot\frac{a-1}{b+2}I(a-2,b+2)\\
&\ \ \vdots\\
&= \frac{a}{b+1}\cdot\frac{a-1}{b+2}\cdots\frac{1}{b+a}I(0,b+a)\\
&= \frac{a(a-1)\cdots(1)}{(b+1)(b+2)\cdots(b+a)}\cdot\frac{1}{b+a+1}\\
&= \frac{b!\,a!}{b!\,(b+1)(b+2)\cdots(b+a)(b+a+1)} = \frac{a!\,b!}{(a+b+1)!}.
\end{aligned}$$

83. Let $I_n = \int x^n\cos(x^2)\,dx$ and $J_n = \int x^n\sin(x^2)\,dx$.

(a) Find a reduction formula that expresses I_n in terms of J_{n-2}. *Hint:* Write $x^n\cos(x^2)$ as $x^{n-1}(x\cos(x^2))$.

(b) Use the result of (a) to show that I_n can be evaluated explicitly if n is odd.

(c) Evaluate I_3.

SOLUTION

(a) Integration by Parts with $u = x^{n-1}$ and $v' = x\cos(x^2)\,dx$ yields

$$I_n = \frac{1}{2}x^{n-1}\sin(x^2) - \frac{n-1}{2}\int x^{n-2}\sin(x^2)\,dx = \frac{1}{2}x^{n-1}\sin(x^2) - \frac{n-1}{2}J_{n-2}.$$

(b) If n is odd, the reduction process will eventually lead to either

$$\int x\cos(x^2)\,dx \quad \text{or} \quad \int x\sin(x^2)\,dx,$$

both of which can be evaluated using the substitution $u = x^2$.
(c) Starting with the reduction formula from part (a), we find

$$I_3 = \frac{1}{2}x^2\sin(x^2) - \frac{2}{2}\int x\sin(x^2)\,dx = \frac{1}{2}x^2\sin(x^2) + \frac{1}{2}\cos(x^2) + C.$$

7.2 Trigonometric Integrals

Preliminary Questions

1. Describe the technique used to evaluate $\int \sin^5 x\,dx$.

SOLUTION Because the sine function is raised to an odd power, rewrite $\sin^5 x = \sin x \sin^4 x = \sin x(1-\cos^2 x)^2$ and then substitute $u = \cos x$.

2. Describe a way of evaluating $\int \sin^6 x\;dx$.

SOLUTION Repeatedly use the reduction formula for powers of $\sin x$.

3. Are reduction formulas needed to evaluate $\int \sin^7 x\cos^2 x\,dx$? Why or why not?

SOLUTION No, a reduction formula is not needed because the sine function is raised to an odd power.

4. Describe a way of evaluating $\int \sin^6 x\cos^2 x\,dx$.

SOLUTION Because both trigonometric functions are raised to even powers, write $\cos^2 x = 1 - \sin^2 x$ and then apply the reduction formula for powers of the sine function.

5. Which integral requires more work to evaluate?

$$\int \sin^{798} x\cos x\,dx \qquad \text{or} \qquad \int \sin^4 x\cos^4 x\,dx$$

Explain your answer.

SOLUTION The first integral can be evaluated using the substitution $u = \sin x$, whereas the second integral requires the use of reduction formulas. The second integral therefore requires more work to evaluate.

Exercises

In Exercises 1–6, use the method for odd powers to evaluate the integral.

1. $\int \cos^3 x\,dx$

SOLUTION Use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite the integrand:

$$\int \cos^3 x\,dx = \int \left(1-\sin^2 x\right)\cos x\,dx.$$

Now use the substitution $u = \sin x$, $du = \cos x\,dx$:

$$\int \cos^3 x\,dx = \int \left(1-u^2\right)\,du = u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C.$$

2. $\int \sin^5 x\,dx$

SOLUTION Use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite the integrand:

$$\int \sin^5 x\,dx = \int \left(\sin^2 x\right)^2 \sin x\,dx = \int \left(1-\cos^2 x\right)^2 \sin x\,dx.$$

Now use the substitution $u = \cos x$, $du = -\sin x\,dx$:

$$\int \sin^5 x\,dx = -\int \left(1-u^2\right)^2 du = -\int \left(1-2u^2+u^4\right) du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C$$
$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.$$

3. $\int \sin^3\theta\cos^2\theta\,d\theta$

SOLUTION Write $\sin^3\theta = \sin^2\theta\sin\theta = (1-\cos^2\theta)\sin\theta$. Then

$$\int \sin^3\theta\cos^2\theta\,d\theta = \int \left(1-\cos^2\theta\right)\cos^2\theta\sin\theta\,d\theta.$$

Now use the substitution $u=\cos\theta$, $du=-\sin\theta\,d\theta$:

$$\int \sin^3\theta\cos^2\theta\,d\theta = -\int\left(1-u^2\right)u^2\,du = -\int\left(u^2-u^4\right)du$$
$$= -\frac{1}{3}u^3+\frac{1}{5}u^5+C = -\frac{1}{3}\cos^3\theta+\frac{1}{5}\cos^5\theta+C.$$

4. $\int \sin^5 x\cos x\,dx$

SOLUTION Write $\sin^5 x = \sin^4 x\sin x = (1-\cos^2 x)^2\sin x$. Then

$$\int \cos x\sin^5 x\,dx = \int \cos x\left(1-\cos^2 x\right)^2\sin x\,dx.$$

Now use the substitution $u=\cos x$, $du=-\sin x\,dx$:

$$\int \cos x\sin^5 x\,dx = -\int u\left(1-u^2\right)^2du = -\int u\left(1-2u^2+u^4\right)du = \int\left(-u+2u^3-u^5\right)du$$
$$= -\frac{1}{2}u^2+\frac{1}{2}u^4-\frac{1}{6}u^6+C = -\frac{1}{2}\cos^2 x+\frac{1}{2}\cos^4 x-\frac{1}{6}\cos^6 x+C.$$

5. $\int \sin^3 t\cos^3 t\,dt$

SOLUTION Write $\sin^3 t = (1-\cos^2 t)\sin t\,dt$. Then

$$\int \sin^3 t\cos^3 t\,dt = \int (1-\cos^2 t)\cos^3 t\sin t\,dt = \int\left(\cos^3 t-\cos^5 t\right)\sin t\,dt.$$

Now use the substitution $u=\cos t$, $du=-\sin t\,dt$:

$$\int \sin^3 t\cos^3 t\,dt = -\int\left(u^3-u^5\right)du = -\frac{1}{4}u^4+\frac{1}{6}u^6+C = -\frac{1}{4}\cos^4 t+\frac{1}{6}\cos^6 t+C.$$

6. $\int \sin^2 x\cos^5 x\,dx$

SOLUTION Write $\cos^5 x = \cos^4 x\cos x = (1-\sin^2 x)^2\cos x$. Then

$$\int \sin^2 x\cos^5 x\,dx = \int \sin^2 x\left(1-\sin^2 x\right)^2\cos x\,dx.$$

Now use the substitution $u=\sin x$, $du=\cos x\,dx$:

$$\int \sin^2 x\cos^5 x\,dx = \int u^2\left(1-u^2\right)^2du = \int\left(u^2-2u^4+u^6\right)du$$
$$= \frac{1}{3}u^3-\frac{2}{5}u^5+\frac{1}{7}u^7+C = \frac{1}{3}\sin^3 x-\frac{2}{5}\sin^5 x+\frac{1}{7}\sin^7 x+C.$$

7. Find the area of the shaded region in Figure 1.

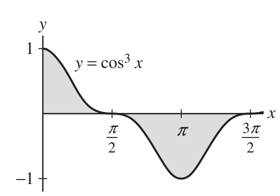


FIGURE 1 Graph of $y=\cos^3 x$.

SOLUTION First evaluate the indefinite integral by writing $\cos^3 x = (1 - \sin^2 x)\cos x$, and using the substitution $u = \sin x$, $du = \cos x\, dx$:

$$\int \cos^3 x\, dx = \int \left(1 - \sin^2 x\right)\cos x\, dx = \int \left(1 - u^2\right) du = u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C.$$

The area is given by

$$A = \int_0^{\pi/2} \cos^3 x\, dx - \int_{\pi/2}^{3\pi/2} \cos^3 x\, dx = \left(\sin x - \frac{1}{3}\sin^3 x\right)\Big|_0^{\pi/2} - \left(\sin x - \frac{1}{3}\sin^3 x\right)\Big|_{\pi/2}^{3\pi/2}$$
$$= \left[\left(\sin\frac{\pi}{2} - \frac{1}{3}\sin^3\frac{\pi}{2}\right) - 0\right] - \left[\left(\sin\frac{3\pi}{2} - \frac{1}{3}\sin^3\frac{3\pi}{2}\right) - \left(\sin\frac{\pi}{2} - \frac{1}{3}\sin^3\frac{\pi}{2}\right)\right]$$
$$= 1 - \frac{1}{3}(1)^3 - (-1) + \frac{1}{3}(-1)^3 + 1 - \frac{1}{3}(1)^3 = 2.$$

8. Use the identity $\sin^2 x = 1 - \cos^2 x$ to write $\int \sin^2 x \cos^2 x\, dx$ as a sum of two integrals, and then evaluate using the reduction formula.

SOLUTION Using the identity $\sin^2 x = 1 - \cos^2 x$, we get

$$\int \sin^2 x \cos^2 x\, d = \int \left(1 - \cos^2 x\right)\cos^2 x\, dx = \int \cos^2 x\, dx - \int \cos^4 x\, dx.$$

Using the reduction formula for $\cos^m x$, we get

$$\int \cos^4 x\, dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\int \cos^2 x\, dx.$$

Thus,

$$\int \sin^2 x \cos^2 x\, dx = \int \cos^2 x - \frac{1}{4}\cos^3 x \sin x - \frac{3}{4}\int \cos^2 x\, dx = -\frac{1}{4}\cos^3 x \sin x + \frac{1}{4}\int \cos^2 x\, dx.$$

Using the reduction formula again, we have

$$\int \sin^2 x \cos^2 x\, dx = -\frac{1}{4}\cos^3 x \sin x + \frac{1}{4}\left[\frac{\cos x \sin x}{2} + \frac{1}{2}\int dx\right] = -\frac{1}{4}\cos^3 x \sin x + \frac{1}{8}\cos x \sin x + \frac{1}{8}x + C.$$

In Exercises 9–12, evaluate the integral using methods employed in Examples 3 and 4.

9. $\int \cos^4 y\, dy$

SOLUTION Using the reduction formula for $\cos^m y$, we get

$$\int \cos^4 y\, dy = \frac{1}{4}\cos^3 y \sin y + \frac{3}{4}\int \cos^2 y\, dy = \frac{1}{4}\cos^3 y \sin y + \frac{3}{4}\left(\frac{1}{2}\cos y \sin y + \frac{1}{2}\int dy\right)$$
$$= \frac{1}{4}\cos^3 y \sin y + \frac{3}{8}\cos y \sin y + \frac{3}{8}y + C.$$

10. $\int \cos^2\theta \sin^2\theta\, d\theta$

SOLUTION First use the identity $\cos^2\theta = 1 - \sin^2\theta$ to write:

$$\int \cos^2\theta \sin^2\theta\, d\theta = \int \left(1 - \sin^2\theta\right)\sin^2\theta\, d\theta = \int \sin^2\theta\, d\theta - \int \sin^4\theta\, d\theta.$$

Using the reduction formula for $\sin^m\theta$, we get

$$\int \cos^2\theta \sin^2\theta\, d\theta = \int \sin^2\theta\, d\theta - \left[-\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}\int \sin^2\theta\, d\theta\right] = \frac{1}{4}\sin^3\theta\cos\theta + \frac{1}{4}\int \sin^2\theta\, d\theta$$
$$= \frac{1}{4}\sin^3\theta\cos\theta + \frac{1}{4}\left(-\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\int d\theta\right) = \frac{1}{4}\sin^3\theta\cos\theta - \frac{1}{8}\sin\theta\cos\theta + \frac{1}{8}\theta + C.$$

11. $\int \sin^4 x \cos^2 x\,dx$

SOLUTION Use the identity $\cos^2 x = 1 - \sin^2 x$ to write:

$$\int \sin^4 x \cos^2 x\,dx = \int \sin^4 x \left(1 - \sin^2 x\right) dx = \int \sin^4 x\,dx - \int \sin^6 x\,dx.$$

Using the reduction formula for $\sin^m x$:

$$\begin{aligned}\int \sin^4 x \cos^2 x\,dx &= \int \sin^4 x\,dx - \left[-\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\int \sin^4 x\,dx\right]\\ &= \frac{1}{6}\sin^5 x\cos x + \frac{1}{6}\int \sin^4 x\,dx = \frac{1}{6}\sin^5 x\cos x + \frac{1}{6}\left(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx\right)\\ &= \frac{1}{6}\sin^5 x\cos x - \frac{1}{24}\sin^3 x\cos x + \frac{1}{8}\int \sin^2 x\,dx\\ &= \frac{1}{6}\sin^5 x\cos x - \frac{1}{24}\sin^3 x\cos x + \frac{1}{8}\left(-\frac{1}{2}\sin x\cos x + \frac{1}{2}\int dx\right)\\ &= \frac{1}{6}\sin^5 x\cos x - \frac{1}{24}\sin^3 x\cos x - \frac{1}{16}\sin x\cos x + \frac{1}{16}x + C.\end{aligned}$$

12. $\int \sin^2 x \cos^6 x\,dx$

SOLUTION Use the identity $\sin^2 x = 1 - \cos^2 x$ to write

$$\int \sin^2 x \cos^6 x\,dx = \int (1 - \cos^2 x)\cos^6 x\,dx = \int \cos^6 x\,dx - \int \cos^8 x\,dx$$

Now use the reduction formula for $\cos^n x$:

$$\begin{aligned}\int \cos^6 x\,dx &= \frac{\cos^5 x\sin x}{6} + \frac{5}{6}\int \cos^4 x\,dx\\ &= \frac{\cos^5 x\sin x}{6} + \frac{5}{6}\left(\frac{\cos^3 x\sin x}{4} + \frac{3}{4}\int \cos^2 x\,dx\right)\\ &= \frac{1}{6}\cos^5 x\sin x + \frac{5}{24}\cos^3 x\sin x + \frac{15}{24}\left(\frac{x}{2} + \frac{\sin 2x}{4}\right) + C\\ &= \frac{1}{6}\cos^5 x\sin x + \frac{5}{24}\cos^3 x\sin x + \frac{15}{48}x + \frac{15}{96}\sin 2x + C\end{aligned}$$

and

$$\begin{aligned}\int \cos^8 x\,dx &= \frac{1}{8}\cos^7 x\sin x + \frac{7}{8}\int \cos^6 x\,dx\\ &= \frac{1}{8}\cos^7 x\sin x + \frac{7}{8}\left(\frac{1}{6}\cos^5 x\sin x + \frac{5}{24}\cos^3 x\sin x + \frac{15}{48}x + \frac{15}{96}\sin 2x\right) + C\\ &= \frac{1}{8}\cos^7 x\sin x + \frac{7}{48}\cos^5 x\sin x + \frac{35}{192}\cos^3 x\sin x + \frac{105}{384}x + \frac{105}{768}\sin 2x + C\end{aligned}$$

so that

$$\int \sin^2 x\cos^6 x\,dx = -\frac{1}{8}\cos^7 x\sin x + \frac{1}{48}\cos^5 x\sin x + \frac{5}{192}\cos^3 x\sin x + \frac{5}{128}x + \frac{5}{256}\sin 2x + C$$

In Exercises 13 and 14, evaluate using Eq. (13).

13. $\int \sin^3 x \cos^2 x\,dx$

SOLUTION First rewrite $\sin^3 x = \sin x \cdot \sin^2 x = \sin x(1 - \cos^2 x)$, so that

$$\int \sin^3 x\cos^2 x\,dx = \int \sin x(1 - \cos^2 x)\cos^2 x\,dx = \int \sin x(\cos^2 x - \cos^4 x)\,dx$$

Now make the substitution $u = \cos x$, $du = -\sin x\,dx$:

$$\int \sin x(\cos^2 x - \cos^4 x)\,dx = -\int u^2 - u^4\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

14. $\int \sin^2 x \cos^4 x \, dx$

SOLUTION Using the formula for $\int \sin^m x \cos^n x \, dx$, we get

$$I = \int \sin^2 x \cos^4 x \, dx = \frac{1}{6} \sin^3 x \cos^3 x + \frac{3}{6} \int \sin^2 x \cos^2 x \, dx = \frac{1}{6} \sin^3 x \cos^3 x + \frac{1}{2} \int \sin^2 x \cos^2 x \, dx.$$

Applying the formula again on the remaining integral, we get

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{4} \sin^3 x \cos x + \frac{1}{4} \int \sin^2 x \cos^0 x \, dx = \frac{1}{4} \sin^3 x \cos x + \frac{1}{4} \int \sin^2 x \, dx.$$

The final result is

$$\begin{aligned} I &= \frac{1}{6} \sin^3 x \cos^3 x + \frac{1}{2} \left(\frac{1}{4} \sin^3 x \cos x + \frac{1}{4} \int \sin^2 x \, dx \right) \\ &= \frac{1}{6} \sin^3 x \cos^3 x + \frac{1}{8} \sin^3 x \cos x + \frac{1}{8} \left(\frac{1}{2} x - \frac{1}{2} \sin x \cos x \right) + C \\ &= \frac{1}{6} \sin^3 x \cos^3 x + \frac{1}{8} \sin^3 x \cos x + \frac{1}{16} x - \frac{1}{16} \sin x \cos x + C. \end{aligned}$$

In Exercises 15–18, evaluate the integral using the method described on page 409 and the reduction formulas on page 410 as necessary.

15. $\int \tan^3 x \sec x \, dx$

SOLUTION Use the identity $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^3 x \sec x = (\sec^2 x - 1) \sec x \tan x$. Then use the substitution $u = \sec x$, $du = \sec x \tan x \, dx$:

$$\int \tan^3 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \tan x \, dx = \int u^2 - 1 \, du = \frac{1}{3} u^3 - u + C = \frac{1}{3} \sec^3 x - \sec x + C$$

16. $\int \tan^2 x \sec x \, dx$

SOLUTION First use the identity $\tan^2 x = \sec^2 x - 1$:

$$\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \, dx = \int \sec^3 x - \sec x \, dx = \int \sec^3 x \, dx - \ln |\sec x + \tan x|$$

To evaluate the remaining integral, we use the reduction formula:

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x|$$

so that finally, putting these together,

$$\int \tan^2 x \sec x \, dx = \int \sec^3 x \, dx - \ln |\sec x + \tan x| = \frac{1}{2} (\sec x \tan x - \ln |\sec x + \tan x|) + C$$

17. $\int \tan^2 x \sec^4 x \, dx$

SOLUTION First use the identity $\tan^2 x = \sec^2 x - 1$:

$$\int \tan^2 x \sec^4 x \, dx = \int (\sec^2 x - 1) \sec^4 x \, dx = \int \sec^6 x - \sec^4 x \, dx = \int \sec^6 x \, dx - \int \sec^4 x, dx$$

We evaluate the second integral using the reduction formula:

$$\begin{aligned} \int \sec^4 x \, dx &= \frac{1}{3} \tan x \sec^2 x + \frac{2}{3} \int \sec^2 x \, dx \\ &= \frac{1}{3} \tan x \sec^2 x + \frac{2}{3} \tan x \end{aligned}$$

Then

$$\int \sec^6 x\,dx = \frac{1}{5}\tan x \sec^4 x + \frac{4}{5}\int \sec^4 x\,dx$$
$$= \frac{1}{5}\tan x \sec^4 x + \frac{4}{5}\left(\frac{1}{3}\tan x \sec^2 x + \frac{2}{3}\tan x\right)$$
$$= \frac{1}{5}\tan x \sec^4 x + \frac{4}{15}\tan x \sec^2 x + \frac{8}{15}\tan x$$

so that

$$\int \tan^2 x \sec^4 x\,dx = \int \sec^6 x\,dx - \int \sec^4 x\,dx$$
$$= \frac{1}{5}\tan x \sec^4 x - \frac{1}{15}\tan x \sec^2 x - \frac{2}{15}\tan x + C$$

18. $\int \tan^8 x \sec^2 x\,dx$

SOLUTION Use the substitution $u = \tan x$, $du = \sec^2 x\,dx$; then

$$\int \tan^8 x \sec^2 x\,dx = \int u^8\,du = \frac{1}{9}u^9 = \frac{1}{9}\tan^9 x + C$$

In Exercises 19–22, evaluate using methods similar to those that apply to integral $\tan^m x \sec^n$.

19. $\int \cot^3 x\,dx$

SOLUTION Using the reduction formula for $\cot^m x$, we get

$$\int \cot^3 x\,dx = -\frac{1}{2}\cot^2 x - \int \cot x\,dx = -\frac{1}{2}\cot^2 x + \ln|\csc x| + C.$$

20. $\int \sec^3 x\,dx$

SOLUTION Using the reduction formula for $\sec^m x$, we get

$$\int \sec^3 x\,dx = \frac{1}{2}\tan x \sec x + \frac{1}{2}\int \sec x\,dx = \frac{1}{2}\tan x \sec x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

21. $\int \cot^5 x \csc^2 x\,dx$

SOLUTION Make the substitution $u = \cot x$, $du = -\csc^2 x\,dx$; then

$$\int \cot^5 x \csc^2 x\,dx = -\int u^5\,du = -\frac{1}{6}u^6 = -\frac{1}{6}\cot^6 x + C$$

22. $\int \cot^4 x \csc x\,dx$

SOLUTION Use the identity $\cot^2 x = \csc^2 x - 1$ to write

$$\int \cot^4 x \csc x\,dx = \int (\csc^2 x - 1)^2 \csc x\,dx = \int \csc^5 x - 2\csc^3 x + \csc x\,dx$$

Now apply the reduction formula:

$$\int \csc^3 x\,dx = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\int \csc x\,dx = -\frac{1}{2}\cot x \csc x - \frac{1}{2}\ln|\csc x + \cot x| + C$$

so that

$$\int \csc^5 x\,dx = -\frac{1}{4}\cot x \csc^3 x + \frac{3}{4}\int \csc^3 x\,dx$$
$$= -\frac{1}{4}\cot x \csc^3 x - \frac{3}{4}\left(\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x + \cot x|\right) + C$$
$$= -\frac{1}{4}\cot x \csc^3 x - \frac{3}{8}\cot x \csc x - \frac{3}{8}\ln|\csc x + \cot x| + C$$

Putting all this together, we get

$$\int \cot^4 x \csc x\,dx = \int \csc^5 x\,dx - 2\int \csc^3 x\,dx + \int \csc x\,dx$$
$$= -\frac{1}{4}\cot x \csc^3 x - \frac{3}{8}\cot x \csc x - \frac{3}{8}\ln|\csc x + \cot x| + \cot x \csc x$$
$$+ \ln|\csc x + \cot x| - \ln|\csc x + \cot x| + C$$
$$= -\frac{1}{4}\cot x \csc^3 x + \frac{5}{8}\cot x \csc x - \frac{3}{8}\ln|\csc x + \cot x| + C$$

In Exercises 23–46, evaluate the integral.

23. $\int \cos^5 x \sin x\,dx$

SOLUTION Use the substitution $u = \cos x$, $du = -\sin x\,dx$. Then

$$\int \cos^5 x \sin x\,dx = -\int u^5\,du = -\frac{1}{6}u^6 + C = -\frac{1}{6}\cos^6 x + C.$$

24. $\int \cos^3(2-x)\sin(2-x)\,dx$

SOLUTION Use the substitution $u = \cos(2-x)$, $du = \sin(2-x)\,dx$. Then

$$\int \cos^3(2-x)\sin(2-x)\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\cos^4(2-x) + C$$

25. $\int \cos^4(3x+2)\,dx$

SOLUTION First use the substitution $u = 3x+2$, $du = 3\,dx$ and then apply the reduction formula for $\cos^n x$:

$$\int \cos^4(3x+2)\,dx = \frac{1}{3}\cos^4 u\,du = \frac{1}{3}\left(\frac{1}{4}\cos^3 u \sin u + \frac{3}{4}\int \cos^2 u\,du\right)$$
$$= \frac{1}{12}\cos^3 u \sin u + \frac{1}{4}\left(\frac{u}{2} + \frac{\sin 2u}{4}\right) + C$$
$$= \frac{1}{12}\cos^3(3x+2)\sin(3x+2) + \frac{1}{8}(3x+2) + \frac{1}{16}\sin(6x+4) + C$$

26. $\int \cos^7 3x\,dx$

SOLUTION Use the substitution $u = 3x$, $du = 3\,dx$, and the reduction formula for $\cos^m x$:

$$\int \cos^7 3x\,dx = \frac{1}{3}\int \cos^7 u\,du = \frac{1}{21}\cos^6 u \sin u + \frac{6}{21}\int \cos^5 u\,du$$
$$= \frac{1}{21}\cos^6 u \sin u + \frac{2}{7}\left(\frac{1}{5}\cos^4 u \sin u + \frac{4}{5}\int \cos^3 u\,du\right)$$
$$= \frac{1}{21}\cos^6 u \sin u + \frac{2}{35}\cos^4 u \sin u + \frac{8}{35}\left(\frac{1}{3}\cos^2 u \sin u + \frac{2}{3}\int \cos u\,du\right)$$
$$= \frac{1}{21}\cos^6 u \sin u + \frac{2}{35}\cos^4 u \sin u + \frac{8}{105}\cos^2 u \sin u + \frac{16}{105}\sin u + C$$
$$= \frac{1}{21}\cos^6 3x \sin 3x + \frac{2}{35}\cos^4 3x \sin 3x + \frac{8}{105}\cos^2 3x \sin 3x + \frac{16}{105}\sin 3x + C.$$

27. $\int \cos^3(\pi\theta)\sin^4(\pi\theta)\,d\theta$

SOLUTION Use the substitution $u = \pi\theta$, $du = \pi\,d\theta$, and the identity $\cos^2 u = 1 - \sin^2 u$ to write

$$\int \cos^3(\pi\theta)\sin^4(\pi\theta)\,d\theta = \frac{1}{\pi}\int \cos^3 u \sin^4 u\,du = \frac{1}{\pi}\int \left(1 - \sin^2 u\right)\sin^4 u \cos u\,du.$$

Now use the substitution $w = \sin u$, $dw = \cos u\,du$:

$$\int \cos^3(\pi\theta)\sin^4(\pi\theta)\,d\theta = \frac{1}{\pi}\int \left(1 - w^2\right) w^4\,dw = \frac{1}{\pi}\int \left(w^4 - w^6\right)\,dw = \frac{1}{5\pi}w^5 - \frac{1}{7\pi}w^7 + C$$
$$= \frac{1}{5\pi}\sin^5(\pi\theta) - \frac{1}{7\pi}\sin^7(\pi\theta) + C.$$

28. $\int \cos^{498} y \sin^3 y\,dy$

SOLUTION Use the identity $\sin^2 y = 1 - \cos^2 y$ to write

$$\int \cos^{498} y \sin^3 y\,dy = \int \cos^{498} y\left(1 - \cos^2 y\right)\sin y\,dy.$$

Now use the substitution $u = \cos y$, $du = -\sin y\,dy$:

$$\int \cos^{498} y \sin^3 y\,dy = -\int u^{498}\left(1 - u^2\right)du = -\int \left(u^{498} - u^{500}\right)du$$
$$= -\frac{1}{499}u^{499} + \frac{1}{501}u^{501} + C = -\frac{1}{499}\cos^{499} y + \frac{1}{501}\cos^{501} y + C.$$

29. $\int \sin^4(3x)\,dx$

SOLUTION Use the substitution $u = 3x$, $du = 3\,dx$ and the reduction formula for $\sin^m x$:

$$\int \sin^4(3x)\,dx = \frac{1}{3}\int \sin^4 u\,du = -\frac{1}{12}\sin^3 u\cos u + \frac{1}{4}\int \sin^2 u\,du$$
$$= -\frac{1}{12}\sin^3 u\cos u + \frac{1}{4}\left(-\frac{1}{2}\sin u\cos u + \frac{1}{2}\int du\right)$$
$$= -\frac{1}{12}\sin^3 u\cos u - \frac{1}{8}\sin u\cos u + \frac{1}{8}u + C$$
$$= -\frac{1}{12}\sin^3(3x)\cos(3x) - \frac{1}{8}\sin(3x)\cos(3x) + \frac{3}{8}x + C.$$

30. $\int \sin^2 x\cos^6 x\,dx$

SOLUTION Use the identity $\sin^2 x = 1 - \cos^2 x$ and the reduction formula for $\cos^m x$:

$$\int \sin^2 x\cos^6 x\,dx = \int \cos^6 x\left(1 - \cos^2 x\right)dx = \int \cos^6 x\,dx - \int \cos^8 x\,dx$$
$$= \int \cos^6 x\,dx - \left(\frac{1}{8}\cos^7 x\sin x + \frac{7}{8}\int \cos^6 x\,dx\right)$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{8}\int \cos^6 x\,dx$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{8}\left(\frac{1}{6}\cos^5 x\sin x + \frac{5}{6}\int \cos^4 x\,dx\right)$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{48}\cos^5 x\sin x + \frac{5}{48}\int \cos^4 x\,dx$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{48}\cos^5 x\sin x + \frac{5}{48}\left(\frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\,dx\right)$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{48}\cos^5 x\sin x + \frac{5}{192}\cos^3 x\sin x + \frac{15}{192}\int \cos^2 x\,dx$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{48}\cos^5 x\sin x + \frac{5}{192}\cos^3 x\sin x + \frac{15}{192}\left(\frac{1}{2}\cos x\sin x + \frac{1}{2}x\right)$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{48}\cos^5 x\sin x + \frac{5}{192}\cos^3 x\sin x + \frac{5}{128}\cos x\sin x + \frac{5}{128}x + C.$$

31. $\int \csc^2(3-2x)\,dx$

SOLUTION First make the substitution $u = 3-2x$, $du = -2\,dx$, so that

$$\int \csc^2(3-2x)\,dx = \frac{1}{2}\int(-\csc^2 u)\,du = \frac{1}{2}\cot u + C = \frac{1}{2}\cot(3-2x) + C$$

32. $\int \csc^3 x\,dx$

SOLUTION Use the reduction formula for $\csc^m x$:

$$\int \csc^3 x\,dx = -\frac{1}{2}\cot x\csc x + \frac{1}{2}\int \csc x\,dx = -\frac{1}{2}\cot x\csc x + \frac{1}{2}\ln|\csc x - \cot x| + C.$$

33. $\int \tan x\sec^2 x\,dx$

SOLUTION Use the substitution $u = \tan x$, $du = \sec^2 x\,dx$. Then

$$\int \tan x\sec^2 x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C.$$

34. $\int \tan^3\theta\sec^3\theta\,d\theta$

SOLUTION Use the identity $\tan^2\theta = \sec^2\theta - 1$ to write

$$\int \tan^3\theta\sec^3\theta\,d\theta = \int\left(\sec^2\theta - 1\right)\sec^2\theta(\sec\theta\tan\theta\,d\theta).$$

Now use the substitution $u = \sec\theta$, $du = \sec\theta\tan\theta\,d\theta$:

$$\int \tan^3\theta\sec^3\theta\,d\theta = \int\left(u^2-1\right)u^2\,du = \int\left(u^4-u^2\right)du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta + C.$$

35. $\int \tan^5 x\sec^4 x\,dx$

SOLUTION Use the identity $\tan^2 x = \sec^2 x - 1$ to write

$$\int \tan^5 x\sec^4 x\,dx = \int\left(\sec^2 x - 1\right)^2\sec^3 x(\sec x\tan x\,dx).$$

Now use the substitution $u = \sec x$, $du = \sec x\tan x\,dx$:

$$\begin{aligned}\int \tan^5 x\sec^4 x\,dx &= \int\left(u^2-1\right)^2 u^3\,du = \int\left(u^7 - 2u^5 + u^3\right)du \\ &= \frac{1}{8}u^8 - \frac{1}{3}u^6 + \frac{1}{4}u^4 + C = \frac{1}{8}\sec^8 x - \frac{1}{3}\sec^6 x + \frac{1}{4}\sec^4 x + C.\end{aligned}$$

36. $\int \tan^4 x\sec x\,dx$

SOLUTION Use the identity $\tan^2 x = \sec^2 x - 1$ to write

$$\int \tan^4 x\sec x\,dx = \int\left(\sec^2 x - 1\right)^2\sec x\,dx = \int \sec^5 x\,dx - 2\int\sec^3 x\,dx + \int\sec x\,dx.$$

Now use the reduction formula for $\sec^m x$:

$$\begin{aligned}\int \tan^4 x\sec x\,dx &= \left(\frac{1}{4}\tan x\sec^3 x + \frac{3}{4}\int\sec^3 x\,dx\right) - 2\int\sec^3 x\,dx + \int\sec x\,dx \\ &= \frac{1}{4}\tan x\sec^3 x - \frac{5}{4}\int\sec^3 x\,dx + \int\sec x\,dx \\ &= \frac{1}{4}\tan x\sec^3 x - \frac{5}{4}\left(\frac{1}{2}\tan x\sec x + \frac{1}{2}\int\sec x\,dx\right) + \int\sec x\,dx \\ &= \frac{1}{4}\tan x\sec^3 x - \frac{5}{8}\tan x\sec x + \frac{3}{8}\int\sec x\,dx \\ &= \frac{1}{4}\tan x\sec^3 x - \frac{5}{8}\tan x\sec x + \frac{3}{8}\ln|\sec x + \tan x| + C.\end{aligned}$$

37. $\int \tan^6 x \sec^4 x\, dx$

SOLUTION Use the identity $\sec^2 x = \tan^2 x + 1$ to write

$$\int \tan^6 x \sec^4 x\, dx = \int \tan^6 x \left(\tan^2 x + 1\right) \sec^2 x\, dx.$$

Now use the substitution $u = \tan x$, $du = \sec^2 x\, dx$:

$$\int \tan^6 x \sec^4 x\, dx = \int u^6 \left(u^2 + 1\right) du = \int \left(u^8 + u^6\right) du = \frac{1}{9}u^9 + \frac{1}{7}u^7 + C = \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C.$$

38. $\int \tan^2 x \sec^3 x\, dx$

SOLUTION Use the identity $\tan^2 x = \sec^2 x - 1$ to write

$$\int \tan^2 x \sec^3 x\, dx = \int \left(\sec^2 x - 1\right) \sec^3 x\, dx = \int \sec^5 x\, dx - \int \sec^3 x\, dx.$$

Now use the reduction formula for $\sec^m x$:

$$\begin{aligned}\int \tan^2 x \sec^3 x\, dx &= \frac{1}{4}\tan x \sec^3 x + \frac{3}{4}\int \sec^3 x\, dx - \int \sec^3 x\, dx\\ &= \frac{1}{4}\tan x \sec^3 x - \frac{1}{4}\int \sec^3 x\, dx\\ &= \frac{1}{4}\tan x \sec^3 x - \frac{1}{4}\left(\frac{1}{2}\tan x \sec x + \frac{1}{2}\int \sec x\, dx\right)\\ &= \frac{1}{4}\tan x \sec^3 x - \frac{1}{8}\tan x \sec x - \frac{1}{8}\ln|\sec x + \tan x| + C.\end{aligned}$$

39. $\int \cot^5 x \csc^5 x\, dx$

SOLUTION First use the identity $\cot^2 x = \csc^2 x - 1$ to rewrite the integral:

$$\int \cot^5 x \csc^5 x\, dx = \int (\csc^2 x - 1)^2 \csc^4 x (\cot x \csc x)\, dx = \int (\csc^8 x - 2\csc^6 x + \csc^4 x)(\cot x \csc x)\, dx$$

Now use the substitution $u = \csc x$ and $du = -\cot x \csc x\, dx$ to get

$$\begin{aligned}\int \cot^5 x \csc^5 x\, dx &= -\int u^8 - 2u^6 + u^4\, du = -\frac{1}{9}u^9 + \frac{2}{7}u^7 - \frac{1}{5}u^5 + C\\ &= -\frac{1}{9}\csc^9 x + \frac{2}{7}\csc^7 x - \frac{1}{5}\csc^5 x + C\end{aligned}$$

40. $\int \cot^2 x \csc^4 x\, dx$

SOLUTION First rewrite using $\cot^2 x = \csc^2 x - 1$ and then use the reduction formula:

$$\begin{aligned}\int \cot^2 x \csc^4 x\, dx &= \int (\csc^2 x - 1)\csc^4 x\, dx = \int \csc^6 x\, dx - \int \csc^4 x\, dx\\ &= -\frac{1}{5}\cot x \csc^4 x + \frac{4}{5}\int \csc^4 x\, dx - \int \csc^4 x\, dx\\ &= -\frac{1}{5}\cot x \csc^4 x - \frac{1}{5}\int \csc^4 x\, dx\\ &= -\frac{1}{5}\cot x \csc^4 x - \frac{1}{5}\left(-\frac{1}{3}\cot x \csc^2 x + \frac{2}{3}\int \csc^2 x\, dx\right)\\ &= -\frac{1}{5}\cot x \csc^4 x + \frac{1}{15}\cot x \csc^2 x + \frac{2}{15}\cot x + C\end{aligned}$$

41. $\int \sin 2x \cos 2x\, dx$

SOLUTION Use the substitution $u = \sin 2x$, $du = 2\cos 2x\, dx$:

$$\int \sin 2x \cos 2x\, dx = \frac{1}{2}\int \sin 2x (2\cos 2x\, dx) = \frac{1}{2}\int u\, du = \frac{1}{4}u^2 + C = \frac{1}{4}\sin^2 2x + C.$$

42. $\int \cos 4x \cos 6x\, dx$

SOLUTION Use the formula for $\int \cos mx \cos nx\, dx$:

$$\int \cos 4x \cos 6x\, dx = \frac{\sin(4-6)x}{2(4-6)} + \frac{\sin(4+6)x}{2(4+6)} + C = \frac{\sin(-2x)}{-4} + \frac{\sin(10x)}{20} + C$$
$$= \frac{1}{4}\sin 2x + \frac{1}{20}\sin 10x + C.$$

Here we've used the fact that $\sin x$ is an odd function: $\sin(-x) = -\sin x$.

43. $\int t \cos^3(t^2)\, dt$

SOLUTION Use the substitution $u = t^2$, $du = 2t\, dt$, followed by the reduction formula for $\cos^m x$:

$$\int t\cos^3(t^2)\, dt = \frac{1}{2}\int \cos^3 u\, du = \frac{1}{6}\cos^2 u \sin u + \frac{1}{3}\int \cos u\, du$$
$$= \frac{1}{6}\cos^2 u \sin u + \frac{1}{3}\sin u + C = \frac{1}{6}\cos^2(t^2)\sin(t^2) + \frac{1}{3}\sin(t^2) + C.$$

44. $\int \frac{\tan^3(\ln t)}{t}\, dt$

SOLUTION Use the substitution $u = \ln t$, $du = \frac{1}{t}\, dt$, followed by the reduction formula for $\tan^n x$:

$$\int \frac{\tan^3(\ln t)}{t}\, dt = \int \tan^3 u\, du = \frac{1}{2}\tan^2 u - \int \tan u\, du$$
$$= \frac{1}{2}\tan^2 u - \ln|\sec u| + C = \frac{1}{2}\tan^2(\ln t) - \ln|\sec(\ln t)| + C.$$

45. $\int \cos^2(\sin t)\cos t\, dt$

SOLUTION Use the substitution $u = \sin t$, $du = \cos t\, dt$, followed by the reduction formula for $\cos^m x$:

$$\int \cos^2(\sin t)\cos t\, dt = \int \cos^2 u\, du = \frac{1}{2}\cos u \sin u + \frac{1}{2}\int du$$
$$= \frac{1}{2}\cos u \sin u + \frac{1}{2}u + C = \frac{1}{2}\cos(\sin t)\sin(\sin t) + \frac{1}{2}\sin t + C.$$

46. $\int e^x \tan^2(e^x)\, dx$

SOLUTION Use the substitution $u = e^x$, $du = e^x\, dx$ followed by the reduction formula for $\tan^m x$:

$$\int e^x \tan^2(e^x)\, dx = \int \tan^2 u\, du = \tan u - \int 1\, du = \tan u - u + C = \tan(e^x) - e^x + C$$

In Exercises 47–60, evaluate the definite integral.

47. $\int_0^{2\pi} \sin^2 x\, dx$

SOLUTION Use the formula for $\int \sin^2 x\, dx$:

$$\int_0^{2\pi} \sin^2 x\, dx = \left(\frac{x}{2} - \frac{\sin 2x}{4}\right)\Bigg|_0^{2\pi} = \left(\frac{2\pi}{2} - \frac{\sin 4\pi}{4}\right) - \left(\frac{0}{2} - \frac{\sin 0}{4}\right) = \pi.$$

48. $\int_0^{\pi/2} \cos^3 x\, dx$

SOLUTION Use the reduction formula for $\cos^m x$:

$$\int_0^{\pi/2} \cos^3 x\, dx = \frac{1}{3}\cos^2 x \sin x\Bigg|_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2} \cos x\, dx = \left[\frac{1}{3}(0)(1) - \frac{1}{3}(1)(0)\right] + \frac{2}{3}\sin x\Bigg|_0^{\pi/2}$$
$$= 0 + \frac{2}{3}(1-0) = \frac{2}{3}.$$

49. $\displaystyle\int_0^{\pi/2} \sin^5 x\, dx$

SOLUTION Use the identity $\sin^2 x = 1 - \cos^2 x$ followed by the substitution $u = \cos x$, $du = -\sin x\, dx$ to get

$$\int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x\, dx = \int_0^{\pi/2} (1 - 2\cos^2 x + \cos^4 x) \sin x\, dx$$
$$= -\int_1^0 (1 - 2u^2 + u^4)\, du = -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right)\Big|_1^0 = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$$

50. $\displaystyle\int_0^{\pi/2} \sin^2 x \cos^3 x\, dx$

SOLUTION Use the identity $\sin^2 x = 1 - \cos^2 x$ followed by the substitution $u = \sin x$, $du = \cos x\, dx$ to get

$$\int_0^{\pi/2} \sin^2 x \cos^3 x\, dx = \int_0^{\pi/2} \sin^2 x (1 - \sin^2 x) \cos x\, dx = \int_0^{\pi/2} (\sin^2 x - \sin^4 x) \cos x\, dx$$
$$= \int_0^1 u^2 - u^4\, du = \left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right)\Big|_0^1 = \frac{2}{15}$$

51. $\displaystyle\int_0^{\pi/4} \frac{dx}{\cos x}$

SOLUTION Use the definition of $\sec x$ to simplify the integral:

$$\int_0^{\pi/4} \frac{dx}{\cos x} = \int_0^{\pi/4} \sec x\, dx = \ln|\sec x + \tan x|\Big|_0^{\pi/4} = \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| = \ln\left(\sqrt{2} + 1\right).$$

52. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{dx}{\sin x}$

SOLUTION Use the definition of $\csc x$ to simplify the integral:

$$\int_{\pi/4}^{\pi/2} \frac{dx}{\sin x} = \int_{\pi/4}^{\pi/2} \csc x\, dx = \ln|\csc x - \cot x|\Big|_{\pi/4}^{\pi/2} = \ln|1 - 0| - \ln\left|\sqrt{2} - 1\right| = -\ln\left|\sqrt{2} - 1\right|$$
$$= \ln\left(\frac{1}{\sqrt{2} - 1}\right) = \ln\left(\frac{(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)}\right) = \ln(\sqrt{2} + 1).$$

53. $\displaystyle\int_0^{\pi/3} \tan x\, dx$

SOLUTION Use the formula for $\int \tan x\, dx$:

$$\int_0^{\pi/3} \tan x\, dx = \ln|\sec x|\Big|_0^{\pi/3} = \ln 2 - \ln 1 = \ln 2.$$

54. $\displaystyle\int_0^{\pi/4} \tan^5 x\, dx$

SOLUTION First use the reduction formula for $\tan^m x$ to evaluate the indefinite integral:

$$\int \tan^5 x\, dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\, dx = \frac{1}{4}\tan^4 x - \left(\frac{1}{2}\tan^2 x - \int \tan x\, dx\right)$$
$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C.$$

Now compute the definite integral:

$$\int_0^{\pi/4} \tan^5 x\, dx = \left(\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x|\right)\Big|_0^{\pi/4}$$
$$= \left(\frac{1}{4}\left(1^4\right) - \frac{1}{2}\left(1^2\right) + \ln\sqrt{2}\right) - (0 - 0 + \ln 1)$$
$$= \frac{1}{4} - \frac{1}{2} + \ln\sqrt{2} - 0 = \frac{1}{2}\ln 2 - \frac{1}{4}.$$

55. $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^4 x\,dx$

SOLUTION First use the reduction formula for $\sec^m x$ to evaluate the indefinite integral:

$$\int \sec^4 x\,dx = \frac{1}{3}\tan x\sec^2 x + \frac{2}{3}\int \sec^2 x\,dx = \frac{1}{3}\tan x\sec^2 x + \frac{2}{3}\tan x + C.$$

Now compute the definite integral:

$$\int_{-\pi/4}^{\pi/4} \sec^4 x\,dx = \left(\frac{1}{3}\tan x\sec^2 x + \frac{2}{3}\tan x\right)\Big|_{-\pi/4}^{\pi/4}$$
$$= \left[\frac{1}{3}(1)\left(\sqrt{2}\right)^2 + \frac{2}{3}(1)\right] - \left[\frac{1}{3}(-1)\left(\sqrt{2}\right)^2 + \frac{2}{3}(-1)\right] = \frac{4}{3} - \left(-\frac{4}{3}\right) = \frac{8}{3}.$$

56. $\displaystyle\int_{\pi/4}^{3\pi/4} \cot^4 x\csc^2 x\,dx$

SOLUTION Use the substitution $u = \cot x$, $du = -\csc^2 x\,dx$. $x = \pi/4$ corresponds to $u = 1$, and $x = 3\pi/4$ corresponds to $u = -1$. We get

$$\int_{\pi/4}^{3\pi/4} \cot^4 x\csc^2 x\,dx = -\int_1^{-1} u^4\,du = -\frac{1}{5}u^5\Big|_1^{-1} = \frac{2}{5}$$

57. $\displaystyle\int_0^{\pi} \sin 3x\cos 4x\,dx$

SOLUTION Use the formula for $\int \sin mx\cos nx\,dx$:

$$\int_0^{\pi} \sin 3x\cos 4x\,dx = \left(-\frac{\cos(3-4)x}{2(3-4)} - \frac{\cos(3+4)x}{2(3+4)}\right)\Big|_0^{\pi} = \left(-\frac{\cos(-x)}{-2} - \frac{\cos 7x}{14}\right)\Big|_0^{\pi}$$
$$= \left(\frac{1}{2}\cos x - \frac{1}{14}\cos 7x\right)\Big|_0^{\pi} = \left[\frac{1}{2}(-1) - \frac{1}{14}(-1)\right] - \left[\frac{1}{2}(1) - \frac{1}{14}(1)\right] = -\frac{6}{7}.$$

58. $\displaystyle\int_0^{\pi} \sin x\sin 3x\,dx$

SOLUTION Use the formula for $\int \sin mx\sin nx\,dx$:

$$\int_0^{\pi} \sin x\sin 3x\,dx = \left(\frac{\sin(1-3)x}{2(1-3)} - \frac{\sin(1+3)x}{2(1+3)}\right)\Big|_0^{\pi} = \left(\frac{\sin(-2x)}{-4} - \frac{\sin 4x}{8}\right)\Big|_0^{\pi}$$
$$= \left(\frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x\right)\Big|_0^{\pi} = 0 - 0 = 0.$$

59. $\displaystyle\int_0^{\pi/6} \sin 2x\cos 4x\,dx$

SOLUTION Using the formula for $\displaystyle\int \sin mx\cos nx\,dx$, we have

$$\int_0^{\pi/6} \sin 2x\cos 4x\,dx = \left(-\frac{1}{-4}\cos(-2x) - \frac{1}{2\cdot 6}\cos(6x)\right)\Big|_0^{\pi/6} = \left(\frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x\right)\Big|_0^{\pi/6}$$
$$= \left(\frac{1}{4}\cdot\frac{1}{2} - \frac{1}{12}\cdot(-1)\right) - \left(\frac{1}{4} - \frac{1}{12}\right) = \frac{1}{24}$$

Here we've used the fact that $\cos x$ is an even function: $\cos(-x) = \cos x$.

60. $\displaystyle\int_0^{\pi/4} \sin 7x\cos 2x\,dx$

SOLUTION Using the formula for $\displaystyle\int \sin mx\cos nx\,dx$, we have

$$\int_0^{\pi/4} \sin 7x\cos 2x\,dx = \left(-\frac{1}{10}\cos 5x - \frac{1}{18}\cos 9x\right)\Big|_0^{\pi/4}$$
$$= \left(-\frac{1}{10}\cdot\left(-\frac{\sqrt{2}}{2}\right) - \frac{1}{18}\cdot\frac{\sqrt{2}}{2}\right) - \left(-\frac{1}{10} - \frac{1}{18}\right) = \frac{1}{45}(7+\sqrt{2})$$

61. Use the identities for $\sin 2x$ and $\cos 2x$ on page 407 to verify that the following formulas are equivalent.

$$\int \sin^4 x\,dx = \frac{1}{32}(12x - 8\sin 2x + \sin 4x) + C$$
$$\int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C$$

SOLUTION First, observe

$$\begin{aligned}\sin 4x &= 2\sin 2x\cos 2x = 2\sin 2x(1 - 2\sin^2 x)\\ &= 2\sin 2x - 4\sin 2x\sin^2 x = 2\sin 2x - 8\sin^3 x\cos x.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{32}(12x - 8\sin 2x + \sin 4x) + C &= \frac{3}{8}x - \frac{3}{16}\sin 2x - \frac{1}{4}\sin^3 x\cos x + C\\ &= \frac{3}{8}x - \frac{3}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x + C.\end{aligned}$$

62. Evaluate $\int \sin^2 x\cos^3 x\,dx$ using the method described in the text and verify that your result is equivalent to the following result produced by a computer algebra system.

$$\int \sin^2 x\cos^3 x\,dx = \frac{1}{30}(7 + 3\cos 2x)\sin^3 x + C$$

SOLUTION Use the identity $\cos^2 x = 1 - \sin^2 x$ to write

$$\int \sin^2 x\cos^3 x\,dx = \int \sin^2 x(1 - \sin^2 x)\cos x\,dx.$$

Now use the substitution $u = \sin x$, $du = \cos x\,dx$:

$$\int \sin^2 x\cos^3 x\,dx = \int u^2(1 - u^2)\,du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$$

To show that this result matches that produced by the computer algebra system, we will make use of the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$. We find

$$\begin{aligned}\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C &= \sin^3 x\left(\frac{1}{3} - \frac{1}{5}\sin^2 x\right) + C = \sin^3 x\left(\frac{7}{30} + \frac{1}{10}\cos 2x\right) + C\\ &= \frac{1}{30}\sin^3 x(7 + 3\cos 2x) + C.\end{aligned}$$

63. Find the volume of the solid obtained by revolving $y = \sin x$ for $0 \le x \le \pi$ about the x-axis.

SOLUTION Using the disk method, the volume is given by

$$V = \int_0^{\pi} \pi(\sin x)^2\,dx = \pi\int_0^{\pi}\sin^2 x\,dx = \pi\left(\frac{x}{2} - \frac{\sin 2x}{4}\right)\Big|_0^{\pi} = \pi\left[\left(\frac{\pi}{2} - 0\right) - (0)\right] = \frac{\pi^2}{2}.$$

64. Use Integration by Parts to prove Eqs. (1) and (2).

SOLUTION To prove the reduction formula for $\sin^n x$, use Integration by Parts with $u = \sin^{n-1} x$ and $v' = \sin x$. Then $u' = (n-1)\sin^{n-2} x\cos x$, $v = -\cos x$, and

$$\begin{aligned}\int \sin^n x\,dx &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx\\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\left(1 - \sin^2 x\right)dx\\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.\end{aligned}$$

Solving this equation for $\int \sin^n x\,dx$, we get

$$n\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx$$
$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

To prove the reduction formula for $\cos^n x$, use Integration by Parts with $u = \cos^{n-1} x$ and $v' = \cos x$. Then $u' = -(n-1)\cos^{n-2} x \sin x$, $v = \sin x$, and

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right)dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.\end{aligned}$$

Solving this equation for $\int \cos^n x\,dx$, we get

$$\begin{aligned}n\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx \\ \int \cos^n x\,dx &= \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx\end{aligned}$$

In Exercises 65–68, use the following alternative method for evaluating the integral $J = \int \sin^m x \cos^n x\,dx$ when m and n are both even. Use the identities

$$\sin^2 x = \frac{1}{2}(1-\cos 2x), \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

to write $J = \frac{1}{4}\int (1-\cos 2x)^{m/2}(1+\cos 2x)^{n/2}\,dx$, and expand the right-hand side as a sum of integrals involving smaller powers of sine and cosine in the variable $2x$.

65. $\displaystyle\int \sin^2 x \cos^2 x\,dx$

SOLUTION Using the identities $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, we have

$$\begin{aligned}J &= \int \sin^2 x\cos^2 x\,dx = \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)\,dx \\ &= \frac{1}{4}\int \left(1-\cos^2 2x\right)dx = \frac{1}{4}\int dx - \frac{1}{4}\int \cos^2 2x\,dx.\end{aligned}$$

Now use the substitution $u = 2x$, $du = 2\,dx$, and the formula for $\displaystyle\int \cos^2 u\,du$:

$$\begin{aligned}J &= \frac{1}{4}x - \frac{1}{8}\int \cos^2 u\,du = \frac{1}{4}x - \frac{1}{8}\left(\frac{u}{2} + \frac{1}{2}\sin u\cos u\right) + C \\ &= \frac{1}{4}x - \frac{1}{16}(2x) - \frac{1}{16}\sin 2x\cos 2x + C = \frac{1}{8}x - \frac{1}{16}\sin 2x\cos 2x + C.\end{aligned}$$

66. $\displaystyle\int \cos^4 x\,dx$

SOLUTION Using the identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, we have

$$\begin{aligned}J &= \int \cos^4 x\,dx = \frac{1}{4}\int (1+\cos 2x)^2\,dx = \frac{1}{4}\int \left(1+2\cos 2x+\cos^2 2x\right)dx \\ &= \frac{1}{4}\int dx + \frac{1}{4}\int \cos 2x(2\,dx) + \frac{1}{8}\int \cos^2 2x(2\,dx)\end{aligned}$$

Using the substitution $u = 2x$, $du = 2\,dx$, we get

$$\begin{aligned}J &= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\int \cos^2 u\,du = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\left(\frac{u}{2} + \frac{1}{2}\sin u\cos u\right) + C \\ &= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{16}(2x) + \frac{1}{16}\sin 2x\cos 2x + C = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{16}\sin 2x\cos 2x + C.\end{aligned}$$

67. $\int \sin^4 x \cos^2 x\, dx$

SOLUTION Using the identities $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, we have

$$\begin{aligned} J = \int \sin^4 x \cos^2 x\, dx &= \frac{1}{8}\int (1 - \cos 2x)^2(1 + \cos 2x)\, dx \\ &= \frac{1}{8}\int \left(1 - 2\cos 2x + \cos^2 2x\right)(1 + \cos 2x)\, dx \\ &= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx. \end{aligned}$$

Now use the substitution $u = 2x$, $du = 2\,dx$, together with the reduction formula for $\cos^m x$:

$$\begin{aligned} J &= \frac{1}{8}x - \frac{1}{16}\int \cos u\, du - \frac{1}{16}\int \cos^2 u\, du + \frac{1}{16}\int \cos^3 u\, du \\ &= \frac{1}{8}x - \frac{1}{16}\sin u - \frac{1}{16}\left(\frac{u}{2} + \frac{1}{2}\sin u \cos u\right) + \frac{1}{16}\left(\frac{1}{3}\cos^2 u \sin u + \frac{2}{3}\int \cos u\, du\right) \\ &= \frac{1}{8}x - \frac{1}{16}\sin 2x - \frac{1}{32}(2x) - \frac{1}{32}\sin 2x \cos 2x + \frac{1}{48}\cos^2 2x \sin 2x + \frac{1}{24}\sin 2x + C \\ &= \frac{1}{16}x - \frac{1}{48}\sin 2x - \frac{1}{32}\sin 2x \cos 2x + \frac{1}{48}\cos^2 2x \sin 2x + C. \end{aligned}$$

68. $\int \sin^6 x\, dx$

SOLUTION Using the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, we have

$$\begin{aligned} J = \int \sin^6 x\, dx &= \int \left(\frac{1}{2}(1 - \cos 2x)\right)^3 dx = \frac{1}{8}\int (1 - \cos 2x)^3\, dx \\ &= \frac{1}{8}\int 1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x\, dx \end{aligned}$$

Now use the substitution $u = 2x$, $du = 2\,dx$ together with the reduction formula for $\cos^m x$:

$$\begin{aligned} J &= \frac{1}{8}x - \frac{3}{16}\int \cos u\, du + \frac{3}{16}\int \cos^2 u\, du - \frac{1}{16}\int \cos^3 u\, du \\ &= \frac{1}{8}x - \frac{3}{16}\sin u + \frac{3}{16}\left(\frac{u}{2} + \frac{1}{2}\sin u \cos u\right) - \frac{1}{16}\left(\frac{1}{3}\cos^2 u \sin u + \frac{2}{3}\int \cos u\, du\right) \\ &= \frac{1}{8}x - \frac{3}{16}\sin u + \frac{3}{32}u + \frac{3}{32}\sin u \cos u - \frac{1}{48}\cos^2 u \sin u - \frac{1}{24}\sin u + C \\ &= \frac{1}{8}x - \frac{11}{48}\sin u + \frac{3}{32}u + \frac{3}{32}\sin u \cos u - \frac{1}{48}\cos^2 u \sin u + C \\ &= \frac{1}{8}x - \frac{11}{48}\sin 2x + \frac{3}{32}\cdot 2x + \frac{3}{32}\sin 2x \cos 2x - \frac{1}{48}\cos^2 2x \sin 2x + C \\ &= \frac{5}{16}x - \frac{11}{48}\sin 2x + \frac{3}{32}\sin 2x \cos 2x - \frac{1}{48}\cos^2 2x \sin 2x + C \end{aligned}$$

69. Prove the reduction formula

$$\int \tan^k x\, dx = \frac{\tan^{k-1} x}{k-1} - \int \tan^{k-2} x\, dx$$

Hint: $\tan^k x = (\sec^2 x - 1)\tan^{k-2} x$.

SOLUTION Use the identity $\tan^2 x = \sec^2 x - 1$ to write

$$\int \tan^k x\, dx = \int \tan^{k-2} x\left(\sec^2 x - 1\right) dx = \int \tan^{k-2} x \sec^2 x\, dx - \int \tan^{k-2} x\, dx.$$

Now use the substitution $u = \tan x$, $du = \sec^2 x\, dx$:

$$\int \tan^k x\, dx = \int u^{k-2}\, du - \int \tan^{k-2} x\, dx = \frac{1}{k-1}u^{k-1} - \int \tan^{k-2} x\, dx = \frac{\tan^{k-1} x}{k-1} - \int \tan^{k-2} x\, dx.$$

70. Use the substitution $u = \csc x - \cot x$ to evaluate $\int \csc x\, dx$ (see Example 5).

SOLUTION Using the substitution $u = \csc x - \cot x$,

$$du = \left(-\csc x \cot x + \csc^2 x\right)dx = \csc x(\csc x - \cot x)\, dx,$$

we have

$$\int \csc x\, dx = \int \frac{\csc x(\csc x - \cot x)\, dx}{\csc x - \cot x} = \int \frac{du}{u} = \ln|u| + C = \ln|\csc x - \cot x| + C.$$

71. Let $I_m = \int_0^{\pi/2} \sin^m x\, dx$.

(a) Show that $I_0 = \frac{\pi}{2}$ and $I_1 = 1$.

(b) Prove that, for $m \geq 2$,

$$I_m = \frac{m-1}{m} I_{m-2}$$

(c) Use (a) and (b) to compute I_m for $m = 2, 3, 4, 5$.

SOLUTION

(a) We have

$$I_0 = \int_0^{\pi/2} \sin^0 x\, dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$$

$$I_1 = \int_0^{\pi/2} \sin x\, dx = -\cos x\Big|_0^{\pi/2} = 1$$

(b) Using the reduction formula for $\sin^m x$, we get for $m \geq 2$

$$\begin{aligned} I_m &= \int_0^{\pi/2} \sin^m x\, dx = -\frac{1}{m}\sin^{m-1} x \cos x\Big|_0^{\pi/2} + \frac{m-1}{m}\int_0^{\pi/2} \sin^{m-2} x\, dx \\ &= -\frac{1}{m}\sin^{m-1}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \frac{1}{m}\sin^{m-1}(0)\cos(0) + \frac{m-1}{m} I_{m-2} \\ &= \frac{1}{m}(-1\cdot 0 + 0\cdot 1) + \frac{m-1}{m} I_{m-2} \\ &= \frac{m-1}{m} I_{m-2} \end{aligned}$$

(c)

$$\begin{aligned} I_2 &= \frac{1}{2} I_0 = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4} \\ I_3 &= \frac{2}{3} I_1 = \frac{2}{3} \\ I_4 &= \frac{3}{4} I_2 = \frac{3}{4}\cdot\frac{\pi}{4} = \frac{3}{16}\pi \\ I_5 &= \frac{4}{5} I_3 = \frac{8}{15} \end{aligned}$$

72. Evaluate $\int_0^{\pi} \sin^2 mx\, dx$ for m an arbitrary integer.

SOLUTION Use the substitution $u = mx$, $du = m\, dx$. Then

$$\begin{aligned} \int_0^{\pi} \sin^2 mx\, dx &= \frac{1}{m}\int_{x=0}^{x=\pi} \sin^2 u\, du = \frac{1}{m}\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\Bigg|_{x=0}^{x=\pi} = \frac{1}{m}\left(\frac{mx}{2} - \frac{\sin 2mx}{4}\right)\Bigg|_0^{\pi} \\ &= \left(\frac{x}{2} - \frac{\sin 2mx}{4m}\right)\Bigg|_0^{\pi} = \left(\frac{\pi}{2} - \frac{\sin 2\pi m}{4}\right) - (0). \end{aligned}$$

If m is an arbitrary integer, then $\sin 2m\pi = 0$. Thus

$$\int_0^{\pi} \sin^2 mx\, dx = \frac{\pi}{2}.$$

73. Evaluate $\int \sin x \ln(\sin x)\,dx$. *Hint:* Use Integration by Parts as a first step.

SOLUTION Start by using integration by parts with $u = \ln(\sin x)$ and $v' = \sin x$, so that $u' = \cot x$ and $v = -\cos x$. Then

$$I = \int \sin x \ln(\sin x)\,dx = -\cos x \ln(\sin x) + \int \cot x \cos x\,dx = -\cos x \ln(\sin x) + \int \frac{\cos^2 x}{\sin x}\,dx$$
$$= -\cos x \ln(\sin x) + \int \frac{1-\sin^2 x}{\sin x}\,dx = -\cos x \ln(\sin x) - \int \sin x\,dx + \int \csc x\,dx$$
$$= -\cos x \ln(\sin x) + \cos x + \int \csc x\,dx$$

Using the table, $\int \csc x\,dx = \ln|\csc x - \cot x| + C$, so finally

$$I = -\cos x \ln(\sin x) + \cos x + \ln|\csc x - \cot x| + C$$

74. Total Energy A 100-W light bulb has resistance $R = 144\ \Omega$ (ohms) when attached to household current, where the voltage varies as $V = V_0 \sin(2\pi f t)$ ($V_0 = 110$ V, $f = 60$ Hz). The energy (in joules) expended by the bulb over a period of T seconds is

$$U = \int_0^T P(t)\,dt$$

where $P = V^2/R$ (J/s) is the power. Compute U if the bulb remains on for 5 hours.

SOLUTION After converting to seconds (5 hours = 18,000 seconds), the total energy expended is given by

$$U = \int_0^{18{,}000} P(t)\,dt = \int_0^{18{,}000} \frac{V^2}{R}\,dt = \frac{V_0^2}{R}\int_0^{18{,}000} \sin^2(2\pi f t)\,dt = \frac{110^2}{144}\int_0^{18{,}000} \sin^2(120\pi t)\,dt.$$

Now use the substitution $u = 120\pi t$, $du = 120\pi\,dt$:

$$U = \frac{110^2}{144}\left(\frac{1}{120\pi}\right)\int_{t=0}^{t=18{,}000} \sin^2 u\,du = \frac{110^2}{144\cdot 120\pi}\left[\frac{u}{2} - \frac{1}{2}\sin u \cos u\right]_{t=0}^{t=18{,}000}$$
$$= \frac{110^2}{144\cdot 120\pi}\left[60\pi t - \frac{1}{2}\sin(120\pi t)\cos(120\pi t)\right]_0^{18{,}000} = \frac{110^2}{144\cdot 120\pi}\left[\big(60\pi(18{,}000) - 0\big) - 0\right]$$
$$= \frac{(110^2)(60\pi)(18{,}000)}{(144)(120\pi)} = 756{,}260 \text{ joules.}$$

75. Let m, n be integers with $m \neq \pm n$. Use Eqs. (23)–(25) to prove the so-called **orthogonality relations** that play a basic role in the theory of Fourier Series (Figure 2):

$$\int_0^{\pi} \sin mx \sin nx\,dx = 0$$
$$\int_0^{\pi} \cos mx \cos nx\,dx = 0$$
$$\int_0^{2\pi} \sin mx \cos nx\,dx = 0$$

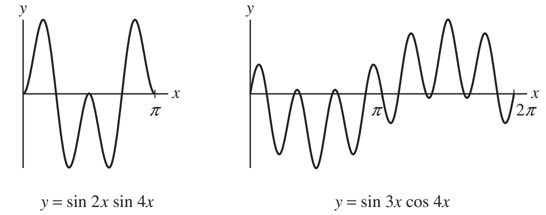


FIGURE 2 The integrals are zero by the orthogonality relations.

SOLUTION If m, n are integers, then $m-n$ and $m+n$ are integers, and therefore $\sin(m-n)\pi = \sin(m+n)\pi = 0$, since $\sin k\pi = 0$ if k is an integer. Thus we have

$$\int_0^{\pi} \sin mx \sin nx\,dx = \left(\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}\right)\Bigg|_0^{\pi} = \left(\frac{\sin(m-n)\pi}{2(m-n)} - \frac{\sin(m+n)\pi}{2(m+n)}\right) - 0 = 0;$$

$$\int_0^{\pi} \cos mx \cos nx\,dx = \left(\frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}\right)\Bigg|_0^{\pi} = \left(\frac{\sin(m-n)\pi}{2(m-n)} + \frac{\sin(m+n)\pi}{2(m+n)}\right) - 0 = 0.$$

If k is an integer, then $\cos 2k\pi = 1$. Using this fact, we have

$$\begin{aligned}\int_0^{2\pi} \sin mx \cos nx\,dx &= \left(-\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}\right)\Bigg|_0^{2\pi} \\ &= \left(-\frac{\cos(m-n)2\pi}{2(m-n)} - \frac{\cos(m+n)2\pi}{2(m+n)}\right) - \left(-\frac{1}{2(m-n)} - \frac{1}{2(m+n)}\right) \\ &= \left(-\frac{1}{2(m-n)} - \frac{1}{2(m+n)}\right) - \left(-\frac{1}{2(m-n)} - \frac{1}{2(m+n)}\right) = 0.\end{aligned}$$

Further Insights and Challenges

76. Use the trigonometric identity

$$\sin mx \cos nx = \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

to prove Eq. (24) in the table of integrals on page 410.

SOLUTION Using the identity $\sin mx \cos nx = \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)$, we get

$$\int \sin mx \cos nx\,dx = \frac{1}{2}\int \sin(m-n)x\,dx + \frac{1}{2}\int \sin(m+n)x\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} + C.$$

77. Use Integration by Parts to prove that (for $m \neq 1$)

$$\int \sec^m x\,dx = \frac{\tan x \sec^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2} x\,dx$$

SOLUTION Using Integration by Parts with $u = \sec^{m-2} x$ and $v' = \sec^2 x$, we have $v = \tan x$ and

$$u' = (m-2)\sec^{m-3} x\big(\sec x \tan x\big) = (m-2)\tan x \sec^{m-2} x.$$

Then,

$$\begin{aligned}\int \sec^m x\,dx &= \tan x \sec^{m-2} x - (m-2)\int \tan^2 x \sec^{m-2} x\,dx \\ &= \tan x \sec^{m-2} x - (m-2)\int \left(\sec^2 x - 1\right)\sec^{m-2} x\,dx \\ &= \tan x \sec^{m-2} x - (m-2)\int \sec^m x\,dx + (m-2)\int \sec^{m-2} x\,dx.\end{aligned}$$

Solving this equation for $\int \sec^m x\,dx$, we get

$$\begin{aligned}(m-1)\int \sec^m x\,dx &= \tan x \sec^{m-2} x + (m-2)\int \sec^{m-2} x\,dx \\ \int \sec^m x\,dx &= \frac{\tan x \sec^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2} x\,dx.\end{aligned}$$

78. Set $I_m = \displaystyle\int_0^{\pi/2} \sin^m x\,dx$. Use Exercise 71 to prove that

$$\begin{aligned}I_{2m} &= \frac{2m-1}{2m}\frac{2m-3}{2m-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} \\ I_{2m+1} &= \frac{2m}{2m+1}\frac{2m-2}{2m-1}\cdots\frac{2}{3}\end{aligned}$$

Conclude that

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2m\cdot 2m}{(2m-1)(2m+1)}\frac{I_{2m}}{I_{2m+1}}$$

SOLUTION We'll use induction to show these results. Recall from Exercise 71 that

$$I_m = \frac{m-1}{m}I_{m-2}$$

when $m \geq 2$. Now, for I_{2m}, the result is true for $m = 1$ and $m = 2$ (again see Exercise 71). Now assume the result is true for $m = k-1$:

$$I_{2(k-1)} = I_{2k-2} = \frac{2k-3}{2k-2}\cdot\frac{2k-5}{2k-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$

Using the relation $I_m = ((m-1)/m)I_{m-2}$, we have

$$I_{2k} = \frac{2k-1}{2k}I_{2k-2} = \frac{2k-1}{2k}\cdot\left(\frac{2k-3}{2k-2}\cdot\frac{2k-5}{2k-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2}\right).$$

For I_{2m+1}, the result is true for $m = 1$. Now assume the result is true for $m = k-1$:

$$I_{2(k-1)+1} = I_{2k-1} = \frac{2k-2}{2k-1}\cdot\frac{2k-4}{2k-3}\cdots\frac{2}{3}$$

Again using the relation $I_m = ((m-1)/m)I_{m-2}$, we have

$$I_{2k+1} = \left(\frac{2k+1-1}{2k+1}\right)I_{2k-1} = \frac{2k}{2k+1}\left(\frac{2k-2}{2k-1}\cdot\frac{2k-4}{2k-3}\cdots\frac{2}{3}\right).$$

This establishes the explicit formulas for I_{2m} and I_{2m+1}. Now, divide these two results to obtain

$$\frac{I_{2m}}{I_{2m+1}} = \frac{(2m-1)(2m+1)}{2m\cdot 2m}\cdot\frac{(2m-3)(2m-1)}{(2m-2)(2m-2)}\cdots\frac{1\cdot 3}{2\cdot 2}\cdot\frac{\pi}{2}.$$

Solving for $\pi/2$, we get the desired result:

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2m\cdot 2m}{(2m-1)(2m+1)}\cdot\frac{I_{2m}}{I_{2m+1}}.$$

79. This is a continuation of Exercise 78.

(a) Prove that $I_{2m+1} \leq I_{2m} \leq I_{2m-1}$. *Hint:* $\sin^{2m+1} x \leq \sin^{2m} x \leq \sin^{2m-1} x$ for $0 \leq x \leq \frac{\pi}{2}$.

(b) Show that $\dfrac{I_{2m-1}}{I_{2m+1}} = 1 + \dfrac{1}{2m}$.

(c) Show that $1 \leq \dfrac{I_{2m}}{I_{2m+1}} \leq 1 + \dfrac{1}{2m}$.

(d) Prove that $\lim\limits_{m\to\infty} \dfrac{I_{2m}}{I_{2m+1}} = 1$.

(e) Finally, deduce the infinite product for $\frac{\pi}{2}$ discovered by English mathematician John Wallis (1616–1703):

$$\frac{\pi}{2} = \lim_{m\to\infty}\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2m\cdot 2m}{(2m-1)(2m+1)}$$

SOLUTION

(a) For $0 \leq x \leq \frac{\pi}{2}$, $0 \leq \sin x \leq 1$. Multiplying this last inequality by $\sin x$, we obtain

$$0 \leq \sin^2 x \leq \sin x.$$

Continuing to multiply this inequality by $\sin x$, we obtain, more generally,

$$\sin^{2m+1} x \leq \sin^{2m} x \leq \sin^{2m-1} x.$$

Integrating these functions over $[0, \frac{\pi}{2}]$, we get

$$\int_0^{\pi/2}\sin^{2m+1} x\,dx \leq \int_0^{\pi/2}\sin^{2m} x\,dx \leq \int_0^{\pi/2}\sin^{2m-1} x\,dx,$$

which is the same as

$$I_{2m+1} \leq I_{2m} \leq I_{2m-1}.$$

(b) Using the relation $I_m = ((m-1)/m)I_{m-2}$, we have

$$\frac{I_{2m-1}}{I_{2m+1}} = \frac{I_{2m-1}}{\left(\frac{2m}{2m+1}\right) I_{2m-1}} = \frac{2m+1}{2m} = \frac{2m}{2m} + \frac{1}{2m} = 1 + \frac{1}{2m}.$$

(c) First start with the inequality of part (a):

$$I_{2m+1} \le I_{2m} \le I_{2m-1}.$$

Divide through by I_{2m+1}:

$$1 \le \frac{I_{2m}}{I_{2m+1}} \le \frac{I_{2m-1}}{I_{2m+1}}.$$

Use the result from part (b):

$$1 \le \frac{I_{2m}}{I_{2m+1}} \le 1 + \frac{1}{2m}.$$

(d) Taking the limit of this inequality, and applying the Squeeze Theorem, we have

$$\lim_{m\to\infty} 1 \le \lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} \le \lim_{m\to\infty} \left(1 + \frac{1}{2m}\right).$$

Because

$$\lim_{m\to\infty} 1 = 1 \quad \text{and} \quad \lim_{m\to\infty} \left(1 + \frac{1}{2m}\right) = 1,$$

we obtain

$$1 \le \lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} \le 1.$$

Therefore

$$\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} = 1.$$

(e) Take the limit of both sides of the equation obtained in Exercise 78(d):

$$\lim_{m\to\infty} \frac{\pi}{2} = \lim_{m\to\infty} \frac{2\cdot 2}{1\cdot 3} \cdot \frac{4\cdot 4}{3\cdot 5} \cdots \frac{2m\cdot 2m}{(2m-1)(2m+1)} \frac{I_{2m}}{I_{2m+1}}$$

$$\frac{\pi}{2} = \left(\lim_{m\to\infty} \frac{2\cdot 2}{1\cdot 3} \cdot \frac{4\cdot 4}{3\cdot 5} \cdots \frac{2m\cdot 2m}{(2m-1)(2m+1)}\right)\left(\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}}\right).$$

Finally, using the result from (d), we have

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{2\cdot 2}{1\cdot 3} \cdot \frac{4\cdot 4}{3\cdot 5} \cdots \frac{2m\cdot 2m}{(2m-1)(2m+1)}.$$

7.3 Trigonometric Substitution

Preliminary Questions

1. State the trigonometric substitution appropriate to the given integral:

(a) $\int \sqrt{9-x^2}\,dx$

(b) $\int x^2(x^2-16)^{3/2}\,dx$

(c) $\int x^2(x^2+16)^{3/2}\,dx$

(d) $\int (x^2-5)^{-2}\,dx$

SOLUTION

(a) $x = 3\sin\theta$ **(b)** $x = 4\sec\theta$ **(c)** $x = 4\tan\theta$ **(d)** $x = \sqrt{5}\sec\theta$

2. Is trigonometric substitution needed to evaluate $\int x\sqrt{9-x^2}\,dx$?

SOLUTION No. There is a factor of x in the integrand outside the radical and the derivative of $9-x^2$ is $-2x$, so we may use the substitution $u = 9-x^2$, $du = -2x\,dx$ to evaluate this integral.

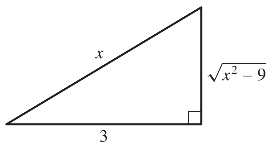
3. Express $\sin 2\theta$ in terms of $x = \sin\theta$.

SOLUTION First note that if $\sin\theta = x$, then $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - x^2}$. Thus,

$$\sin 2\theta = 2\sin\theta\cos\theta = 2x\sqrt{1 - x^2}.$$

4. Draw a triangle that would be used together with the substitution $x = 3\sec\theta$.

SOLUTION



Exercises

In Exercises 1–4, evaluate the integral by following the steps given.

1. $I = \int \frac{dx}{\sqrt{9 - x^2}}$

(a) Show that the substitution $x = 3\sin\theta$ transforms I into $\int d\theta$, and evaluate I in terms of θ.

(b) Evaluate I in terms of x.

SOLUTION

(a) Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\, d\theta$, and

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = 3\sqrt{1 - \sin^2\theta} = 3\sqrt{\cos^2\theta} = 3\cos\theta.$$

Thus,

$$I = \int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{3\cos\theta\, d\theta}{3\cos\theta} = \int d\theta = \theta + C.$$

(b) If $x = 3\sin\theta$, then $\theta = \sin^{-1}(\frac{x}{3})$. Thus,

$$I = \theta + C = \sin^{-1}\left(\frac{x}{3}\right) + C.$$

2. $I = \int \frac{dx}{x^2\sqrt{x^2 - 2}}$

(a) Show that the substitution $x = \sqrt{2}\sec\theta$ transforms the integral I into $\frac{1}{2}\int \cos\theta d\theta$, and evaluate I in terms of θ.

(b) Use a right triangle to show that with the above substitution, $\sin\theta = \sqrt{x^2 - 2}/x$.

(c) Evaluate I in terms of x.

SOLUTION

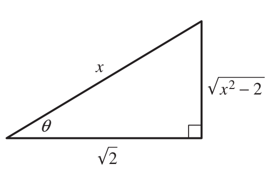
(a) Let $x = \sqrt{2}\sec\theta$. Then $dx = \sqrt{2}\sec\theta\tan\theta\, d\theta$, and

$$\sqrt{x^2 - 2} = \sqrt{2\sec^2\theta - 2} = \sqrt{2(\sec^2\theta - 1)} = \sqrt{2\tan^2\theta} = \sqrt{2}\tan\theta.$$

Thus,

$$I = \int \frac{dx}{x^2\sqrt{x^2 - 2}} = \int \frac{\sqrt{2}\sec\theta\tan\theta\, d\theta}{(2\sec^2\theta)(\sqrt{2}\tan\theta)} = \frac{1}{2}\int \frac{d\theta}{\sec\theta} = \frac{1}{2}\int \cos\theta\, d\theta = \frac{1}{2}\sin\theta + C.$$

(b) Since $x = \sqrt{2}\sec\theta$, $\sec\theta = \frac{x}{\sqrt{2}}$, and we construct the following right triangle:



From this triangle we see that $\sin\theta = \sqrt{x^2-2}/x$.

(c) Combining the results from parts (a) and (b),

$$I = \frac{1}{2}\sin\theta + C = \frac{\sqrt{x^2-2}}{2x} + C.$$

3. $I = \displaystyle\int \frac{dx}{\sqrt{4x^2+9}}$

(a) Show that the substitution $x = \frac{3}{2}\tan\theta$ transforms I into $\frac{1}{2}\int \sec\theta\, d\theta$.

(b) Evaluate I in terms of θ (refer to the table of integrals on page 410 in Section 7.2 if necessary).

(c) Express I in terms of x.

SOLUTION

(a) If $x = \frac{3}{2}\tan\theta$, then $dx = \frac{3}{2}\sec^2\theta\, d\theta$, and

$$\sqrt{4x^2+9} = \sqrt{4\cdot\left(\frac{3}{2}\tan\theta\right)^2 + 9} = \sqrt{9\tan^2\theta + 9} = 3\sqrt{\sec^2\theta} = 3\sec\theta$$

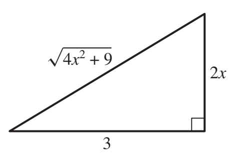
Thus,

$$I = \int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{3\sec\theta} = \frac{1}{2}\int \sec\theta\, d\theta$$

(b)

$$I = \frac{1}{2}\int \sec\theta\, d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

(c) Since $x = \frac{3}{2}\tan\theta$, we construct a right triangle with $\tan\theta = \frac{2x}{3}$:



From this triangle, we see that $\sec\theta = \frac{1}{3}\sqrt{4x^2+9}$, and therefore

$$I = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln\left|\frac{1}{3}\sqrt{4x^2+9} + \frac{2x}{3}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{4x^2+9}+2x}{3}\right| + C = \frac{1}{2}\ln|\sqrt{4x^2+9}+2x| - \frac{1}{2}\ln 3 + C = \frac{1}{2}\ln|\sqrt{4x^2+9}+2x| + C$$

4. $I = \displaystyle\int \frac{dx}{(x^2+4)^2}$

(a) Show that the substitution $x = 2\tan\theta$ transforms the integral I into $\frac{1}{8}\int \cos^2\theta\, d\theta$.

(b) Use the formula $\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$ to evaluate I in terms of θ.

(c) Show that $\sin\theta = \dfrac{x}{\sqrt{x^2+4}}$ and $\cos\theta = \dfrac{2}{\sqrt{x^2+4}}$.

(d) Express I in terms of x.

SOLUTION

(a) If $x = 2\tan\theta$, then $dx = 2\sec^2\theta\, d\theta$, and

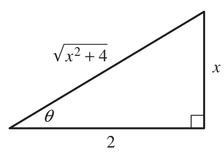
$$I = \int \frac{dx}{(x^2+4)^2} = \int \frac{2\sec^2\theta\, d\theta}{(4\tan^2\theta+4)^2} = \frac{2}{16}\int \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^2}$$

$$= \frac{1}{8}\int \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^2} = \frac{1}{8}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{8}\int \cos^2\theta\, d\theta.$$

(b) Using the formula $\int \cos^2 d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$, we get

$$I = \frac{1}{8}\int \cos^2\theta\, d\theta = \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C.$$

(c) Since $x = 2\tan\theta$, we construct a right triangle with $\tan\theta = \frac{x}{2}$:



From this triangle we see that

$$\sin\theta = \frac{x}{\sqrt{x^2+4}} \quad \text{and} \quad \cos\theta = \frac{2}{\sqrt{x^2+4}}.$$

(d) Since $x = 2\tan\theta$, then $\theta = \tan^{-1}(\frac{x}{2})$, and

$$I = \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{16}\left(\frac{x}{\sqrt{x^2+4}}\right)\left(\frac{2}{\sqrt{x^2+4}}\right) + C = \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{x}{8(x^2+4)} + C.$$

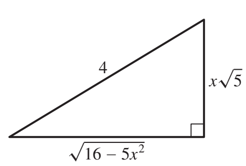
In Exercises 5–10, use the indicated substitution to evaluate the integral.

5. $\int \sqrt{16-5x^2}\,dx, \quad x = \frac{4}{\sqrt{5}}\sin\theta$

SOLUTION Let $x = \frac{4}{\sqrt{5}}\sin\theta$. Then $dx = \frac{4}{\sqrt{5}}\cos\theta\,d\theta$, and

$$\begin{aligned} I &= \int \sqrt{16-5x^2}\,dx = \int \sqrt{16 - 5\left(\frac{4}{\sqrt{5}}\sin\theta\right)^2} \cdot \frac{4}{\sqrt{5}}\cos\theta\,d\theta = \frac{4}{\sqrt{5}}\int \sqrt{16-16\sin^2\theta}\cdot\cos\theta\,d\theta \\ &= \frac{4}{\sqrt{5}}\cdot 4\int \cos\theta\cdot\cos\theta\,d\theta = \frac{16}{\sqrt{5}}\int \cos^2\theta\,d\theta \\ &= \frac{16}{\sqrt{5}}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C = \frac{8}{\sqrt{5}}(\theta + \sin\theta\cos\theta) + C \end{aligned}$$

Since $x = \frac{4}{\sqrt{5}}\sin\theta$, we construct a right triangle with $\sin\theta = \frac{x\sqrt{5}}{4}$:



From this triangle we see that $\cos\theta = \frac{1}{4}\sqrt{16-5x^2}$, so we have

$$\begin{aligned} I &= \frac{8}{\sqrt{5}}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{8}{\sqrt{5}}\left(\sin^{-1}\left(\frac{x\sqrt{5}}{4}\right) + \frac{x\sqrt{5}}{4}\cdot\frac{1}{4}\sqrt{16-5x^2}\right) + C \\ &= \frac{8}{\sqrt{5}}\sin^{-1}\left(\frac{x\sqrt{5}}{4}\right) + \frac{1}{2}x\sqrt{16-5x^2} + C \end{aligned}$$

6. $\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx, \quad x = \sin\theta$

SOLUTION Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$, and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta.$$

Converting the limits of integration to θ, we find

$$x = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$x = 0 \Rightarrow \theta = \sin^{-1}(0) = 0$$

Therefore

$$I = \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} \frac{\sin^2\theta}{\cos\theta}(\cos\theta\,d\theta) = \int_0^{\pi/6} \sin^2\theta\,d\theta = \left(\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta\right)\Big|_0^{\pi/6}$$

$$= \left[\frac{\pi}{12} - \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\right] - [0-0] = \frac{\pi}{12} - \frac{\sqrt{3}}{8} = \frac{2\pi - 3\sqrt{3}}{24}.$$

7. $\displaystyle\int \frac{dx}{x\sqrt{x^2-9}}, \quad x = 3\sec\theta$

SOLUTION Let $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\,d\theta$, and

$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\sqrt{\sec^2\theta - 1} = 3\sqrt{\tan^2\theta} = 3\tan\theta.$$

Thus,

$$\int \frac{dx}{x\sqrt{x^2-9}} = \int \frac{(3\sec\theta\tan\theta\,d\theta)}{(3\sec\theta)(3\tan\theta)} = \frac{1}{3}\int d\theta = \frac{1}{3}\theta + C.$$

Since $x = 3\sec\theta$, $\theta = \sec^{-1}(\frac{x}{3})$, and

$$\int \frac{dx}{x\sqrt{x^2-9}} = \frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right) + C.$$

8. $\displaystyle\int_{1/2}^{1} \frac{dx}{x^2\sqrt{x^2+4}}, \quad x = 2\tan\theta$

SOLUTION Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$, and

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = 2\sqrt{\tan^2\theta + 1} = 2\sqrt{\sec^2\theta} = 2\sec\theta.$$

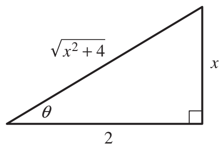
This gives us

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta(2\sec\theta)} = \frac{1}{4}\int \frac{\sec\theta\,d\theta}{\tan^2\theta} = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta.$$

Now use substitution, with $u = \sin\theta$ and $du = \cos\theta\,d\theta$. Then

$$\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int u^{-2}\,du = \frac{1}{4}\left(-u^{-1}\right) + C = -\frac{1}{4\sin\theta} + C.$$

Since $x = 2\tan\theta$, we construct a right triangle with $\tan\theta = \frac{x}{2}$:



From this triangle we see that $\sin\theta = \frac{x}{\sqrt{x^2+4}}$. Thus

$$\int_{1/2}^{1} \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x}\Bigg|_{1/2}^{1} = -\frac{1}{4}\left[\sqrt{5} - \frac{\sqrt{\frac{1}{4}+4}}{\frac{1}{2}}\right] = \frac{1}{4}\left[\sqrt{17} - \sqrt{5}\right].$$

9. $\displaystyle\int \frac{dx}{(x^2-4)^{3/2}}, \quad x = 2\sec\theta$

SOLUTION Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$, and

$$x^2 - 4 = 4\sec^2\theta - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta.$$

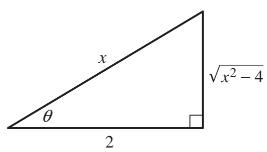
This gives

$$I = \int \frac{dx}{(x^2-4)^{3/2}} = \int \frac{2\sec\theta\tan\theta\,d\theta}{(4\tan^2\theta)^{3/2}} = \int \frac{2\sec\theta\tan\theta\,d\theta}{8\tan^3\theta} = \frac{1}{4}\int \frac{\sec\theta\,d\theta}{\tan^2\theta} = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta.$$

Now use substitution with $u = \sin\theta$ and $du = \cos\theta\,d\theta$. Then

$$I = \frac{1}{4}\int u^{-2}\,du = -\frac{1}{4}u^{-1} + C = \frac{-1}{4\sin\theta} + C.$$

Since $x = 2\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x}{2}$:



From this triangle we see that $\sin\theta = \sqrt{x^2-4}/x$, so therefore

$$I = \frac{-1}{4(\sqrt{x^2-4}/x)} + C = \frac{-x}{4\sqrt{x^2-4}} + C.$$

10. $\displaystyle\int_0^1 \frac{dx}{(4+9x^2)^2}, \quad x = \frac{2}{3}\tan\theta$

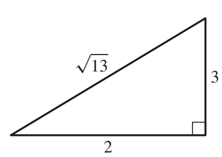
SOLUTION Let $x = \frac{2}{3}\tan\theta$. Then $dx = \frac{2}{3}\sec^2\theta\,d\theta$, and

$$4 + 9x^2 = 4 + 9\left(\frac{2}{3}\tan\theta\right)^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta$$

This gives

$$\begin{aligned}\int \frac{dx}{(4+9x^2)^2} &= \int \frac{\frac{2}{3}\sec^2\theta\,d\theta}{16\sec^4\theta} = \frac{1}{24}\int \frac{d\theta}{\sec^2\theta} \\ &= \frac{1}{24}\int \cos^2\theta\,d\theta = \frac{1}{24}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C \\ &= \frac{1}{48}(\theta + \sin\theta\cos\theta) + C\end{aligned}$$

The limits of integration are from $x = 0$ to $x = 1$. $x = 0$ corresponds to $\theta = 0$, while $x = 1$ corresponds to the angle θ with $\tan\theta = \frac{3}{2}$. So we construct a right triangle with $\tan\theta = \frac{3}{2}$:



From this triangle we see that $\sin\theta = \frac{3}{\sqrt{13}}$ and $\cos\theta = \frac{2}{\sqrt{13}}$, so that

$$\begin{aligned}\int_0^1 \frac{dx}{(4+9x^2)^2} &= \frac{1}{48}(\theta + \sin\theta\cos\theta)\Big|_0^{\tan^{-1}(3/2)} \\ &= \frac{1}{48}\left(\tan^{-1}\left(\frac{3}{2}\right) + \frac{3}{\sqrt{13}}\cdot\frac{2}{\sqrt{13}} - 0 - 0\right) = \frac{1}{48}\tan^{-1}\left(\frac{3}{2}\right) + \frac{1}{104}\end{aligned}$$

11. Evaluate $\int \frac{x\,dx}{\sqrt{x^2-4}}$ in two ways: using the direct substitution $u = x^2 - 4$ and by trigonometric substitution.

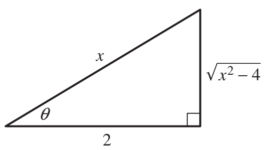
SOLUTION Let $u = x^2 - 4$. Then $du = 2x\,dx$, and

$$I_1 = \int \frac{x\,dx}{\sqrt{x^2-4}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\left(2u^{1/2}\right) + C = \sqrt{u} + C = \sqrt{x^2-4} + C.$$

To use trigonometric substitution, let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$, $x^2 - 4 = 4\sec^2\theta - 4 = 4\tan^2\theta$, and

$$I_1 = \int \frac{x\,dx}{\sqrt{x^2-4}} = \int \frac{2\sec\theta(2\sec\theta\tan\theta\,d\theta)}{2\tan\theta} = 2\int \sec^2\theta\,d\theta = 2\tan\theta + C.$$

Since $x = 2\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x}{2}$:



From this triangle we see that

$$I_1 = 2\left(\frac{\sqrt{x^2-4}}{2}\right) + C = \sqrt{x^2-4} + C.$$

12. Is the substitution $u = x^2 - 4$ effective for evaluating the integral $\int \frac{x^2\,dx}{\sqrt{x^2-4}}$? If not, evaluate using trigonometric substitution.

SOLUTION If $u = x^2 - 4$, then $du = 2x\,dx$, $x^2 = u + 4$, $dx = du/2x = du/2\sqrt{u+4}$, and

$$I = \int \frac{x^2\,dx}{\sqrt{x^2-4}} = \int \frac{(u+4)}{\sqrt{u}}\left(\frac{du}{2\sqrt{u+4}}\right) = \frac{1}{2}\int \frac{u+4}{\sqrt{u^2+4u}}\,du$$

This substitution is clearly not effective for evaluating this integral.

Instead, use the trigonometric substitution $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta$,

$$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\tan\theta,$$

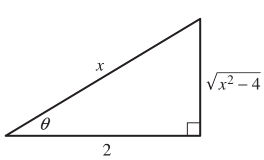
and we have

$$I = \int \frac{x^2\,dx}{\sqrt{x^2-4}} = \int \frac{4\sec^2\theta(2\sec\theta\tan\theta\,d\theta)}{2\tan\theta} = 4\int \sec^3\theta\,d\theta.$$

Now use the reduction formula for $\int \sec^m x\,dx$ from Section 8.7.2:

$$4\int \sec^3\theta\,d\theta = 4\left[\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\,d\theta\right] = 2\tan\theta\sec\theta + 2\left[\ln|\sec\theta + \tan\theta|\right] + C.$$

Since $x = 2\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x}{2}$:



From this triangle we see that $\tan\theta = \frac{1}{2}\sqrt{x^2-4}$. Therefore

$$I = 2\left(\frac{1}{2}\sqrt{x^2-4}\right)\left(\frac{x}{2}\right) + 2\ln\left|\frac{x}{2} + \frac{1}{2}\sqrt{x^2-4}\right| + C = \frac{1}{2}x\sqrt{x^2-4} + 2\ln\left|\frac{1}{2}\left(x + \sqrt{x^2-4}\right)\right| + C.$$

Finally, since

$$\ln\left|\frac{1}{2}(x + \sqrt{x^2-4})\right| = \ln\left(\frac{1}{2}\right) + \ln|x + \sqrt{x^2-4}|,$$

and $\ln(\frac{1}{2})$ is a constant, we can "absorb" this constant into the constant of integration, so that

$$I = \frac{1}{2}x\sqrt{x^2-4} + 2\ln|x + \sqrt{x^2-4}| + C.$$

13. Evaluate using the substitution $u = 1 - x^2$ or trigonometric substitution.

(a) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

(b) $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

(c) $\displaystyle\int x^3\sqrt{1-x^2}\,dx$

(d) $\displaystyle\int \frac{x^4}{\sqrt{1-x^2}}\,dx$

SOLUTION

(a) Let $u = 1 - x^2$. Then $du = -2x\,dx$, and we have

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{du}{u^{1/2}}.$$

(b) Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$, $1 - x^2 = \cos^2\theta$, and so

$$\int x^2\sqrt{1-x^2}\,dx = \int \sin^2\theta(\cos\theta)\cos\theta\,d\theta = \int \sin^2\theta\cos^2\theta\,d\theta.$$

(c) Use the substitution $u = 1 - x^2$. Then $du = -2x\,dx$, $x^2 = 1 - u$, and so

$$\int x^3\sqrt{1-x^2}\,dx = -\frac{1}{2}\int x^2\sqrt{1-x^2}(-2x\,dx) = -\frac{1}{2}\int (1-u)u^{1/2}\,du.$$

(d) Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$, $1 - x^2 = \cos^2\theta$, and so

$$\int \frac{x^4}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^4\theta}{\cos\theta}\cos\theta\,d\theta = \int \sin^4\theta\,d\theta.$$

14. Evaluate:

(a) $\displaystyle\int \frac{dt}{(t^2+1)^{3/2}}$

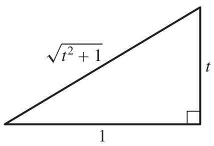
(b) $\displaystyle\int \frac{t\,dt}{(t^2+1)^{3/2}}$

SOLUTION

(a) Use the substitution $t = \tan\theta$, so that $dt = \sec^2\theta\,d\theta$. Then

$$\int \frac{dt}{(t^2+1)^{3/2}} = \int \frac{\sec^2\theta}{(\tan^2\theta+1)^{3/2}}\,d\theta = \int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C$$

Since $t = \tan\theta$, we construct a right triangle with $\tan\theta = t$:



From this we see that $\sin\theta = \frac{t}{\sqrt{t^2+1}}$, so that the integral is

$$\int \frac{dt}{(t^2+1)^{3/2}} = \sin\theta + C = \frac{t}{\sqrt{t^2+1}} + C$$

(b) Use the substitution $u = t^2 + 1$, $du = 2t\,dt$; then

$$\int \frac{t\,dt}{(t^2+1)^{3/2}} = \frac{1}{2}\int u^{-3/2}\,du = -u^{-1/2} + C = -\frac{1}{\sqrt{t^2+1}} + C$$

In Exercises 15–32, evaluate using trigonometric substitution. Refer to the table of trigonometric integrals as necessary.

15. $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}}$

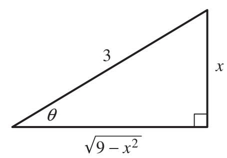
SOLUTION Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$,

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1 - \sin^2\theta) = 9\cos^2\theta,$$

and

$$I = \int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta(3\cos\theta\,d\theta)}{3\cos\theta} = 9\int \sin^2\theta\,d\theta = 9\left[\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta\right] + C.$$

Since $x = 3\sin\theta$, we construct a right triangle with $\sin\theta = \frac{x}{3}$:



From this we see that $\cos\theta = \sqrt{9-x^2}/3$, and so

$$I = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C.$$

16. $\displaystyle\int \frac{dt}{(16-t^2)^{3/2}}$

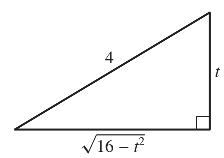
SOLUTION Let $t = 4\sin\theta$. Then $dt = 4\cos\theta\,d\theta$, and

$$(16-t^2)^{3/2} = (16-16\sin^2\theta)^{3/2} = (16\cos^2\theta)^{3/2} = (4\cos\theta)^3 = 64\cos^3\theta$$

so that

$$I = \int \frac{dt}{(16-t^2)^{3/2}} = \int \frac{4\cos\theta}{64\cos^3\theta}\,d\theta = \frac{1}{16}\int \sec^2\theta\,d\theta + C = \frac{1}{16}\tan\theta + C$$

Since $t = 4\sin\theta$, we construct a right triangle with $\sin\theta = \frac{t}{4}$:



From this, we see that $\tan\theta = \frac{t}{\sqrt{16-t^2}}$, so that

$$I = \frac{1}{16}\tan\theta + C = \frac{t}{16\sqrt{16-t^2}} + C$$

17. $\displaystyle\int \frac{dx}{x\sqrt{x^2+16}}$

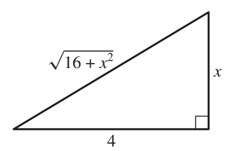
SOLUTION Use the substitution $x = 4\tan\theta$, so that $dx = 4\sec^2\theta\,d\theta$. Then

$$x\sqrt{x^2+16} = 4\tan\theta\sqrt{(4\tan\theta)^2+16} = 4\tan\theta\sqrt{16(\tan^2\theta+1)} = 16\tan\theta\sec\theta$$

so that

$$I = \int \frac{dx}{x\sqrt{x^2+16}} = \int \frac{4\sec^2\theta}{16\tan\theta\sec\theta}\,d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{4}\int \csc\theta\,d\theta = -\frac{1}{4}\ln|\csc x + \cot x| + C$$

Since $x = 4\tan\theta$, we construct a right triangle with $\tan\theta = \frac{x}{4}$:



From this, we see that $\csc x = \frac{\sqrt{x^2+16}}{x}$ and $\cot x = \frac{4}{x}$, so that

$$I = -\frac{1}{4}\ln|\csc x + \cot x| + C = -\frac{1}{4}\ln\left|\frac{\sqrt{x^2+16}}{x} + \frac{4}{x}\right| + C = -\frac{1}{4}\ln\left|\frac{4+\sqrt{x^2+16}}{x}\right| + C$$

18. $\int \sqrt{12 + 4t^2}\, dt$

SOLUTION First simplify the integral:

$$I = \int \sqrt{12 + 4t^2}\, dt = 2\int \sqrt{3 + t^2}\, dt$$

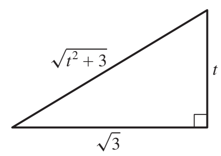
Now let $t = \sqrt{3}\tan\theta$. Then $dt = \sqrt{3}\sec^2\theta\, d\theta$,

$$3 + t^2 = 3 + 3\tan^2\theta = 3(1 + \tan^2\theta) = 3\sec^2\theta,$$

and

$$I = 2\int \sqrt{3\sec^2\theta}\left(\sqrt{3}\sec^2\theta\, d\theta\right) = 6\int \sec^3\theta\, d\theta = 6\left[\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\, d\theta\right]$$
$$= 3\tan\theta\sec\theta + 3\ln|\sec\theta + \tan\theta| + C.$$

Since $t = \sqrt{3}\tan\theta$, we construct a right triangle with $\tan\theta = \frac{t}{\sqrt{3}}$:



From this we see that $\sec\theta = \sqrt{t^2 + 3}/\sqrt{3}$. Therefore,

$$I = 3\left(\frac{t}{\sqrt{3}}\right)\left(\frac{\sqrt{t^2+3}}{\sqrt{3}}\right) + 3\ln\left|\frac{\sqrt{t^2+3}}{\sqrt{3}} + \frac{t}{\sqrt{3}}\right| + C_1 = t\sqrt{t^2+3} + 3\ln\left|\sqrt{t^2+3} + t\right| + 3\ln\left(\frac{1}{\sqrt{3}}\right) + C_1$$
$$= t\sqrt{t^2+3} + 3\ln\left|\sqrt{t^2+3} + t\right| + C,$$

where $C = 3\ln(\frac{1}{\sqrt{3}}) + C_1$.

19. $\int \frac{dx}{\sqrt{x^2 - 9}}$

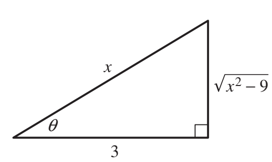
SOLUTION Let $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\, d\theta$,

$$x^2 - 9 = 9\sec^2\theta - 9 = 9(\sec^2\theta - 1) = 9\tan^2\theta,$$

and

$$I = \int \frac{dx}{\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta\, d\theta}{3\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Since $x = 3\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x}{3}$:



From this we see that $\tan\theta = \sqrt{x^2 - 9}/3$, and so

$$I = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C_1 = \ln\left|x + \sqrt{x^2-9}\right| + \ln\left(\frac{1}{3}\right) + C_1 = \ln\left|x + \sqrt{x^2-9}\right| + C,$$

where $C = \ln\left(\frac{1}{3}\right) + C_1$.

20. $\int \frac{dt}{t^2\sqrt{t^2-25}}$

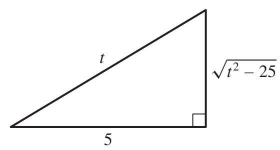
SOLUTION Let $t = 5\sec\theta$. Then $dt = 5\sec\theta\tan\theta\,d\theta$,

$$t^2 - 25 = 25\sec^2\theta - 25 = 25(\sec^2\theta - 1) = 25\tan^2\theta,$$

and

$$I = \int \frac{dt}{t^2\sqrt{t^2-25}} = \int \frac{5\sec\theta\tan\theta\,d\theta}{(25\sec^2\theta)(5\tan\theta)} = \frac{1}{25}\int \frac{d\theta}{\sec\theta} = \frac{1}{25}\int \cos\theta\,d\theta = \frac{1}{25}\sin\theta + C.$$

Since $t = 5\sec\theta$, we construct a right triangle with $\sec\theta = \frac{t}{5}$:



From this we see that $\sin\theta = \sqrt{t^2-25}/t$, and so

$$I = \frac{1}{25}\left(\frac{\sqrt{t^2-25}}{t}\right) + C = \frac{\sqrt{t^2-25}}{25t} + C.$$

21. $\int \frac{dy}{y^2\sqrt{5-y^2}}$

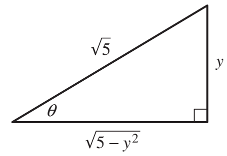
SOLUTION Let $y = \sqrt{5}\sin\theta$. Then $dy = \sqrt{5}\cos\theta\,d\theta$,

$$5 - y^2 = 5 - 5\sin^2\theta = 5(1-\sin^2\theta) = 5\cos^2\theta,$$

and

$$I = \int \frac{dy}{y^2\sqrt{5-y^2}} = \int \frac{\sqrt{5}\cos\theta\,d\theta}{(5\sin^2\theta)(\sqrt{5}\cos\theta)} = \frac{1}{5}\int \frac{d\theta}{\sin^2\theta} = \frac{1}{5}\int \csc^2\theta\,d\theta = \frac{1}{5}(-\cot\theta) + C.$$

Since $y = \sqrt{5}\sin\theta$, we construct a right triangle with $\sin\theta = \frac{y}{\sqrt{5}}$:



From this we see that $\cot\theta = \sqrt{5-y^2}/y$, which gives us

$$I = \frac{1}{5}\left(\frac{-\sqrt{5-y^2}}{y}\right) + C = -\frac{\sqrt{5-y^2}}{5y} + C.$$

22. $\int x^3\sqrt{9-x^2}\,dx$

SOLUTION Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$,

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta,$$

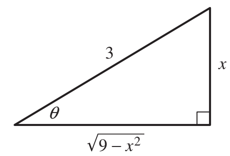
and

$$\begin{aligned} I &= \int x^3\sqrt{9-x^2}\,dx = \int (27\sin^3\theta)(3\cos\theta)(3\cos\theta\,d\theta) \\ &= 243\int \sin^3\theta\cos^2\theta\,d\theta = 243\int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta \\ &= 243\left[\int \cos^2\theta\sin\theta\,d\theta - \int \cos^4\theta\sin\theta\,d\theta\right]. \end{aligned}$$

Now use substitution, with $u = \cos\theta$ and $du = -\sin\theta\, d\theta$ for both integrals:

$$I = 243\left[-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right] + C.$$

Since $x = 3\sin\theta$, we construct a right triangle with $\sin\theta = \frac{x}{3}$:



From this we see that $\cos\theta = \sqrt{9-x^2}/3$. Thus

$$I = 243\left[-\frac{1}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5\right] + C = -3(9-x^2)^{3/2} + \frac{1}{5}(9-x^2)^{5/2} + C.$$

Alternately, let $u = 9 - x^2$. Then

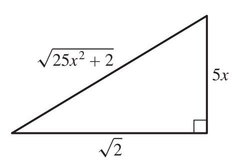
$$I = \int x^3\sqrt{9-x^2}\,dx = -\frac{1}{2}\int (9-u)\sqrt{u}\,du = -\frac{1}{2}\left(6u^{3/2} - \frac{2}{5}u^{5/2}\right) + C$$
$$= \frac{1}{5}u^{5/2} - 3u^{3/2} + C = \frac{1}{5}(9-x^2)^{5/2} - 3(9-x^2)^{3/2} + C.$$

23. $\displaystyle\int \frac{dx}{\sqrt{25x^2+2}}$

SOLUTION Let $x = \frac{\sqrt{2}}{5}\tan\theta$. Then $dx = \frac{\sqrt{2}}{5}\sec^2\theta\,d\theta$, $25x^2 + 2 = 2\tan^2\theta + 2 = 2\sec^2\theta$, and

$$I = \int \frac{dx}{\sqrt{25x^2+2}} = \int \frac{\frac{\sqrt{2}}{5}\sec^2\theta\,d\theta}{\sqrt{2}\sec\theta} = \frac{1}{5}\int \sec\theta\,d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C.$$

Since $x = \frac{\sqrt{2}}{5}\tan\theta$, we construct a right triangle with $\tan\theta = \frac{5x}{\sqrt{2}}$:



From this we see that $\sec\theta = \frac{1}{\sqrt{2}}\sqrt{25x^2+2}$, so that

$$I = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C = \frac{1}{5}\ln\left|\frac{\sqrt{25x^2+2}}{\sqrt{2}} + \frac{5x}{\sqrt{2}}\right| + C$$
$$= \frac{1}{5}\ln\left|\frac{5x + \sqrt{25x^2+2}}{\sqrt{2}}\right| + C = \frac{1}{5}\ln\left|5x + \sqrt{25x^2+2}\right| - \frac{1}{5}\ln\sqrt{2} + C$$
$$= \frac{1}{5}\ln\left|5x + \sqrt{25x^2+2}\right| + C$$

24. $\displaystyle\int \frac{dt}{(9t^2+4)^2}$

SOLUTION First factor out the t^2-coefficient:

$$I = \int \frac{dt}{(9t^2+4)^2} = \int \frac{dt}{\left[9\left(t^2+\frac{4}{9}\right)\right]^2} = \frac{1}{81}\int \frac{dt}{\left(t^2+\frac{4}{9}\right)^2}.$$

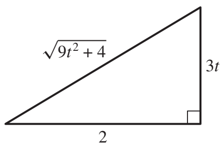
Now let $t = \frac{2}{3}\tan\theta$. Then $dt = \frac{2}{3}\sec^2\theta\,d\theta$,

$$t^2 + \frac{4}{9} = \frac{4}{9}\tan^2\theta + \frac{4}{9} = \frac{4}{9}(\tan^2\theta + 1) = \frac{4}{9}\sec^2\theta,$$

and

$$I = \frac{1}{81}\int \frac{\frac{2}{3}\sec^2 d\theta}{\frac{16}{81}\sec^4\theta\, d\theta} = \frac{1}{24}\int \cos^2\theta\, d\theta = \frac{1}{24}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] + C.$$

Since $t = \frac{2}{3}\tan\theta$, we construct a right triangle with $\tan\theta = \frac{3t}{2}$:



From this we see that $\sin\theta = 3t/\sqrt{9t^2+4}$ and $\cos\theta = 2/\sqrt{9t^2+4}$. Thus

$$I = \frac{1}{48}\tan^{-1}\left(\frac{3t}{2}\right) + \frac{1}{48}\left(\frac{3t}{\sqrt{9t^2+4}}\right)\left(\frac{2}{\sqrt{9t^2+4}}\right) + C = \frac{1}{48}\tan^{-1}\left(\frac{3t}{2}\right) + \frac{t}{8(9t^2+4)} + C.$$

25. $\displaystyle\int \frac{dz}{z^3\sqrt{z^2-4}}$

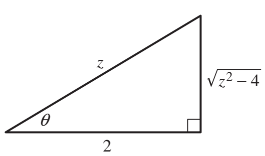
SOLUTION Let $z = 2\sec\theta$. Then $dz = 2\sec\theta\tan\theta\, d\theta$,

$$z^2 - 4 = 4\sec^2\theta - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta,$$

and

$$I = \int \frac{dz}{z^3\sqrt{z^2-4}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{(8\sec^3\theta)(2\tan\theta)} = \frac{1}{8}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{8}\int \cos^2\theta\, d\theta$$
$$= \frac{1}{8}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] + C = \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C.$$

Since $z = 2\sec\theta$, we construct a right triangle with $\sec\theta = \frac{z}{2}$:



From this we see that $\sin\theta = \sqrt{z^2-4}/z$ and $\cos\theta = 2/z$. Then

$$I = \frac{1}{16}\sec^{-1}\left(\frac{z}{2}\right) + \frac{1}{16}\left(\frac{\sqrt{z^2-4}}{z}\right)\left(\frac{2}{z}\right) + C = \frac{1}{16}\sec^{-1}\left(\frac{z}{2}\right) + \frac{\sqrt{z^2-4}}{8z^2} + C.$$

26. $\displaystyle\int \frac{dy}{\sqrt{y^2-9}}$

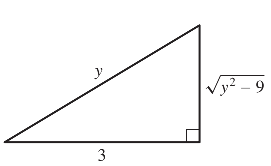
SOLUTION Let $y = 3\sec\theta$, so that $dy = 3\sec\theta\tan\theta\, d\theta$ and

$$y^2 - 9 = (3\sec\theta)^2 - 9 = 9(\sec^2\theta - 1) = 9\tan^2\theta$$

so that

$$I = \int \frac{dy}{\sqrt{y^2-9}} = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Since $y = 3\sec\theta$, we construct a right triangle with $\sec\theta = \frac{y}{3}$:



From this, we see that $\tan\theta = \frac{1}{3}\sqrt{y^2-9}$, so that

$$I = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{y}{3} + \frac{\sqrt{y^2-9}}{3}\right| + C$$
$$= \ln\left|\frac{y+\sqrt{y^2-9}}{3}\right| + C = \ln\left|y+\sqrt{y^2-9}\right| - \ln 3 + C = \ln\left|y+\sqrt{y^2-9}\right| + C$$

27. $\displaystyle\int \frac{x^2\,dx}{(6x^2-49)^{1/2}}$

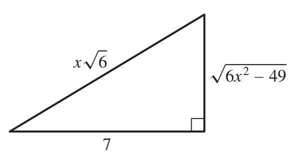
SOLUTION Let $x = \frac{7}{\sqrt{6}}\sec\theta$; then $dx = \frac{7}{\sqrt{6}}\sec\theta\tan\theta\,d\theta$, and

$$6x^2 - 49 = 6\left(\frac{7}{\sqrt{6}}\sec\theta\right)^2 - 49 = 49(\sec^2\theta - 1) = 49\tan^2\theta$$

so that

$$I = \int \frac{x^2\,dx}{(6x^2-49)^{1/2}} = \int \frac{\frac{49}{6}\sec^2\theta(\frac{7}{\sqrt{6}}\sec\theta\tan\theta)}{7\tan\theta}\,d\theta$$
$$= \frac{49}{6\sqrt{6}}\int \sec^3\theta\,d\theta = \frac{49}{6\sqrt{6}}\left(\frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\int\sec\theta\,d\theta\right)$$
$$= \frac{49}{12\sqrt{6}}(\tan\theta\sec\theta + \ln|\sec\theta+\tan\theta|) + C$$

Since $x = \frac{7}{\sqrt{6}}\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x\sqrt{6}}{7}$:



From this we see that $\tan\theta = \frac{1}{7}\sqrt{6x^2-49}$, so that

$$I = \frac{49}{12\sqrt{6}}\left(\frac{x\sqrt{6}\sqrt{6x^2-49}}{49} + \ln\left|\frac{x\sqrt{6}+\sqrt{6x^2-49}}{7}\right|\right) + C$$
$$= \frac{49}{12\sqrt{6}}\left(\frac{x\sqrt{6}\sqrt{6x^2-49}}{49} + \ln\left|x\sqrt{6}+\sqrt{6x^2-49}\right| - \ln 7\right) + C$$
$$= \frac{1}{12\sqrt{6}}\left(x\sqrt{6}\sqrt{6x^2-49} + 49\ln\left|x\sqrt{6}+\sqrt{6x^2-49}\right|\right) + C$$

28. $\displaystyle\int \frac{dx}{(x^2-4)^2}$

SOLUTION Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$,

$$x^2 - 4 = 4\sec^2\theta - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta,$$

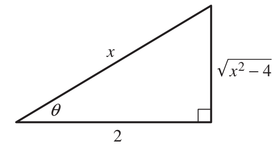
and

$$I = \int\frac{dx}{(x^2-4)^2} = \int\frac{2\sec\theta\tan\theta\,d\theta}{16\tan^4\theta} = \frac{1}{8}\int\frac{\sec\theta\,d\theta}{\tan^3\theta}$$
$$= \frac{1}{8}\int\frac{\cos^2\theta}{\sin^3\theta}\,d\theta = \frac{1}{8}\int\frac{1-\sin^2\theta}{\sin^3\theta}\,d\theta = \frac{1}{8}\int\csc^3\theta\,d\theta - \frac{1}{8}\int\csc\theta\,d\theta.$$

Now use the reduction formula for $\int \csc^3\theta\,d\theta$:

$$I = \frac{1}{8}\left[-\frac{\cot\theta\csc\theta}{2} + \frac{1}{2}\int\csc\theta\,d\theta\right] - \frac{1}{8}\int\csc\theta\,d\theta = -\frac{1}{16}\cot\theta\csc\theta - \frac{1}{16}\int\csc\theta\,d\theta$$
$$= -\frac{1}{16}\cot\theta\csc\theta - \frac{1}{16}\ln|\csc\theta - \cot\theta| + C.$$

Since $x = 2\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x}{2}$:



From this we see that $\cot\theta = 2/\sqrt{x^2-4}$ and $\csc\theta = x/\sqrt{x^2-4}$. Thus

$$I = -\frac{1}{16}\left(\frac{2}{\sqrt{x^2-4}}\right)\left(\frac{x}{\sqrt{x^2-4}}\right) - \frac{1}{16}\ln\left|\frac{x}{\sqrt{x^2-4}} - \frac{2}{\sqrt{x^2-4}}\right| + C$$
$$= \frac{-x}{8(x^2-4)} - \frac{1}{16}\ln\left|\frac{x-2}{\sqrt{x^2-4}}\right| + C.$$

29. $\displaystyle\int \frac{dt}{(t^2+9)^2}$

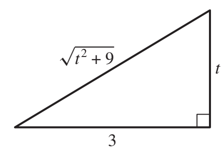
SOLUTION Let $t = 3\tan\theta$. Then $dt = 3\sec^2\theta\,d\theta$,

$$t^2 + 9 = 9\tan^2\theta + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta,$$

and

$$I = \int \frac{dt}{(t^2+9)^2} = \int \frac{3\sec^2\theta\,d\theta}{81\sec^4\theta} = \frac{1}{27}\int \cos^2\theta\,d\theta = \frac{1}{27}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] + C.$$

Since $t = 3\tan\theta$, we construct a right triangle with $\tan\theta = \frac{t}{3}$:



From this we see that $\sin\theta = t/\sqrt{t^2+9}$ and $\cos\theta = 3/\sqrt{t^2+9}$. Thus

$$I = \frac{1}{54}\tan^{-1}\left(\frac{t}{3}\right) + \frac{1}{54}\left(\frac{t}{\sqrt{t^2+9}}\right)\left(\frac{3}{\sqrt{t^2+9}}\right) + C = \frac{1}{54}\tan^{-1}\left(\frac{t}{3}\right) + \frac{t}{18(t^2+9)} + C.$$

30. $\displaystyle\int \frac{dx}{(x^2+1)^3}$

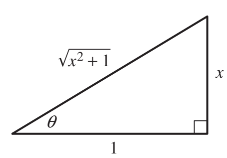
SOLUTION Let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, $x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$, and

$$I = \int \frac{dx}{(x^2+1)^3} = \int \frac{\sec^2\theta\,d\theta}{\sec^6\theta} = \int \cos^4\theta\,d\theta.$$

Using the reduction formula for $\displaystyle\int \cos^4\theta\,d\theta$, we get

$$I = \frac{\cos^3\theta\sin\theta}{4} + \frac{3}{4}\int \cos^2\theta\,d\theta = \frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C.$$

Since $x = \tan\theta$, we construct the following right triangle:



From this we see that $\sin\theta = x/\sqrt{x^2+1}$ and $\cos\theta = 1/\sqrt{x^2+1}$. Thus

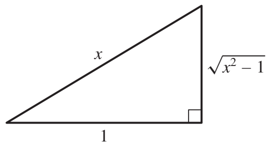
$$I = \frac{1}{4}\left(\frac{1}{\sqrt{x^2+1}}\right)^3\left(\frac{x}{\sqrt{x^2+1}}\right) + \frac{3}{8}\tan^{-1}x + \frac{3}{8}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right) + C$$
$$= \frac{x}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8}\tan^{-1}x + C.$$

31. $\int \frac{x^2\,dx}{(x^2-1)^{3/2}}$

SOLUTION Let $x = \sec\theta$. Then $dx = \sec\theta\tan\theta\,d\theta$, and $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$. Thus

$$\begin{aligned} I &= \int \frac{x^2}{(x^2-1)^{3/2}}\,dx = \int \frac{\sec^2\theta}{(\tan^2\theta)^{3/2}}\sec\theta\tan\theta\,d\theta \\ &= \int \frac{\sec^2\theta\sec\theta\tan\theta}{\tan^3\theta}\,d\theta = \int \frac{\sec^3\theta}{\tan^2\theta}\,d\theta \\ &= \int \frac{\sec^2\theta}{\tan^2\theta}\sec\theta\,d\theta = \int \csc^2\theta\sec\theta\,d\theta = \int (1+\cot^2\theta)\sec\theta\,d\theta \\ &= \int \sec\theta + \cot\theta\csc\theta\,d\theta = \ln|\sec\theta + \tan\theta| - \csc\theta + C \end{aligned}$$

Since $x = \sec\theta$, we construct the following right triangle:



From this we see that $\tan\theta = \sqrt{x^2-1}$ and that $\csc\theta = \frac{x}{\sqrt{x^2-1}}$, so that

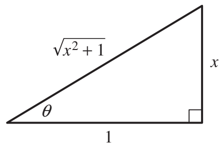
$$I = \ln\left|x + \sqrt{x^2-1}\right| - \frac{x}{\sqrt{x^2-1}} + C$$

32. $\int \frac{x^2\,dx}{(x^2+1)^{3/2}}$

SOLUTION Let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, $x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$, and

$$\begin{aligned} I &= \int \frac{x^2\,dx}{(x^2+1)^{3/2}} = \int \frac{\tan^2\theta(\sec^2\theta\,d\theta)}{(\sec^2\theta)^{3/2}} = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \int \frac{\sin^2\theta}{\cos\theta}\,d\theta = \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta \\ &= \int \frac{1}{\cos\theta}\,d\theta - \int \frac{\cos^2\theta}{\cos\theta}\,d\theta = \int \sec\theta\,d\theta - \int \cos\theta\,d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C. \end{aligned}$$

Since $x = \tan\theta$, we construct the following right triangle:



From this we see that $\sec\theta = \sqrt{x^2+1}$ and $\sin\theta = x/\sqrt{x^2+1}$. Thus

$$I = \ln\left|\sqrt{x^2+1} + x\right| - \frac{x}{\sqrt{x^2+1}} + C.$$

33. Prove for $a > 0$:

$$\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}} + C$$

SOLUTION Let $x = \sqrt{a}\,u$. Then, $x^2 = au^2$, $dx = \sqrt{a}\,du$, and

$$\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}}\int \frac{du}{u^2+1} = \frac{1}{\sqrt{a}}\tan^{-1}u + C = \frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + C.$$

34. Prove for $a > 0$:

$$\int \frac{dx}{(x^2+a)^2} = \frac{1}{2a}\left(\frac{x}{x^2+a} + \frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}}\right) + C$$

SOLUTION Let $x = \sqrt{a}\,u$. Then, $x^2 = au^2$, $dx = \sqrt{a}\,du$, and

$$\int \frac{dx}{(x^2+a)^2} = \frac{1}{a^{3/2}} \int \frac{du}{(u^2+1)^2}.$$

Now, let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$, and

$$\int \frac{dx}{(x^2+a)^2} = \frac{1}{a^{3/2}} \int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = \frac{1}{a^{3/2}} \int \cos^2\theta\,d\theta = \frac{1}{a^{3/2}} \left(\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right) + C$$

$$= \frac{1}{2a^{3/2}} \left(\frac{u}{1+u^2} + \tan^{-1} u\right) + C = \frac{1}{2a^{3/2}} \left(\frac{x/\sqrt{a}}{1+(x/\sqrt{a})^2} + \tan^{-1}\left(\frac{x}{\sqrt{a}}\right)\right) + C$$

$$= \frac{1}{2a} \left(\frac{x}{x^2+a} + \frac{1}{\sqrt{a}} \tan^{-1}\left(\frac{x}{\sqrt{a}}\right)\right) + C.$$

35. Let $I = \int \frac{dx}{\sqrt{x^2-4x+8}}$.

(a) Complete the square to show that $x^2 - 4x + 8 = (x-2)^2 + 4$.

(b) Use the substitution $u = x - 2$ to show that $I = \int \frac{du}{\sqrt{u^2+2^2}}$. Evaluate the u-integral.

(c) Show that $I = \ln\left|\sqrt{(x-2)^2+4} + x - 2\right| + C$.

SOLUTION

(a) Completing the square, we get

$$x^2 - 4x + 8 = x^2 - 4x + 4 + 4 = (x-2)^2 + 4.$$

(b) Let $u = x - 2$. Then $du = dx$, and

$$I = \int \frac{dx}{\sqrt{x^2-4x+8}} = \int \frac{dx}{\sqrt{(x-2)^2+4}} = \int \frac{du}{\sqrt{u^2+4}}.$$

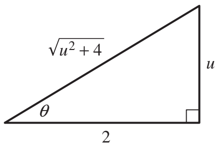
Now let $u = 2\tan\theta$. Then $du = 2\sec^2\theta\,d\theta$,

$$u^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta,$$

and

$$I = \int \frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Since $u = 2\tan\theta$, we construct a right triangle with $\tan\theta = \frac{u}{2}$:



From this we see that $\sec\theta = \sqrt{u^2+4}/2$. Thus

$$I = \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C_1 = \ln\left|\sqrt{u^2+4} + u\right| + \left(\ln\frac{1}{2} + C_1\right) = \ln\left|\sqrt{u^2+4} + u\right| + C.$$

(c) Substitute back for x in the result of part (b):

$$I = \ln\left|\sqrt{(x-2)^2+4} + x - 2\right| + C.$$

36. Evaluate $\int \frac{dx}{\sqrt{12x - x^2}}$. First complete the square to write $12x - x^2 = 36 - (x-6)^2$.

SOLUTION First complete the square:

$$12x - x^2 = -\left(x^2 - 12x + 36 - 36\right) = -\left(x^2 - 12x + 36\right) + 36 = 36 - (x-6)^2.$$

Now let $u = x - 6$, and $du = dx$. This gives us

$$I = \int \frac{dx}{\sqrt{12x - x^2}} = \int \frac{dx}{\sqrt{36 - (x-6)^2}} = \int \frac{du}{\sqrt{36 - u^2}}.$$

Next, let $u = 6\sin\theta$. Then $du = 6\cos\theta\, d\theta$,

$$36 - u^2 = 36 - 36\sin^2\theta = 36(1 - \sin^2\theta) = 36\cos^2\theta,$$

and

$$I = \int \frac{6\cos\theta\, d\theta}{6\cos\theta} = \int d\theta = \theta + C.$$

Substituting back, we find

$$I = \sin^{-1}\left(\frac{u}{6}\right) + C = \sin^{-1}\left(\frac{x-6}{6}\right) + C.$$

In Exercises 37–42, evaluate the integral by completing the square and using trigonometric substitution.

37. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 4x + 13}}$

SOLUTION First complete the square:

$$x^2 + 4x + 13 = x^2 + 4x + 4 + 9 = (x+2)^2 + 9.$$

Let $u = x + 2$. Then $du = dx$, and

$$I = \int \frac{dx}{\sqrt{x^2 + 4x + 13}} = \int \frac{dx}{\sqrt{(x+2)^2 + 9}} = \int \frac{du}{\sqrt{u^2 + 9}}.$$

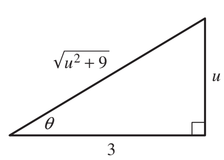
Now let $u = 3\tan\theta$. Then $du = 3\sec^2\theta\, d\theta$,

$$u^2 + 9 = 9\tan^2\theta + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta,$$

and

$$I = \int \frac{3\sec^2\theta\, d\theta}{3\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Since $u = 3\tan\theta$, we construct the following right triangle:



From this we see that $\sec\theta = \sqrt{u^2 + 9}/3$. Thus

$$I = \ln\left|\frac{\sqrt{u^2+9}}{3} + \frac{u}{3}\right| + C_1 = \ln\left|\sqrt{u^2+9} + u\right| + \left(\ln\frac{1}{3} + C_1\right)$$

$$= \ln\left|\sqrt{(x+2)^2 + 9} + x + 2\right| + C = \ln\left|\sqrt{x^2 + 4x + 13} + x + 2\right| + C.$$

38. $\displaystyle\int \frac{dx}{\sqrt{2 + x - x^2}}$

SOLUTION First complete the square:

$$2 + x - x^2 = -\left(x^2 - x\right) + 2 = -\left(x^2 - x + \frac{1}{4}\right) + 2 + \frac{1}{4} = \frac{9}{4} - \left(x - \frac{1}{2}\right)^2.$$

Let $u = x - \frac{1}{2}$ and $du = dx$. This gives us

$$I = \int \frac{dx}{\sqrt{2 + x - x^2}} = \int \frac{dx}{\sqrt{\frac{9}{4} - (x - \frac{1}{2})^2}} = \int \frac{du}{\sqrt{\frac{9}{4} - u^2}}.$$

Now let $u = \frac{3}{2}\sin\theta$. Then $du = \frac{3}{2}\cos\theta\, d\theta$,

$$\frac{9}{4} - u^2 = \frac{9}{4} - \frac{9}{4}\sin^2\theta = \frac{9}{4}(1-\sin^2\theta) = \frac{9}{4}\cos^2\theta,$$

and

$$I = \int \frac{\frac{3}{2}\cos\theta\, d\theta}{\frac{3}{2}\cos\theta} = \int d\theta = \theta + C = \sin^{-1}\left(\frac{2u}{3}\right) + C = \sin^{-1}\left(\frac{2(x-\frac{1}{2})}{3}\right) + C = \sin^{-1}\left(\frac{2x-1}{3}\right) + C.$$

39. $\displaystyle\int \frac{dx}{\sqrt{x+6x^2}}$

SOLUTION First complete the square:

$$6x^2 + x = \left(6x^2 + x + \frac{1}{24}\right) - \frac{1}{24} = \left(\sqrt{6}x + \frac{1}{2\sqrt{6}}\right)^2 - \frac{1}{24}$$

Let $u = \sqrt{6}x + \frac{1}{2\sqrt{6}}$ so that $du = \sqrt{6}\,dx$. Then

$$I = \int \frac{1}{\sqrt{x+6x^2}}\,dx = \int \frac{1}{\sqrt{\left(\sqrt{6}x + \frac{1}{2\sqrt{6}}\right)^2 - \frac{1}{24}}}\,dx = \frac{1}{\sqrt{6}}\int \frac{1}{\sqrt{u^2 - \frac{1}{24}}}\,du$$

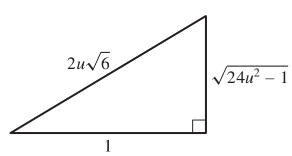
Now let $u = \frac{1}{2\sqrt{6}}\sec\theta$. Then $du = \frac{1}{2\sqrt{6}}\sec\theta\tan\theta$, and

$$u^2 - \frac{1}{24} = \frac{1}{24}(\sec^2\theta - 1) = \frac{1}{24}\tan^2\theta$$

so that

$$I = \frac{1}{\sqrt{6}}\int \frac{1}{\frac{1}{2\sqrt{6}}\tan\theta}\frac{1}{2\sqrt{6}}\sec\theta\tan\theta\, d\theta = \frac{1}{\sqrt{6}}\int \sec\theta\, d\theta = \frac{1}{\sqrt{6}}\ln|\sec\theta + \tan\theta| + C$$

Since $u = \frac{1}{2\sqrt{6}}\sec\theta$, we construct the following right triangle:



from which we see that $\tan\theta = \sqrt{24u^2-1}$ and $\sec\theta = 2u\sqrt{6}$. Thus

$$I = \frac{1}{\sqrt{6}}\ln\left|2u\sqrt{6} + \sqrt{24u^2-1}\right| + C = \frac{1}{\sqrt{6}}\ln\left|2\sqrt{6}\left(\sqrt{6}x + \frac{1}{2\sqrt{6}}\right) + \sqrt{24\left(6x^2+x+\frac{1}{24}\right) - 1}\right| + C$$

$$= \frac{1}{\sqrt{6}}\ln\left|12x + 1 + \sqrt{144x^2 + 24x}\right| + C$$

40. $\displaystyle\int \sqrt{x^2-4x+7}\,dx$

SOLUTION First complete the square:

$$x^2 - 4x + 7 = x^2 - 4x + 4 + 3 = (x-2)^2 + 3.$$

Let $u = x - 2$. Then $du = dx$, and

$$I = \int \sqrt{x^2-4x+7}\,dx = \int \sqrt{(x-2)^2+3}\,dx = \int \sqrt{u^2+3}\,du.$$

Now let $u = \sqrt{3}\tan\theta$. Then $du = \sqrt{3}\sec^2\theta\, d\theta$,

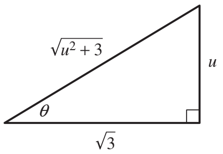
$$u^2 + 3 = 3\tan^2\theta + 3 = 3(\tan^2\theta + 1) = 3\sec^2\theta,$$

and

$$I = \int \sqrt{3\sec^2\theta}\sqrt{3}\sec^2\theta\, d\theta = 3\int \sec^3\theta\, d\theta = 3\left[\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\, d\theta\right]$$

$$= \frac{3}{2}\tan\theta\sec\theta + \frac{3}{2}\ln|\sec\theta + \tan\theta| + C.$$

Since $u = \sqrt{3}\tan\theta$, we construct a right triangle with $\tan\theta = \frac{u}{\sqrt{3}}$:



From this we see that $\sec\theta = \sqrt{u^2+3}/3$. Thus

$$\begin{aligned} I &= \frac{3}{2}\left(\frac{u}{\sqrt{3}}\right)\left(\frac{\sqrt{u^2+3}}{\sqrt{3}}\right) + \frac{3}{2}\ln\left|\frac{\sqrt{u^2+3}}{\sqrt{3}} + \frac{u}{\sqrt{3}}\right| + C_1 \\ &= \frac{1}{2}u\sqrt{u^2+3} + \frac{3}{2}\ln\left|\sqrt{u^2+3}+u\right| + \left(\frac{3}{2}\ln\frac{1}{\sqrt{3}} + C_1\right) \\ &= \frac{1}{2}(x-2)\sqrt{(x-2)^2+3} + \frac{3}{2}\ln\left|\sqrt{(x-2)^2+3}+x-2\right| + C \\ &= \frac{1}{2}(x-2)\sqrt{x^2-4x+7} + \frac{3}{2}\ln\left|\sqrt{x^2-4x+7}+x-2\right| + C. \end{aligned}$$

41. $\int \sqrt{x^2-4x+3}\,dx$

SOLUTION First complete the square:

$$x^2-4x+3 = x^2-4x+4-1 = (x-2)^2-1.$$

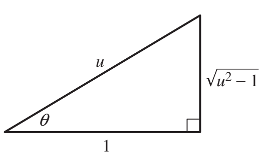
Let $u = x-2$. Then $du = dx$, and

$$I = \int \sqrt{x^2-4x+3}\,dx = \int \sqrt{(x-2)^2-1}\,dx = \int \sqrt{u^2-1}\,du.$$

Now let $u = \sec\theta$. Then $du = \sec\theta\tan\theta\,d\theta$, $u^2-1 = \sec^2\theta - 1 = \tan^2\theta$, and

$$\begin{aligned} I &= \int \sqrt{\tan^2\theta}(\sec\theta\tan\theta\,d\theta) = \int \tan^2\theta\sec\theta\,d\theta = \int \left(\sec^2\theta-1\right)\sec\theta\,d\theta \\ &= \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta = \left(\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\,d\theta\right) - \int \sec\theta\,d\theta \\ &= \frac{1}{2}\tan\theta\sec\theta - \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\tan\theta\sec\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C. \end{aligned}$$

Since $u = \sec\theta$, we construct the following right triangle:



From this we see that $\tan\theta = \sqrt{u^2-1}$. Thus

$$\begin{aligned} I &= \frac{1}{2}u\sqrt{u^2-1} - \frac{1}{2}\ln\left|u+\sqrt{u^2-1}\right| + C = \frac{1}{2}(x-2)\sqrt{(x-2)^2-1} - \frac{1}{2}\ln\left|x-2+\sqrt{(x-2)^2-1}\right| + C \\ &= \frac{1}{2}(x-2)\sqrt{x^2-4x+3} - \frac{1}{2}\ln\left|x-2+\sqrt{x^2-4x+3}\right| + C. \end{aligned}$$

42. $\int \frac{dx}{(x^2+6x+6)^2}$

SOLUTION First complete the square:

$$x^2+6x+6 = x^2+6x+9-3 = (x+3)^2-3.$$

Let $u = x+3$. Then $du = dx$, and

$$I = \int \frac{dx}{(x^2+6x+6)^2} = \int \frac{dx}{((x+3)^2-3)^2} = \int \frac{du}{(u^2-3)^2}.$$

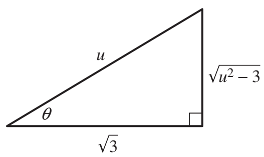
Now let $u = \sqrt{3}\sec\theta$. Then $du = \sqrt{3}\sec\theta\tan\theta$,

$$u^2 - 3 = 3\sec^2\theta - 3 = 3(\sec^2\theta - 1) = 3\tan^2\theta,$$

and

$$I = \int \frac{\sqrt{3}\sec\theta\tan\theta\,d\theta}{9\tan^4\theta} = \frac{\sqrt{3}}{9}\int\frac{\sec\theta\,d\theta}{\tan^3\theta} = \frac{\sqrt{3}}{9}\int\frac{\cos^2\theta}{\sin^3\theta}\,d\theta = \frac{\sqrt{3}}{9}\int\frac{(1-\sin^2\theta)\,d\theta}{\sin^3\theta}$$
$$= \frac{\sqrt{3}}{9}\left[\int\csc^3\theta\,d\theta - \int\csc\theta\,d\theta\right] = \frac{\sqrt{3}}{9}\left[\left(-\frac{\cot\theta\csc\theta}{2} + \frac{1}{2}\int\csc\theta\,d\theta\right) - \int\csc\theta\,d\theta\right]$$
$$= \frac{\sqrt{3}}{9}\left[-\frac{1}{2}\cot\theta\csc\theta - \frac{1}{2}\int\csc\theta\,d\theta\right] = -\frac{\sqrt{3}}{18}\cot\theta\csc\theta - \frac{\sqrt{3}}{18}\ln|\csc\theta - \cot\theta| + C.$$

Since $u = \sqrt{3}\sec\theta$, we construct a right triangle with $\sec\theta = \frac{u}{\sqrt{3}}$:



From this we see that $\cot\theta = \sqrt{3}/\sqrt{u^2-3}$ and $\csc\theta = u/\sqrt{u^2-3}$. Thus

$$I = -\frac{\sqrt{3}}{18}\left(\frac{\sqrt{3}}{\sqrt{u^2-3}}\right)\left(\frac{u}{\sqrt{u^2-3}}\right) - \frac{\sqrt{3}}{18}\ln\left|\frac{u}{\sqrt{u^2-3}} - \frac{\sqrt{3}}{\sqrt{u^2-3}}\right| + C$$
$$= \frac{-u}{6(u^2-3)} - \frac{\sqrt{3}}{18}\ln\left|\frac{u-\sqrt{3}}{\sqrt{u^2-3}}\right| + C = \frac{-(x+3)}{6((x+3)^2-3)} - \frac{\sqrt{3}}{18}\ln\left|\frac{x+3-\sqrt{3}}{\sqrt{(x+3)^2-3}}\right| + C$$
$$= \frac{-(x+3)}{6(x^2+6x+6)} - \frac{\sqrt{3}}{18}\ln\left|\frac{x+3-\sqrt{3}}{\sqrt{x^2+6x+6}}\right| + C.$$

In Exercises 43–52, indicate a good method for evaluating the integral (but do not evaluate). Your choices are: substitution (specify u and du), Integration by Parts (specify u and v′), a trigonometric method, or trigonometric substitution (specify). If it appears that these techniques are not sufficient, state this.

43. $\displaystyle\int \frac{x\,dx}{\sqrt{12-6x-x^2}}$

SOLUTION Complete the square so the the denominator is $\sqrt{15-(x+3)^2}$ and then use trigonometric substitution with $x + 3 = \sin\theta$.

44. $\displaystyle\int \sqrt{4x^2-1}\,dx$

SOLUTION Use trigonometric substitution, with $x = \frac{1}{2}\sec\theta$.

45. $\displaystyle\int \sin^3 x\cos^3 x\,dx$

SOLUTION Use one of the following trigonometric methods: rewrite $\sin^3 x = (1-\cos^2 x)\sin x$ and let $u = \cos x$, or rewrite $\cos^3 x = (1-\sin^2 x)\cos x$ and let $u = \sin x$.

46. $\displaystyle\int x\sec^2 x\,dx$

SOLUTION Use Integration by Parts, with $u = x$ and $v' = \sec^2 x$.

47. $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$

SOLUTION Either use the substitution $x = 3u$ and then recognize the formula for the inverse sine:

$$\int\frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C,$$

or use trigonometric substitution, with $x = 3\sin\theta$.

48. $\int \sqrt{1-x^3}\,dx$

SOLUTION Not solvable by any method yet considered. (In fact, this has no antiderivative using elementary functions).

49. $\int \sin^{3/2} x\,dx$

SOLUTION Not solvable by any method yet considered.

50. $\int x^2\sqrt{x+1}\,dx$

SOLUTION Use integration by parts twice, first with $u = x^2$ and then with $u = x$.

51. $\int \frac{dx}{(x+1)(x+2)^3}$

SOLUTION The techniques we have covered thus far are not sufficient to treat this integral. This integral requires a technique known as partial fractions.

52. $\int \frac{dx}{(x+12)^4}$

SOLUTION Use the substitution $u = x + 12$, and then recognize the formula

$$\int u^{-4}\,du = -\frac{1}{3u^3} + C.$$

In Exercises 53–56, evaluate using Integration by Parts as a first step.

53. $\int \sec^{-1} x\,dx$

SOLUTION Let $u = \sec^{-1} x$ and $v' = 1$. Then $v = x$, $u' = 1/x\sqrt{x^2-1}$, and

$$I = \int \sec^{-1} x\,dx = x\sec^{-1} x - \int \frac{x}{x\sqrt{x^2-1}}\,dx = x\sec^{-1} x - \int \frac{dx}{\sqrt{x^2-1}}.$$

To evaluate the integral on the right, let $x = \sec\theta$. Then $dx = \sec\theta\tan\theta\,d\theta$, $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$, and

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\tan\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|x + \sqrt{x^2-1}\right| + C.$$

Thus, the final answer is

$$I = x\sec^{-1} x - \ln\left|x + \sqrt{x^2-1}\right| + C.$$

54. $\int \frac{\sin^{-1} x}{x^2}\,dx$

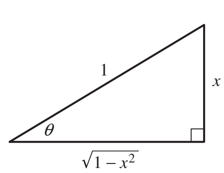
SOLUTION Let $u = \sin^{-1} x$ and $v' = x^{-2}$. Then $u' = 1/\sqrt{1-x^2}$, $v = -x^{-1}$, and

$$I = \int \frac{\sin^{-1} x}{x^2}\,dx = -\frac{\sin^{-1} x}{x} + \int \frac{dx}{x\sqrt{1-x^2}}.$$

To evaluate the integral on the right, let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$, $1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$, and

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{(\sin\theta)(\cos\theta)} = \int \csc\theta\,d\theta = \ln|\csc\theta - \cot\theta| + C.$$

Since $x = \sin\theta$, we construct the following right triangle:



From this we see that $\csc\theta = 1/x$ and $\cot\theta = \sqrt{1-x^2}/x$. Thus

$$\int \frac{dx}{x\sqrt{1-x^2}} = \ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + C = \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + C.$$

The final answer is

$$I = -\frac{\sin^{-1} x}{x} + \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + C.$$

55. $\int \ln(x^2+1)\,dx$

SOLUTION Start by using integration by parts, with $u = \ln(x^2+1)$ and $v' = 1$; then $u' = \frac{2x}{x^2+1}$ and $v = x$, so that

$$\begin{aligned} I = \int \ln(x^2+1)\,dx &= x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\,dx = x\ln(x^2+1) - 2\int \left(1 - \frac{1}{x^2+1}\right)dx \\ &= x\ln(x^2+1) - 2x + 2\int \frac{1}{x^2+1}\,dx \end{aligned}$$

To deal with the remaining integral, use the substitution $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$ and

$$\int \frac{1}{x^2+1}\,dx = \int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta = \int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \int 1\,d\theta = \theta = \tan^{-1} x + C$$

so that finally

$$I = x\ln(x^2+1) - 2x + 2\tan^{-1} x + C$$

56. $\int x^2\ln(x^2+1)\,dx$

SOLUTION Start by using integration by parts with $u = \ln(x^2+1)$, $v' = x^2$; then $u' = \frac{2x}{x^2+1}$ and $v = \frac{1}{3}x^3$, so that

$$I = \int x^2\ln(x^2+1)\,dx = \frac{1}{3}x^3\ln(x^2+1) - \frac{2}{3}\int \frac{x^4}{x^2+1}\,dx$$

To deal with the remaining integral, use the substitution $x = \tan\theta$; then $dx = \sec^2\theta\,d\theta$ and

$$\int \frac{x^4}{x^2+1}\,dx = \int \frac{\tan^4\theta}{\tan^2\theta+1}\sec^2\theta\,d\theta = \int \frac{\tan^4\theta}{\sec^2\theta}\sec^2\theta\,d\theta = \int \tan^4\theta\,d\theta$$

Using the reduction formula for $\tan^n$ gives

$$\int \tan^4\theta\,d\theta = \frac{1}{3}\tan^3\theta - \int \tan^2\theta\,d\theta = \frac{1}{3}\tan^3\theta - \tan\theta + \theta + C$$

so that, substituting back for $x = \tan\theta$, we get

$$I = \frac{1}{3}x^3\ln(x^2+1) - \frac{2}{3}\left(\frac{1}{3}x^3 - x + \tan^{-1} x\right) + C = \frac{1}{3}x^3\ln(x^2+1) - \frac{2}{9}x^3 + \frac{2}{3}x - \frac{2}{3}\tan^{-1} x + C$$

57. Find the average height of a point on the semicircle $y = \sqrt{1-x^2}$ for $-1 \le x \le 1$.

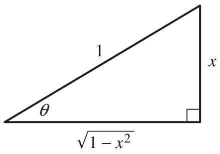
SOLUTION The average height is given by the formula

$$y_{\text{ave}} = \frac{1}{1-(-1)}\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}\,dx$$

Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$, $1 - x^2 = \cos^2\theta$, and

$$\int \sqrt{1-x^2}\,dx = \int (\cos\theta)(\cos\theta\,d\theta) = \int \cos^2\theta\,d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C.$$

Since $x = \sin\theta$, we construct the following right triangle:



From this we see that $\cos\theta = \sqrt{1-x^2}$. Therefore,

$$y_{\text{ave}} = \frac{1}{2}\left(\frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2}\right)\Big|_{-1}^{1} = \frac{1}{2}\left[\left(\frac{1}{2}\pi + 0\right) - \left(-\frac{1}{2}\pi + 0\right)\right] = \frac{\pi}{4}.$$

58. Find the volume of the solid obtained by revolving the graph of $y = x\sqrt{1-x^2}$ over [0, 1] about the y-axis.

SOLUTION Using the method of cylindrical shells, the volume is given by

$$V = 2\pi\int_0^1 x\left(x\sqrt{1-x^2}\right)dx = 2\pi\int_0^1 x^2\sqrt{1-x^2}\,dx.$$

To evaluate this integral, let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$,

$$1 - x^2 = 1 - \sin^2\theta = \cos^2\theta,$$

and

$$I = \int x^2\sqrt{1-x^2}\,dx = \int \sin^2\theta\cos^2\theta\,d\theta = \int\left(1-\cos^2\theta\right)\cos^2\theta\,d\theta = \int\cos^2\theta\,d\theta - \int\cos^4\theta\,d\theta.$$

Now use the reduction formula for $\int\cos^4\theta\,d\theta$:

$$I = \int\cos^2\theta\,d\theta - \left[\frac{\cos^3\theta\sin\theta}{4} + \frac{3}{4}\int\cos^2\theta\,d\theta\right] = -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{4}\int\cos^2\theta\,d\theta$$

$$= -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{4}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] + C = -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{8}\theta + \frac{1}{8}\sin\theta\cos\theta + C.$$

Since $\sin\theta = x$, we know that $\cos\theta = \sqrt{1-x^2}$. Then we have

$$I = -\frac{1}{4}(1-x^2)^{3/2}x + \frac{1}{8}\sin^{-1}x + \frac{1}{8}x\sqrt{1-x^2} + C.$$

Now we can complete the volume:

$$V = 2\pi\left(-\frac{1}{4}x(1-x^2)^{3/2} + \frac{1}{8}\sin^{-1}x + \frac{1}{8}x\sqrt{1-x^2}\right)\Big|_0^1 = 2\pi\left[\left(0 + \frac{\pi}{16} + 0\right) - (0)\right] = \frac{\pi^2}{8}.$$

59. Find the volume of the solid obtained by revolving the region between the graph of $y^2 - x^2 = 1$ and the line $y = 2$ about the line $y = 2$.

SOLUTION First solve the equation $y^2 - x^2 = 1$ for y:

$$y = \pm\sqrt{x^2+1}.$$

The region in question is bounded in part by the top half of this hyperbola, which is the equation

$$y = \sqrt{x^2+1}.$$

The limits of integration are obtained by finding the points of intersection of this equation with $y = 2$:

$$2 = \sqrt{x^2+1} \Rightarrow x = \pm\sqrt{3}.$$

The radius of each disk is given by $2 - \sqrt{x^2+1}$; the volume is therefore given by

$$V = \int_{-\sqrt{3}}^{\sqrt{3}}\pi r^2\,dx = 2\pi\int_0^{\sqrt{3}}\left(2-\sqrt{x^2+1}\right)^2dx = 2\pi\int_0^{\sqrt{3}}\left[4 - 4\sqrt{x^2+1} + (x^2+1)\right]dx$$

$$= 8\pi\int_0^{\sqrt{3}}dx - 8\pi\int_0^{\sqrt{3}}\sqrt{x^2+1}\,dx + 2\pi\int_0^{\sqrt{3}}(x^2+1)\,dx.$$

To evaluate the integral $\int \sqrt{x^2+1}\,dx$, let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, $x^2+1 = \sec^2\theta$, and

$$\int \sqrt{x^2+1}\,dx = \int \sec^3\theta\,d\theta = \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\int \sec\theta\,d\theta$$
$$= \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left|\sqrt{x^2+1}+x\right| + C.$$

Now we can compute the volume:

$$V = \left[8\pi x - 8\pi\left(\frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left|\sqrt{x^2+1}+x\right|\right) + \frac{2}{3}\pi x^3 + 2\pi x\right]\Bigg|_0^{\sqrt{3}}$$
$$= \left(10\pi x + \frac{2}{3}\pi x^3 - 4\pi x\sqrt{x^2+1} - 4\pi\ln\left|\sqrt{x^2+1}+x\right|\right)\Bigg|_0^{\sqrt{3}}$$
$$= \left(10\pi\sqrt{3} + 2\pi\sqrt{3} - 8\pi\sqrt{3} - 4\pi\ln\left|2+\sqrt{3}\right|\right) - (0) = 4\pi\left[\sqrt{3} - \ln\left|2+\sqrt{3}\right|\right].$$

60. Find the volume of revolution for the region in Exercise 59, but revolve around $y = 3$.

SOLUTION Using the washer method, the volume is given by

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi\left(R^2 - r^2\right)dx = 2\pi\int_0^{\sqrt{3}}\left[\left(3-\sqrt{x^2+1}\right)^2 - 1^2\right]dx$$
$$= 2\pi\int_0^{\sqrt{3}}\left(9 - 6\sqrt{x^2+1} + \left(x^2+1\right) - 1\right)dx = 2\pi\int_0^{\sqrt{3}}\left(9 - 6\sqrt{x^2+1} + x^2\right)dx$$
$$= 2\pi\left[9x - 6\left(\frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left|\sqrt{x^2+1}+x\right|\right) + \frac{1}{3}x^3\right]\Bigg|_0^{\sqrt{3}}$$
$$= 2\pi\left[\left(9\sqrt{3} - 3\sqrt{3}(2) - 3\ln\left|2+\sqrt{3}\right| + \sqrt{3}\right) - (0)\right] = 8\pi\sqrt{3} - 6\pi\ln\left|2+\sqrt{3}\right|.$$

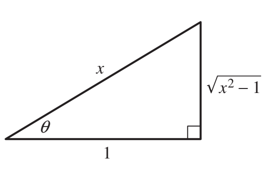
61. Compute $\int \frac{dx}{x^2-1}$ in two ways and verify that the answers agree: first via trigonometric substitution and then using the identity

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

SOLUTION Using trigonometric substitution, let $x = \sec\theta$. Then $dx = \sec\theta\tan\theta d\theta$, $x^2-1 = \sec^2\theta - 1 = \tan^2\theta$, and

$$I = \int\frac{dx}{x^2-1} = \int\frac{\sec\theta\tan\theta\,d\theta}{\tan^2\theta} = \int\frac{\sec\theta}{\tan\theta}\,d\theta = \int\frac{d\theta}{\sin\theta} = \int\csc\theta\,d\theta = \ln|\csc\theta - \cot\theta| + C.$$

Since $x = \sec\theta$, we construct the following right triangle:



From this we see that $\csc\theta = x/\sqrt{x^2-1}$ and $\cot\theta = 1/\sqrt{x^2-1}$. This gives us

$$I = \ln\left|\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right| + C = \ln\left|\frac{x-1}{\sqrt{x^2-1}}\right| + C.$$

Using the given identity, we get

$$I = \int\frac{dx}{x^2-1} = \frac{1}{2}\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx = \frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{dx}{x+1} = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

To confirm that these answers agree, note that

$$\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| = \ln\sqrt{\left|\frac{x-1}{x+1}\right|} = \ln\left|\frac{\sqrt{x-1}}{\sqrt{x+1}}\cdot\frac{\sqrt{x-1}}{\sqrt{x-1}}\right| = \ln\left|\frac{x-1}{\sqrt{x^2-1}}\right|.$$

62. CAS You want to divide an 18-inch pizza equally among three friends using vertical slices at $\pm x$ as in Figure 6. Find an equation satisfied by x and find the approximate value of x using a computer algebra system.

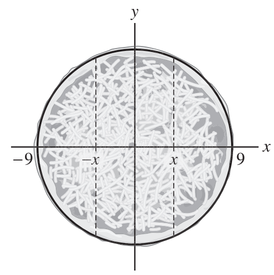


FIGURE 6 Dividing a pizza into three equal parts.

SOLUTION First find the value of x which divides evenly a pizza with a 1-inch radius. By proportionality, we can then take this answer and multiply by 9 to get the answer for the 18-inch pizza. The total area of a 1-inch radius pizza is $\pi \cdot 1^2 = \pi$ (in square inches). The three equal pieces will have an area of $\pi/3$. The center piece is further divided into 4 equal pieces, each of area $\pi/12$. From Example 1, we know that

$$\int_0^x \sqrt{1-x^2}\,dx = \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2}.$$

Setting this expression equal to $\pi/12$ and solving for x using a computer algebra system, we find $x = 0.265$. For the 18-inch pizza, the value of x should be

$$x = 9(0.265) = 2.385 \text{ inches.}$$

63. A charged wire creates an electric field at a point P located at a distance D from the wire (Figure 7). The component $E_\perp$ of the field perpendicular to the wire (in N/C) is

$$E_\perp = \int_{x_1}^{x_2} \frac{k\lambda D}{(x^2+D^2)^{3/2}}\,dx$$

where λ is the charge density (coulombs per meter), $k = 8.99 \times 10^9$ N·m^2/C^2 (Coulomb constant), and x_1, x_2 are as in the figure. Suppose that $\lambda = 6 \times 10^{-4}$ C/m, and $D = 3$ m. Find $E_\perp$ if (a) $x_1 = 0$ and $x_2 = 30$ m, and (b) $x_1 = -15$ m and $x_2 = 15$ m.

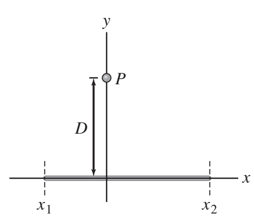


FIGURE 7

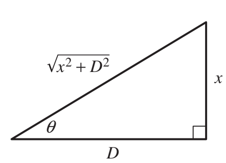
SOLUTION Let $x = D\tan\theta$. Then $dx = D\sec^2\theta\,d\theta$,

$$x^2 + D^2 = D^2\tan^2\theta + D^2 = D^2(\tan^2\theta + 1) = D^2\sec^2\theta,$$

and

$$E_\perp = \int_{x_1}^{x_2} \frac{k\lambda D}{(x^2+D^2)^{3/2}}\,dx = k\lambda D\int_{x_1}^{x_2} \frac{D\sec^2\theta\,d\theta}{(D^2\sec^2\theta)^{3/2}}$$
$$= \frac{k\lambda D^2}{D^3}\int_{x_1}^{x_2} \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \frac{k\lambda}{D}\int_{x_1}^{x_2}\cos\theta\,d\theta = \frac{k\lambda}{D}\sin\theta\Big|_{x_1}^{x_2}$$

Since $x = D\tan\theta$, we construct a right triangle with $\tan\theta = x/D$:



From this we see that $\sin\theta = x/\sqrt{x^2+D^2}$. Then

$$E_\perp = \frac{k\lambda}{D}\left(\frac{x}{\sqrt{x^2+D^2}}\right)\Bigg|_{x_1}^{x_2}$$

(a) Plugging in the values for the constants k, λ, D, and evaluating the antiderivative for $x_1 = 0$ and $x_2 = 30$, we get

$$E_\perp = \frac{(8.99\times 10^9)(6\times 10^{-4})}{3}\left[\frac{30}{\sqrt{30^2+3^2}} - 0\right] \approx 1.789\times 10^6\ \frac{V}{\text{m}}$$

(b) If $x_1 = -15$ m and $x_2 = 15$ m, we get

$$E_\perp = \frac{(8.99\times 10^9)(6\times 10^{-4})}{3}\left[\frac{15}{\sqrt{15^2+3^2}} - \frac{-15}{\sqrt{(-15)^2+3^2}}\right] \approx 3.526\times 10^6\ \frac{V}{\text{m}}$$

Further Insights and Challenges

64. Let $J_n = \int \frac{dx}{(x^2+1)^n}$. Use Integration by Parts to prove

$$J_{n+1} = \left(1 - \frac{1}{2n}\right)J_n + \left(\frac{1}{2n}\right)\frac{x}{(x^2+1)^n}$$

Then use this recursion relation to calculate J_2 and J_3.

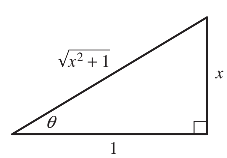
SOLUTION Let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, $x^2+1 = \tan^2\theta + 1 = \sec^2\theta$, and

$$J_{n+1} = \int \frac{dx}{(x^2+1)^{n+1}} = \int \frac{\sec^2\theta\,d\theta}{\sec^{2n+2}\theta} = \int \sec^{-2n}\theta\,d\theta = \int \cos^{2n}\theta\,d\theta.$$

Using the reduction formula for $\int \cos^m\theta\,d\theta$, we get

$$J_{n+1} = \frac{\cos^{2n-1}\theta\sin\theta}{2n} + \frac{2n-1}{2n}\int \cos^{2n-2}\theta\,d\theta.$$

Since $x = \tan\theta$, we construct the following right triangle:



From this we see that $\cos\theta = 1/\sqrt{x^2+1}$, and $\sin\theta = x/\sqrt{x^2+1}$. This gives us

$$J_{n+1} = \frac{1}{2n}\left(\frac{1}{\sqrt{x^2+1}}\right)^{2n-1}\left(\frac{x}{\sqrt{x^2+1}}\right) + \frac{2n-1}{2n}\int\left(\frac{1}{\sqrt{x^2+1}}\right)^{2n-2}\left(\frac{1}{\sqrt{x^2+1}}\right)^2 dx.$$

Here we've used the fact that

$$d\theta = \frac{dx}{\sec^2\theta} = \cos^2\theta\,dx = \left(\frac{1}{\sqrt{x^2+1}}\right)^2 dx.$$

Simplifying, we get

$$J_{n+1} = \left(\frac{1}{2n}\right)\frac{x}{(\sqrt{x^2+1})^{2n}} + \frac{2n-1}{2n}\int\frac{dx}{(\sqrt{x^2+1})^{2n}} = \frac{1}{2n}\frac{x}{(x^2+1)^n} + \frac{2n-1}{2n}\int\frac{dx}{(x^2+1)^n}$$
$$= \frac{1}{2n}\frac{x}{(x^2+1)^n} + \left(1 - \frac{1}{2n}\right)J_n.$$

To use this formula, we first compute J_1:

$$J_1 = \int\frac{dx}{x^2+1} = \tan^{-1}x + C.$$

Now use the formula to compute J_2 and J_3:

$$J_2 = \frac{1}{2}\frac{x}{x^2+1} + \left(1 - \frac{1}{2}\right) J_1 = \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1} x + C;$$

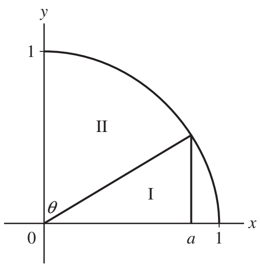
$$J_3 = \frac{1}{4}\frac{x}{(x^2+1)^2} + \left(1 - \frac{1}{4}\right) J_2 = \frac{1}{4}\left[\frac{x}{(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8}\tan^{-1} x\right] + C.$$

65. Prove the formula

$$\int \sqrt{1-x^2}\, dx = \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} + C$$

using geometry by interpreting the integral as the area of part of the unit circle.

SOLUTION The integral $\int_0^a \sqrt{1-x^2}\, dx$ is the area bounded by the unit circle, the x-axis, the y-axis, and the line $x = a$. This area can be divided into two regions as follows:



Region I is a triangle with base a and height $\sqrt{1-a^2}$. Region II is a sector of the unit circle with central angle $\theta = \frac{\pi}{2} - \cos^{-1} a = \sin^{-1} a$. Thus,

$$\int_0^a \sqrt{1-x^2}\, dx = \frac{1}{2}a\sqrt{1-a^2} + \frac{1}{2}\sin^{-1} a = \left(\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right)\Big|_0^a.$$

7.4 Integrals Involving Hyperbolic and Inverse Hyperbolic Functions

Preliminary Questions

1. Which hyperbolic substitution can be used to evaluate the following integrals?

(a) $\int \frac{dx}{\sqrt{x^2+1}}$ **(b)** $\int \frac{dx}{\sqrt{x^2+9}}$ **(c)** $\int \frac{dx}{\sqrt{9x^2+1}}$

SOLUTION The appropriate hyperbolic substitutions are

(a) $x = \sinh t$
(b) $x = 3\sinh t$
(c) $3x = \sinh t$

2. Which two of the hyperbolic integration formulas differ from their trigonometric counterparts by a minus sign?

SOLUTION The integration formulas for $\sinh x$ and $\tanh x$ differ from their trigonometric counterparts by a minus sign.

3. Which antiderivative of $y = (1-x^2)^{-1}$ should we use to evaluate the integral $\int_3^5 (1-x^2)^{-1}\, dx$?

SOLUTION Because the integration interval lies outside $-1 < x < 1$, the appropriate antiderivative of $y = (1-x^2)^{-1}$ is $\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$.

Exercises

In Exercises 1–16, calculate the integral.

1. $\int \cosh(3x)\, dx$

SOLUTION $\int \cosh(3x)\, dx = \frac{1}{3}\sinh 3x + C.$

2. $\int \sinh(x+1)\, dx$

SOLUTION $\int \sinh(x+1)\, dx = \cosh(x+1) + C.$

3. $\int x \sinh(x^2+1)\,dx$

SOLUTION $\int x \sinh(x^2+1)\,dx = \frac{1}{2}\cosh(x^2+1) + C.$

4. $\int \sinh^2 x \cosh x\,dx$

SOLUTION Let $u = \sinh x$. Then $du = \cosh x\,dx$ and

$$\int \sinh^2 x \cosh x\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\sinh x)^3 + C.$$

5. $\int \operatorname{sech}^2(1-2x)\,dx$

SOLUTION $\int \operatorname{sech}^2(1-2x)\,dx = -\frac{1}{2}\tanh(1-2x) + C.$

6. $\int \tanh(3x)\operatorname{sech}(3x)\,dx$

SOLUTION $\int \tanh(3x)\operatorname{sech}(3x)\,dx = -\frac{1}{3}\operatorname{sech} 3x + C.$

7. $\int \tanh x \operatorname{sech}^2 x\,dx$

SOLUTION Let $u = \tanh x$. Then $du = \operatorname{sech}^2 x\,dx$ nd

$$\int \tanh x \operatorname{sech}^2 x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{\tanh^2 x}{2} + C.$$

8. $\int \frac{\cosh x}{3\sinh x + 4}\,dx$

SOLUTION Let $u = 3\sinh x + 4$. Then $du = 3\cosh x\,dx$ and

$$\int \frac{\cosh x}{3\sinh x + 4}\,dx = \int \frac{du}{3u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3\sinh x + 4| + C.$$

9. $\int \tanh x\,dx$

SOLUTION $\int \tanh x\,dx = \ln\cosh x + C.$

10. $\int x \operatorname{csch}(x^2)\coth(x^2)\,dx$

SOLUTION Let $u = x^2$. Then $du = 2x\,dx$ and

$$\int x \operatorname{csch}(x^2)\coth(x^2)\,dx = \frac{1}{2}\int \operatorname{csch} u \coth u\,du = -\frac{1}{2}\operatorname{csch} u + C = -\frac{1}{2}\operatorname{csch}(x^2) + C.$$

11. $\int \frac{\cosh x}{\sinh x}\,dx$

SOLUTION $\int \frac{\cosh x}{\sinh x}\,dx = \ln|\sinh x| + C.$

12. $\int \frac{\cosh x}{\sinh^2 x}\,dx$

SOLUTION $\int \frac{\cosh x}{\sinh^2 x}\,dx = \int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C.$

13. $\int \sinh^2(4x-9)\,dx$

SOLUTION $\int \sinh^2(4x-9)\,dx = \frac{1}{2}\int (\cosh(8x-18) - 1)\,dx = \frac{1}{16}\sinh(8x-18) - \frac{1}{2}x + C.$

14. $\int \sinh^3 x \cosh^6 x\,dx$

SOLUTION Let $u = \cosh x$. Then $du = \sinh x\,dx$ and

$$\int \sinh^3 x \cosh^6 x\,dx = \int (\cosh^2 x - 1)\cosh^6 x \sinh x\,dx = \int (u^2-1)u^6\,du = \int (u^8 - u^6)\,du$$
$$= \frac{1}{9}u^9 - \frac{1}{7}u^7 + C = \frac{1}{9}\cosh^9 x - \frac{1}{7}\cosh^7 x + C.$$

15. $\int \sinh^2 x \cosh^2 x\,dx$

SOLUTION

$$\int \sinh^2 x \cosh^2 x\,dx = \frac{1}{4}\int \sinh^2 2x\,dx = \frac{1}{8}\int (\cosh 4x - 1)\,dx = \frac{1}{32}\sinh 4x - \frac{1}{8}x + C.$$

16. $\int \tanh^3 x\,dx$

SOLUTION

$$\int \tanh^3 x\,dx = \int (1 - \operatorname{sech}^2 x)\tanh x\,dx = \ln\cosh x - \int \tanh x \operatorname{sech}^2 x\,dx.$$

To evaluate the remaining integral, let $u = \tanh x$. Then $du = \operatorname{sech}^2 x\,dx$ and

$$\int \tanh x \operatorname{sech}^2 x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\tanh^2 x + C.$$

Therefore,

$$\int \tanh^3 x\,dx = \ln\cosh x - \frac{1}{2}\tanh^2 x + C.$$

In Exercises 17–30, calculate the integral in terms of the inverse hyperbolic functions.

17. $\int \frac{dx}{\sqrt{x^2-1}}$

SOLUTION $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C.$

18. $\int \frac{dx}{\sqrt{9x^2-4}}$

SOLUTION $\int \frac{dx}{\sqrt{9x^2-4}} = \frac{1}{3}\cosh^{-1}\left(\frac{3x}{2}\right) + C.$

19. $\int \frac{dx}{\sqrt{16+25x^2}}$

SOLUTION $\int \frac{dx}{\sqrt{16+25x^2}} = \frac{1}{5}\sinh^{-1}\left(\frac{5x}{4}\right) + C.$

20. $\int \frac{dx}{\sqrt{1+3x^2}}$

SOLUTION $\int \frac{dx}{\sqrt{1+3x^2}} = \frac{1}{\sqrt{3}}\sinh^{-1}(\sqrt{3}x) + C.$

21. $\int \sqrt{x^2-1}\,dx$

SOLUTION Let $x = \cosh t$. Then $dx = \sinh t\,dt$ and

$$\int \sqrt{x^2-1}\,dx = \int \sinh^2 t\,dt = \frac{1}{2}\int (\cosh 2t - 1)\,dt = \frac{1}{4}\sinh 2t - \frac{1}{2}t + C$$
$$= \frac{1}{2}\sinh t\cosh t - \frac{1}{2}t + C = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\cosh^{-1} x + C.$$

22. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+1}}$

SOLUTION Let $x = \sinh t$. Then $dx = \cosh t\,dt$ and

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \int \sinh^2 t\,dt = \frac{1}{2}\int (\cosh 2t - 1)\,dt = \frac{1}{4}\sinh 2t - \frac{1}{2}t + C = \frac{1}{2}\sinh t\cosh t - \frac{1}{2}t + C$$
$$= \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\sinh^{-1} x + C.$$

23. $\displaystyle\int_{-1/2}^{1/2} \frac{dx}{1-x^2}$

SOLUTION

$$\int_{-1/2}^{1/2} \frac{dx}{1-x^2} = \tanh^{-1} x\Big|_{-1/2}^{1/2} = \tanh^{-1}\left(\frac{1}{2}\right) - \tanh^{-1}\left(-\frac{1}{2}\right) = 2\tanh^{-1}\left(\frac{1}{2}\right).$$

24. $\displaystyle\int_4^5 \frac{dx}{1-x^2}$

SOLUTION

$$\int_4^5 \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\Big|_4^5 = \frac{1}{2}\left(\ln\frac{3}{2} - \ln\frac{5}{3}\right) = \frac{1}{2}\ln\frac{9}{10}.$$

25. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^2}}$

SOLUTION $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}\Big|_0^1 = \sinh^{-1}(1) - \sinh^{-1}(0) = \sinh^{-1} 1.$

26. $\displaystyle\int_2^{10} \frac{dx}{4x^2-1}$

SOLUTION $\displaystyle\int_2^{10} \frac{dx}{4x^2-1} = -\frac{1}{2}\coth^{-1}(2x)\Big|_2^{10} = \frac{1}{2}(\coth^{-1} 4 - \coth^{-1} 20).$

27. $\displaystyle\int_{-3}^{-1} \frac{dx}{x\sqrt{x^2+16}}$

SOLUTION $\displaystyle\int_{-3}^{-1} \frac{dx}{x\sqrt{x^2+16}} = \frac{1}{4}\operatorname{csch}^{-1}\left(\frac{x}{4}\right)\Big|_{-3}^{-1} = \frac{1}{4}\left(\operatorname{csch}^{-1}\left(-\frac{1}{4}\right) - \operatorname{csch}^{-1}\left(-\frac{3}{4}\right)\right).$

28. $\displaystyle\int_{0.2}^{0.8} \frac{dx}{x\sqrt{1-x^2}}$

SOLUTION $\displaystyle\int_{0.2}^{0.8} \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1} x\Big|_{0.2}^{0.8} = \operatorname{sech}^{-1}(0.2) - \operatorname{sech}^{-1}(0.8)$

29. $\displaystyle\int \frac{\sqrt{x^2-1}\,dx}{x^2}$

SOLUTION Let $x = \cosh t$. Then $dx = \sinh t\,dt$ and

$$\int \frac{\sqrt{x^2-1}\,dx}{x^2} = \int \frac{\sinh^2 t}{\cosh^2 t}\,dt = \int \tanh^2 t\,dt = \int (1 - \operatorname{sech}^2 t)\,dt$$
$$= t - \tanh t + C = \cosh^{-1} x - \frac{\sqrt{x^2-1}}{x} + C.$$

30. $\displaystyle\int_1^9 \frac{dx}{x\sqrt{x^4+1}}$

SOLUTION Let $u = x^2$. Then $du = 2x\,dx$ or $\frac{dx}{x} = \frac{1}{2}\frac{du}{x^2} = \frac{1}{2}\frac{du}{u}$. Hence,

$$\int_1^9 \frac{dx}{x\sqrt{x^4+1}} = \frac{1}{2}\int_1^{81} \frac{du}{u\sqrt{u^2+1}} = -\operatorname{csch}^{-1} u\Big|_1^{81} = \operatorname{csch}^{-1} 1 - \operatorname{csch}^{-1} 81.$$

31. Verify the formulas

$$\sinh^{-1} x = \ln|x + \sqrt{x^2+1}|$$
$$\cosh^{-1} x = \ln|x + \sqrt{x^2-1}| \qquad (\text{for } x \geq 1)$$

SOLUTION Let $x = \sinh t$. Then

$$\cosh t = \sqrt{1 + \sinh^2 t} = \sqrt{1+x^2}.$$

Moreover, because

$$\sinh t + \cosh t = \frac{e^t - e^{-t}}{2} + \frac{e^t + e^{-t}}{2} = e^t,$$

it follows that

$$\sinh^{-1} x = t = \ln(\sinh t + \cosh t) = \ln(x + \sqrt{x^2+1}).$$

Now, Let $x = \cosh t$. Then

$$\sinh t = \sqrt{\cosh^2 t - 1} = \sqrt{x^2 - 1}.$$

and

$$\cosh^{-1} x = t = \ln(\sinh t + \cosh t) = \ln(x + \sqrt{x^2-1}).$$

Because $\cosh t \geq 1$ for all t, this last expression is only valid for $x = \cosh t \geq 1$.

32. Verify that $\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$ for $|x| < 1$.

SOLUTION Let $A = \tanh^{-1} x$. Then

$$x = \tanh A = \frac{\sinh A}{\cosh A} = \frac{e^A - e^{-A}}{e^A + e^{-A}}.$$

Solving for A yields

$$A = \frac{1}{2}\ln\frac{x+1}{1-x};$$

hence,

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{x+1}{1-x} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|.$$

for $|x| < 1$ (so that both $1+x$ and $1-x$ are positive).

33. Evaluate $\int \sqrt{x^2+16}\,dx$ using trigonometric substitution. Then use Exercise 31 to verify that your answer agrees with the answer in Example 3.

SOLUTION Let $x = 4\tan\theta$. Then $dx = 4\sec^2\theta\,d\theta$ and

$$\int \sqrt{x^2+16}\,dx = 16\int \sec^3\theta\,d\theta = 8\tan\theta\sec\theta + 8\int\sec\theta\,d\theta = 8\tan\theta\sec\theta + 8\ln|\sec\theta + \tan\theta| + C$$
$$= 8\cdot\frac{x}{4}\cdot\frac{\sqrt{x^2+16}}{4} + 8\ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$$
$$= \frac{1}{2}x\sqrt{x^2+16} + 8\ln\left|\frac{x}{4} + \sqrt{\left(\frac{x}{4}\right)^2 + 1}\right| + C.$$

Using Exercise 31,

$$\ln\left|\frac{x}{4} + \sqrt{\left(\frac{x}{4}\right)^2 + 1}\right| = \sinh^{-1}\left(\frac{x}{4}\right),$$

so we can write the antiderivative as

$$\frac{1}{2}x\sqrt{x^2+16} + 8\sinh^{-1}\left(\frac{x}{4}\right) + C,$$

which agrees with the answer in Example 3.

34. Evaluate $\int \sqrt{x^2 - 9}\, dx$ in two ways: using trigonometric substitution and using hyperbolic substitution. Then use Exercise 31 to verify that the two answers agree.

SOLUTION First, let $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\, d\theta$ and

$$\begin{aligned}
\int \sqrt{x^2 - 9}\, dx &= 9\int \tan^2\theta \sec\theta\, d\theta = 9\int \sec^3\theta\, d\theta - 9\int \sec\theta\, d\theta \\
&= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\int \sec\theta\, d\theta - 9\int \sec\theta\, d\theta \\
&= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| + C \\
&= \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{x^2-9}}{3} - \frac{9}{2}\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C \\
&= \frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\ln\left|\frac{x}{3} + \sqrt{\left(\frac{x}{3}\right)^2 - 1}\right| + C.
\end{aligned}$$

Alternately, let $x = 3\cosh t$. Then $dx = 3\sinh t\, dt$ and

$$\begin{aligned}
\int \sqrt{x^2 - 9}\, dx &= 9\int \sinh^2 t\, dt = \frac{9}{2}\int (\cosh 2t - 1)\, dt = \frac{9}{2}\sinh t\cosh t - \frac{9}{2}t + C \\
&= \frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\cosh^{-1}\left(\frac{x}{3}\right) + C.
\end{aligned}$$

Using Exercise 31,

$$\cosh^{-1}\left(\frac{x}{3}\right) = \ln\left|\frac{x}{3} + \sqrt{\left(\frac{x}{3}\right)^2 - 1}\right|,$$

so our two answers agree.

35. Prove the reduction formula for $n \geq 2$:

$$\int \cosh^n x\, dx = \frac{1}{n}\cosh^{n-1} x\sinh x + \frac{n-1}{n}\int \cosh^{n-2} x\, dx \qquad \boxed{5}$$

SOLUTION Using Integration by Parts with $u = \cosh^{n-1} x$ and $v' = \cosh x$, we have

$$\begin{aligned}
\int \cosh^n x\, dx &= \cosh^{n-1} x\sinh x - (n-1)\int \cosh^{n-2} x\sinh^2 x\, dx \\
&= \cosh^{n-1} x\sinh x - (n-1)\int \cosh^n x\, dx + (n-1)\int \cosh^{n-2} x\, dx.
\end{aligned}$$

Adding $(n-1)\int \cosh^n x\, dx$ to both sides then yields

$$n\int \cosh^n x\, dx = \cosh^{n-1} x\sinh x + (n-1)\int \cosh^{n-2} x\, dx.$$

Finally,

$$\int \cosh^n x\, dx = \frac{1}{n}\cosh^{n-1} x\sinh x + \frac{n-1}{n}\int \cosh^{n-2} x\, dx.$$

36. Use Eq. (5) to evaluate $\int \cosh^4 x\, dx$.

SOLUTION Using Eq. (5) twice,

$$\begin{aligned}
\int \cosh^4 x\, dx &= \frac{1}{4}\cosh^3 x\sinh x + \frac{3}{4}\int \cosh^2 x\, dx \\
&= \frac{1}{4}\cosh^3 x\sinh x + \frac{3}{4}\left(\frac{1}{2}\cosh x\sinh x + \frac{1}{2}\int dx\right) \\
&= \frac{1}{4}\cosh^3 x\sinh x + \frac{3}{8}\cosh x\sinh x + \frac{3}{8}x + C.
\end{aligned}$$

In Exercises 37–40, evaluate the integral.

37. $\displaystyle\int \frac{\tanh^{-1} x\,dx}{x^2-1}$

SOLUTION Let $u = \tanh^{-1} x$. Then $du = \dfrac{1}{1-x^2}\,dx = -\dfrac{1}{x^2-1}\,dx$ and

$$\int \frac{\tanh^{-1} x}{x^2-1}\,dx = -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\left(\tanh^{-1} x\right)^2 + C.$$

38. $\displaystyle\int \sinh^{-1} x\,dx$

SOLUTION Using Integration by Parts with $u = \sinh^{-1} x$ and $v' = 1$,

$$\int \sinh^{-1} x\,dx = x\sinh^{-1} x - \int \frac{x}{\sqrt{x^2+1}}\,dx = x\sinh^{-1} x - \sqrt{x^2+1} + C.$$

39. $\displaystyle\int \tanh^{-1} x\,dx$

SOLUTION Using Integration by Parts with $u = \tanh^{-1} x$ and $v' = 1$,

$$\int \tanh^{-1} x\,dx = x\tanh^{-1} x - \int \frac{x}{1-x^2}\,dx = x\tanh^{-1} x + \frac{1}{2}\ln|1-x^2| + C.$$

40. $\displaystyle\int x\tanh^{-1} x\,dx$

SOLUTION Using Integration by Parts with $u = \tanh^{-1} x$ and $v' = x$,

$$\int x\tanh^{-1} x\,dx = \frac{1}{2}x^2\tanh^{-1} x - \frac{1}{2}\int \frac{x^2}{1-x^2}\,dx = \frac{1}{2}x^2\tanh^{-1} x - \frac{1}{2}\int \left(\frac{1}{1-x^2} - 1\right)dx$$
$$= \frac{1}{2}x^2\tanh^{-1} x - \frac{1}{2}\tanh^{-1} x + \frac{1}{2}x + C.$$

Further Insights and Challenges

41. Show that if $u = \tanh(x/2)$, then

$$\cosh x = \frac{1+u^2}{1-u^2}, \qquad \sinh x = \frac{2u}{1-u^2}, \qquad dx = \frac{2du}{1-u^2}$$

Hint: For the first relation, use the identities

$$\sinh^2\left(\frac{x}{2}\right) = \frac{1}{2}(\cosh x - 1), \qquad \cosh^2\left(\frac{x}{2}\right) = \frac{1}{2}(\cosh x + 1)$$

SOLUTION Let $u = \tanh(x/2)$. Then

$$u = \frac{\sinh(x/2)}{\cosh(x/2)} = \sqrt{\frac{\cosh x - 1}{\cosh x + 1}}.$$

Solving for $\cosh x$ yields

$$\cosh x = \frac{1+u^2}{1-u^2}.$$

Next,

$$\sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\frac{(1+u^2)^2 - (1-u^2)^2}{(1-u^2)^2}} = \frac{2u}{1-u^2}.$$

Finally, if $u = \tanh(x/2)$, then $x = 2\tanh^{-1} u$ and

$$dx = \frac{2\,du}{1-u^2}.$$

Exercises 42 and 43: evaluate using the substitution of Exercise 41.

42. $\int \operatorname{sech} x\, dx$

SOLUTION Let $u = \tanh(x/2)$. Then, by Exercise 41,

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{1-u^2}{1+u^2} \quad \text{and} \quad dx = \frac{2\,du}{1-u^2},$$

so

$$\int \operatorname{sech} x\, dx = 2\int \frac{du}{1+u^2} = 2\tan^{-1} u + C = 2\tan^{-1}\left(\tanh\frac{x}{2}\right) + C.$$

43. $\int \frac{dx}{1+\cosh x}$

SOLUTION Let $u = \tanh(x/2)$. Then, by Exercise 41,

$$1+\cosh x = 1 + \frac{1+u^2}{1-u^2} = \frac{2}{1-u^2} \quad \text{and} \quad dx = \frac{2\,du}{1-u^2},$$

so

$$\int \frac{dx}{1+\cosh x} = \int du = u + C = \tanh\frac{x}{2} + C.$$

44. Suppose that $y = f(x)$ satisfies $y'' = y$. Prove:
(a) $f(x)^2 - (f'(x))^2$ is constant.
(b) If $f(0) = f'(0) = 0$, then $f(x)$ is the zero function.
(c) $f(x) = f(0)\cosh x + f'(0)\sinh x$.

SOLUTION
(a)

$$\frac{d}{dx}\left[f(x)^2 - (f'(x))^2\right] = 2f(x)f'(x) - 2f'(x)f''(x) = 2f(x)f'(x) - 2f'(x)f(x) = 0$$

so that $f(x)^2 - (f'(x))^2$ must be constant, since it has zero derivative everywhere.
(b) If $f(0) = f'(0) = 0$, then part (a) implies that $f(x)^2 - (f'(x))^2$ is the zero function, since it is constant and vanishes at 0. Thus $f(x) = \pm f'(x)$. But Theorem 1 in Section 5.8 states that the only function $y = f(x)$ with $y' = ky$ is $y = Ce^{kx}$; thus either $f(x) = Ce^x$ or $f(x) = Ce^{-x}$. But in either case, $f(0) = C = 0$, so we must have $C = 0$ and $f(x)$ is the zero function.
(c) Let $g(x) = f(x) - f(0)\cosh x - f'(0)\sinh x$. Then

$$\begin{aligned} g'(x) &= f'(x) - f(0)(\cosh x)' - f'(0)(\sinh x)' = f'(x) - f(0)\sinh x - f'(0)\cosh x \\ g''(x) &= f''(x) - f(0)(\sinh x)' - f'(0)(\cosh x)' = f''(x) - f(0)\cosh x - f'(0)\sinh x \\ &= f(x) - f(0)\cosh x - f'(0)\sinh x = g(x) \end{aligned}$$

since $f''(x) = f(x)$. But also

$$\begin{aligned} g(0) &= f(0) - f(0)\cosh 0 - f'(0)\sinh 0 = f(0) - f(0) = 0 \\ g'(0) &= f'(0) - f(0)\sinh 0 - f'(0)\cosh 0 = f'(0) - f'(0) = 0 \end{aligned}$$

Thus $g(x)$ satisfies the conditions the problem, and in particular of part (b) [replace f by g], so that $g(x)$ must be the zero function. But this means that $f(x) - f(0)\cosh x - f'(0)\sinh x = 0$ so that

$$f(x) = f(0)\cosh x + f'(0)\sinh x$$

Exercises 45–48 refer to the function $gd(y) = \tan^{-1}(\sinh y)$, called the ***gudermannian****. In a map of the earth constructed by Mercator projection, points located y radial units from the equator correspond to points on the globe of latitude $gd(y)$.*

45. Prove that $\frac{d}{dy} gd(y) = \operatorname{sech} y$.

SOLUTION Let $gd(y) = \tan^{-1}(\sinh y)$. Then

$$\frac{d}{dy} gd(y) = \frac{1}{1+\sinh^2 y}\cosh y = \frac{1}{\cosh y} = \operatorname{sech} y,$$

where we have used the identity $1 + \sinh^2 y = \cosh^2 y$.

46. Let $f(y) = 2\tan^{-1}(e^y) - \pi/2$. Prove that $gd(y) = f(y)$. *Hint:* Show that $gd'(y) = f'(y)$ and $f(0) = g(0)$.

SOLUTION Let $f(y) = 2\tan^{-1}\left(e^y\right) - \frac{\pi}{2}$. Then

$$f'(y) = \frac{2e^y}{1+e^{2y}} = \frac{2}{e^{-y}+e^y} = \frac{1}{\frac{e^y+e^{-y}}{2}} = \frac{1}{\cosh y} = \operatorname{sech} y.$$

In the previous exercise we found that $\frac{d}{dy}gd(y) = \operatorname{sech} y$; therefore, $gd'(y) = f'(y)$. Now, since the two functions have equal derivatives, they differ by a constant; that is,

$$gd(y) = f(y) + C.$$

To find C we substitute $y = 0$:

$$\begin{aligned} \tan^{-1}(\sinh 0) &= 2\tan^{-1}(e^0) - \frac{\pi}{2} + C \\ \tan^{-1}0 &= 2\tan^{-1}(1) - \frac{\pi}{2} + C \\ 0 &= 2\cdot\frac{\pi}{4} - \frac{\pi}{2} + C \\ C &= 0. \end{aligned}$$

Therefore,

$$gd(y) = f(y).$$

47. Let $t(y) = \sinh^{-1}(\tan y)$. Show that $t(y)$ is the inverse of $gd(y)$ for $0 \le y < \pi/2$.

SOLUTION Let $x = gd(y) = \tan^{-1}(\sinh y)$. Solving for y yields $y = \sinh^{-1}(\tan x)$. Therefore,

$$gd^{-1}(y) = \sinh^{-1}(\tan y).$$

48. Verify that $t(y)$ in Exercise 47 satisfies $t'(y) = \sec y$, and find a value of a such that

$$t(y) = \int_a^y \frac{dt}{\cos t}$$

SOLUTION Let $t(y) = \sinh^{-1}(\tan y)$. Then

$$t'(y) = \frac{1}{\cos^2 y\sqrt{\tan^2 y + 1}} = \frac{1}{\cos^2 y\sqrt{\frac{1}{\cos^2 y}}} = \frac{1}{\cos^2 y \cdot \frac{1}{|\cos y|}} = \frac{1}{|\cos y|} = |\sec y|.$$

For $0 \le y < \frac{\pi}{2}$, $\sec y > 0$; therefore $t'(y) = \sec y$. Integrating this last relation yields

$$t(y) - t(a) = \int_a^y \frac{1}{\cos t}\,dt.$$

For this to be of the desired form, we must have $t(a) = \sinh^{-1}(\tan a) = 0$. The only value for a that satisfies this equation is $a = 0$.

49. The relations $\cosh(it) = \cos t$ and $\sinh(it) = i\sin t$ were discussed in the Excursion. Use these relations to show that the identity $\cos^2 t + \sin^2 t = 1$ results from setting $x = it$ in the identity $\cosh^2 x - \sinh^2 x = 1$.

SOLUTION Let $x = it$. Then

$$\cosh^2 x = (\cosh(it))^2 = \cos^2 t$$

and

$$\sinh^2 x = (\sinh(it))^2 = i^2\sin^2 t = -\sin^2 t.$$

Thus,

$$1 = \cosh^2(it) - \sinh^2(it) = \cos^2 t - (-\sin^2 t) = \cos^2 t + \sin^2 t,$$

as desired.

7.5 The Method of Partial Fractions

Preliminary Questions

1. Suppose that $\int f(x)\,dx = \ln x + \sqrt{x+1} + C$. Can $f(x)$ be a rational function? Explain.

SOLUTION No, $f(x)$ cannot be a rational function because the integral of a rational function cannot contain a term with a non-integer exponent such as $\sqrt{x+1}$.

2. Which of the following are *proper* rational functions?

(a) $\dfrac{x}{x-3}$

(b) $\dfrac{4}{9-x}$

(c) $\dfrac{x^2+12}{(x+2)(x+1)(x-3)}$

(d) $\dfrac{4x^3-7x}{(x-3)(2x+5)(9-x)}$

SOLUTION

(a) No, this is not a proper rational function because the degree of the numerator is not less than the degree of the denominator.

(b) Yes, this is a proper rational function.

(c) Yes, this is a proper rational function.

(d) No, this is not a proper rational function because the degree of the numerator is not less than the degree of the denominator.

3. Which of the following quadratic polynomials are irreducible? To check, complete the square if necessary.

(a) x^2+5

(b) x^2-5

(c) x^2+4x+6

(d) x^2+4x+2

SOLUTION

(a) Square is already completed; irreducible.

(b) Square is already completed; factors as $(x-\sqrt{5})(x+\sqrt{5})$.

(c) $x^2+4x+6=(x+2)^2+2$; irreducible.

(d) $x^2+4x+2=(x+2)^2-2$; factors as $(x+2-\sqrt{2})(x+2+\sqrt{2})$.

4. Let $P(x)/Q(x)$ be a proper rational function where $Q(x)$ factors as a product of distinct linear factors $(x-a_i)$. Then

$$\int \frac{P(x)\,dx}{Q(x)}$$

(choose the correct answer):

(a) is a sum of logarithmic terms $A_i \ln(x-a_i)$ for some constants A_i.

(b) may contain a term involving the arctangent.

SOLUTION The correct answer is **(a)**: the integral is a sum of logarithmic terms $A_i \ln(x-a_i)$ for some constants A_i.

Exercises

1. Match the rational functions (a)–(d) with the corresponding partial fraction decompositions (i)–(iv).

(a) $\dfrac{x^2+4x+12}{(x+2)(x^2+4)}$

(b) $\dfrac{2x^2+8x+24}{(x+2)^2(x^2+4)}$

(c) $\dfrac{x^2-4x+8}{(x-1)^2(x-2)^2}$

(d) $\dfrac{x^4-4x+8}{(x+2)(x^2+4)}$

(i) $x-2+\dfrac{4}{x+2}-\dfrac{4x-4}{x^2+4}$

(ii) $\dfrac{-8}{x-2}+\dfrac{4}{(x-2)^2}+\dfrac{8}{x-1}+\dfrac{5}{(x-1)^2}$

(iii) $\dfrac{1}{x+2}+\dfrac{2}{(x+2)^2}+\dfrac{-x+2}{x^2+4}$

(iv) $\dfrac{1}{x+2}+\dfrac{4}{x^2+4}$

SOLUTION

(a) $\dfrac{x^2+4x+12}{(x+2)(x^2+4)} = \dfrac{1}{x+2} + \dfrac{4}{x^2+4}$.

(b) $\dfrac{2x^2+8x+24}{(x+2)^2(x^2+4)} = \dfrac{1}{x+2} + \dfrac{2}{(x+2)^2} + \dfrac{-x+2}{x^2+4}$.

(c) $\dfrac{x^2-4x+8}{(x-1)^2(x-2)^2} = \dfrac{-8}{x-2} + \dfrac{4}{(x-2)^2} + \dfrac{8}{x-1} + \dfrac{5}{(x-1)^2}$.

(d) $\dfrac{x^4-4x+8}{(x+2)(x^2+4)} = x-2 + \dfrac{4}{x+2} - \dfrac{4x-4}{x^2+4}$.

2. Determine the constants A, B:

$$\frac{2x-3}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

SOLUTION Clearing denominators gives

$$2x-3 = A(x-4) + B(x-3).$$

Setting $x=4$ then yields

$$8-3 = A(0) + B(1) \qquad \text{or} \qquad B = 5,$$

while setting $x=3$ yields

$$6-3 = A(-1) + 0 \qquad \text{or} \qquad A = -3.$$

3. Clear denominators in the following partial fraction decomposition and determine the constant B (substitute a value of x or use the method of undetermined coefficients).

$$\frac{3x^2+11x+12}{(x+1)(x+3)^2} = \frac{1}{x+1} - \frac{B}{x+3} - \frac{3}{(x+3)^2}$$

SOLUTION Clearing denominators gives

$$3x^2+11x+12 = (x+3)^2 - B(x+1)(x+3) - 3(x+1).$$

Setting $x=0$ then yields

$$12 = 9 - B(1)(3) - 3(1) \qquad \text{or} \qquad B = -2.$$

To use the method of undetermined coefficients, expand the right-hand side and gather like terms:

$$3x^2+11x+12 = (1-B)x^2 + (3-4B)x + (6-3B).$$

Equating x^2-coefficients on both sides, we find

$$3 = 1 - B \qquad \text{or} \qquad B = -2.$$

4. Find the constants in the partial fraction decomposition

$$\frac{2x+4}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

SOLUTION Clearing denominators gives

$$2x+4 = A(x^2+4) + (Bx+C)(x-2).$$

Setting $x=2$ then yields

$$4+4 = A(4+4) + 0 \qquad \text{or} \qquad A = 1.$$

To find B and C, expand the right side, gather like terms, and use the method of undetermined coefficients:

$$2x+4 = (B+1)x^2 + (-2B+C)x + (4-2C).$$

Equating x^2-coefficients, we find

$$0 = B+1 \qquad \text{or} \qquad B = -1,$$

while equating constants yields

$$4 = 4 - 2C \qquad \text{or} \qquad C = 0.$$

Thus, $A = 1$, $B = -1$, $C = 0$.

In Exercises 5–8, evaluate using long division first to write $f(x)$ as the sum of a polynomial and a proper rational function.

5. $\displaystyle\int \frac{x\,dx}{3x-4}$

SOLUTION Long division gives us

$$\frac{x}{3x-4} = \frac{1}{3} + \frac{4/3}{3x-4}$$

Therefore the integral is

$$\int \frac{x}{3x-4}\,dx = \int \frac{1}{3} - \frac{4}{9x-12}\,dx = \frac{1}{3}x - \frac{4}{9}\ln|9x-12| + C$$

6. $\displaystyle\int \frac{(x^2+2)\,dx}{x+3}$

SOLUTION Long division gives us

$$\frac{x^2+2}{x+3} = x - 3 + \frac{11}{x+3}.$$

Therefore the integral is

$$\int \frac{x^2+2}{x+3}\,dx = \int (x-3)\,dx + 11\int \frac{dx}{x+3} = \frac{x^2}{2} - 3x + 11\ln|x+3| + C.$$

7. $\displaystyle\int \frac{(x^3+2x^2+1)\,dx}{x+2}$

SOLUTION Long division gives us

$$\frac{x^3+2x^2+1}{x+2} = x^2 + \frac{1}{x+2}$$

Therefore the integral is

$$\int \frac{x^3+2x^2+1}{x+2}\,dx = \int x^2 + \frac{1}{x+2}\,dx = \frac{1}{3}x^3 + \ln|x+2| + C$$

8. $\displaystyle\int \frac{(x^3+1)\,dx}{x^2+1}$

SOLUTION Long division gives

$$\frac{x^3+1}{x^2+1} = x - \frac{x-1}{x^2+1}$$

Therefore the integral is

$$\begin{aligned}\int \frac{x^3+1}{x^2+1}\,dx &= \int x - \frac{x-1}{x^2+1}\,dx = \frac{1}{2}x^2 - \int \frac{x}{x^2+1}\,dx + \frac{1}{x^2+1}\,dx \\ &= \frac{1}{2}x^2 - \frac{1}{2}\int \frac{2x\,dx}{x^2+1} + \frac{1}{x^2+1}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C\end{aligned}$$

In Exercises 9–44, evaluate the integral.

9. $\displaystyle\int \frac{dx}{(x-2)(x-4)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4}.$$

Clearing denominators gives us

$$1 = A(x-4) + B(x-2).$$

Setting $x = 2$ then yields

$$1 = A(2-4) + 0 \qquad \text{or} \qquad A = -\frac{1}{2},$$

while setting $x = 4$ yields

$$1 = 0 + B(4-2) \qquad \text{or} \qquad B = \frac{1}{2}.$$

The result is:

$$\frac{1}{(x-2)(x-4)} = \frac{-\frac{1}{2}}{x-2} + \frac{\frac{1}{2}}{x-4}.$$

Thus,

$$\int \frac{dx}{(x-2)(x-4)} = -\frac{1}{2}\int \frac{dx}{x-2} + \frac{1}{2}\int \frac{dx}{x-4} = -\frac{1}{2}\ln|x-2| + \frac{1}{2}\ln|x-4| + C.$$

10. $\displaystyle\int \frac{(x+3)\,dx}{x+4}$

SOLUTION Start with long division:

$$\frac{x+3}{x+4} = 1 - \frac{1}{x+4}$$

so that

$$\int \frac{x+3}{x+4}\,dx = \int 1 - \frac{1}{x+4}\,dx = x - \ln|x+4| + C$$

11. $\displaystyle\int \frac{dx}{x(2x+1)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}.$$

Clearing denominators gives us

$$1 = A(2x+1) + Bx.$$

Setting $x = 0$ then yields

$$1 = A(1) + 0 \qquad \text{or} \qquad A = 1,$$

while setting $x = -\frac{1}{2}$ yields

$$1 = 0 + B\left(-\frac{1}{2}\right) \qquad \text{or} \qquad B = -2.$$

The result is:

$$\frac{1}{x(2x+1)} = \frac{1}{x} + \frac{-2}{2x+1}.$$

Thus,

$$\int \frac{dx}{x(2x+1)} = \int \frac{dx}{x} - \int \frac{2\,dx}{2x+1} = \ln|x| - \ln|2x+1| + C.$$

For the integral on the right, we have used the substitution $u = 2x+1$, $du = 2\,dx$.

12. $\displaystyle\int \frac{(2x-1)\,dx}{x^2-5x+6}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{2x-1}{x^2-5x+6} = \frac{2x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Clearing denominators gives us

$$2x-1 = A(x-3) + B(x-2).$$

Setting $x = 2$ then yields

$$3 = A(-1) + 0 \qquad \text{or} \qquad A = -3,$$

while setting $x = 3$ yields

$$5 = 0 + B(1) \qquad \text{or} \qquad B = 5.$$

The result is:

$$\frac{2x-1}{x^2-5x+6} = \frac{-3}{x-2} + \frac{5}{x-3}.$$

Thus,

$$\int \frac{(2x-1)\,dx}{x^2-5x+6} = -3\int \frac{dx}{x-2} + 5\int \frac{dx}{x-3} = -3\ln|x-2| + 5\ln|x-3| + C.$$

13. $\displaystyle\int \frac{x^2\,dx}{x^2+9}$

SOLUTION

$$\int \frac{x^2}{x^2+9}\,dx = \int 1 - \frac{9}{x^2+9}\,dx = x - 3\tan^{-1}\left(\frac{x}{3}\right) + C$$

14. $\displaystyle\int \frac{dx}{(x-2)(x-3)(x+2)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x-2)(x-3)(x+2)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x+2}.$$

Clearing denominators gives us

$$1 = A(x-3)(x+2) + B(x-2)(x+2) + C(x-2)(x-3).$$

Setting $x = 2$ then yields

$$1 = A(-1)(4) + 0 + 0 \qquad \text{or} \qquad A = -\frac{1}{4},$$

while setting $x = 3$ yields

$$1 = 0 + B(1)(5) + 0 \qquad \text{or} \qquad B = \frac{1}{5},$$

and setting $x = -2$ yields

$$1 = 0 + 0 + C(-4)(-5) \qquad \text{or} \qquad C = \frac{1}{20}.$$

The result is:

$$\frac{1}{(x-2)(x-3)(x+2)} = \frac{-\frac{1}{4}}{x-2} + \frac{\frac{1}{5}}{x-3} + \frac{\frac{1}{20}}{x+2}.$$

Thus,

$$\begin{aligned}\int \frac{dx}{(x-2)(x-3)(x+2)} &= -\frac{1}{4}\int \frac{dx}{x-2} + \frac{1}{5}\int \frac{dx}{x-3} + \frac{1}{20}\int \frac{dx}{x+2} \\ &= -\frac{1}{4}\ln|x-2| + \frac{1}{5}\ln|x-3| + \frac{1}{20}\ln|x+2| + C.\end{aligned}$$

15. $\displaystyle\int \frac{(x^2+3x-44)\,dx}{(x+3)(x+5)(3x-2)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{x^2+3x-44}{(x+3)(x+5)(3x-2)} = \frac{A}{x+3} + \frac{B}{x+5} + \frac{C}{3x-2}.$$

Clearing denominators gives us

$$x^2+3x-44 = A(x+5)(3x-2) + B(x+3)(3x-2) + C(x+3)(x+5).$$

Setting $x=-3$ then yields

$$9-9-44 = A(2)(-11)+0+0 \qquad \text{or} \qquad A=2,$$

while setting $x=-5$ yields

$$25-15-44 = 0+B(-2)(-17)+0 \qquad \text{or} \qquad B=-1,$$

and setting $x=\frac{2}{3}$ yields

$$\frac{4}{9}+2-44 = 0+0+C\left(\frac{11}{3}\right)\left(\frac{17}{3}\right) \qquad \text{or} \qquad C=-2.$$

The result is:

$$\frac{x^2+3x-44}{(x+3)(x+5)(3x-2)} = \frac{2}{x+3} + \frac{-1}{x+5} + \frac{-2}{3x-2}.$$

Thus,

$$\int \frac{(x^2+3x-44)\,dx}{(x+3)(x+5)(3x-2)} = 2\int \frac{dx}{x+3} - \int \frac{dx}{x+5} - 2\int \frac{dx}{3x-2}$$
$$= 2\ln|x+3| - \ln|x+5| - \frac{2}{3}\ln|3x-2| + C.$$

To evaluate the last integral, we have made the substitution $u=3x-2$, $du=3\,dx$.

16. $\displaystyle\int \frac{3\,dx}{(x+1)(x^2+x)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{3}{(x+1)(x^2+x)} = \frac{3}{(x+1)(x)(x+1)} = \frac{3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Clearing denominators gives us

$$3 = A(x+1)^2 + Bx(x+1) + Cx.$$

Setting $x=0$ then yields

$$3 = A(1)+0+0 \qquad \text{or} \qquad A=3,$$

while setting $x=-1$ yields

$$3 = 0+0+C(-1) \qquad \text{or} \qquad C=-3.$$

Now plug in $A=3$ and $C=-3$:

$$3 = 3(x+1)^2 + Bx(x+1) - 3x.$$

The constant B can be determined by plugging in for x any value other than 0 or -1. Plugging in $x=1$ gives us

$$3 = 3(4) + B(1)(2) - 3 \qquad \text{or} \qquad B=-3.$$

The result is

$$\frac{3}{(x+1)(x^2+x)} = \frac{3}{x} + \frac{-3}{x+1} + \frac{-3}{(x+1)^2}.$$

Thus,

$$\int \frac{3\,dx}{(x+1)(x^2+x)} = 3\int \frac{dx}{x} - 3\int \frac{dx}{x+1} - 3\int \frac{dx}{(x+1)^2} = 3\ln|x| - 3\ln|x+1| + \frac{3}{x+1} + C.$$

17. $\displaystyle\int \frac{(x^2+11x)\,dx}{(x-1)(x+1)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{x^2+11x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Clearing denominators gives us

$$x^2+11x = A(x+1)^2 + B(x-1)(x+1) + C(x-1).$$

Setting $x=1$ then yields

$$12 = A(4)+0+0 \qquad \text{or} \qquad A=3,$$

while setting $x=-1$ yields

$$-10 = 0+0+C(-2) \qquad \text{or} \qquad C=5.$$

Plugging in these values results in

$$x^2+11x = 3(x+1)^2 + B(x-1)(x+1) + 5(x-1).$$

The constant B can be determined by plugging in for x any value other than 1 or -1. If we plug in $x=0$, we get

$$0 = 3 + B(-1)(1) + 5(-1) \qquad \text{or} \qquad B=-2.$$

The result is

$$\frac{x^2+11x}{(x-1)(x+1)^2} = \frac{3}{x-1} + \frac{-2}{x+1} + \frac{5}{(x+1)^2}.$$

Thus,

$$\int \frac{(x^2+11x)\,dx}{(x-1)(x+1)^2} = 3\int\frac{dx}{x-1} - 2\int\frac{dx}{x+1} + 5\int\frac{dx}{(x+1)^2} = 3\ln|x-1| - 2\ln|x+1| - \frac{5}{x+1} + C.$$

18. $\displaystyle\int \frac{(4x^2-21x)\,dx}{(x-3)^2(2x+3)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{4x^2-21x}{(x-3)^2(2x+3)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{2x+3}.$$

Clearing denominators gives us

$$4x^2-21x = A(x-3)(2x+3) + B(2x+3) + C(x-3)^2.$$

Setting $x=3$ then yields

$$-27 = 0 + B(9) + 0 \qquad \text{or} \qquad B=-3,$$

while setting $x=-\frac{3}{2}$ yields

$$9+\frac{63}{2} = 0+0+C\left(\frac{81}{4}\right) \qquad \text{or} \qquad C=2.$$

Plugging in these values results in

$$4x^2-21x = A(x-3)(2x+3) - 3(2x+3) + 2(x-3)^2.$$

Setting $x=0$ gives us

$$0 = A(-3)(3) - 9 + 18 \qquad \text{or} \qquad A=1.$$

The result is

$$\frac{4x^2-21x}{(x-3)^2(2x+3)} = \frac{1}{x-3} + \frac{-3}{(x-3)^2} + \frac{2}{2x+3}.$$

Thus,

$$\int \frac{(4x^2-21x)\,dx}{(x-3)^2(2x+3)} = \int\frac{dx}{x-3} - 3\int\frac{dx}{(x-3)^2} + \int\frac{2\,dx}{2x+3} = \ln|x-3| + \frac{3}{x-3} + \ln|2x+3| + C.$$

19. $\displaystyle\int \frac{dx}{(x-1)^2(x-2)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x-1)^2(x-2)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}.$$

Clearing denominators gives us

$$1 = A(x-1)(x-2)^2 + B(x-2)^2 + C(x-2)(x-1)^2 + D(x-1)^2.$$

Setting $x = 1$ then yields

$$1 = B(1) \qquad \text{or} \qquad B = 1,$$

while setting $x = 2$ yields

$$1 = D(1) \qquad \text{or} \qquad D = 1.$$

Plugging in these values gives us

$$1 = A(x-1)(x-2)^2 + (x-2)^2 + C(x-2)(x-1)^2 + (x-1)^2.$$

Setting $x = 0$ now yields

$$1 = A(-1)(4) + 4 + C(-2)(1) + 1 \qquad \text{or} \qquad -4 = -4A - 2C,$$

while setting $x = 3$ yields

$$1 = A(2)(1) + 1 + C(1)(4) + 4 \qquad \text{or} \qquad -4 = 2A + 4C.$$

Solving this system of two equations in two unknowns gives $A = 2$ and $C = -2$. The result is

$$\frac{1}{(x-1)^2(x-2)^2} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{-2}{x-2} + \frac{1}{(x-2)^2}.$$

Thus,

$$\int \frac{dx}{(x-1)^2(x-2)^2} = 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} - 2\int \frac{dx}{x-2} + \int \frac{dx}{(x-2)^2}$$
$$= 2\ln|x-1| - \frac{1}{x-1} - 2\ln|x-2| - \frac{1}{x-2} + C.$$

20. $\displaystyle\int \frac{(x^2-8x)\,dx}{(x+1)(x+4)^3}$

SOLUTION The partial fraction decomposition is

$$\frac{x^2-8x}{(x+1)(x+4)^3} = \frac{A}{x+1} + \frac{B}{x+4} + \frac{C}{(x+4)^2} + \frac{D}{(x+4)^3}$$

Clearing fractions gives

$$x^2 - 8x = A(x+4)^3 + B(x+4)^2(x+1) + C(x+4)(x+1) + D(x+1)$$

Setting $x = -4$ gives $48 = -3D$ so that $D = -16$. Setting $x = -1$ gives $9 = 27A$ so that $A = \frac{1}{3}$. Thus

$$x^2 - 8x = \frac{1}{3}(x+4)^3 + B(x+4)^2(x+1) + C(x+4)(x+1) - 16(x+1)$$

The coefficient of x^3 on the right hand side must be zero; it is $\frac{1}{3} + B$, so that $B = -\frac{1}{3}$. Finally, the constant term on the right must be zero as well; substituting the known values of A, B, and D gives for the constant term

$$\frac{1}{3}\cdot 64 - \frac{1}{3}\cdot 16 + 4C - 16 = 4C$$

so that $C = 0$, and the partial fraction decomposition is

$$\frac{x^2-8x}{(x+1)(x+4)^3} = \frac{1}{3(x+1)} - \frac{1}{3(x+4)} - \frac{16}{(x+4)^3}$$

Thus

$$\int \frac{x^2-8x}{(x+1)(x+4)^3}\,dx = \frac{1}{3}\int \frac{1}{x+1}\,dx - \frac{1}{3}\int \frac{1}{x+4}\,dx - 16\int \frac{1}{(x+4)^3}\,dx$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x+4| + 8(x+4)^{-2} + C = \frac{1}{3}\ln\left|\frac{x+1}{x+4}\right| + 8(x+4)^{-2} + C$$

21. $\int \frac{8\,dx}{x(x+2)^3}$

SOLUTION The partial fraction decomposition is

$$\frac{8}{x(x+2)^3} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

Clearing fractions gives

$$8 = A(x+2)^3 + Bx(x+2)^2 + Cx(x+2) + Dx$$

Setting $x = 0$ gives $8 = 8A$ so $A = 1$; setting $x = -2$ gives $8 = -2D$ so that $D = -4$; the result is

$$8 = (x+2)^3 + Bx(x+2)^2 + Cx(x+2) - 4x$$

The coefficient of x^3 on the right-hand side must be zero, since it is zero on the left. We compute it to be $1 + B$, so that $B = -1$. Finally, we look at the coefficient of x^2 on the right-hand side; it must be zero as well. We compute it to be

$$3 \cdot 2 - 4 + C = C + 2$$

so that $C = -2$ and the partial fraction decomposition is

$$\frac{8}{x(x+2)^3} = \frac{1}{x} - \frac{1}{x+2} - \frac{2}{(x+2)^2} - \frac{4}{(x+2)^3}$$

and

$$\int \frac{8}{x(x+2)^3}\,dx = \int \frac{1}{x}\,dx - \frac{1}{x+2}\,dx - 2\int (x+2)^{-2}\,dx - 4\int (x+2)^{-3}\,dx$$

$$= \ln|x| - \ln|x+2| + 2(x+2)^{-1} + 2(x+2)^{-2} + C = \ln\left|\frac{x}{x+2}\right| + \frac{2}{x+2} + \frac{2}{(x+2)^2} + C$$

22. $\int \frac{x^2\,dx}{x^2+3}$

SOLUTION

$$\int \frac{x^2}{x^2+3}\,dx = \int 1 - \frac{3}{x^2+3}\,dx = \int 1\,dx - 3\int \frac{1}{x^2+3}\,dx = x - \sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

23. $\int \frac{dx}{2x^2-3}$

SOLUTION The partial fraction decomposition has the form

$$\frac{1}{2x^2-3} = \frac{1}{(\sqrt{2}x-\sqrt{3})(\sqrt{2}x+\sqrt{3})} = \frac{A}{\sqrt{2}x-\sqrt{3}} + \frac{B}{\sqrt{2}x+\sqrt{3}}.$$

Clearing denominators, we get

$$1 = A\left(\sqrt{2}x+\sqrt{3}\right) + B\left(\sqrt{2}x-\sqrt{3}\right).$$

Setting $x = \sqrt{3}/\sqrt{2}$ then yields

$$1 = A\left(\sqrt{3}+\sqrt{3}\right) + 0 \qquad \text{or} \qquad A = \frac{1}{2\sqrt{3}},$$

while setting $x = -\sqrt{3}/\sqrt{2}$ yields

$$1 = 0 + B\left(-\sqrt{3}-\sqrt{3}\right) \qquad \text{or} \qquad B = \frac{-1}{2\sqrt{3}}.$$

The result is

$$\frac{1}{2x^2-3} = \frac{1/2\sqrt{3}}{\sqrt{2}x-\sqrt{3}} - \frac{1/2\sqrt{3}}{\sqrt{2}x+\sqrt{3}}.$$

Thus,

$$\int \frac{dx}{2x^2-3} = \frac{1}{2\sqrt{3}}\int \frac{dx}{\sqrt{2}x-\sqrt{3}} - \frac{1}{2\sqrt{3}}\int \frac{dx}{\sqrt{2}x+\sqrt{3}}.$$

For the first integral, let $u = \sqrt{2}x - \sqrt{3}$, $du = \sqrt{2}\,dx$, and for the second, let $w = \sqrt{2}x + \sqrt{3}$, $dw = \sqrt{2}\,dx$. Then we have

$$\int \frac{dx}{2x^2-3} = \frac{1}{2\sqrt{3}(\sqrt{2})}\int \frac{du}{u} - \frac{1}{2\sqrt{3}(\sqrt{2})}\int \frac{dw}{w} = \frac{1}{2\sqrt{6}}\ln\left|\sqrt{2}x-\sqrt{3}\right| - \frac{1}{2\sqrt{6}}\ln\left|\sqrt{2}x+\sqrt{3}\right| + C.$$

24. $\displaystyle\int \frac{dx}{(x-4)^2(x-1)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x-4)^2(x-1)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C}{(x-1)}.$$

Clearing denominators, we get

$$1 = A(x-4)(x-1) + B(x-1) + C(x-4)^2.$$

Setting $x = 1$ then yields

$$1 = 0 + 0 + C(9) \qquad \text{or} \qquad C = \frac{1}{9},$$

while setting $x = 4$ yields

$$1 = 0 + B(3) + 0 \qquad \text{or} \qquad B = \frac{1}{3}.$$

Plugging in $B = \frac{1}{3}$ and $C = \frac{1}{9}$, and setting $x = 5$, we find

$$1 = A(1)(4) + \frac{1}{3}(4) + \frac{1}{9}(1) \qquad \text{or} \qquad A = -\frac{1}{9}.$$

The result is

$$\frac{1}{(x-4)^2(x-1)} = \frac{-\frac{1}{9}}{x-4} + \frac{\frac{1}{3}}{(x-4)^2} + \frac{\frac{1}{9}}{x-1}.$$

Thus,

$$\int \frac{dx}{(x-4)^2(x-1)} = -\frac{1}{9}\int \frac{dx}{x-4} + \frac{1}{3}\int \frac{dx}{(x-4)^2} + \frac{1}{9}\int \frac{dx}{x-1}$$
$$= -\frac{1}{9}\ln|x-4| - \frac{1}{3(x-4)} + \frac{1}{9}\ln|x-1| + C.$$

25. $\displaystyle\int \frac{4x^2-20}{(2x+5)^3}\,dx$

SOLUTION The partial fraction decomposition is

$$\frac{4x^2-20}{(2x+5)^3} = \frac{A}{2x+5} + \frac{B}{(2x+5)^2} + \frac{C}{(2x+5)^3}$$

Clearing fractions gives

$$4x^2 - 20 = A(2x+5)^2 + B(2x+5) + C$$

Setting $x = -5/2$ gives $5 = C$ so that $C = 5$. The coefficient of x^2 on the left-hand side is 4, and on the right-hand side is $4A$, so that $A = 1$ and we have

$$4x^2 - 20 = (2x+5)^2 + B(2x+5) + 5$$

Considering the constant terms now gives $-20 = 25 + 5B + 5$ so that $B = -10$. Thus

$$\int \frac{4x^2 - 20}{(2x+5)^3} = \int \frac{1}{2x+5}\,dx - 10\int \frac{1}{(2x+5)^2}\,dx + 5\int \frac{1}{(2x+5)^3}\,dx$$
$$= \frac{1}{2}\ln|2x+5| + \frac{5}{2x+5} - \frac{5}{4(2x+5)^2} + C$$

26. $\displaystyle\int \frac{3x+6}{x^2(x-1)(x-3)}\,dx$

SOLUTION The partial fraction decomposition has the form:

$$\frac{3x+6}{x^2(x-1)(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x-3}.$$

Clearing denominators gives us

$$3x + 6 = Ax(x-1)(x-3) + B(x-1)(x-3) + Cx^2(x-3) + Dx^2(x-1).$$

Setting $x = 0$, then yields

$$6 = 0 + B(-1)(-3) + 0 + 0 \qquad \text{or} \qquad B = 2,$$

while setting $x = 1$ yields

$$9 = 0 + 0 + C(1)(-2) + 0 \qquad \text{or} \qquad C = -\frac{9}{2},$$

and setting $x = 3$ yields

$$15 = 0 + 0 + 0 + D(9)(2) \qquad \text{or} \qquad D = \frac{5}{6}.$$

In order to find A, let's look at the x^3-coefficient on the right-hand side (which must equal 0, since there's no x^3 term on the left):

$$0 = A + C + D = A - \frac{9}{2} + \frac{5}{6}, \qquad \text{so} \qquad A = \frac{11}{3}.$$

The result is

$$\frac{3x+6}{x^2(x-1)(x-3)} = \frac{\frac{11}{3}}{x} + \frac{2}{x^2} + \frac{-\frac{9}{2}}{x-1} + \frac{\frac{5}{6}}{x-3}.$$

Thus,

$$\int \frac{(3x+6)\,dx}{x^2(x-1)(x-3)} = \frac{11}{3}\int \frac{dx}{x} + 2\int \frac{dx}{x^2} - \frac{9}{2}\int \frac{dx}{x-1} + \frac{5}{6}\int \frac{dx}{x-3}$$
$$= \frac{11}{3}\ln|x| - \frac{2}{x} - \frac{9}{2}\ln|x-1| + \frac{5}{6}\ln|x-3| + C.$$

27. $\displaystyle\int \frac{dx}{x(x-1)^3}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}.$$

Clearing denominators, we get

$$1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx.$$

Setting $x = 0$ then yields

$$1 = A(-1) + 0 + 0 + 0 \qquad \text{or} \qquad A = -1,$$

while setting $x = 1$ yields

$$1 = 0 + 0 + 0 + D(1) \qquad \text{or} \qquad D = 1.$$

Plugging in $A = -1$ and $D = 1$ gives us

$$1 = -(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + x.$$

Now, setting $x = 2$ yields

$$1 = -1 + 2B + 2C + 2 \qquad \text{or} \qquad 2B + 2C = 0,$$

and setting $x = 3$ yields

$$1 = -8 + 12B + 6C + 3 \qquad \text{or} \qquad 2B + C = 1.$$

Solving these two equations in two unknowns, we find $B = 1$ and $C = -1$. The result is

$$\frac{1}{x(x-1)^3} = \frac{-1}{x} + \frac{1}{x-1} + \frac{-1}{(x-1)^2} + \frac{1}{(x-1)^3}.$$

Thus,

$$\int \frac{dx}{x(x-1)^3} = -\int \frac{dx}{x} + \int \frac{dx}{x-1} - \int \frac{dx}{(x-1)^2} + \int \frac{dx}{(x-1)^3}$$
$$= -\ln|x| + \ln|x-1| + \frac{1}{x-1} - \frac{1}{2(x-1)^2} + C.$$

28. $\displaystyle\int \frac{(3x^2-2)\,dx}{x-4}$

SOLUTION First we use long division to write

$$\frac{3x^2-2}{x-4} = 3x + 12 + \frac{46}{x-4}.$$

Then the integral becomes

$$\int \frac{(3x^2-2)\,dx}{x-4} = \int (3x+12)\,dx + 46\int \frac{dx}{x-4} = \frac{3}{2}x^2 + 12x + 46\ln|x-4| + C.$$

29. $\displaystyle\int \frac{(x^2-x+1)\,dx}{x^2+x}$

SOLUTION First use long division to write

$$\frac{x^2-x+1}{x^2+x} = 1 + \frac{-2x+1}{x^2+x} = 1 + \frac{-2x+1}{x(x+1)}.$$

The partial fraction decomposition of the term on the right has the form:

$$\frac{-2x+1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}.$$

Clearing denominators gives us

$$-2x + 1 = A(x+1) + Bx.$$

Setting $x = 0$ then yields

$$1 = A(1) + 0 \qquad \text{or} \qquad A = 1,$$

while setting $x = -1$ yields

$$3 = 0 + B(-1) \qquad \text{or} \qquad B = -3.$$

The result is

$$\frac{-2x+1}{x(x+1)} = \frac{1}{x} + \frac{-3}{x+1}.$$

Thus,

$$\int \frac{(x^2-x+1)\,dx}{x^2+x} = \int dx + \int \frac{dx}{x} - 3\int \frac{dx}{x+1} = x + \ln|x| - 3\ln|x+1| + C.$$

30. $\displaystyle\int \frac{dx}{x(x^2+1)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Clearing denominators, we get

$$1 = A(x^2+1) + (Bx+C)x.$$

Setting $x = 0$ then yields

$$1 = A(1) + 0 \qquad \text{or} \qquad A = 1.$$

This gives us

$$1 = x^2 + 1 + Bx^2 + Cx = (B+1)x^2 + Cx + 1.$$

Equating x^2-coefficients, we find

$$B + 1 = 0 \qquad \text{or} \qquad B = -1;$$

while equating x-coefficients yields $C = 0$. The result is

$$\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}.$$

Thus,

$$\int \frac{dx}{x(x^2+1)} = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1}.$$

For the integral on the right, use the substitution $u = x^2 + 1$, $du = 2x\,dx$. Then we have

$$\int \frac{dx}{x(x^2+1)} = \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} = \ln|x| - \frac{1}{2}\ln|x^2+1| + C.$$

31. $\displaystyle\int \frac{(3x^2-4x+5)\,dx}{(x-1)(x^2+1)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{3x^2-4x+5}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Clearing denominators, we get

$$3x^2 - 4x + 5 = A(x^2+1) + (Bx+C)(x-1).$$

Setting $x = 1$ then yields

$$3 - 4 + 5 = A(2) + 0 \qquad \text{or} \qquad A = 2.$$

This gives us

$$3x^2 - 4x + 5 = 2(x^2+1) + (Bx+C)(x-1) = (B+2)x^2 + (C-B)x + (2-C).$$

Equating x^2-coefficients, we find

$$3 = B + 2 \qquad \text{or} \qquad B = 1;$$

while equating constant coefficients yields

$$5 = 2 - C \qquad \text{or} \qquad C = -3.$$

The result is

$$\frac{3x^2-4x+5}{(x-1)(x^2+1)} = \frac{2}{x-1} + \frac{x-3}{x^2+1}.$$

Thus,

$$\int \frac{(3x^2-4x+5)\,dx}{(x-1)(x^2+1)} = 2\int \frac{dx}{x-1} + \int \frac{(x-3)\,dx}{x^2+1} = 2\int \frac{dx}{x-1} + \int \frac{x\,dx}{x^2+1} - 3\int \frac{dx}{x^2+1}.$$

For the second integral, use the substitution $u = x^2+1$, $du = 2x\,dx$. The final answer is

$$\int \frac{(3x^2-4x+5)\,dx}{(x-1)(x^2+1)} = 2\ln|x-1| + \frac{1}{2}\ln|x^2+1| - 3\tan^{-1}x + C.$$

32. $\displaystyle\int \frac{x^2}{(x+1)(x^2+1)}\,dx$

SOLUTION The partial fraction decomposition has the form

$$\frac{x^2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

Clearing denominators, we get

$$x^2 = A(x^2+1) + (Bx+C)(x+1).$$

Setting $x=-1$ then yields

$$1 = A(2)+0 \qquad \text{or} \qquad A = \frac{1}{2}.$$

This gives us

$$x^2 = \frac{1}{2}x^2 + \frac{1}{2} + Bx^2 + Bx + Cx + C = \left(B+\frac{1}{2}\right)x^2 + (B+C)x + \left(C+\frac{1}{2}\right).$$

Equating x^2-coefficients, we find

$$1 = B + \frac{1}{2} \qquad \text{or} \qquad B = \frac{1}{2},$$

while equating constant coefficients yields

$$0 = C + \frac{1}{2} \qquad \text{or} \qquad C = -\frac{1}{2}.$$

The result is

$$\frac{x^2}{(x+1)(x^2+1)} = \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x - \frac{1}{2}}{x^2+1}.$$

Thus,

$$\int \frac{x^2\,dx}{(x+1)(x^2+1)} = \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{(x-1)\,dx}{x^2+1} = \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{x\,dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$
$$= \frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + C.$$

Here we used $u = x^2+1$, $du = 2x\,dx$ for the second integral.

33. $\displaystyle\int \frac{dx}{x(x^2+25)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x(x^2+25)} = \frac{A}{x} + \frac{Bx+C}{x^2+25}.$$

Clearing denominators, we get

$$1 = A(x^2+25) + (Bx+C)x.$$

Setting $x=0$ then yields

$$1 = A(25)+0 \qquad \text{or} \qquad A = \frac{1}{25}.$$

This gives us

$$1 = \frac{1}{25}x^2 + 1 + Bx^2 + Cx = \left(B + \frac{1}{25}\right)x^2 + Cx + 1.$$

Equating x^2-coefficients, we find

$$0 = B + \frac{1}{25} \qquad \text{or} \qquad B = -\frac{1}{25},$$

while equating x-coefficients yields $C = 0$. The result is

$$\frac{1}{x(x^2+25)} = \frac{\frac{1}{25}}{x} + \frac{-\frac{1}{25}x}{x^2+25}.$$

Thus,

$$\int \frac{dx}{x(x^2+25)} = \frac{1}{25}\int \frac{dx}{x} - \frac{1}{25}\int \frac{x\,dx}{x^2+25}.$$

For the integral on the right, use $u = x^2 + 25$, $du = 2x\,dx$. Then we have

$$\int \frac{dx}{x(x^2+25)} = \frac{1}{25}\ln|x| - \frac{1}{50}\ln|x^2+25| + C.$$

34. $\displaystyle\int \frac{dx}{x^2(x^2+25)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x^2(x^2+25)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+25}.$$

Clearing denominators, we get

$$1 = Ax(x^2+25) + B(x^2+25) + (Cx+D)x^2.$$

Setting $x = 0$ then yields

$$1 = 0 + B(25) + 0 \qquad \text{or} \qquad B = \frac{1}{25}.$$

This gives us

$$1 = Ax^3 + 25Ax + \frac{1}{25}x^2 + 1 + Cx^3 + Dx^2 = (A+C)x^3 + \left(D + \frac{1}{25}\right)x^2 + 25Ax + 1.$$

Equating x-coefficients yields

$$0 = 25A \qquad \text{or} \qquad A = 0,$$

while equating x^3-coefficients yields

$$0 = A + C = 0 + C \qquad \text{or} \qquad C = 0,$$

and equating x^2-coefficients yields

$$0 = D + \frac{1}{25} \qquad \text{or} \qquad D = \frac{-1}{25}.$$

The result is

$$\frac{1}{x^2(x^2+25)} = \frac{\frac{1}{25}}{x^2} + \frac{\frac{-1}{25}}{x^2+25}.$$

Thus,

$$\int \frac{dx}{x^2(x^2+25)} = \frac{1}{25}\int \frac{dx}{x^2} - \frac{1}{25}\int \frac{dx}{x^2+25} = -\frac{1}{25x} - \frac{1}{125}\tan^{-1}\left(\frac{x}{5}\right) + C.$$

35. $\displaystyle\int \frac{(6x^2+2)\,dx}{x^2+2x-3}$

SOLUTION Long division gives

$$\frac{6x^2+2}{x^2+2x-3} = 6 - \frac{12x-20}{x^2+2x-3} = 6 - \frac{12x-20}{(x+3)(x-1)}$$

The partial fraction decomposition of the second term is

$$\frac{12x-20}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

Clear fractions to get

$$12x - 20 = A(x-1) + B(x+3)$$

Set $x = 1$ to get $-8 = 4B$ so that $B = -2$. Set $x = -3$ to get $-56 = -4A$ so that $A = 14$, and we have

$$\int \frac{6x^2+2}{x^2+2x-3} = \int 6 - \frac{14}{x+3} + \frac{2}{x-1}\,dx = \int 6\,dx - 14\int \frac{1}{x+3}\,dx + 2\int \frac{1}{x-1}\,dx$$
$$= 6x - 14\ln|x+3| + 2\ln|x-1| + C$$

36. $\displaystyle\int \frac{6x^2+7x-6}{(x^2-4)(x+2)}\,dx$

SOLUTION The partial fraction decomposition has the form:

$$\frac{6x^2+7x-6}{(x^2-4)(x+2)} = \frac{6x^2+7x-6}{(x-2)(x+2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

Clearing denominators, we get

$$6x^2 + 7x - 6 = A(x+2)^2 + B(x-2)(x+2) + C(x-2).$$

Setting $x = 2$ then yields

$$24 + 14 - 6 = A(16) + 0 + 0 \qquad \text{or} \qquad A = 2,$$

while setting $x = -2$ yields

$$24 - 14 - 6 = 0 + 0 + C(-4) \qquad \text{or} \qquad C = -1.$$

This gives us

$$6x^2 + 7x - 6 = 2(x+2)^2 + B(x-2)(x+2) - (x-2).$$

Now, setting $x = 1$ yields

$$6 + 7 - 6 = 2(9) + B(-1)(3) - (-1) \qquad \text{or} \qquad B = 4.$$

The result is

$$\frac{6x^2+7x-6}{(x^2-4)(x+2)} = \frac{2}{x-2} + \frac{4}{x+2} + \frac{-1}{(x+2)^2}.$$

Thus,

$$\int \frac{(6x^2+7x-6)\,dx}{(x^2-4)(x+2)} = 2\int \frac{dx}{x-2} + 4\int \frac{dx}{x+2} - \int \frac{dx}{(x+2)^2} = 2\ln|x-2| + 4\ln|x+2| + \frac{1}{x+2} + C.$$

37. $\displaystyle\int \frac{10\,dx}{(x-1)^2(x^2+9)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{10}{(x-1)^2(x^2+9)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+9}.$$

Clearing denominators, we get

$$10 = A(x-1)(x^2+9) + B(x^2+9) + (Cx+D)(x-1)^2.$$

Setting $x = 1$ then yields

$$10 = 0 + B(10) + 0 \qquad \text{or} \qquad B = 1.$$

Expanding the right-hand side, we have

$$10 = (A+C)x^3 + (1 - A - 2C + D)x^2 + (9A + C - 2D)x + (9 - 9A + D).$$

Equating coefficients of like powers of x then yields

$$\begin{aligned} A + C &= 0 \\ 1 - A - 2C + D &= 0 \\ 9A + C - 2D &= 0 \\ 9 - 9A + D &= 10 \end{aligned}$$

From the first equation, we have $C = -A$, and from the fourth equation we have $D = 1 + 9A$. Substituting these into the second equation, we get

$$1 - A - 2(-A) + (1 + 9A) = 0 \qquad \text{or} \qquad A = -\frac{1}{5}.$$

Finally, $C = \frac{1}{5}$ and $D = -\frac{4}{5}$. The result is

$$\frac{10}{(x-1)^2(x^2+9)} = \frac{-\frac{1}{5}}{x-1} + \frac{1}{(x-1)^2} + \frac{\frac{1}{5}x - \frac{4}{5}}{x^2+9}.$$

Thus,

$$\begin{aligned} \int \frac{10\,dx}{(x-1)^2(x^2+9)} &= -\frac{1}{5}\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} + \frac{1}{5}\int \frac{x\,dx}{x^2+9} - \frac{4}{5}\int \frac{dx}{x^2+9} \\ &= -\frac{1}{5}\ln|x-1| - \frac{1}{x-1} + \frac{1}{10}\ln|x^2+9| - \frac{4}{15}\tan^{-1}\left(\frac{x}{3}\right) + C. \end{aligned}$$

38. $\displaystyle\int \frac{10\,dx}{(x+1)(x^2+9)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{10}{(x+1)(x^2+9)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9} + \frac{Dx+E}{(x^2+9)^2}.$$

Clearing denominators gives us

$$10 = A(x^2+9)^2 + (Bx+C)(x+1)(x^2+9) + (Dx+E)(x+1).$$

Setting $x = -1$ then yields

$$10 = A(100) + 0 + 0 \qquad \text{or} \qquad A = \frac{1}{10}.$$

Expanding the right-hand side, we find

$$10 = \left(B + \frac{1}{10}\right)x^4 + (B+C)x^3 + \left(9B + C + D + \frac{18}{10}\right)x^2(9B + 9C + D + E)x + \left(9C + E + \frac{81}{10}\right).$$

Equating x^4-coefficients yields

$$B + \frac{1}{10} = 0 \qquad \text{or} \qquad B = -\frac{1}{10},$$

while equating x^3-coefficients yields

$$-\frac{1}{10} + C = 0 \qquad \text{or} \qquad C = \frac{1}{10},$$

and equating x^2-coefficients yields

$$-\frac{9}{10} + \frac{1}{10} + D + \frac{18}{10} = 0 \qquad \text{or} \qquad D = -1.$$

Finally, equating constant coefficients, we find

$$10 = \frac{9}{10} + E + \frac{81}{10} \qquad \text{or} \qquad E = 1.$$

The result is

$$\frac{10}{(x+1)(x^2+9)^2} = \frac{\frac{1}{10}}{x+1} + \frac{-\frac{1}{10}x + \frac{1}{10}}{x^2+9} + \frac{-x+1}{(x^2+9)^2}.$$

Thus,

$$\int \frac{10\,dx}{(x+1)(x^2+9)^2} = \frac{1}{10}\int \frac{dx}{x+1} - \frac{1}{10}\int \frac{x\,dx}{x^2+9} + \frac{1}{10}\int \frac{dx}{x^2+9} - \int \frac{x\,dx}{(x^2+9)^2} + \int \frac{dx}{(x^2+9)^2}.$$

For the second and fourth integrals, use the substitution $u = x^2+9$, $du = 2x\,dx$. Then we have

$$\int \frac{10\,dx}{(x+1)(x^2+9)^2} = \frac{1}{10}\ln|x+1| - \frac{1}{20}\ln|x^2+9| + \frac{1}{30}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{2(x^2+9)} + \int \frac{dx}{(x^2+9)^2}.$$

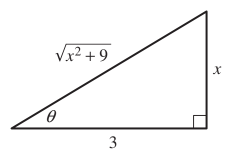
For the last integral, use the trigonometric substitution

$$x = 3\tan\theta, \qquad dx = 3\sec^2\theta\,d\theta, \qquad x^2+9 = 9\tan^2\theta + 9 = 9\sec^2\theta.$$

Then,

$$\int \frac{dx}{(x^2+9)^2} = \int \frac{3\sec^2\theta\,d\theta}{(9\sec^2\theta)^2} = \frac{1}{27}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{27}\int \cos^2\theta\,d\theta = \frac{1}{27}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] + C.$$

Now we construct a right triangle with $\tan\theta = \frac{x}{3}$:



From this we see that $\sin\theta = x/\sqrt{x^2+9}$ and $\cos\theta = 3/\sqrt{x^2+9}$. Thus

$$\int \frac{dx}{(x^2+9)^2} = \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{54}\left(\frac{x}{\sqrt{x^2+9}}\right)\left(\frac{3}{\sqrt{x^2+9}}\right) + C = \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{x}{18(x^2+9)} + C.$$

Collecting all the terms, we obtain

$$\begin{aligned}\int \frac{10\,dx}{(x+1)(x^2+9)^2} &= \frac{1}{10}\ln|x+1| - \frac{1}{20}\ln|x^2+9| + \frac{1}{30}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{2(x^2+9)} \\ &\quad + \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{x}{18(x^2+9)} + C \\ &= \frac{1}{10}\ln|x+1| - \frac{1}{20}\ln|x^2+9| + \frac{7}{135}\tan^{-1}\left(\frac{x}{3}\right) + \frac{x+9}{18(x^2+9)} + C.\end{aligned}$$

39. $\displaystyle\int \frac{dx}{x(x^2+8)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x(x^2+8)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+8} + \frac{Dx+E}{(x^2+8)^2}.$$

Clearing denominators, we get

$$1 = A(x^2+8)^2 + (Bx+C)x(x^2+8) + (Dx+E)x.$$

Expanding the right-hand side gives us

$$1 = (A+B)x^4 + Cx^3 + (16A+8B+D)x^2 + (8C+E)x + 64A.$$

Equating coefficients of like powers of x yields

$$\begin{aligned} A+B &= 0 \\ C &= 0 \\ 16A+8B+D &= 0 \\ 8C+E &= 0 \\ 64A &= 1 \end{aligned}$$

The solution to this system of equations is

$$A = \frac{1}{64}, \qquad B = -\frac{1}{64}, \qquad C = 0, \qquad D = -\frac{1}{8}, \qquad E = 0.$$

Therefore

$$\frac{1}{x(x^2+8)^2} = \frac{\frac{1}{64}}{x} + \frac{-\frac{1}{64}x}{x^2+8} + \frac{-\frac{1}{8}x}{(x^2+8)^2},$$

and

$$\int \frac{dx}{x(x^2+8)^2} = \frac{1}{64}\int \frac{dx}{x} - \frac{1}{64}\int \frac{x\,dx}{x^2+8} - \frac{1}{8}\int \frac{x\,dx}{(x^2+8)^2}.$$

For the second and third integrals, use the substitution $u = x^2 + 8$, $du = 2x\,dx$. Then we have

$$\int \frac{dx}{x(x^2+8)^2} = \frac{1}{64}\ln|x| - \frac{1}{128}\ln|x^2+8| + \frac{1}{16(x^2+8)} + C.$$

40. $\displaystyle\int \frac{100x\,dx}{(x-3)(x^2+1)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{100x}{(x-3)(x^2+1)^2} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Clearing denominators, we get

$$100x = A(x^2+1)^2 + (Bx+C)(x-3)(x^2+1) + (Dx+E)(x-3).$$

Setting $x = 3$ then yields

$$300 = A(100) + 0 + 0 \qquad \text{or} \qquad A = 3.$$

Expanding the right-hand side, we find

$$100x = (B+3)x^4 + (C-3B)x^3 + (B-3C+D+6)x^2 + (C-3B-3D+E)x + (3-3C-3E).$$

Equating coefficients of like powers of x then yields

$$\begin{aligned} B+3 &= 0 \\ C-3B &= 0 \\ B-3C+D+6 &= 0 \\ C-3B-3D+E &= 100 \\ 3-3C-3E &= 0 \end{aligned}$$

The solution to this system of equations is

$$B = -3, \qquad C = -9, \qquad D = -30, \qquad E = 10.$$

Therefore

$$\frac{100x}{(x-3)(x^2+1)^2} = \frac{3}{x-3} + \frac{-3x-9}{x^2+1} + \frac{-30x+10}{(x^2+1)^2},$$

and

$$\begin{aligned} \int \frac{100x\,dx}{(x-3)(x^2+1)^2} &= 3\int \frac{dx}{x-3} + \int \frac{(-3x-9)\,dx}{x^2+1} + \int \frac{(-30x+10)\,dx}{(x^2+1)^2} \\ &= 3\int \frac{dx}{x-3} - 3\int \frac{x\,dx}{x^2+1} - 9\int \frac{dx}{x^2+1} - 30\int \frac{x\,dx}{(x^2+1)^2} + 10\int \frac{dx}{(x^2+1)^2}. \end{aligned}$$

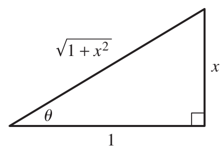
For the second and fourth integrals, use the substitution $u = x^2 + 1$, $du = 2x\,dx$. Then we have

$$\int \frac{100x\,dx}{(x-3)(x^2+1)^2} = 3\ln|x-3| - \frac{3}{2}\ln|x^2+1| - 9\tan^{-1}x + \frac{15}{x^2+1} + 10\int \frac{dx}{(x^2+1)^2}.$$

For the last integral, use the trigonometric substitution $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$. Then $x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$, and

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int \cos^2\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C.$$

We construct the following right triangle with $\tan\theta = x$:



From this we see that $\sin\theta = x/\sqrt{1+x^2}$ and $\cos\theta = 1/\sqrt{1+x^2}$. Thus

$$\int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\tan^{-1}x + \frac{1}{2}\left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right) + C = \frac{1}{2}\tan^{-1}x + \frac{x}{2(x^2+1)} + C.$$

Collecting all the terms, we obtain

$$\int \frac{100x\,dx}{(x-3)(x^2+1)^2} = 3\ln|x-3| - \frac{3}{2}\ln|x^2+1| - 9\tan^{-1}x + \frac{15}{x^2+1} + 10\left(\frac{1}{2}\tan^{-1}x + \frac{x}{2(x^2+1)}\right) + C$$
$$= 3\ln|x-3| - \frac{3}{2}\ln|x^2+1| - 4\tan^{-1}x + \frac{5x+15}{x^2+1} + C.$$

41. $\displaystyle\int \frac{dx}{(x+2)(x^2+4x+10)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x+2)(x^2+4x+10)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4x+10}.$$

Clearing denominators, we get

$$1 = A(x^2+4x+10) + (Bx+C)(x+2).$$

Setting $x = -2$ then yields

$$1 = A(6) + 0 \qquad \text{or} \qquad A = \frac{1}{6}.$$

Expanding the right-hand side gives us

$$1 = \left(\frac{1}{6} + B\right)x^2 + \left(\frac{2}{3} + 2B + C\right)x + \left(\frac{5}{3} + 2C\right).$$

Equating x^2-coefficients yields

$$0 = \frac{1}{6} + B \qquad \text{or} \qquad B = -\frac{1}{6},$$

while equating constant coefficients yields

$$1 = \frac{5}{3} + 2C \qquad \text{or} \qquad C = -\frac{1}{3}.$$

The result is

$$\frac{1}{(x+2)(x^2+4x+10)} = \frac{\frac{1}{6}}{x+2} + \frac{-\frac{1}{6}x - \frac{1}{3}}{x^2+4x+10}.$$

Thus,

$$\int \frac{dx}{(x+2)(x^2+4x+10)} = \frac{1}{6}\int \frac{dx}{x+2} - \frac{1}{6}\int \frac{(x+2)\,dx}{x^2+4x+10}.$$

For the second integral, let $u = x^2 + 4x + 10$. Then $du = (2x+4)\,dx$, and

$$\int \frac{dx}{(x+2)(x^2+4x+10)} = \frac{1}{6}\ln|x+2| - \frac{1}{12}\int \frac{(2x+4)\,dx}{x^2+4x+10}$$
$$= \frac{1}{6}\ln|x+2| - \frac{1}{12}\ln|x^2+4x+10| + C.$$

42. $\int \frac{9\,dx}{(x+1)(x^2-2x+6)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{9}{(x+1)(x^2-2x+6)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+6}.$$

Clearing denominators gives us

$$9 = A(x^2-2x+6) + (Bx+C)(x+1).$$

Setting $x=-1$ then yields

$$9 = A(9) + 0 \qquad \text{or} \qquad A = 1.$$

Expanding the right-hand side gives us

$$9 = (1+B)x^2 + (-2+B+C)x + (6+C).$$

Equating x^2-coefficients yields

$$0 = 1+B \qquad \text{or} \qquad B = -1,$$

while equating constant coefficients yields

$$9 = 6+C \qquad \text{or} \qquad C = 3.$$

The result is

$$\frac{9}{(x+1)(x^2-2x+6)} = \frac{1}{x+1} + \frac{-x+3}{x^2-2x+6}.$$

Thus,

$$\int \frac{9\,dx}{(x+1)(x^2-2x+6)} = \int \frac{dx}{x+1} + \int \frac{(-x+3)\,dx}{x^2-2x+6}.$$

To evaluate the integral on the right, we first write

$$\int \frac{(-x+3)\,dx}{x^2-2x+6} = -\int \frac{(x-1-2)\,dx}{x^2-2x+6} = -\int \frac{(x-1)\,dx}{x^2-2x+6} + 2\int \frac{dx}{x^2-2x+6}.$$

For the first integral, use the substitution $u = x^2-2x+6$, $du = (2x-2)\,dx$. Then

$$-\int \frac{(x-1)\,dx}{x^2-2x+6} = -\frac{1}{2}\int \frac{(2x-2)\,dx}{x^2-2x+6} = -\frac{1}{2}\ln|x^2-2x+6| + C.$$

For the second integral, we first complete the square:

$$2\int \frac{dx}{x^2-2x+6} = 2\int \frac{dx}{(x^2-2x+1)+5} = 2\int \frac{dx}{(x-1)^2+5}.$$

Now let $u = x-1$, $du = dx$. Then

$$2\int \frac{dx}{(x-1)^2+5} = 2\int \frac{du}{u^2+5} = 2\left(\frac{1}{\sqrt{5}}\right)\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + C = \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x-1}{\sqrt{5}}\right) + C.$$

Collecting all the terms, we have

$$\int \frac{9\,dx}{(x+1)(x^2-2x+6)} = \ln|x+1| - \frac{1}{2}\ln|x^2-2x+6| + \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x-1}{\sqrt{5}}\right) + C.$$

43. $\int \frac{25\,dx}{x(x^2+2x+5)^2}$

SOLUTION The partial fraction decomposition has the form

$$\frac{25}{x(x^2+2x+5)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+5} + \frac{Dx+E}{(x^2+2x+5)^2}.$$

Clearing denominators yields:

$$\begin{aligned} 25 &= A(x^2+2x+5)^2 + x(Bx+C)(x^2+2x+5) + x(Dx+E) \\ &= (Ax^4+4Ax^3+14Ax^2+20Ax+25A) + (Bx^4+Cx^3+2Bx^3+2Cx^2+5Bx^2+5Cx) + Dx^2+Ex. \end{aligned}$$

Equating constant terms yields

$$25A = 25 \qquad \text{or} \qquad A = 1,$$

while equating x^4-coefficients yields

$$A + B = 0 \qquad \text{or} \qquad B = -A = -1.$$

Equating x^3-coefficients yields

$$4A + C + 2B = 0 \qquad \text{or} \qquad C = -2,$$

and equating x^2-coefficients yields

$$14A + 2C + 5B + D = 0 \qquad \text{or} \qquad D = -5.$$

Finally, equating x-coefficients yields

$$20A + 5C + E = 0 \qquad \text{or} \qquad E = -10.$$

Thus,

$$\begin{aligned}\int \frac{25\,dx}{x(x^2+2x+5)^2} &= \int \left(\frac{1}{x} - \frac{x+2}{x^2+2x+5} - 5\frac{x+2}{(x^2+2x+5)^2}\right) dx \\ &= \ln|x| - \int \frac{x+2}{x^2+2x+5}\,dx - 5\int \frac{x+2}{(x^2+2x+5)^2}\,dx.\end{aligned}$$

The two integrals on the right both require the substitution $u = x+1$, so that $x^2+2x+5 = (x+1)^2+4 = u^2+4$ and $du = dx$. This means:

$$\begin{aligned}\int \frac{25\,dx}{x(x^2+2x+5)^2} &= \ln|x| - \int \frac{u+1}{u^2+4}\,du - 5\int \frac{u+1}{(u^2+4)^2}\,du \\ &= \ln|x| - \int \frac{u}{u^2+4}\,du - \int \frac{1}{u^2+4}\,du - 5\int \frac{u}{(u^2+4)^2}\,du - 5\int \frac{1}{(u^2+4)^2}\,du.\end{aligned}$$

For the first and third integrals, we make the substitution $w = u^2+4$, $dw = 2u\,du$. Then we have

$$\begin{aligned}\int \frac{25\,dx}{x(x^2+2x+5)^2} &= \ln|x| - \frac{1}{2}\ln|u^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + \frac{5}{2(u^2+4)} - 5\int \frac{du}{(u^2+4)^2} \\ &= \ln|x| - \frac{1}{2}\ln|x^2+2x+5| - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + \frac{5}{2(x^2+2x+5)} - 5\int \frac{du}{(u^2+4)^2}.\end{aligned}$$

For the remaining integral, we use the trigonometric substitution $2\tan w = u$, so that $u^2+4 = 4\tan^2 w + 4 = 4\sec^2 w$ and $du = 2\sec^2 w\,dw$. This means

$$\begin{aligned}\int \frac{1}{(u^2+4)^2}\,du &= \frac{1}{8}\int \frac{1}{\sec^4 w}\sec^2 w\,dw = \frac{1}{8}\int \cos^2 w\,dw \\ &= \frac{1}{8}\left(\frac{1}{4}\sin 2w + \frac{w}{2}\right) + C = \left(\frac{1}{16}\sin w\cos w + \frac{w}{16}\right) + C \\ &= \frac{1}{16}\frac{u}{\sqrt{u^2+4}}\frac{2}{\sqrt{u^2+4}} + \frac{1}{16}\tan^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{8}\frac{u}{u^2+4} + \frac{1}{16}\tan^{-1}\left(\frac{u}{2}\right) + C \\ &= \frac{1}{8}\frac{x+1}{x^2+2x+5} + \frac{1}{16}\tan^{-1}\left(\frac{x+1}{2}\right).\end{aligned}$$

Hence, the integral is

$$\begin{aligned}\int \frac{25\,dx}{x(x^2+2x+5)^2} &= \ln|x| - \frac{1}{2}\ln|x^2+2x+5| - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) \\ &\quad + \frac{5}{2(x^2+2x+5)} - \frac{5}{8}\frac{x+1}{x^2+2x+5} - \frac{5}{16}\tan^{-1}\left(\frac{x+1}{2}\right) \\ &= \ln|x| + \frac{15-5x}{8(x^2+2x+5)} - \frac{13}{16}\tan^{-1}\left(\frac{x+1}{2}\right) - \frac{1}{2}\ln|x^2+2x+5| + C.\end{aligned}$$

44. $\int \frac{(x^2+3)\,dx}{(x^2+2x+3)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{x^2+3}{(x^2+2x+3)^2} = \frac{Ax+B}{x^2+2x+3} + \frac{Cx+D}{(x^2+2x+3)^2}.$$

Clearing denominators gives us

$$x^2+3 = (Ax+B)(x^2+2x+3) + Cx + D.$$

Expanding the right-hand side, we get

$$x^2+3 = Ax^3 + (2A+B)x^2 + (3A+2B+C)x + (3B+D).$$

Equating coefficients of like powers of x then yields

$$\begin{aligned} A &= 0 \\ 2A + B &= 1 \\ 3A + 2B + C &= 0 \\ 3B + D &= 3 \end{aligned}$$

The solution to this system of equations is

$$A = 0, \qquad B = 1, \qquad C = -2, \qquad D = 0.$$

Therefore

$$\frac{x^2+3}{(x^2+2x+3)^2} = \frac{1}{x^2+2x+3} + \frac{-2x}{(x^2+2x+3)^2},$$

and

$$\int \frac{(x^2+3)\,dx}{(x^2+2x+3)^2} = \int \frac{dx}{x^2+2x+3} - \int \frac{2x\,dx}{(x^2+2x+3)^2}.$$

The first integral can be evaluated by completing the square:

$$\int \frac{dx}{x^2+2x+3} = \int \frac{dx}{x^2+2x+1+2} = \int \frac{dx}{(x+1)^2+2}.$$

Now use the substitution $u = x+1$, $du = dx$. Then

$$\int \frac{dx}{x^2+2x+3} = \int \frac{du}{u^2+2} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C.$$

For the second integral, let $u = x^2+2x+3$. We want $du = (2x+2)\,dx$ to appear in the numerator, so we write

$$\begin{aligned} \int \frac{2x\,dx}{(x^2+2x+3)^2} &= \int \frac{(2x+2-2)\,dx}{(x^2+2x+3)^2} = \int \frac{(2x+2)\,dx}{(x^2+2x+3)^2} - 2\int \frac{dx}{(x^2+2x+3)^2} \\ &= \int \frac{du}{u^2} - 2\int \frac{dx}{(x^2+2x+3)^2} = -\frac{1}{u} - 2\int \frac{dx}{(x^2+2x+3)^2} \\ &= \frac{-1}{x^2+2x+3} - 2\int \frac{dx}{(x^2+3x+3)^2}. \end{aligned}$$

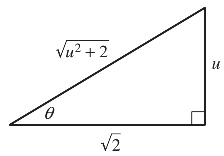
Finally, for this last integral, complete the square, then substitute $u = x+1$, $du = dx$:

$$\int \frac{dx}{(x^2+2x+3)^2} = \int \frac{dx}{((x+1)^2+2)^2} = \int \frac{du}{(u^2+2)^2}.$$

Now use the trigonometric substitution $u = \sqrt{2}\tan\theta$. Then $du = \sqrt{2}\sec^2\theta\,d\theta$, and $u^2+2 = 2\tan^2\theta + 2 = 2\sec^2\theta$. Thus

$$\int \frac{du}{(u^2+2)^2} = \int \frac{\sqrt{2}\sec^2\theta\,d\theta}{4\sec^4\theta} = \frac{\sqrt{2}}{4}\int \cos^2\theta\,d\theta = \frac{\sqrt{2}}{4}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] = \frac{\sqrt{2}}{8}\theta + \frac{\sqrt{2}}{8}\sin\theta\cos\theta + C.$$

We construct a right triangle with $\tan\theta = u/\sqrt{2}$:



From this we see that $\sin\theta = u/\sqrt{u^2+2}$ and $\cos\theta = \sqrt{2}/\sqrt{u^2+2}$. Therefore

$$\int \frac{du}{(u^2+2)^2} = \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8}\left(\frac{u}{\sqrt{u^2+2}}\right)\left(\frac{\sqrt{2}}{\sqrt{u^2+2}}\right) + C$$

$$= \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{u}{4(u^2+2)} + C = \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{x+1}{4(x^2+2x+3)} + C.$$

Collecting all the terms, we have

$$\int \frac{(x^2+3)\,dx}{(x^2+2x+3)^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) - \left[\frac{-1}{x^2+2x+3} - 2\left(\frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{x+1}{4(x^2+2x+3)}\right)\right] + C$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{4}\right)\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{2+(x+1)}{2(x^2+2x+3)} + C$$

$$= \frac{3\sqrt{2}}{4}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{x+3}{2(x^2+2x+3)} + C.$$

In Exercises 45–48, evaluate by using first substitution and then partial fractions if necessary.

45. $\displaystyle\int \frac{x\,dx}{x^4+1}$

SOLUTION Use the substitution $u = x^2$ so that $du = 2x\,dx$, and

$$\int \frac{x}{x^4+1}\,dx = \frac{1}{2}\int \frac{1}{u^2+1}\,du = \frac{1}{2}\tan^{-1}u = \frac{1}{2}\tan^{-1}(x^2)$$

46. $\displaystyle\int \frac{x\,dx}{(x+2)^4}$

SOLUTION Use the substitution $u = x+2$ and $du = dx$; then

$$\int \frac{x}{(x+2)^4}\,dx = \int \frac{u-2}{u^4}\,du = \int \frac{1}{u^3}\,du - 2\int \frac{1}{u^4}\,du$$

$$= -\frac{1}{2u^2} + \frac{2}{3u^3} + C = \frac{2}{3(x+2)^3} - \frac{1}{2(x+2)^2} + C$$

47. $\displaystyle\int \frac{e^x\,dx}{e^{2x}-e^x}$

SOLUTION Use the substitution $u = e^x$. Then $du = e^x\,dx = u\,dx$ so that $dx = \frac{1}{u}\,du$. Then

$$\int \frac{e^x\,dx}{e^{2x}-e^x} = \int \frac{u\cdot\frac{1}{u}\,du}{u^2-u} = \int \frac{1}{u(u-1)}\,du$$

Using partial fractions, we have

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} = \frac{(A+B)u - A}{u(u-1)}$$

Upon equating coefficients in the numerators, we have $A+B=0$, $A=-1$ so that $B=1$. Then

$$\int \frac{e^x\,dx}{e^{2x}-e^x} = -\int \frac{1}{u}\,du + \int \frac{1}{u-1}\,du = \ln|u-1| - \ln|u| + C = \ln|e^x-1| - \ln e^x + C$$

48. $\displaystyle\int \frac{\sec^2\theta\,d\theta}{\tan^2\theta-1}$

SOLUTION Let $u = \tan\theta$; then $du = \sec^2\theta\,d\theta$ and

$$\int \frac{\sec^2\theta\,d\theta}{\tan^2\theta-1} = \int \frac{1}{u^2-1}\,du = -\int \frac{1}{1-u^2}\,du = -\tanh^{-1}(u) + C = -\tanh^{-1}(\tan\theta) + C$$

49. Evaluate $\int \frac{\sqrt{x}\,dx}{x-1}$. *Hint:* Use the substitution $u = \sqrt{x}$ (sometimes called a **rationalizing substitution**).

SOLUTION Let $u = \sqrt{x}$. Then $du = (1/2\sqrt{x})\,dx = (1/2u)\,dx$. Thus

$$\int \frac{\sqrt{x}\,dx}{x-1} = \int \frac{u(2u\,du)}{u^2-1} = 2\int \frac{u^2\,du}{u^2-1} = 2\int \frac{(u^2-1+1)\,du}{u^2-1}$$
$$= 2\int \left(\frac{u^2-1}{u^2-1} + \frac{1}{u^2-1}\right) du = 2\int du + \int \frac{2\,du}{u^2-1} = 2u + \int \frac{2\,du}{u^2-1}.$$

The partial fraction decomposition of the remaining integral has the form:

$$\frac{2}{u^2-1} = \frac{2}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}.$$

Clearing denominators gives us

$$2 = A(u+1) + B(u-1).$$

Setting $u = 1$ yields $2 = A(2) + 0$ or $A = 1$, while setting $u = -1$ yields $2 = 0 + B(-2)$ or $B = -1$. The result is

$$\frac{2}{u^2-1} = \frac{1}{u-1} + \frac{-1}{u+1}.$$

Thus,

$$\int \frac{2\,du}{u^2-1} = \int \frac{du}{u-1} - \int \frac{du}{u+1} = \ln|u-1| - \ln|u+1| + C.$$

The final answer is

$$\int \frac{\sqrt{x}\,dx}{x-1} = 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{x} + \ln|\sqrt{x}-1| - \ln|\sqrt{x}+1| + C.$$

50. Evaluate $\int \frac{dx}{x^{1/2} - x^{1/3}}$.

SOLUTION First use the substitution $u = x^{1/6}$. Then

$$du = \frac{1}{6}x^{-5/6}\,dx \quad \Rightarrow \quad 6x^{5/6}\,du = dx \quad \Rightarrow \quad 6u^5\,du = dx$$

and we have (using long division)

$$\int \frac{dx}{x^{1/2}-x^{1/3}} = \int \frac{6u^5}{u^3-u^2}\,du = 6\int \frac{u^3}{u-1}\,du = 6\int u^2 + u + 1 + \frac{1}{u-1}\,du$$
$$= 6\left(\frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|u-1|\right) + C = 2u^3 + 3u^2 + 6u + 6\ln|u-1| + C$$
$$= 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + 6\ln\left|x^{1/6} - 1\right| + C$$

51. Evaluate $\int \frac{dx}{x^2-1}$ in two ways: using partial fractions and using trigonometric substitution. Verify that the two answers agree.

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Clearing denominators gives us

$$1 = A(x+1) + B(x-1).$$

Setting $x = 1$, we get $1 = A(2)$ or $A = \frac{1}{2}$; while setting $x = -1$, we get $1 = B(-2)$ or $B = -\frac{1}{2}$. The result is

$$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}.$$

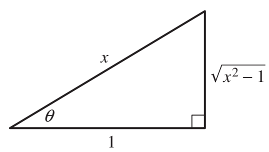
Thus,

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

Using trigonometric substitution, let $x = \sec\theta$. Then $dx = \tan\theta\sec\theta\,d\theta$, and $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$. Thus

$$\int \frac{dx}{x^2-1} = \int \frac{\tan\theta\sec\theta\,d\theta}{\tan^2\theta} = \int \frac{\sec\theta\,d\theta}{\tan\theta} = \int \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta}$$
$$= \int \csc\theta\,d\theta = \ln|\csc\theta - \cot\theta| + C.$$

Now we construct a right triangle with $\sec\theta = x$:



From this we see that $\csc\theta = x/\sqrt{x^2-1}$ and $\cot\theta = 1/\sqrt{x^2-1}$. Thus

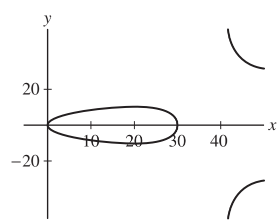
$$\int \frac{dx}{x^2-1} = \ln\left|\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right| + C = \ln\left|\frac{x-1}{\sqrt{x^2-1}}\right| + C.$$

To check that these two answers agree, we write

$$\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| = \frac{1}{2}\left|\frac{x-1}{x+1}\right| = \ln\left|\sqrt{\frac{x-1}{x+1}}\right| = \ln\left|\frac{\sqrt{x-1}}{\sqrt{x+1}}\cdot\frac{\sqrt{x-1}}{\sqrt{x-1}}\right| = \ln\left|\frac{x-1}{\sqrt{x^2-1}}\right|.$$

52. GU Graph the equation $(x-40)y^2 = 10x(x-30)$ and find the volume of the solid obtained by revolving the region between the graph and the x-axis for $0 \le x \le 30$ around the x-axis.

SOLUTION The graph of $(x-40)y^2 = 10x(x-30)$ is shown below



Using the disk method, the volume is given by

$$V = \int_0^{30} \pi r^2\,dx = \pi\int_0^{30}\left(\sqrt{\frac{10x(x-30)}{x-40}}\right)^2 dx = \pi\int_0^{30}\frac{10x(x-30)\,dx}{x-40}.$$

To find the anti-derivative, expand the numerator and then use long division:

$$\frac{10x(x-30)}{x-40} = \frac{10x^2-300x}{x-40} = 10x + 100 + \frac{4000}{x-40}.$$

Thus,

$$\pi\int_0^{30}\frac{10x(x-30)\,dx}{x-40} = \pi\left[10\int_0^{30}x\,dx + 100\int_0^{30}dx + 4000\int_0^{30}\frac{dx}{x-40}\right]$$
$$= \pi\left(5x^2 + 100x + 4000\ln|x-40|\right)\Big|_0^{30}$$
$$= \pi\left[\left(4500 + 3000 + 4000\ln(10)\right) - \left(0 + 4000\ln(40)\right)\right]$$
$$= (7500 - 4000\ln 4)\pi.$$

In Exercises 53–66, evaluate the integral using the appropriate method or combination of methods covered thus far in the text.

53. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

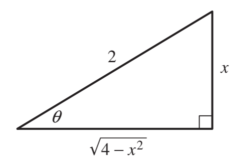
SOLUTION Use the trigonometric substitution $x = 2\sin\theta$. Then $dx = 2\cos\theta\, d\theta$,

$$4 - x^2 = 4 - 4\sin^2\theta = 4(1-\sin^2\theta) = 4\cos^2\theta,$$

and

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(4\sin^2\theta)(2\cos\theta)} = \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C.$$

Now construct a right triangle with $\sin\theta = x/2$:



From this we see that $\cot\theta = \sqrt{4-x^2}/x$. Thus

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\left(\frac{\sqrt{4-x^2}}{x}\right) + C = -\frac{\sqrt{4-x^2}}{4x} + C.$$

54. $\displaystyle\int \frac{dx}{x(x-1)^2}$

SOLUTION Using partial fractions, we first write

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Clearing denominators gives us

$$1 = A(x-1)^2 + Bx(x-1) + Cx.$$

Setting $x = 0$ yields

$$1 = A(1) + 0 + 0 \qquad \text{or} \qquad A = 1,$$

while setting $x = 1$ yields

$$1 = 0 + 0 + C \qquad \text{or} \qquad C = 1,$$

and setting $x = 2$ yields

$$1 = 1 + 2B + 2 \qquad \text{or} \qquad B = -1.$$

The result is

$$\frac{1}{x(x-1)^2} = \frac{1}{x} + \frac{-1}{x-1} + \frac{1}{(x-1)^2}.$$

Thus,

$$\int \frac{dx}{x(x-1)^2} = \int \frac{dx}{x} - \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} = \ln|x| - \ln|x-1| - \frac{1}{x-1} + C.$$

55. $\displaystyle\int \cos^2 4x\, dx$

SOLUTION Use the substitution $u = 4x$, $du = 4\,dx$. Then we have

$$\int \cos^2(4x)\, dx = \frac{1}{4}\int \cos^2(4x) 4\, dx = \frac{1}{4}\int \cos^2 u\, du = \frac{1}{4}\left[\frac{1}{2}u + \frac{1}{2}\sin u \cos u\right] + C$$

$$= \frac{1}{8}u + \frac{1}{8}\sin u\cos u + C = \frac{1}{2}x + \frac{1}{8}\sin 4x \cos 4x + C.$$

56. $\int x \sec^2 x \, dx$

SOLUTION Use integration by parts, with $u = x$ and $v' = \sec^2 x$. Then $u' = 1$, $v = \tan x$, and

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx = x \tan x - \big(-\ln|\cos x| \big) + C = x \tan x + \ln|\cos x| + C.$$

57. $\int \frac{dx}{(x^2+9)^2}$

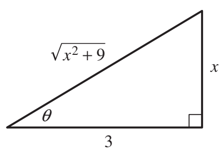
SOLUTION Use the trigonometric substitution $x = 3\tan\theta$. Then $dx = 3\sec^2\theta \, d\theta$,

$$x^2 + 9 = 9\tan^2\theta + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta,$$

and

$$\int \frac{dx}{(x^2+9)^2} = \int \frac{3\sec^2\theta \, d\theta}{(9\sec^2\theta)^2} = \frac{3}{81}\int \frac{\sec^2\theta \, d\theta}{\sec^4\theta} = \frac{1}{27}\int \cos^2\theta \, d\theta = \frac{1}{27}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C.$$

Now construct a right triangle with $\tan\theta = x/3$:



From this we see that $\sin\theta = x/\sqrt{x^2+9}$ and $\cos\theta = 3/\sqrt{x^2+9}$. Thus

$$\int \frac{dx}{\sqrt{x^2+9}^2} = \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{54}\left(\frac{x}{\sqrt{x^2+9}}\right)\left(\frac{3}{\sqrt{x^2+9}}\right) + C = \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{x}{18(x^2+9)} + C.$$

58. $\int \theta \sec^{-1}\theta \, d\theta$

SOLUTION Use Integration by Parts, with $u = \sec^{-1}\theta$ and $v' = \theta$. Then $u' = 1/\theta\sqrt{\theta^2-1}$, $v = \theta^2/2$, and

$$\int \theta \sec^{-1}\theta \, d\theta = \frac{\theta^2}{2}\sec^{-1}\theta - \int \frac{\theta^2 \, d\theta}{2\theta\sqrt{\theta^2-1}} = \frac{\theta^2}{2}\sec^{-1}\theta - \frac{1}{2}\int \frac{\theta \, d\theta}{\sqrt{\theta^2-1}}.$$

To evaluate the remaining integral, use the substitution $w = \theta^2 - 1$, $dw = 2\theta \, d\theta$. Then

$$\int \frac{\theta \, d\theta}{\sqrt{\theta^2-1}} = \frac{1}{2}\int \frac{2\theta \, d\theta}{\sqrt{\theta^2-1}} = \frac{1}{2}\int \frac{dw}{\sqrt{w}} = \frac{1}{2}\left(2\sqrt{w}\right) + C = \sqrt{\theta^2-1} + C.$$

The final answer is

$$\int \theta \sec^{-1}\theta \, d\theta = \frac{\theta^2}{2}\sec^{-1}\theta - \frac{1}{2}\sqrt{\theta^2-1} + C.$$

59. $\int \tan^5 x \sec x \, dx$

SOLUTION Use the trigonometric identity $\tan^2 x = \sec^2 x - 1$ to write

$$\int \tan^5 x \sec x \, dx = \int \left(\sec^2 x - 1\right)^2 \tan x \sec x \, dx.$$

Now use the substitution $u = \sec x$, $du = \sec x \tan x \, dx$:

$$\int \tan^5 x \sec x \, dx = \int (u^2-1)^2 \, du = \int \left(u^4 - 2u^2 + 1\right) du$$
$$= \frac{1}{5}u^5 - \frac{2}{3}u^3 + u + C = \frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C.$$

60. $\int \frac{(3x^2-1)\,dx}{x(x^2-1)}$

SOLUTION The denominator expands to $x^3 - x$, so if we let $u = x^3 - x$, then $du = (3x^2-1)\,dx$, which is the numerator. Thus

$$\int \frac{(3x^2-1)\,dx}{x(x^2-1)} = \int \frac{du}{u} = \ln|u| + C = \ln(x(x^2-1)) + C$$

61. $\int \ln(x^4 - 1)\, dx$

SOLUTION Apply integration by parts with $u = \ln(x^4 - 1)$, $v' = 1$; then $u' = \frac{4x^3}{x^4-1}$ and $v = x$, so after simplification,

$$\begin{aligned}\int \ln(x^4 - 1)\, dx &= x\ln(x^4 - 1) - 4\int \frac{x^4}{x^4 - 1}\, dx = x\ln(x^4 - 1) - 4\int 1 + \frac{1}{x^4 - 1}\, dx \\ &= x\ln(x^4 - 1) - 4\int 1\, dx - 4\int \frac{1}{x^4 - 1}\, dx \\ &= x\ln(x^4 - 1) - 4x - 4\int \frac{1}{2}\left(\frac{1}{x^2 - 1} - \frac{1}{x^2 + 1}\right) dx \\ &= x\ln(x^4 - 1) - 4x - 2\int \frac{1}{x^2 - 1}\, dx + 2\int \frac{1}{x^2 + 1}\, dx \\ &= x\ln(x^4 - 1) - 4x + 2\tanh^{-1} x + 2\tan^{-1} x + C\end{aligned}$$

62. $\int \frac{x\, dx}{(x^2 - 1)^{3/2}}$

SOLUTION Use the substitution $u = x^2 - 1$, $du = 2x\, dx$. Then we have

$$\int \frac{x\, dx}{(x^2 - 1)^{3/2}} = \frac{1}{2}\int \frac{2x\, dx}{(x^2 - 1)^{3/2}} = \frac{1}{2}\int \frac{du}{u^{3/2}} = \frac{1}{2}(-2)u^{-1/2} + C = \frac{-1}{\sqrt{u}} + C = \frac{-1}{\sqrt{x^2 - 1}} + C.$$

63. $\int \frac{x^2\, dx}{(x^2 - 1)^{3/2}}$

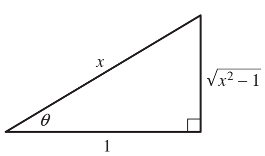
SOLUTION Use the trigonometric substitution $x = \sec\theta$. Then $dx = \sec\theta\tan\theta\, d\theta$,

$$x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta,$$

and

$$\begin{aligned}\int \frac{x^2\, dx}{(x^2 - 1)^{3/2}} &= \int \frac{(\sec^2\theta)\sec\theta\tan\theta\, d\theta}{(\tan^2\theta)^{3/2}} = \int \frac{\sec^3\theta\, d\theta}{\tan^2\theta} = \int \frac{(\tan^2\theta + 1)\sec\theta\, d\theta}{\tan^2\theta} \\ &= \int \frac{\tan^2\theta\sec\theta\, d\theta}{\tan^2\theta} + \int \frac{\sec\theta\, d\theta}{\tan^2\theta} = \int \sec\theta\, d\theta + \int \csc\theta\cot\theta\, d\theta \\ &= \ln|\sec\theta + \tan\theta| - \csc\theta + C.\end{aligned}$$

Now construct a right triangle with $\sec\theta = x$:



From this we see that $\tan\theta = \sqrt{x^2 - 1}$ and $\csc\theta = x/\sqrt{x^2 - 1}$. So the final answer is

$$\int \frac{x^2\, dx}{(x^2 - 1)^{3/2}} = \ln\left|x + \sqrt{x^2 - 1}\right| - \frac{x}{\sqrt{x^2 - 1}} + C.$$

64. $\int \frac{(x + 1)\, dx}{(x^2 + 4x + 8)^2}$

SOLUTION At first it might appear that one would use partial fractions to simplify this problem, but in fact it's already in simplified form. Instead, use the substitution $u = x^2 + 4x + 8$, $du = (2x + 4)\, dx$. Then we have

$$\begin{aligned}\int \frac{(x + 1)\, dx}{(x^2 + 4x + 8)^2} &= \frac{1}{2}\int \frac{(2x + 2)\, dx}{(x^2 + 4x + 8)^2} = \frac{1}{2}\int \frac{(2x + 2 + 2 - 2)\, dx}{(x^2 + 4x + 8)^2} \\ &= \frac{1}{2}\int \frac{(2x + 4)\, dx}{(x^2 + 4x + 8)^2} - \int \frac{dx}{(x^2 + 4x + 8)^2} \\ &= \frac{1}{2}\int \frac{du}{u^2} - \int \frac{dx}{(x^2 + 4x + 8)^2} = \frac{-1}{2u} - \int \frac{dx}{(x^2 + 4x + 8)^2}.\end{aligned}$$

To evaluate the remaining integral, complete the square, then let $w = x + 2$, $dw = dx$:

$$\int \frac{dx}{(x^2+4x+8)^2} = \int \frac{dx}{(x^2+4x+4+4)^2} = \int \frac{dx}{((x+2)^2+4)^2} = \int \frac{dw}{(w^2+4)^2}.$$

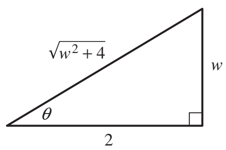
Next, let $w = 2\tan\theta$, $dw = 2\sec^2\theta\, d\theta$. Then

$$w^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta,$$

and we have

$$\int \frac{dw}{(w^2+4)^2} = \int \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta} = \frac{1}{8}\cos^2\theta\, d\theta = \frac{1}{8}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C = \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C.$$

Now construct a right triangle with $\tan\theta = w/2$:



From this we see that $\sin\theta = w/\sqrt{w^2+4}$ and $\cos\theta = 2/\sqrt{w^2+4}$. Thus

$$\int \frac{dw}{(w^2+4)^2} = \frac{1}{16}\tan^{-1}\left(\frac{w}{2}\right) + \frac{1}{16}\left(\frac{w}{\sqrt{w^2+4}}\right)\left(\frac{2}{\sqrt{w^2+4}}\right) + C = \frac{1}{16}\tan^{-1}\left(\frac{w}{2}\right) + \frac{w}{8(w^2+4)} + C.$$

In terms of x, we have

$$\int \frac{dx}{(x^2+4x+8)^2} = \int \frac{dw}{(w^2+4)^2} = \frac{1}{16}\tan^{-1}\left(\frac{x+2}{2}\right) + \frac{x+2}{8((x+2)^2+4)} + C.$$

Collecting all the terms, we have

$$\begin{aligned}\int \frac{(x+1)\,dx}{(x^2+4x+8)^2} &= \frac{-1}{2(x^2+4x+8)} - \frac{1}{16}\tan^{-1}\left(\frac{x+2}{2}\right) - \frac{x+2}{8(x^2+4x+8)} + C\\ &= -\frac{1}{16}\tan^{-1}\left(\frac{x+2}{2}\right) - \frac{x+6}{8(x^2+4x+8)} + C.\end{aligned}$$

65. $\displaystyle\int \frac{\sqrt{x}\,dx}{x^3+1}$

SOLUTION Use the substitution $u = x^{3/2}$, $du = \frac{3}{2}x^{1/2}\,dx$. Then $x^3 = (x^{3/2})^2 = u^2$, so we have

$$\int \frac{\sqrt{x}\,dx}{x^3+1} = \frac{2}{3}\int \frac{du}{u^2+1} = \frac{2}{3}\tan^{-1}u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

66. $\displaystyle\int \frac{x^{1/2}\,dx}{x^{1/3}+1}$

SOLUTION Use the substitution $u = x^{1/6}$, $du = \frac{1}{6}x^{-5/6}\,dx$. Then $dx = 6x^{5/6}\,du = 6u^5\,du$, and we get

$$\int \frac{x^{1/2}\,dx}{x^{1/3}+1} = \int \frac{u^3(6u^5\,du)}{u^2+1} = 6\int \frac{u^8\,du}{u^2+1}.$$

By long division

$$\frac{u^8}{u^2+1} = u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1},$$

thus

$$\int \frac{u^8}{u^2+1}\,du = \int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1}\right)du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + \frac{1}{3}u^3 - u + \tan^{-1}u + C.$$

The final answer is

$$\int \frac{x^{1/2}}{x^{1/3}+1} = \frac{6}{7}x^{7/6} - \frac{6}{5}x^{5/6} + 2x^{1/2} - 6x^{1/6} + 6\tan^{-1}(x^{1/6}) + C.$$

67. Show that the substitution $\theta = 2\tan^{-1} t$ (Figure 2) yields the formulas

$$\cos\theta = \frac{1-t^2}{1+t^2}, \qquad \sin\theta = \frac{2t}{1+t^2}, \qquad d\theta = \frac{2\,dt}{1+t^2} \tag{10}$$

This substitution transforms the integral of any rational function of $\cos\theta$ and $\sin\theta$ into an integral of a rational function of t (which can then be evaluated using partial fractions). Use it to evaluate $\int \frac{d\theta}{\cos\theta + \frac{3}{4}\sin\theta}$.

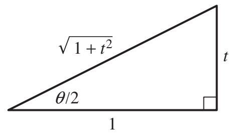


FIGURE 2

SOLUTION If $\theta = 2\tan^{-1} t$, then $d\theta = 2\,dt/(1+t^2)$. We also have that $\cos(\frac{\theta}{2}) = 1/\sqrt{1+t^2}$ and $\sin(\frac{\theta}{2}) = t/\sqrt{1+t^2}$. To find $\cos\theta$, we use the double angle identity $\cos\theta = 1 - 2\sin^2(\frac{\theta}{2})$. This gives us

$$\cos\theta = 1 - 2\left(\frac{t}{\sqrt{1+t^2}}\right)^2 = 1 - \frac{2t^2}{1+t^2} = \frac{1+t^2-2t^2}{1+t^2} = \frac{1-t^2}{1+t^2}.$$

To find $\sin\theta$, we use the double angle identity $\sin\theta = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$. This gives us

$$\sin\theta = 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right) = \frac{2t}{1+t^2}.$$

With these formulas, we have

$$\int \frac{d\theta}{\cos\theta + (3/4)\sin\theta} = \int \frac{\frac{2\,dt}{1+t^2}}{\left(\frac{1-t^2}{1+t^2}\right) + \frac{3}{4}\left(\frac{2t}{1+t^2}\right)} = \int \frac{8\,dt}{4(1-t^2)+3(2t)} = \int \frac{8\,dt}{4+6t-4t^2} = \int \frac{4\,dt}{2+3t-2t^2}.$$

The partial fraction decomposition has the form

$$\frac{4}{2+3t-2t^2} = \frac{A}{2-t} + \frac{B}{1+2t}.$$

Clearing denominators gives us

$$4 = A(1+2t) + B(2-t).$$

Setting $t = 2$ then yields

$$4 = A(5) + 0 \qquad \text{or} \qquad A = \frac{4}{5},$$

while setting $t = -\frac{1}{2}$ yields

$$4 = 0 + B\left(\frac{5}{2}\right) \qquad \text{or} \qquad B = \frac{8}{5}.$$

The result is

$$\frac{4}{2+3t-2t^2} = \frac{\frac{4}{5}}{2-t} + \frac{\frac{8}{5}}{1+2t}.$$

Thus,

$$\int \frac{4}{2+3t-2t^2}\,dt = \frac{4}{5}\int \frac{dt}{2-t} + \frac{8}{5}\int \frac{dt}{1+2t} = -\frac{4}{5}\ln|2-t| + \frac{4}{5}\ln|1+2t| + C.$$

The original substitution was $\theta = 2\tan^{-1} t$, which means that $t = \tan(\frac{\theta}{2})$. The final answer is then

$$\int \frac{d\theta}{\cos\theta + \frac{3}{4}\sin\theta} = -\frac{4}{5}\ln\left|2 - \tan\left(\frac{\theta}{2}\right)\right| + \frac{4}{5}\ln\left|1 + 2\tan\left(\frac{\theta}{2}\right)\right| + C.$$

68. Use the substitution of Exercise 67 to evaluate $\int \frac{d\theta}{\cos\theta + \sin\theta}$.

SOLUTION Using the substitution $\theta = 2\tan^{-1} t$, we get

$$\int \frac{d\theta}{\cos\theta + \sin\theta} = \int \frac{2\,dt/(1+t^2)}{(1-t^2)/(1+t^2) + 2t/(1+t^2)} = \int \frac{2\,dt}{1-t^2+2t} = -2\int \frac{dt}{t^2-2t-1}.$$

The partial fraction decomposition has the form

$$\frac{-2}{t^2-2t-1} = \frac{A}{t-1-\sqrt{2}} + \frac{B}{t-1+\sqrt{2}}.$$

Clearing denominators gives us

$$-2 = A(t-1+\sqrt{2}) + B(t-1-\sqrt{2}).$$

Setting $t = 1+\sqrt{2}$ then yields $A = -\frac{1}{\sqrt{2}}$, while setting $t = 1-\sqrt{2}$ yields $B = \frac{1}{\sqrt{2}}$. Thus,

$$\int \frac{d\theta}{\cos\theta + \sin\theta} = \frac{1}{\sqrt{2}}\int \frac{dt}{t-1+\sqrt{2}} - \frac{1}{\sqrt{2}}\int \frac{dt}{t-1-\sqrt{2}} = \frac{1}{\sqrt{2}}\ln|t-1+\sqrt{2}| - \frac{1}{\sqrt{2}}\ln|t-1-\sqrt{2}| + C$$
$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{\theta}{2}\right) - 1 + \sqrt{2}}{\tan\left(\frac{\theta}{2}\right) - 1 - \sqrt{2}}\right| + C.$$

Further Insights and Challenges

69. Prove the general formula

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b}\ln\frac{x-a}{x-b} + C$$

where a, b are constants such that $a \neq b$.

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

Clearing denominators, we get

$$1 = A(x-b) + B(x-a).$$

Setting $x = a$ then yields

$$1 = A(a-b) + 0 \qquad \text{or} \qquad A = \frac{1}{a-b},$$

while setting $x = b$ yields

$$1 = 0 + B(b-a) \qquad \text{or} \qquad B = \frac{1}{b-a}.$$

The result is

$$\frac{1}{(x-a)(x-b)} = \frac{\frac{1}{a-b}}{x-a} + \frac{\frac{1}{b-a}}{x-b}.$$

Thus,

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b}\int \frac{dx}{x-a} + \frac{1}{b-a}\int \frac{dx}{x-b} = \frac{1}{a-b}\ln|x-a| + \frac{1}{b-a}\ln|x-b| + C$$
$$= \frac{1}{a-b}\ln|x-a| - \frac{1}{a-b}\ln|x-b| + C = \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right| + C.$$

70. The method of partial fractions shows that

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

The computer algebra system Mathematica evaluates this integral as $-\tanh^{-1} x$, where $\tanh^{-1} x$ is the inverse hyperbolic tangent function. Can you reconcile the two answers?

SOLUTION Let

$$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Solving for x in terms of y, we find

$$\begin{aligned}(e^x + e^{-x})y &= e^x - e^{-x}\\ e^{-x}(1+y) &= e^x(1-y)\\ e^{2x} &= \frac{1+y}{1-y}\\ x &= \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right|\end{aligned}$$

Thus,

$$\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|,$$

so

$$-\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1-x}{1+x}\right| = \frac{1}{2}\ln|1-x| - \frac{1}{2}\ln|1+x|,$$

as desired.

71. Suppose that $Q(x) = (x-a)(x-b)$, where $a \neq b$, and let $P(x)/Q(x)$ be a proper rational function so that

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

(a) Show that $A = \dfrac{P(a)}{Q'(a)}$ and $B = \dfrac{P(b)}{Q'(b)}$.

(b) Use this result to find the partial fraction decomposition for $P(x) = 3x - 2$ and $Q(x) = x^2 - 4x - 12$.

SOLUTION

(a) Clearing denominators gives us

$$P(x) = A(x-b) + B(x-a).$$

Setting $x = a$ then yields

$$P(a) = A(a-b) + 0 \qquad \text{or} \qquad A = \frac{P(a)}{a-b},$$

while setting $x = b$ yields

$$P(b) = 0 + B(b-a) \qquad \text{or} \qquad B = \frac{P(b)}{b-a}.$$

Now use the product rule to differentiate $Q(x)$:

$$Q'(x) = (x-a)(1) + (1)(x-b) = x - a + x - b = 2x - a - b;$$

therefore,

$$\begin{aligned}Q'(a) &= 2a - a - b = a - b\\ Q'(b) &= 2b - a - b = b - a\end{aligned}$$

Substituting these into the above results, we find

$$A = \frac{P(a)}{Q'(a)} \quad \text{and} \quad B = \frac{P(b)}{Q'(b)}.$$

(b) The partial fraction decomposition has the form:

$$\frac{P(x)}{Q(x)} = \frac{3x-2}{x^2-4x-12} = \frac{3x-2}{(x-6)(x+2)} = \frac{A}{x-6} + \frac{B}{x+2};$$

$$A = \frac{P(6)}{Q'(6)} = \frac{3(6)-2}{2(6)-4} = \frac{16}{8} = 2;$$

$$B = \frac{P(-2)}{Q'(-2)} = \frac{3(-2)-2}{2(-2)-4} = \frac{-8}{-8} = 1.$$

The result is

$$\frac{3x-2}{x^2-4x-12} = \frac{2}{x-6} + \frac{1}{x+2}.$$

72. Suppose that $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$, where the roots a_j are all distinct. Let $P(x)/Q(x)$ be a proper rational function so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_n}{(x-a_n)}$$

(a) Show that $A_j = \dfrac{P(a_j)}{Q'(a_j)}$ for $j = 1, \ldots, n$.

(b) Use this result to find the partial fraction decomposition for $P(x) = 2x^2 - 1$, $Q(x) = x^3 - 4x^2 + x + 6 = (x+1)(x-2)(x-3)$.

SOLUTION

(a) To differentiate $Q(x)$, first take the logarithm of both sides, and then differentiate:

$$\ln\big(Q(x)\big) = \ln\big[(x-a_1)(x-a_2)\cdots(x-a_n)\big] = \ln(x-a_1) + \ln(x-a_2) + \cdots + \ln(x-a_n)$$

$$\frac{d}{dx}\ln\big(Q(x)\big) = \frac{Q'(x)}{Q(x)} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_n}$$

Multiplying both sides by $Q(x)$ gives us

$$\begin{aligned} Q'(x) &= Q(x)\left[\frac{1}{x-a_1} + \cdots + \frac{1}{x-a_n}\right] \\ &= (x-a_2)(x-a_3)\cdots(x-a_n) + (x-a_1)(x-a_3)\cdots(x-a_n) + \cdots + (x-a_1)(x-a_2)\cdots(x-a_{n-1}). \end{aligned}$$

In other words, the ith product in the formula for $Q'(x)$ has the $(x-a_i)$ factor removed. This means that

$$Q'(a_j) = (a_j - a_1)\cdots(a_j - a_{j-1})(a_j - a_{j+1})\cdots(a_j - a_n).$$

Now clear denominators in the expression for $P(x)/Q(x)$:

$$\begin{aligned} P(x) &= \frac{A_1 Q(x)}{x-a_1} + \frac{A_2 Q(x)}{x-a_2} + \cdots + \frac{A_n Q(x)}{x-a_n} \\ &= A_1(x-a_2)\cdots(x-a_n) + (x-a_1)A_2(x-a_3)\cdots(x-a_n) + \cdots + (x-a_1)(x-a_2)\cdots(x-a_{n-1})A_n. \end{aligned}$$

Setting $x = a_j$, we get

$$P(a_j) = (a_j - a_1)(a_j - a_2)\cdots(a_j - a_{j-1})A_j(a_j - a_{j+1})\cdots(a_j - a_n),$$

so that

$$A_j = \frac{P(a_j)}{(a_j - a_1)\cdots(a_j - a_{j-1})(a_j - a_{j+1})\cdots(a_j - a_n)} = \frac{P(a_j)}{Q'(a_j)}.$$

(b) Let $P(x) = 2x^2 - 1$ and $Q(x) = (x+1)(x-2)(x-3)$, so that $Q'(x) = 3x^2 - 8x + 1$. Then $a_1 = -1$, $a_2 = 2$, and $a_3 = 3$, so that

$$A_1 = P(-1)/Q'(-1) = \frac{1}{12};$$

$$A_2 = P(2)/Q'(2) = -\frac{7}{3};$$

$$A_3 = P(3)/Q'(3) = \frac{17}{4}.$$

Thus

$$\frac{P(x)}{Q(x)} = \frac{1}{12(x+1)} - \frac{7}{1 = 3(x-2)} + \frac{17}{4(x-3)}.$$

7.6 Improper Integrals

Preliminary Questions

1. State whether the integral converges or diverges:

(a) $\int_1^\infty x^{-3}\,dx$

(b) $\int_0^1 x^{-3}\,dx$

(c) $\int_1^\infty x^{-2/3}\,dx$

(d) $\int_0^1 x^{-2/3}\,dx$

SOLUTION

(a) The integral is improper because one of the limits of integration is infinite. Because the power of x in the integrand is less than -1, this integral converges.

(b) The integral is improper because the integrand is undefined at $x = 0$. Because the power of x in the integrand is less than -1, this integral diverges.

(c) The integral is improper because one of the limits of integration is infinite. Because the power of x in the integrand is greater than -1, this integral diverges.

(d) The integral is improper because the integrand is undefined at $x = 0$. Because the power of x in the integrand is greater than -1, this integral converges.

2. Is $\int_0^{\pi/2} \cot x\,dx$ an improper integral? Explain.

SOLUTION Because the integrand $\cot x$ is undefined at $x = 0$, this is an improper integral.

3. Find a value of $b > 0$ that makes $\int_0^b \frac{1}{x^2 - 4}\,dx$ an improper integral.

SOLUTION Any value of b satisfying $|b| \geq 2$ will make this an improper integral.

4. Which comparison would show that $\int_0^\infty \frac{dx}{x + e^x}$ converges?

SOLUTION Note that, for $x > 0$,

$$\frac{1}{x + e^x} < \frac{1}{e^x} = e^{-x}.$$

Moreover

$$\int_0^\infty e^{-x}\,dx$$

converges. Therefore,

$$\int_0^\infty \frac{1}{x + e^x}\,dx$$

converges by the comparison test.

5. Explain why it is not possible to draw any conclusions about the convergence of $\int_1^\infty \frac{e^{-x}}{x}\,dx$ by comparing with the integral $\int_1^\infty \frac{dx}{x}$.

SOLUTION For $1 \leq x < \infty$,

$$\frac{e^{-x}}{x} < \frac{1}{x},$$

but

$$\int_1^\infty \frac{dx}{x}$$

diverges. Knowing that an integral is smaller than a divergent integral does not allow us to draw any conclusions using the comparison test.

Exercises

1. Which of the following integrals is improper? Explain your answer, but do not evaluate the integral.

(a) $\int_0^2 \frac{dx}{x^{1/3}}$

(b) $\int_1^\infty \frac{dx}{x^{0.2}}$

(c) $\int_{-1}^\infty e^{-x}\,dx$

(d) $\int_0^1 e^{-x}\,dx$

(e) $\int_0^{\pi/2} \sec x\,dx$

(f) $\int_0^\infty \sin x\,dx$

(g) $\int_0^1 \sin x\,dx$ **(h)** $\int_0^1 \frac{dx}{\sqrt{3-x^2}}$ **(i)** $\int_1^\infty \ln x\,dx$

(j) $\int_0^3 \ln x\,dx$

SOLUTION

(a) Improper. The function $x^{-1/3}$ is infinite at 0.
(b) Improper. Infinite interval of integration.
(c) Improper. Infinite interval of integration.
(d) Proper. The function e^{-x} is continuous on the finite interval [0, 1].
(e) Improper. The function $\sec x$ is infinite at $\frac{\pi}{2}$.
(f) Improper. Infinite interval of integration.
(g) Proper. The function $\sin x$ is continuous on the finite interval [0, 1].
(h) Proper. The function $1/\sqrt{3-x^2}$ is continuous on the finite interval [0, 1].
(i) Improper. Infinite interval of integration.
(j) Improper. The function $\ln x$ is infinite at 0.

2. Let $f(x) = x^{-4/3}$.

(a) Evaluate $\int_1^R f(x)\,dx$.

(b) Evaluate $\int_1^\infty f(x)\,dx$ by computing the limit

$$\lim_{R\to\infty} \int_1^R f(x)\,dx$$

SOLUTION

(a) $\int_1^R x^{-4/3}\,dx = -3x^{-1/3}\Big|_1^R = -3R^{-1/3} - \big(-3(1)\big) = 3\left(1 - \frac{1}{R^{1/3}}\right)$.

(b) $\int_1^\infty x^{-4/3}\,dx = \lim_{R\to\infty} \int_1^R x^{-4/3}\,dx = \lim_{R\to\infty} 3\left(1 - \frac{1}{R^{1/3}}\right) = 3(1-0) = 3.$

3. Prove that $\int_1^\infty x^{-2/3}\,dx$ diverges by showing that

$$\lim_{R\to\infty} \int_1^R x^{-2/3}\,dx = \infty$$

SOLUTION First compute the proper integral:

$$\int_1^R x^{-2/3}\,dx = 3x^{1/3}\Big|_1^R = 3R^{1/3} - 3 = 3\left(R^{1/3} - 1\right).$$

Then show divergence:

$$\int_1^\infty x^{-2/3}\,dx = \lim_{R\to\infty} \int_1^R x^{-2/3}\,dx = \lim_{R\to\infty} 3\left(R^{1/3} - 1\right) = \infty.$$

4. Determine whether $\int_0^3 \frac{dx}{(3-x)^{3/2}}$ converges by computing

$$\lim_{R\to 3-} \int_0^R \frac{dx}{(3-x)^{3/2}}$$

SOLUTION First evaluate the integral on the interval $[0, R]$ for $0 < R < 3$:

$$\int_0^R \frac{dx}{(3-x)^{3/2}} = 2(3-x)^{-1/2}\Big|_0^R = \frac{2}{\sqrt{3-R}} - \frac{2}{\sqrt{3}}.$$

Now compute the limit as $R \to 3^-$:

$$\int_0^3 \frac{dx}{(3-x)^{3/2}} = \lim_{R\to 3^-} \int_0^R \frac{dx}{(3-x)^{3/2}} = \lim_{R\to 3^-} \left(\frac{2}{\sqrt{3-R}} - \frac{2}{\sqrt{3}}\right) = \infty;$$

thus, the integral diverges.

In Exercises 5–40, determine whether the improper integral converges and, if so, evaluate it.

5. $\int_1^\infty \frac{dx}{x^{19/20}}$

SOLUTION First evaluate the integral over the finite interval $[1, R]$ for $R > 1$:

$$\int_1^R \frac{dx}{x^{19/20}} = 20x^{1/20}\Big|_1^R = 20R^{1/20} - 20.$$

Now compute the limit as $R \to \infty$:

$$\int_1^\infty \frac{dx}{x^{19/20}} = \lim_{R\to\infty}\int_1^R \frac{dx}{x^{19/20}} = \lim_{R\to\infty}\left(20R^{1/20} - 20\right) = \infty.$$

The integral does not converge.

6. $\int_1^\infty \frac{dx}{x^{20/19}}$

SOLUTION First evaluate the integral over the finite interval $[1, R]$ for $R > 1$:

$$\int_1^R \frac{dx}{x^{20/19}} = -19x^{-1/19}\Big|_1^R = \frac{-19}{R^{1/19}} - (-19) = 19 - \frac{19}{R^{1/19}}.$$

Now compute the limit as $R \to \infty$:

$$\int_1^\infty \frac{dx}{x^{20/19}} = \lim_{R\to\infty}\int_1^R \frac{dx}{x^{20/19}} = \lim_{R\to\infty}\left(19 - \frac{19}{R^{1/19}}\right) = 19 - 0 = 19.$$

7. $\int_{-\infty}^4 e^{0.0001t}\, dt$

SOLUTION First evaluate the integral over the finite interval $[R, 4]$ for $R < 4$:

$$\int_R^4 e^{(0.0001)t}\, dt = \frac{e^{(0.0001)t}}{0.0001}\Big|_R^4 = 10{,}000\left(e^{0.0004} - e^{(0.0001)R}\right).$$

Now compute the limit as $R \to -\infty$:

$$\int_{-\infty}^4 e^{(0.0001)t}\, dt = \lim_{R\to-\infty}\int_R^4 e^{(0.0001)t}\, dt = \lim_{R\to-\infty} 10{,}000\left(e^{0.0004} - e^{(0.0001)R}\right)$$
$$= 10{,}000\left(e^{0.0004} - 0\right) = 10{,}000e^{0.0004}.$$

8. $\int_{20}^\infty \frac{dt}{t}$

SOLUTION First evaluate the integral over the finite interval $[20, R]$ for $20 < R$:

$$\int_{20}^R \frac{dt}{t} = \ln|t|\big|_{20}^R = \ln R - \ln 20.$$

Now compute the limit as $R \to \infty$:

$$\int_{20}^\infty \frac{dt}{t} = \lim_{R\to\infty}\int_{20}^R \frac{dt}{t} = \lim_{R\to\infty}(\ln R - \ln 20) = \infty;$$

thus, the integral does not converge.

9. $\int_0^5 \frac{dx}{x^{20/19}}$

SOLUTION The function $x^{-20/19}$ is infinite at the endpoint 0, so we'll first evaluate the integral on the finite interval $[R, 5]$ for $0 < R < 5$:

$$\int_R^5 \frac{dx}{x^{20/19}} = -19x^{-1/19}\Big|_R^5 = -19\left(5^{-1/19} - R^{-1/19}\right) = 19\left(\frac{1}{R^{1/19}} - \frac{1}{5^{1/19}}\right).$$

Now compute the limit as $R \to 0^+$:

$$\int_0^5 \frac{dx}{x^{20/19}} = \lim_{R\to 0^+} \int_R^5 \frac{dx}{x^{20/19}} = \lim_{R\to 0^+} 19\left(\frac{1}{R^{1/19}} - \frac{1}{5^{1/19}}\right) = \infty;$$

thus, the integral does not converge.

10. $\displaystyle\int_0^5 \frac{dx}{x^{19/20}}$

SOLUTION The function $x^{-19/20}$ is infinite at the endpoint 0, so we'll first evaluate the integral on the finite interval $[R, 5]$ for $0 < R < 5$:

$$\int_R^5 \frac{dx}{x^{19/20}} = 20x^{1/20}\Big|_R^5 = 20\left(5^{1/20} - R^{1/20}\right).$$

Now compute the limit as $R \to 0^+$:

$$\int_0^5 \frac{dx}{x^{19/20}} = \lim_{R\to 0^+} \int_R^5 \frac{dx}{x^{19/20}} = \lim_{R\to 0^+} 20\left(5^{1/20} - R^{1/20}\right) = 20\left(5^{1/20} - 0\right) = 20 \cdot 5^{1/20}.$$

11. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

SOLUTION The function $1/\sqrt{4-x}$ is infinite at $x = 4$, but is left-continuous at $x = 4$, so we'll first evaluate the integral on the interval $[0, R]$ for $0 < R < 4$:

$$\int_0^R \frac{dx}{\sqrt{4-x}} = -2\sqrt{4-x}\Big|_0^R = -2\sqrt{4-R} - (-2)\sqrt{4} = 4 - 2\sqrt{4-R}.$$

Now compute the limit as $R \to 4^-$:

$$\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{R\to 4^-} \int_0^R \frac{dx}{\sqrt{4-x}} = \lim_{R\to 4^-} \left(4 - 2\sqrt{4-R}\right) = 4 - 0 = 4.$$

12. $\displaystyle\int_5^6 \frac{dx}{(x-5)^{3/2}}$

SOLUTION The function $(x-5)^{-3/2}$ is infinite at $x = 5$, but is right-continuous at $x = 5$, so we'll first evaluate the integral on the interval $[R, 6]$ for $5 < R < 6$:

$$\int_R^6 \frac{dx}{(x-5)^{3/2}} = 2(x-5)^{-1/2}\Big|_R^6 = \frac{-2}{\sqrt{1}} - \frac{-2}{\sqrt{R-5}} = \frac{2}{\sqrt{R-5}} - 2.$$

Now compute the limit as $R \to 5^+$:

$$\int_5^6 \frac{dx}{(x-5)^{-3/2}} = \lim_{R\to 5^+} \int_R^6 \frac{dx}{(x-5)^{3/2}} = \lim_{R\to 5^+} \left(\frac{2}{\sqrt{R-5}} - 2\right) = \infty;$$

thus, the integral does not converge.

13. $\displaystyle\int_2^\infty x^{-3}\,dx$

SOLUTION First evaluate the integral on the finite interval $[2, R]$ for $2 < R$:

$$\int_2^R x^{-3}\,dx = \frac{x^{-2}}{-2}\Big|_2^R = \frac{-1}{2R^2} - \frac{-1}{2(2^2)} = \frac{1}{8} - \frac{1}{2R^2}.$$

Now compute the limit as $R \to \infty$:

$$\int_2^\infty x^{-3}\,dx = \lim_{R\to\infty} \int_2^R x^{-3}\,dx = \lim_{R\to\infty} \left(\frac{1}{8} - \frac{1}{2R^2}\right) = \frac{1}{8}.$$

14. $\displaystyle\int_0^\infty \frac{dx}{(x+1)^3}$

SOLUTION First evaluate the integral on the finite interval $[0, R]$ for $R > 0$:

$$\int_0^R \frac{dx}{(x+1)^3} = \frac{(x+1)^{-2}}{-2}\Big|_0^R = \frac{-1}{2(R+1)^2} - \frac{-1}{2(1)^2} = \frac{1}{2} - \frac{1}{2(R+1)^2}.$$

Now compute the limit as $R \to \infty$:

$$\int_0^\infty \frac{dx}{(x+1)^3} = \lim_{R\to\infty} \int_0^R \frac{dx}{(x+1)^3} = \lim_{R\to\infty} \left(\frac{1}{2} - \frac{1}{2(R+1)^2}\right) = \frac{1}{2}.$$

15. $\displaystyle\int_{-3}^\infty \frac{dx}{(x+4)^{3/2}}$

SOLUTION First evaluate the integral on the finite interval $[-3, R]$ for $R > -3$:

$$\int_{-3}^R \frac{dx}{(x+4)^{3/2}} = -2(x+4)^{-1/2}\Big|_{-3}^R = \frac{-2}{\sqrt{R+4}} - \frac{-2}{\sqrt{1}} = 2 - \frac{2}{\sqrt{R+4}}.$$

Now compute the limit as $R \to \infty$:

$$\int_{-3}^\infty \frac{dx}{(x+4)^{3/2}} = \lim_{R\to\infty} \int_{-3}^R \frac{dx}{(x+4)^{3/2}} = \lim_{R\to\infty} \left(2 - \frac{2}{\sqrt{R+4}}\right) = 2 - 0 = 2.$$

16. $\displaystyle\int_2^\infty e^{-2x}\,dx$

SOLUTION First evaluate the integral on the finite interval $[2, R]$ for $R > 2$:

$$\int_2^R e^{-2x}\,dx = \frac{e^{-2x}}{-2}\Big|_2^R = -\frac{1}{2}\left(e^{-2R} - e^{-4}\right) = \frac{1}{2}\left(e^{-4} - e^{-2R}\right).$$

Now compute the limit as $R \to \infty$:

$$\int_2^\infty e^{-2x}\,dx = \lim_{R\to\infty} \int_2^R e^{-2x}\,dx = \lim_{R\to\infty} \left(e^{-4} - e^{-2R}\right) = \frac{1}{2}\left(e^{-4} - 0\right) = \frac{1}{2e^4}.$$

17. $\displaystyle\int_0^1 \frac{dx}{x^{0.2}}$

SOLUTION The function $x^{-0.2}$ is infinite at $x = 0$ and right-continuous at $x = 0$, so we'll first evaluate the integral on the interval $[R, 1]$ for $0 < R < 1$:

$$\int_R^1 \frac{dx}{x^{0.2}} = \frac{x^{0.8}}{0.8}\Big|_R^1 = 1.25\left(1 - R^{0.8}\right).$$

Now compute the limit as $R \to 0^+$:

$$\int_0^1 \frac{dx}{x^{0.2}} = \lim_{R\to 0^+} \int_R^1 \frac{dx}{x^{0.2}} = \lim_{R\to 0^+} 1.25\left(1 - R^{0.8}\right) = 1.25(1-0) = 1.25.$$

18. $\displaystyle\int_2^\infty x^{-1/3}\,dx$

SOLUTION First evaluate the integral on the finite interval $[2, R]$ for $R > 2$:

$$\int_2^R x^{-1/3}\,dx = \frac{3}{2}x^{2/3}\Big|_2^R = \frac{3}{2}\left(R^{2/3} - 2^{2/3}\right).$$

Now compute the limit as $R \to \infty$:

$$\int_2^\infty x^{-1/3}\,dx = \lim_{R\to\infty} \int_2^R x^{-1/3}\,dx = \lim_{R\to\infty} \frac{3}{2}\left(R^{2/3} - 2^{2/3}\right) = \infty;$$

thus, the integral does not converge.

19. $\displaystyle\int_4^\infty e^{-3x}\,dx$

SOLUTION First evaluate the integral on the finite interval $[4, R]$ for $R > 4$:

$$\int_4^R e^{-3x}\,dx = \frac{e^{-3x}}{-3}\Big|_4^R = -\frac{1}{3}\left(e^{-3R} - e^{-12}\right) = \frac{1}{3}\left(e^{-12} - e^{-3R}\right).$$

Now compute the limit as $R \to \infty$:

$$\int_4^{\infty} e^{-3x}\,dx = \lim_{R\to\infty}\int_4^R e^{-3x}\,dx = \lim_{R\to\infty}\frac{1}{3}\left(e^{-12} - e^{-3R}\right) = \frac{1}{3}\left(e^{-12} - 0\right) = \frac{1}{3e^{12}}.$$

20. $\displaystyle\int_4^{\infty} e^{3x}\,dx$

SOLUTION First evaluate the integral on the finite interval $[4, R]$ for $R > 4$:

$$\int_4^R e^{3x}\,dx = \left.\frac{e^{3x}}{3}\right|_4^R = \frac{1}{3}\left(e^{3R} - e^{12}\right).$$

Now compute the limit as $R \to \infty$:

$$\int_4^{\infty} e^{3x}\,dx = \lim_{R\to\infty}\int_4^R e^{3x}\,dx = \lim_{R\to\infty}\frac{1}{3}\left(e^{3R} - e^{12}\right) = \infty;$$

thus, the integral does not converge.

21. $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx$

SOLUTION First evaluate the integral on the finite interval $[R, 0]$ for $R < 0$:

$$\int_R^0 e^{3x}\,dx = \left.\frac{e^{3x}}{3}\right|_R^0 = \frac{1}{3} - \frac{e^{3R}}{3}.$$

Now compute the limit as $R \to -\infty$:

$$\int_{-\infty}^0 e^{3x}\,dx = \lim_{R\to-\infty}\int_R^0 e^{3x}\,dx = \lim_{R\to-\infty}\left(\frac{1}{3} - \frac{e^{3R}}{3}\right) = \frac{1}{3} - 0 = \frac{1}{3}.$$

22. $\displaystyle\int_1^2 \frac{dx}{(x-1)^2}$

SOLUTION The function $(x-1)^{-2}$ is infinite at $x = 1$ and is right-continuous at $x = 1$, so we first evaluate the integral on the interval $[R, 2]$ for $1 < R < 2$:

$$\int_R^2 \frac{dx}{(x-1)^2} = \left.\frac{(x-1)^{-1}}{-1}\right|_R^2 = \frac{-1}{1} - \frac{-1}{R-1} = \frac{1}{R-1} - 1.$$

Now compute the limit as $R \to 1^+$:

$$\int_1^2 \frac{dx}{(x-1)^2} = \lim_{R\to1^+}\int_R^2 \frac{dx}{(x-1)^2} = \lim_{R\to1^+}\left(\frac{1}{R-1} - 1\right) = \infty;$$

thus, the integral does not converge.

23. $\displaystyle\int_1^3 \frac{dx}{\sqrt{3-x}}$

SOLUTION The function $f(x) = 1/\sqrt{3-x}$ is infinite at $x = 3$ and is left continuous at $x = 3$, so we first evaluate the integral on the interval $[1, R]$ for $1 < R < 3$:

$$\int_1^R \frac{dx}{\sqrt{3-x}} = \left.-2\sqrt{3-x}\right|_1^R = -2\sqrt{3-R} + 2\sqrt{2}.$$

Now compute the limit as $R \to 3^-$:

$$\int_1^3 \frac{dx}{\sqrt{3-x}} = \lim_{R\to3-}\int_1^R \frac{dx}{\sqrt{3-x}} = 0 + 2\sqrt{2} = 2\sqrt{2}.$$

24. $\displaystyle\int_{-2}^4 \frac{dx}{(x+2)^{1/3}}$

SOLUTION The function $(x+2)^{-1/3}$ is infinite at $x = -2$ and right-continuous at $x = -2$, so we'll first evaluate the integral on the interval $[R, 4]$ for $-2 < R < 4$:

$$\int_R^4 \frac{dx}{(x+2)^{1/3}} = \left.\frac{3}{2}(x+2)^{2/3}\right|_R^4 = \frac{3}{2}\left(6^{2/3} - (R+2)^{2/3}\right).$$

Now compute the limit as $R \to -2^+$:

$$\int_{-2}^{4} \frac{dx}{(x+2)^{1/3}} = \lim_{R\to -2^+} \int_{R}^{4} \frac{dx}{(x+2)^{1/3}} = \lim_{R\to 2^+} \frac{3}{2}\left(6^{2/3} - (R+2)^{2/3}\right) = \frac{3}{2}\left(6^{2/3} - 0\right) = \frac{3\cdot 6^{2/3}}{2}.$$

25. $\displaystyle\int_0^\infty \frac{dx}{1+x}$

SOLUTION First evaluate the integral on the finite interval $[0, R]$ for $R > 0$:

$$\int_0^R \frac{dx}{1+x} = \ln|1+x|\Big|_0^R = \ln|1+R| - \ln 1 = \ln|1+R|.$$

Now compute the limit as $R \to \infty$:

$$\int_0^\infty \frac{dx}{1+x} = \lim_{R\to\infty} \int_0^R \frac{dx}{1+x} = \lim_{R\to\infty} \ln|1+R| = \infty;$$

thus, the integral does not converge.

26. $\displaystyle\int_{-\infty}^0 xe^{-x^2}\,dx$

SOLUTION First evaluate the indefinite integral using substitution, with $u = -x^2$, $du = -2x\,dx$. This gives us

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^{-x^2}(-2x\,dx) = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C.$$

Next, evaluate the integral on the finite interval $[R, 0]$ for $R < 0$:

$$\int_R^0 xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}\Big|_R^0 = -\frac{1}{2}\left(1 - e^{-R^2}\right).$$

Finally, compute the limit as $R \to -\infty$:

$$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{R\to-\infty}\int_R^0 xe^{-x^2}\,dx = \lim_{R\to-\infty}\frac{1}{2}\left(e^{-R^2} - 1\right) = \frac{1}{2}(0-1) = -\frac{1}{2}.$$

27. $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x^2)^2}$

SOLUTION First evaluate the indefinite integral, using the substitution $u = x^2$, $du = 2x\,dx$; then

$$\int \frac{x\,dx}{(1+x^2)^2} = \frac{1}{2}\int \frac{1}{(1+u)^2}\,du = -\frac{1}{2(u+1)} + C = -\frac{1}{2(x^2+1)} + C$$

Thus, for $R > 0$,

$$\int_0^R \frac{x\,dx}{(x^2+1)^2} = \left(-\frac{1}{2(x^2+1)}\right)\Bigg|_0^R = -\frac{1}{2(R^2+1)} + \frac{1}{2}$$

and thus in the limit

$$\int_0^\infty \frac{x\,dx}{(x^2+1)^2} = \lim_{R\to\infty}\int_0^R \frac{x\,dx}{(x^2+1)^2} = \frac{1}{2} - \lim_{R\to\infty}\frac{1}{2(R^2+1)} = \frac{1}{2}$$

28. $\displaystyle\int_3^6 \frac{x\,dx}{\sqrt{x-3}}$

SOLUTION First, evaluate the indefinite integral using the substitution $u = x - 3$, $du = dx$:

$$\int \frac{x}{\sqrt{x-3}}\,dx = \int \frac{u+3}{\sqrt{u}}\,du = \frac{2}{3}u^{3/2} + 6u^{1/2} + C = \frac{2}{3}(x-3)^{3/2} + 6(x-3)^{1/2} + C.$$

Next, evaluate the definite integral over the interval $[R, 6]$ for $R > 3$:

$$\int_R^6 \frac{x}{\sqrt{x-3}}\,dx = \left(\frac{2}{3}(x-3)^{3/2} + 6(x-3)^{1/2}\right)\Big|_R^6 = \frac{2}{3}3^{3/2} + 6\sqrt{3} - \frac{2}{3}(R-3)^{3/2} - 6(R-3)^{1/2}$$
$$= 8\sqrt{3} - \frac{2}{3}(R-3)^{3/2} - 6(R-3)^{1/2}.$$

Finally, we compute the limit as $R \to 3^+$:

$$\int_3^6 \frac{x}{\sqrt{x-3}}\,dx = \lim_{R\to 3^+}\int_R^6 \frac{x}{\sqrt{x-3}}\,dx = \lim_{R\to 3^+}\left(8\sqrt{3} - \frac{2}{3}(R-3)^{3/2} - 6(R-3)^{1/2}\right) = 8\sqrt{3}.$$

29. $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$

SOLUTION First evaluate the indefinite integral using Integration by Parts, with $u = e^{-x}$, $v' = \cos x$. Then $u' = -e^{-x}$, $v = \sin x$, and

$$\int e^{-x}\cos x\,dx = e^{-x}\sin x - \int \sin x(-e^{-x})\,dx = e^{-x}\sin x + \int e^{-x}\sin x\,dx.$$

Now use Integration by Parts again, with $u = e^{-x}$, $v' = \sin x$. Then $u' = -e^{-x}$, $v = -\cos x$, and

$$\int e^{-x}\cos x\,dx = e^{-x}\sin x + \left[-e^{-x}\cos x - \int e^{-x}\cos x\,dx\right].$$

Solving this equation for $\int e^{-x}\cos x\,dx$, we find

$$\int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x) + C.$$

Thus,

$$\int_0^R e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x)\Big|_0^R = \frac{\sin R - \cos R}{2e^R} - \frac{\sin 0 - \cos 0}{2} = \frac{\sin R - \cos R}{2e^R} + \frac{1}{2},$$

and

$$\int_0^\infty e^{-x}\cos x\,dx = \lim_{R\to\infty}\left(\frac{\sin R - \cos R}{2e^R} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}.$$

30. $\displaystyle\int_1^\infty xe^{-2x}\,dx$

SOLUTION First evaluate the indefinite integral using Integration by Parts, with $u = x$ and $v' = e^{-2x}$. Then $u' = 1$, $v = -\frac{1}{2}e^{-2x}$, and

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \int\left(-\frac{1}{2}\right)e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$$
$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C = -\frac{1}{4}e^{-2x}(2x+1) + C = \frac{-(2x+1)}{4e^{2x}} + C.$$

Therefore,

$$\int_1^\infty xe^{-2x}\,dx = \lim_{R\to-\infty}\int_1^R xe^{-2x}\,dx = \lim_{R\to\infty}\left(\frac{-(2x+1)}{4e^{2x}}\Big|_1^R\right) = \lim_{R\to\infty}\left[\frac{-(2R+1)}{4e^{2R}} + \frac{3}{4e^2}\right].$$

Use L'Hôpital's Rule to evaluate the limit:

$$\int_1^\infty xe^{-2x}\,dx = \frac{3}{4e^2} - \lim_{R\to\infty}\frac{2}{8e^{2R}} = \frac{3}{4e^2} - 0 = \frac{3}{4e^2}.$$

31. $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

SOLUTION The function $(9-x^2)^{-1/2}$ is infinite at $x = 3$, and is left-continuous at $x = 3$, so we'll first evaluate the integral on the interval $[0, R]$ for $0 < R < 3$:

$$\int_0^R \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3}\Big|_0^R = \sin^{-1}\frac{R}{3} - \sin^{-1}0 = \sin^{-1}\frac{R}{3}.$$

Thus,

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{R\to 3^-} \sin^{-1}\frac{R}{3} = \sin^{-1} 1 = \frac{\pi}{2}.$$

32. $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$

SOLUTION Let $u = \sqrt{x}$, $du = \frac{1}{2}x^{-1/2}\,dx$. Then

$$\int \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = 2\int e^{\sqrt{x}}\left(\frac{dx}{2\sqrt{x}}\right) = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C.$$

The function $e^{\sqrt{x}}/\sqrt{x}$ is infinite and right-continuous at $x = 0$, so we first evaluate the integral on $[R, 1]$ for $0 < R < 1$:

$$\int_R^1 \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = 2e^{\sqrt{x}}\Big|_R^1 = 2e - 2e^{\sqrt{R}}.$$

Now we compute the limit as $R \to 0+$:

$$\int_0^1 \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = \lim_{R\to 0^+}\left(2e - 2e^{\sqrt{x}}\right) = 2e - 2(1) = 2(e-1).$$

33. $\displaystyle\int_1^\infty \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$

SOLUTION Let $u = \sqrt{x}$, $du = \frac{1}{2}x^{-1/2}\,dx$. Then

$$\int \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = 2\int e^{\sqrt{x}}\left(\frac{dx}{2\sqrt{x}}\right) = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C,$$

and

$$\int_1^\infty \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = \lim_{R\to\infty}\int_1^R \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = \lim_{R\to\infty} 2e^{\sqrt{x}}\Big|_1^R = \lim_{R\to\infty}\left(2e^{\sqrt{R}} - 2e\right) = \infty.$$

The integral does not converge.

34. $\displaystyle\int_0^{\pi/2} \sec\theta\,d\theta$

SOLUTION First, evaluate the integral on the interval $[0, R]$ for $0 < R < \frac{\pi}{2}$:

$$\int_0^R \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big|_0^R = \ln|\sec R + \tan R|.$$

Now we compute the limit as $R \to \frac{\pi}{2}^-$:

$$\int_0^{\pi/2} \sec\theta\,d\theta = \lim_{R\to\pi/2^-}\int_0^R \sec\theta\,d\theta = \lim_{R\to\pi/2^-} \ln|\sec R + \tan R| = \infty.$$

The integral does not converge.

35. $\displaystyle\int_0^\infty \sin x\,dx$

SOLUTION First evaluate the integral on the finite interval $[0, R]$ for $R > 0$:

$$\int_0^R \sin x\,dx = -\cos x\Big|_0^R = -\cos R + \cos 0 = 1 - \cos R.$$

Thus,

$$\int_0^R \sin x\,dx = \lim_{R\to\infty}(1 - \cos R) = 1 - \lim_{R\to\infty}\cos R.$$

This limit does not exist, since the value of $\cos R$ oscillates between 1 and -1 as R approaches infinity. Hence the integral does not converge.

36. $\displaystyle\int_0^{\pi/2} \tan x\, dx$

SOLUTION The function $\tan x$ is infinite and left-continuous at $x = \frac{\pi}{2}$, so we'll first evaluate the integral on $[0, R]$ for $0 < R < \frac{\pi}{2}$:

$$\int_0^R \tan x\, dx = \ln|\sec x|\Big|_0^R = \ln|\sec R|.$$

Thus,

$$\int_0^{\pi/2} \tan x\, dx = \lim_{R\to\frac{\pi}{2}-} \int_0^R \tan x\, dx = \lim_{R\to\frac{\pi}{2}-} \big(\ln|\sec R|\big) = \infty.$$

The integral does not converge.

37. $\displaystyle\int_0^1 \ln x\, dx$

SOLUTION The function $\ln x$ is infinite and right-continuous at $x = 0$, so we'll first evaluate the integral on $[R, 1]$ for $0 < R < 1$. Use Integration by Parts with $u = \ln x$ and $v' = 1$. Then $u' = 1/x$, $v = x$, and we have

$$\int_R^1 \ln x\, dx = x\ln x\Big|_R^1 - \int_R^1 dx = (x\ln x - x)\Big|_R^1 = (\ln 1 - 1) - (R\ln R - R) = R - 1 - R\ln R.$$

Thus,

$$\int_0^1 \ln x\, dx = \lim_{R\to 0^+}(R - 1 - R\ln R) = -1 - \lim_{R\to 0^+} R\ln R.$$

To compute the limit, rewrite the function as a quotient and apply L'Hôpital's Rule:

$$\int_0^1 \ln x\, dx = -1 - \lim_{R\to 0^+} \frac{\ln R}{\frac{1}{R}} = -1 - \lim_{R\to 0^+} \frac{\frac{1}{R}}{\frac{-1}{R^2}} = -1 - \lim_{R\to 0^+}(-R) = -1 - 0 = -1.$$

38. $\displaystyle\int_1^2 \frac{dx}{x\ln x}$

SOLUTION Evaluate the indefinite integral using substitution, with $u = \ln x$, $du = (1/x)\, dx$. Then

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C.$$

Thus,

$$\int_R^2 \frac{dx}{x\ln x} = \ln|\ln x|\Big|_R^2 = \ln(\ln 2) - \ln(\ln R),$$

and

$$\int_1^2 \frac{dx}{x\ln x} = \lim_{R\to 1^+}\big[\ln(\ln 2) - \ln(\ln R)\big] = \ln(\ln 2) - \lim_{R\to 1^+} \ln(\ln R) = \infty.$$

The integral does not converge.

39. $\displaystyle\int_0^1 \frac{\ln x}{x^2}\, dx$

SOLUTION Use Integration by Parts, with $u = \ln x$ and $v' = x^{-2}$. Then $u' = 1/x$, $v = -x^{-1}$, and

$$\int \frac{\ln x}{x^2}\, dx = -\frac{1}{x}\ln x + \int \frac{dx}{x^2} = -\frac{1}{x}\ln x - \frac{1}{x} + C.$$

The function is infinite and right-continuous at $x = 0$, so we'll first evaluate the integral on $[R, 1]$ for $0 < R < 1$:

$$\int_a^1 \frac{\ln x}{x^2}\, dx = \left(-\frac{1}{x}\ln x - \frac{1}{x}\right)\Big|_R^1 = \left(-\frac{1}{1}\ln 1 - \frac{1}{1}\right) - \left(-\frac{1}{R}\ln R - \frac{1}{R}\right) = \frac{1}{R}\ln R + \frac{1}{R} - 1.$$

Thus,

$$\int_0^1 \frac{\ln x}{x^2}\, dx = \lim_{R\to 0^+} \frac{1}{R}\ln R + \frac{1}{R} - 1 = -1 + \lim_{R\to 0^+} \frac{\ln R + 1}{R} = -\infty.$$

The integral does not converge.

40. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

SOLUTION Use Integration by Parts, with $u = \ln x$ and $v' = x^{-2}$. Then $u' = x^{-1}$, $v = -x^{-1}$, and

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x + \int x^{-2}\,dx = -\frac{1}{x}\ln x - \frac{1}{x} + C.$$

Thus,

$$\int_1^R \frac{\ln x}{x^2}\,dx = \left(-\frac{1}{x}\ln x - \frac{1}{x}\right)\Bigg|_1^R = \left(-\frac{1}{R}\ln R - \frac{1}{R}\right) - \left(-\frac{1}{1}\ln 1 - \frac{1}{1}\right) = 1 - \frac{1}{R}\ln R - \frac{1}{R}.$$

Use L'Hôpital's Rule to compute the limit:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{R\to\infty}\left(1 - \frac{1}{R}\ln R - \frac{1}{R}\right) = 1 - \lim_{R\to\infty}\left(\frac{\ln R}{R}\right) - 0 = 1 - \lim_{R\to\infty}\frac{\frac{1}{R}}{1} = 1 - \frac{0}{1} = 1.$$

41. Let $I = \displaystyle\int_4^\infty \frac{dx}{(x-2)(x-3)}$.

(a) Show that for $R > 4$,

$$\int_4^R \frac{dx}{(x-2)(x-3)} = \ln\left|\frac{R-3}{R-2}\right| - \ln\frac{1}{2}$$

(b) Then show that $I = \ln 2$.

SOLUTION

(a) The partial fraction decomposition takes the form

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Clearing denominators gives us

$$1 = A(x-3) + B(x-2).$$

Setting $x = 2$ then yields $A = -1$, while setting $x = 3$ yields $B = 1$. Thus,

$$\int \frac{dx}{(x-2)(x-3)} = \int \frac{dx}{x-3} - \int \frac{dx}{x-2} = \ln|x-3| - \ln|x-2| + C = \ln\left|\frac{x-3}{x-2}\right| + C,$$

and, for $R > 4$,

$$\int_4^R \frac{dx}{(x-2)(x-3)} = \ln\left|\frac{x-3}{x-2}\right|\Bigg|_4^R = \ln\left|\frac{R-3}{R-2}\right| - \ln\frac{1}{2}.$$

(b) Using the result from part (a),

$$I = \lim_{R\to\infty}\left(\ln\left|\frac{R-3}{R-2}\right| - \ln\frac{1}{2}\right) = \ln 1 - \ln\frac{1}{2} = \ln 2.$$

42. Evaluate the integral $I = \displaystyle\int_1^\infty \frac{dx}{x(2x+5)}$.

SOLUTION The partial fraction decomposition takes the form

$$\frac{1}{x(2x+5)} = \frac{A}{x} + \frac{B}{2x+5}.$$

Clearing denominators gives us

$$1 = A(2x+5) + Bx.$$

Setting $x = 0$ then yields $A = \frac{1}{5}$, while setting $x = -\frac{5}{2}$ yields $B = -\frac{2}{5}$. Thus,

$$\int \frac{dx}{x(2x+5)} = \frac{1}{5}\int \frac{dx}{x} - \frac{2}{5}\int \frac{dx}{2x+5} = \frac{1}{5}\ln|x| - \frac{1}{5}\ln|2x+5| + C = \frac{1}{5}\ln\left|\frac{x}{2x+5}\right| + C,$$

and, for $R > 1$,

$$\int_1^R \frac{dx}{x(2x+5)} = \frac{1}{5}\ln\left|\frac{x}{2x+5}\right|\Bigg|_1^R = \frac{1}{5}\ln\left|\frac{R}{2R+5}\right| - \frac{1}{5}\ln\frac{1}{7}.$$

Thus,

$$I = \lim_{R\to\infty}\left(\frac{1}{5}\ln\left|\frac{R}{2R+5}\right| - \frac{1}{5}\ln\frac{1}{7}\right) = \frac{1}{5}\ln\frac{1}{2} - \frac{1}{5}\ln\frac{1}{7} = \frac{1}{5}\ln\frac{7}{2}.$$

43. Evaluate $I = \int_0^1 \frac{dx}{x(2x+5)}$ or state that it diverges.

SOLUTION The partial fraction decomposition takes the form

$$\frac{1}{x(2x+5)} = \frac{A}{x} + \frac{B}{2x+5}.$$

Clearing denominators gives us

$$1 = A(2x+5) + Bx.$$

Setting $x = 0$ then yields $A = \frac{1}{5}$, while setting $x = -\frac{5}{2}$ yields $B = -\frac{2}{5}$. Thus,

$$\int \frac{dx}{x(2x+5)} = \frac{1}{5}\int\frac{dx}{x} - \frac{2}{5}\int\frac{dx}{2x+5} = \frac{1}{5}\ln|x| - \frac{1}{5}\ln|2x+5| + C = \frac{1}{5}\ln\left|\frac{x}{2x+5}\right| + C,$$

and, for $0 < R < 1$,

$$\int_R^1 \frac{dx}{x(2x+5)} = \frac{1}{5}\ln\left|\frac{x}{2x+5}\right|\Bigg|_R^1 = \frac{1}{5}\ln\frac{1}{7} - \frac{1}{5}\ln\left|\frac{R}{2R+5}\right|.$$

Thus,

$$I = \lim_{R\to 0+}\left(\frac{1}{5}\ln\frac{1}{7} - \frac{1}{5}\ln\left|\frac{R}{2R+5}\right|\right) = \infty.$$

The integral does not converge.

44. Evaluate $I = \int_2^\infty \frac{dx}{(x+3)(x+1)^2}$ or state that it diverges.

SOLUTION The partial fraction decomposition takes the form

$$\frac{1}{(x+3)(x+1)^2} = \frac{A}{x+3} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Clearing denominators gives us

$$1 = A(x+1)^2 + B(x+1)(x+3) + C(x+3).$$

Setting $x = -3$ then yields $A = \frac{1}{4}$, while setting $x = -1$ yields $C = \frac{1}{2}$. Setting $x = 0$ gives $1 = \frac{1}{4} + 3B + \frac{3}{2}$ or $B = -\frac{1}{4}$. Thus,

$$\begin{aligned}\int \frac{dx}{(x+3)(x+1)^2} &= \frac{1}{4}\int\frac{dx}{x+3} - \frac{1}{4}\int\frac{dx}{x+1} + \frac{1}{2}\int\frac{dx}{(x+1)^2}\\ &= \frac{1}{4}\ln|x+3| - \frac{1}{4}\ln|x+1| - \frac{1}{2(x+1)} + C = \frac{1}{4}\ln\left|\frac{x+3}{x+1}\right| - \frac{1}{2(x+1)} + C,\end{aligned}$$

and, for $R > 2$,

$$\int_2^R \frac{dx}{(x+3)(x+1)^2} = \left(\frac{1}{4}\ln\left|\frac{x+3}{x+1}\right| - \frac{1}{2(x+1)}\right)\Bigg|_2^R = \frac{1}{4}\ln\left|\frac{R+3}{R+1}\right| - \frac{1}{2(R+1)} - \frac{1}{4}\ln\frac{5}{3} + \frac{1}{6}.$$

Thus

$$I = \lim_{R\to\infty}\left(\frac{1}{4}\ln\left|\frac{R+3}{R+1}\right| - \frac{1}{2(R+1)} - \frac{1}{4}\ln\frac{5}{3} + \frac{1}{6}\right) = \frac{1}{6} - \frac{1}{4}\ln\frac{5}{3}.$$

In Exercises 45–48, determine whether the doubly infinite improper integral converges and, if so, evaluate it. Use definition (2).

45. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^2}$

SOLUTION Using the substitution $u = x^2 + 1$, $du = 2x\,dx$, we obtain

$$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(x^2+1) + C.$$

Thus,

$$\int_0^{\infty} \frac{x\,dx}{1+x^2} = \lim_{R\to\infty} \int_0^R \frac{x\,dx}{1+x^2} = \lim_{R\to\infty} \frac{1}{2}\ln(R^2+1) = \infty;$$

$$\int_{-\infty}^{0} \frac{x\,dx}{1+x^2} = \lim_{R\to-\infty} \int_R^0 \frac{x\,dx}{1+x^2} = \lim_{R\to-\infty} \frac{1}{2}\ln(R^2+1) = \infty;$$

It follows that

$$\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^2}$$

diverges.

46. $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\,dx$

SOLUTION First, we find

$$\int_0^{\infty} e^{-|x|}\,dx = \int_0^{\infty} e^{-x}\,dx = \lim_{R\to\infty} \int_0^R e^{-x}\,dx = \lim_{R\to\infty} \left(1 - e^{-R}\right) = 1;$$

$$\int_{-\infty}^{0} e^{-|x|}\,dx = \int_{\infty}^{0} e^{x}\,dx = \lim_{R\to-\infty} \int_R^0 e^{x}\,dx = \lim_{R\to-\infty} \left(1 - e^{R}\right) = 1;$$

and

$$\int_{\infty}^{\infty} e^{-|x|}\,dx = 1 + 1 = 2.$$

47. $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

SOLUTION First note that

$$\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + C.$$

Thus,

$$\int_0^{\infty} xe^{-x^2}\,dx = \lim_{R\to\infty} \int_0^R xe^{-x^2}\,dx = \lim_{R\to\infty} \left(\frac{1}{2} - \frac{1}{2}e^{-R^2}\right) = \frac{1}{2};$$

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{R\to-\infty} \int_R^0 xe^{-x^2}\,dx = \lim_{R\to-\infty} \left(-\frac{1}{2} + \frac{1}{2}e^{-R^2}\right) = -\frac{1}{2};$$

and

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \frac{1}{2} - \frac{1}{2} = 0.$$

48. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^{3/2}}$

SOLUTION First, we evaluate the indefinite integral using the trigonometric substitution $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$. Then

$$\int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C = \frac{x}{\sqrt{1+x^2}} + C.$$

Thus,

$$\int_0^{\infty} \frac{dx}{(1+x^2)^{3/2}} = \lim_{R\to\infty} \int_0^R \frac{dx}{(1+x^2)^{3/2}} = \lim_{R\to\infty} \frac{R}{\sqrt{1+R^2}} = 1;$$

$$\int_{-\infty}^{0} \frac{dx}{(1+x^2)^{3/2}} = \lim_{R\to-\infty} \int_R^0 \frac{dx}{(1+x^2)^{3/2}} = \lim_{R\to-\infty} -\frac{R}{\sqrt{1+R^2}} = 1;$$

and

$$\int_{\infty}^{\infty} \frac{dx}{(1+x^2)^{(3/2)}} = 1+1 = 2.$$

49. Define $J = \int_{-1}^{1} \frac{dx}{x^{1/3}}$ as the sum of the two improper integrals $\int_{-1}^{0} \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}}$. Show that J converges and that $J = 0$.

SOLUTION Note that since $x^{-1/3}$ is an odd function, one might expect this integral over a symmetric interval to be zero. To prove this, we start by evaluating the indefinite integral:

$$\int \frac{dx}{x^{1/3}} = \frac{3}{2}x^{2/3} + C$$

Then

$$\int_{-1}^{0} \frac{dx}{x^{1/3}} = \lim_{R\to 0^-} \int_{-1}^{R} \frac{dx}{x^{1/3}} = \lim_{R\to 0^-} \frac{3}{2}x^{2/3}\Big|_{-1}^{R} = \lim_{R\to 0^-} \frac{3}{2}R^{2/3} - \frac{3}{2} = -\frac{3}{2}$$

$$\int_{0}^{1} \frac{dx}{x^{1/3}} = \lim_{R\to 0^+} \int_{R}^{1} \frac{dx}{x^{1/3}} = \lim_{R\to 0^+} \frac{3}{2}x^{2/3}\Big|_{R}^{1} = \frac{3}{2} - \lim_{R\to 0^+} \frac{3}{2}R^{2/3} = \frac{3}{2}$$

so that

$$J = \int_{-1}^{1} \frac{dx}{x^{1/3}} = \int_{-1}^{0} \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} = -\frac{3}{2} + \frac{3}{2} = 0$$

50. Determine whether $J = \int_{-1}^{1} \frac{dx}{x^2}$ (defined as in Exercise 49) converges.

SOLUTION We have

$$\int \frac{dx}{x^2} = -\frac{1}{x} + C$$

so that

$$\int_{-1}^{0} \frac{dx}{x^2} = \lim_{R\to 0^-} \int_{-1}^{R} \frac{dx}{x^2} = \lim_{R\to 0^-} \left(-\frac{1}{x}\Big|_{-1}^{R}\right) = \lim_{R\to 0^-} \left(-\frac{1}{R} + 1\right) = 1 - \lim_{R\to 0^-} \frac{1}{R} = \infty$$

$$\int_{0}^{1} \frac{dx}{x^2} = \lim_{R\to 0^+} \int_{R}^{1} \frac{dx}{x^2} = \lim_{R\to 0^+} \left(-\frac{1}{x}\Big|_{R}^{1}\right) = \lim_{R\to 0^+} \left(-1 + \frac{1}{R}\right) = -1 + \lim_{R\to 0^+} \frac{1}{R} = \infty$$

so that the integral diverges.

51. For which values of a does $\int_0^{\infty} e^{ax}\,dx$ converge?

SOLUTION First evaluate the integral on the finite interval $[0, R]$ for $R > 0$:

$$\int_0^R e^{ax}\,dx = \frac{1}{a}e^{ax}\Big|_0^R = \frac{1}{a}\left(e^{aR} - 1\right).$$

Thus,

$$\int_0^{\infty} e^{ax}\,dx = \lim_{R\to\infty} \frac{1}{a}\left(e^{aR} - 1\right).$$

If $a > 0$, then $e^{aR} \to \infty$ as $R \to \infty$. If $a < 0$, then $e^{aR} \to 0$ as $R \to \infty$, and

$$\int_0^{\infty} e^{ax}\,dx = \lim_{R\to\infty} \frac{1}{a}\left(e^{aR} - 1\right) = -\frac{1}{a}.$$

The integral converges for $a < 0$.

52. Show that $\int_0^1 \frac{dx}{x^p}$ converges if $p < 1$ and diverges if $p \geq 1$.

SOLUTION The function x^{-p} is infinite and right-continuous at $x = 0$, so we'll first evaluate the integral on $[R, 1]$ for $0 < R < 1$:

$$\int_R^1 \frac{dx}{x^p} = \left.\frac{x^{-p+1}}{-p+1}\right|_R^1 = \frac{1}{-p+1}\left(1 - R^{-p+1}\right).$$

If $p < 1$, then $-p+1 = 1-p > 0$, and

$$\int_0^1 \frac{dx}{x^p} = \lim_{R\to 0^+} \frac{1}{1-p}\left(1 - R^{1-p}\right) = \frac{1}{1-p}(1-0) = \frac{1}{1-p}.$$

If $p > 1$, then $-p+1 < 0$, and

$$\int_0^1 \frac{dx}{x^p} = \lim_{R\to 0^+} \frac{1}{1-p}\left(1 - R^{1-p}\right) = \lim_{R\to 0^+} \frac{1}{1-p}\left(1 - \frac{1}{a^{p-1}}\right) = \infty.$$

If $p = 1$, then

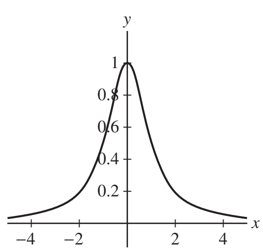
$$\int_R^1 \frac{dx}{x^p} = \int_R^1 \frac{dx}{x} = \ln x\Big|_R^1 = \ln 1 - \ln R = -\ln R; \text{ and}$$

$$\int_0^1 \frac{dx}{x} = \lim_{R\to 0^+} (-\ln R) = \infty.$$

Thus, the integral converges for $p < 1$ and diverges for $p \geq 1$.

53. Sketch the region under the graph of $f(x) = \frac{1}{1+x^2}$ for $-\infty < x < \infty$, and show that its area is π.

SOLUTION The graph is shown below.



Since $(1+x^2)^{-1}$ is an even function, we can first compute the area under the graph for $x > 0$:

$$\int_0^R \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^R = \tan^{-1} R - \tan^{-1} 0 = \tan^{-1} R.$$

Thus,

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{R\to\infty} \tan^{-1} R = \frac{\pi}{2}.$$

By symmetry, we have

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

54. Show that $\frac{1}{\sqrt{x^4+1}} \leq \frac{1}{x^2}$ for all x, and use this to prove that $\int_1^\infty \frac{dx}{\sqrt{x^4+1}}$ converges.

SOLUTION Since $\sqrt{x^4+1} \geq \sqrt{x^4} = x^2$, it follows that

$$\frac{1}{\sqrt{x^4+1}} \leq \frac{1}{x^2}.$$

The integral

$$\int_1^\infty \frac{dx}{x^2}$$

converges by Theorem 2, since $2 > 1$. Therefore, by the comparison test,

$$\int_1^\infty \frac{dx}{\sqrt{x^4+1}} \text{ converges.}$$

55. Show that $\int_1^\infty \frac{dx}{x^3+4}$ converges by comparing with $\int_1^\infty x^{-3}\,dx$.

SOLUTION The integral $\int_1^\infty x^{-3}\,dx$ converges because $3 > 1$. Since $x^3+4 \ge x^3$, it follows that

$$\frac{1}{x^3+4} \le \frac{1}{x^3}.$$

Therefore, by the comparison test,

$$\int_1^\infty \frac{dx}{x^3+4} \text{ converges.}$$

56. Show that $\int_2^\infty \frac{dx}{x^3-4}$ converges by comparing with $\int_2^\infty 2x^{-3}\,dx$.

SOLUTION The integral $\int_1^\infty x^{-3}\,dx$ converges because $3 > 1$. If $\int_1^\infty x^{-3}\,dx = M < \infty$, then

$$\int_1^\infty 2x^{-3}\,dx = 2\int_1^\infty x^{-3}\,dx = 2M$$

also converges. If $x \ge 2$, then $x^3 \ge 8$ so $2x^3 - 8 \ge x^3$ and $x^3 - 4 \ge \frac{1}{2}x^3$. Then we have, for $x \ge 2$,

$$\frac{1}{x^3-4} \le \frac{2}{x^3}.$$

Therefore, by the comparison test:

$$\int_2^\infty \frac{2}{x^3-4} \text{ converges.}$$

57. Show that $0 \le e^{-x^2} \le e^{-x}$ for $x \ge 1$ (Figure 10). Use the Comparison Test to show that $\int_0^\infty e^{-x^2}\,dx$ converges. *Hint:* It suffices (why?) to make the comparison for $x \ge 1$ because

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

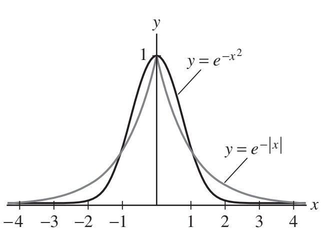


FIGURE 10 Comparison of $y = e^{-|x|}$ and $y = e^{-x^2}$.

SOLUTION For $x \ge 1$, $x^2 \ge x$, so $-x^2 \le -x$ and $e^{-x^2} \le e^{-x}$. Now

$$\int_1^\infty e^{-x}\,dx \text{ converges,} \quad \text{so} \quad \int_1^\infty e^{-x^2}\,dx \text{ converges}$$

by the comparison test. Finally, because e^{-x^2} is continuous on [0, 1],

$$\int_0^\infty e^{-x^2}\,dx \text{ converges.}$$

We conclude that our integral converges by writing it as a sum:

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

58. Prove that $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges by comparing with $\int_{-\infty}^{\infty} e^{-|x|}\,dx$ (Figure 10).

SOLUTION From Figure 10, we see that for $|x| \ge 1$, $e^{-x^2} \le e^{-|x|}$. Now

$$\int_{-\infty}^{-1} e^{-|x|}\,dx \quad \text{and} \quad \int_{1}^{\infty} e^{-|x|}\,dx$$

both converge, so

$$\int_{-\infty}^{-1} e^{-x^2}\,dx \quad \text{and} \quad \int_{1}^{\infty} e^{-x^2}\,dx$$

must also converge by the comparison test. Because e^{-x^2} is continuous on $[-1, 1]$, it follows that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{-1} e^{-x^2}\,dx + \int_{-1}^{1} e^{-x^2}\,dx + \int_{1}^{\infty} e^{-x^2}\,dx$$

converges.

59. Show that $\int_1^{\infty} \frac{1-\sin x}{x^2}\,dx$ converges.

SOLUTION Let $f(x) = \frac{1-\sin x}{x^2}$. Since $f(x) \le \frac{2}{x^2}$ and $\int_1^{\infty} 2x^{-2}\,dx = 2$, it follows that

$$\int_1^{\infty} \frac{1-\sin x}{x^2}\,dx \text{ converges}$$

by the comparison test.

60. Let $a > 0$. Recall that $\lim_{x\to\infty} \frac{x^a}{\ln x} = \infty$ (by Exercise 64 in Section 4.5).

(a) Show that $x^a > 2\ln x$ for all x sufficiently large.
(b) Show that $e^{-x^a} < x^{-2}$ for all x sufficiently large.
(c) Show that $\int_1^{\infty} e^{-x^a}\,dx$ converges.

SOLUTION

(a) Since $\lim_{x\to\infty} x^a/\ln x = \infty$, there must be some number $M > 0$ such that, for all $x > M$,

$$\frac{x^a}{\ln x} > 2.$$

But this means that, for all $x > M$,

$$x^a > 2\ln x.$$

(b) For all $x > M$, we have $x^a > 2\ln x$. Then

$$-x^a < -2\ln x = \ln x^{-2}$$

so that

$$e^{-x^a} < e^{\ln x^{-2}} = x^{-2}.$$

(c) By the above calculations, we can use the comparison test on the interval $[M, \infty)$:

$$\int_M^{\infty} \frac{dx}{x^2} \text{ converges} \Rightarrow \int_M^{\infty} e^{-x^a}\,dx \text{ also converges.}$$

Since e^{-x^a} is continuous on $[1, M]$, we have that

$$\int_M^{\infty} e^{-x^a}\,dx \text{ converges} \Rightarrow \int_1^{\infty} e^{-x^a}\,dx \text{ also converges.}$$

In Exercises 61–74, use the Comparison Test to determine whether or not the integral converges.

61. $\int_1^{\infty} \frac{1}{\sqrt{x^5+2}}\,dx$

SOLUTION Since $\sqrt{x^5+2} \ge \sqrt{x^5} = x^{5/2}$, it follows that

$$\frac{1}{\sqrt{x^5+2}} \le \frac{1}{x^{5/2}}.$$

The integral $\int_1^\infty dx/x^{5/2}$ converges because $\frac{5}{2} > 1$. Therefore, by the comparison test:

$$\int_1^\infty \frac{dx}{\sqrt{x^5+2}} \text{ also converges.}$$

62. $\displaystyle\int_1^\infty \frac{dx}{(x^3+2x+4)^{1/2}}$

SOLUTION For all $x \geq 1$, $\sqrt{x^3+2x+4} \geq \sqrt{x^3} = x^{3/2}$. Thus

$$\frac{1}{\sqrt{x^3+2x+4}} \leq \frac{1}{x^{3/2}}.$$

The integral $\int_1^\infty dx/x^{3/2}$ converges because $\frac{3}{2} > 1$. Therefore, by the comparison test,

$$\int_1^\infty \frac{dx}{\sqrt{x^3+2x+4}} \text{ also converges.}$$

63. $\displaystyle\int_3^\infty \frac{dx}{\sqrt{x}-1}$

SOLUTION Since $\sqrt{x} \geq \sqrt{x} - 1$, we have (for $x > 1$)

$$\frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{x}-1}.$$

The integral $\int_1^\infty dx/\sqrt{x} = \int_1^\infty dx/x^{1/2}$ diverges because $\frac{1}{2} < 1$. Since the function $x^{-1/2}$ is continuous (and therefore finite) on $[1, 3]$, we also know that $\int_3^\infty dx/x^{1/2}$ diverges. Therefore, by the comparison test,

$$\int_3^\infty \frac{dx}{\sqrt{x}-1} \text{ also diverges.}$$

64. $\displaystyle\int_0^5 \frac{dx}{x^{1/3}+x^3}$

SOLUTION For $0 \leq x \leq 5$, $x^{1/3} + x^3 \geq x^{1/3}$, so that

$$\frac{1}{x^{1/3}+x^3} \leq \frac{1}{x^{1/3}}.$$

The integral $\int_0^5 x^{-1/3}\,dx$ converges; therefore, by the comparison test

$$\int_0^5 \frac{dx}{x^{1/3}+x^3} \text{ also converges.}$$

65. $\displaystyle\int_1^\infty e^{-(x+x^{-1})}\,dx$

SOLUTION For all $x \geq 1$, $\frac{1}{x} > 0$ so $x + \frac{1}{x} \geq x$. Then

$$-\left(x + x^{-1}\right) \leq -x \qquad \text{and} \qquad e^{-(x+x^{-1})} \leq e^{-x}.$$

The integral $\int_1^\infty e^{-x}\,dx$ converges by direct computation:

$$\int_1^\infty e^{-x}\,dx = \lim_{R\to\infty} \int_1^R e^{-x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_1^R = \lim_{R\to\infty} -e^{-R} + e^{-1} = 0 + e^{-1} = e^{-1}.$$

Therefore, by the comparison test,

$$\int_1^\infty e^{-(x+x^{-1})} \text{ also converges.}$$

66. $\displaystyle\int_0^1 \frac{|\sin x|}{\sqrt{x}}\,dx$

SOLUTION For all x, $|\sin x| \le 1$. Therefore, for $x \ne 0$,

$$\frac{|\sin x|}{\sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

The integral

$$\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 \frac{dx}{x^{1/2}}$$

converges, since $\frac{1}{2} < 1$. Therefore, by the comparison test,

$$\int_0^1 \frac{|\sin x|}{\sqrt{x}}\,dx \text{ also converges.}$$

67. $\displaystyle\int_0^1 \frac{e^x}{x^2}\,dx$

SOLUTION For $0 < x < 1$, $e^x > 1$, and therefore

$$\frac{1}{x^2} < \frac{e^x}{x^2}.$$

The integral $\displaystyle\int_0^1 dx/x^2$ diverges since $2 > 1$. Therefore, by the comparison test,

$$\int_0^1 \frac{e^x}{x^2} \text{ also diverges.}$$

68. $\displaystyle\int_1^\infty \frac{1}{x^4 + e^x}\,dx$

SOLUTION For $x > 1$, $x^4 + e^x \ge x^4$, and

$$\frac{1}{x^4 + e^x} \le \frac{1}{x^4}.$$

The integral $\displaystyle\int_0^1 dx/x^4$ converges, since $4 > 1$. Therefore, by the comparison test,

$$\int_1^\infty \frac{dx}{x^4 + e^x} \text{ also converges.}$$

69. $\displaystyle\int_0^1 \frac{1}{x^4 + \sqrt{x}}\,dx$

SOLUTION For $0 < x < 1$, $x^4 + \sqrt{x} \ge \sqrt{x}$, and

$$\frac{1}{x^4 + \sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

The integral $\displaystyle\int_0^1 (1/\sqrt{x})\,dx$ converges, since $p = \frac{1}{2} < 1$. Therefore, by the comparison test,

$$\int_0^1 \frac{dx}{x^4 + \sqrt{x}} \text{ also converges.}$$

70. $\displaystyle\int_1^\infty \frac{\ln x}{\sinh x}\,dx$

SOLUTION For $x > 1$, $e^{-x} < \frac{1}{2}e^x$, so

$$\sinh x = \frac{e^x - e^{-x}}{2} \ge \frac{1}{4}e^x.$$

Similarly, $\ln x < x$ for all $x > 1$, so

$$\frac{\ln x}{\sinh x} \le \frac{4x}{e^x} \quad \text{for all } x \ge 1.$$

Because

$$\int_1^\infty 4xe^{-x}\,dx = -4xe^{-x}\Big|_1^\infty + \int_1^\infty 4e^{-x}\,dx = \frac{8}{e},$$

it follows by the comparison test that

$$\int_1^\infty \frac{\ln x}{\sinh x}\,dx \text{ converges.}$$

71. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^{1/3}+x^3}}$

SOLUTION For $x \ge 0$, $\sqrt{x^{1/3}+x^3} \ge \sqrt{x^3} = x^{3/2}$, so that

$$\frac{1}{\sqrt{x^{1/3}+x^3}} \le \frac{1}{x^{3/2}}$$

The integral $\displaystyle\int_1^\infty x^{-3/2}\,dx$ converges since $p = 3/2 > 1$. Therefore, by the comparison test,

$$\int \frac{1}{\sqrt{x^{1/3}+x^3}}\,dx \quad \text{also converges.}$$

72. $\displaystyle\int_0^1 \frac{dx}{(8x^2+x^4)^{1/3}}$

SOLUTION Clearly $8x^2 + x^4 \ge 8x^2$, so that

$$\frac{1}{(8x^2+x^4)^{1/3}} \le \frac{1}{(8x^2)^{1/3}}$$

Thus

$$\int_0^1 \frac{1}{(8x^2+x^4)}\,dx \le \int_0^1 \frac{1}{(8x^2)^{1/3}}\,dx = \frac{1}{2}\int_0^1 \frac{1}{x^{2/3}}\,dx$$

But $\displaystyle\int_0^1 x^{-2/3}\,dx$ converges since $p = 2/3 < 1$. Therefore, by the comparison test,

$$\int_0^1 \frac{1}{(8x^2+x^4)^{1/3}}\,dx \quad \text{also converges.}$$

73. $\displaystyle\int_1^\infty \frac{dx}{(x+x^2)^{1/3}}$

SOLUTION For $x > 1$, $x < x^2$ so that $x + x^2 < 2x^2$; then

$$\int_1^\infty \frac{1}{(x+x^2)^{1/3}}\,dx \ge \int_1^\infty \frac{1}{(2x^2)^{1/3}}\,dx = \frac{1}{2^{1/3}}\int_1^\infty \frac{1}{x^{2/3}}\,dx$$

But $\displaystyle\int_1^\infty \frac{1}{x^{2/3}}\,dx$ diverges since $p = 2/3 < 1$. Therefore, by the comparison test,

$$\int_1^\infty \frac{1}{(x+x^2)^{1/3}}\,dx \quad \text{diverges as well.}$$

74. $\displaystyle\int_0^1 \frac{dx}{xe^x+x^2}$

SOLUTION $xe^x + x^2 = x(e^x + x)$; for $0 \le x \le 1$, $e^x \le e^1 = e$ and $x \le 1$, so that $x(e^x + x) \le x(e+1)$. It follows that

$$\int_0^1 \frac{1}{xe^x+x^2}\,dx \ge \int_0^1 \frac{1}{x(e+1)}\,dx = \frac{1}{e+1}\int_0^1 \frac{1}{x}\,dx$$

But $\displaystyle\int_0^1 \frac{1}{x}\,dx$ diverges since $p = 1$. Therefore, by the comparison test,

$$\int_0^1 \frac{1}{xe^x+x^2}\,dx \quad \text{diverges as well.}$$

Hint for Exercise 73: Show that for $x \geq 1$,

$$\frac{1}{(x+x^2)^{1/3}} \geq \frac{1}{2^{1/3}x^{2/3}}$$

Hint for Exercise 74: Show that for $0 \leq x \leq 1$,

$$\frac{1}{xe^x + x^2} \geq \frac{1}{(e+1)x}$$

75. Define $J = \int_0^\infty \frac{dx}{x^{1/2}(x+1)}$ as the sum of the two improper integrals

$$\int_0^1 \frac{dx}{x^{1/2}(x+1)} + \int_1^\infty \frac{dx}{x^{1/2}(x+1)}$$

Use the Comparison Test to show that J converges.

SOLUTION For the first integral, note that for $0 \leq x \leq 1$, we have $1 \leq 1 + x$, so that $x^{1/2}(x+1) \geq x^{1/2}$. It follows that

$$\int_0^1 \frac{1}{x^{1/2}(x+1)}\,dx \leq \int_0^1 \frac{1}{x^{1/2}}\,dx$$

which converges since $p = 1/2 < 1$. Thus the first integral converges by the comparison test. For the second integral, for $1 \leq x$, we have $x^{1/2}(x+1) = x^{3/2} + x^{1/2} \geq x^{3/2}$, so that

$$\int_1^\infty \frac{1}{x^{1/2}(x+1)}\,dx = \int_1^\infty \frac{1}{x^{3/2}+x^{1/2}}\,dx \leq \int_1^\infty \frac{1}{x^{3/2}}\,dx$$

which converges since $p = 3/2 > 1$. Thus the second integral converges as well by the comparison test, and therefore J, which is the sum of the two, converges.

76. Determine whether $J = \int_0^\infty \frac{dx}{x^{3/2}(x+1)}$ (defined as in Exercise 75) converges.

SOLUTION We have $x^{3/2}(x+1) = x^{5/2} + x^{3/2}$. For $0 \leq x \leq 1$, $x^{5/2} \leq x^{3/2}$, so that $x^{5/2} + x^{3/2} \leq 2x^{3/2}$. Then

$$\int_0^1 \frac{1}{x^{3/2}(x+1)}\,dx = \int_0^1 \frac{1}{x^{5/2}+x^{3/2}}\,dx \geq \int_0^1 \frac{1}{2x^{3/2}}\,dx = \frac{1}{2}\int_0^1 \frac{1}{x^{3/2}}\,dx$$

But this integral diverges since $p = 3/2 > 1$. By the comparison test, $\int_0^1 \frac{1}{x^{3/2}(x+1)}\,dx$ diverges as well, so that J diverges.

77. An investment pays a dividend of \$250/year continuously forever. If the interest rate is 7%, what is the present value of the entire income stream generated by the investment?

SOLUTION The present value of the income stream after T years is

$$\int_0^T 250e^{-0.07t}\,dt = \left.\frac{250e^{-0.07t}}{-0.07}\right|_0^T = \frac{-250}{0.07}\left(e^{-0.07T} - 1\right) = \frac{250}{0.07}\left(1 - e^{-0.07T}\right).$$

Therefore the present value of the entire income stream is

$$\int_0^\infty 250e^{-0.07t} = \lim_{T\to\infty}\int_0^T 250e^{-0.07t} = \lim_{T\to\infty}\frac{250}{0.07}\left(1 - e^{-0.07T}\right) = \frac{250}{0.07}(1-0) = \frac{250}{0.07} = \$3571.43.$$

78. An investment is expected to earn profits at a rate of $10{,}000e^{0.01t}$ dollars per year forever. Find the present value of the income stream if the interest rate is 4%.

SOLUTION The present value of the income stream after T years is

$$\int_0^T \left(10{,}000e^{0.01t}\right)e^{-0.04t}\,dt = 10{,}000\int_0^T e^{-0.03t}\,dt = \left.\frac{10{,}000}{-0.03}e^{-0.03t}\right|_0^T = -333{,}333.33\left(e^{-0.03t} - 1\right).$$

Therefore the present value of the entire income stream is

$$\int_0^\infty 10{,}000e^{-0.03t} = \lim_{T\to\infty} 333{,}333.33\left(1 - e^{-0.03t}\right) = \$333{,}333.33.$$

79. Compute the present value of an investment that generates income at a rate of $5000te^{0.01t}$ dollars per year forever, assuming an interest rate of 6%.

SOLUTION The present value of the income stream after T years is

$$\int_0^T \left(5000te^{0.01t}\right) e^{-0.06t}\, dt = 5000 \int_0^T te^{-0.05t}\, dt$$

Compute the indefinite integral using Integration by Parts, with $u = t$ and $v' = e^{-0.05t}$. Then $u' = 1$, $v = (-1/0.05)e^{-0.05t}$, and

$$\int te^{-0.05t}\, dt = \frac{-t}{0.05} e^{-0.05t} + \frac{1}{0.05} \int e^{-0.05t}\, dt = -20te^{-0.05t} + \frac{20}{-0.05} e^{-0.05t} + C$$
$$= e^{-0.05t}(-20t - 400) + C.$$

Thus,

$$5000 \int_0^T te^{-0.05t}\, dt = 5000e^{-0.05t}(-20t - 400)\Big|_0^T = 5000e^{-0.05T}(-20T - 400) - 5000(-400)$$
$$= 2{,}000{,}000 - 5000e^{-0.05T}(20T + 400).$$

Use L'Hôpital's Rule to compute the limit:

$$\lim_{T\to\infty} \left(2{,}000{,}000 - \frac{5000(20T + 400)}{e^{0.05T}}\right) = 2{,}000{,}000 - \lim_{T\to\infty} \frac{5000(20)}{0.05e^{0.05T}} = 2{,}000{,}000 - 0 = \$2{,}000{,}000.$$

80. Find the volume of the solid obtained by rotating the region below the graph of $y = e^{-x}$ about the x-axis for $0 \le x < \infty$.

SOLUTION Using the disk method, the volume is given by

$$V = \int_0^\infty \pi \left(e^{-x}\right)^2 dx = \pi \int_0^\infty e^{-2x}\, dx.$$

First compute the volume over a finite interval:

$$\pi \int_0^R e^{-2x}\, dx = \frac{-\pi}{2} e^{-2x}\Big|_0^R = \frac{-\pi}{2}\left(e^{-2R} - 1\right) = \frac{\pi}{2}\left(1 - e^{-2R}\right).$$

Thus,

$$V = \lim_{R\to\infty} \pi \int_0^R e^{-2x}\, dx = \lim_{R\to\infty} \frac{\pi}{2}\left(1 - e^{-2R}\right) = \frac{\pi}{2}(1 - 0) = \frac{\pi}{2}.$$

81. The solid S obtained by rotating the region below the graph of $y = x^{-1}$ about the x-axis for $1 \le x < \infty$ is called **Gabriel's Horn** (Figure 11).

(a) Use the Disk Method (Section 6.3) to compute the volume of S. Note that the volume is finite even though S is an infinite region.

(b) It can be shown that the surface area of S is

$$A = 2\pi \int_1^\infty x^{-1}\sqrt{1 + x^{-4}}\, dx$$

Show that A is infinite. If S were a container, you could fill its interior with a finite amount of paint, but you could not paint its surface with a finite amount of paint.

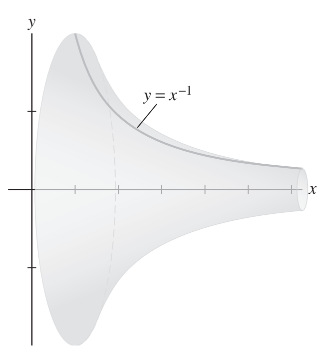


FIGURE 11

SOLUTION

(a) The volume is given by

$$V = \int_1^{\infty} \pi \left(\frac{1}{x}\right)^2 dx.$$

First compute the volume over a finite interval:

$$\int_1^R \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^R x^{-2}\, dx = \pi \left.\frac{x^{-1}}{-1}\right|_1^R = \pi \left(\frac{-1}{R} - \frac{-1}{1}\right) = \pi \left(1 - \frac{1}{R}\right).$$

Thus,

$$V = \lim_{R\to\infty} \int_1^{\infty} \pi x^{-2}\, dx = \lim_{R\to\infty} \pi \left(1 - \frac{1}{R}\right) = \pi.$$

(b) For $x > 1$, we have

$$\frac{1}{x}\sqrt{1 + \frac{1}{x^4}} = \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}} = \frac{\sqrt{x^4+1}}{x^3} \geq \frac{\sqrt{x^4}}{x^3} = \frac{x^2}{x^3} = \frac{1}{x}.$$

The integral $\int_1^{\infty} \frac{1}{x}\, dx$ diverges, since $p = 1 \geq 1$. Therefore, by the comparison test,

$$\int_1^{\infty} \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx \text{ also diverges.}$$

Finally,

$$A = 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx$$

diverges.

82. Compute the volume of the solid obtained by rotating the region below the graph of $y = e^{-|x|/2}$ about the x-axis for $-\infty < x < \infty$.

SOLUTION The graph of y is symmetric around the y-axis, so it suffices to compute the volume for $0 \leq x \leq \infty$, where we have $y = e^{-x/2}$. Using the disk method,

$$V = 2\int_0^{\infty} \pi \left(e^{-x/2}\right)^2 dx = 2\pi \int_0^{\infty} e^{-x}\, dx = 2\pi \lim_{R\to\infty} \int_0^R e^{-x}\, dx$$

$$= -\lim_{R\to\infty} \left. 2\pi e^{-x}\right|_0^R = -2\pi \lim_{R\to\infty} (e^{-R} - 1) = 2\pi$$

Therefore $V = 2\pi$.

83. When a capacitor of capacitance C is charged by a source of voltage V, the power expended at time t is

$$P(t) = \frac{V^2}{R}(e^{-t/RC} - e^{-2t/RC})$$

where R is the resistance in the circuit. The total energy stored in the capacitor is

$$W = \int_0^{\infty} P(t)\, dt$$

Show that $W = \frac{1}{2}CV^2$.

SOLUTION The total energy contained after the capacitor is fully charged is

$$W = \frac{V^2}{R} \int_0^{\infty} \left(e^{-t/RC} - e^{-2t/RC}\right) dt.$$

The energy after a finite amount of time ($t = T$) is

$$\frac{V^2}{R}\int_0^T \left(e^{-t/RC} - e^{-2t/RC}\right) dt = \frac{V^2}{R}\left(-RCe^{-t/RC} + \frac{RC}{2}e^{-2t/RC}\right)\Bigg|_0^T$$
$$= V^2C\left[\left(-e^{-T/RC} + \frac{1}{2}e^{-2T/RC}\right) - \left(-1 + \frac{1}{2}\right)\right]$$
$$= CV^2\left(\frac{1}{2} - e^{-T/RC} + \frac{1}{2}e^{-2T/RC}\right).$$

Thus,

$$W = \lim_{T\to\infty} CV^2\left(\frac{1}{2} - e^{-T/RC} + \frac{1}{2}e^{-2T/RC}\right) = CV^2\left(\frac{1}{2} - 0 + 0\right) = \frac{1}{2}CV^2.$$

84. For which integers p does $\displaystyle\int_0^{1/2} \frac{dx}{x(\ln x)^p}$ converge?

SOLUTION If $p = 1$, the integral diverges. By substituting $u = \ln x$ and $du = dx/x$, we get

$$\int \frac{dx}{x(\ln x)} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C,$$

so

$$\int_0^{1/2} \frac{dx}{x(\ln x)} = \lim_{R\to 0+} (\ln|\ln x|)\Big|_R^{1/2} = \lim_{R\to 0+} (\ln|\ln(1/2)| - \ln|\ln R|),$$

which is infinite.

Now, suppose $p \neq 1$. Using the substitution $u = \ln x$, so that $du = \frac{1}{x}dx$, the integral becomes

$$\int_R^{1/2} \frac{dx}{x(\ln x)^p} = \int_{x=R}^{x=1/2} \frac{du}{u^p} = \int_{x=R}^{x=1/2} u^{-p}\,du = \frac{1}{p-1}u^{-p+1}\Big|_{x=R}^{x=1/2}$$
$$= \frac{1}{p-1}(\ln x)^{-p+1}\Big|_R^{1/2} = \frac{1}{p-1}(\ln(1/2))^{-p+1} - \frac{1}{p-1}(\ln(R))^{-p+1}.$$

By definition,

$$\int_0^{1/2} \frac{dx}{x(\ln x)^p} = \lim_{R\to 0+}\int_R^{1/2} \frac{dx}{x(\ln x)^p} = \lim_{R\to 0+}\left[\frac{1}{p-1}(\ln(1/2))^{-p+1} - \frac{1}{p-1}(\ln R)^{-p+1}\right].$$

If $p > 1$, $\lim_{R\to 0+} (\ln R)^{-p+1} = \lim_{R\to 0} \frac{1}{(\ln R)^{p-1}} = 0$. If $p < 1$, $\lim_{R\to 0+} (\ln R)^{1-p} = \infty$. Therefore, the integral diverges if $p < 1$ or $p = 1$, and converges if $p > 1$.

85. Conservation of Energy can be used to show that when a mass m oscillates at the end of a spring with spring constant k, the period of oscillation is

$$T = 4\sqrt{m}\int_0^{\sqrt{2E/k}} \frac{dx}{\sqrt{2E - kx^2}}$$

where E is the total energy of the mass. Show that this is an improper integral with value $T = 2\pi\sqrt{m/k}$.

SOLUTION The integrand is infinite at the upper limit of integration, $x = \sqrt{2E/k}$, so the integral is improper. Now, let

$$T(R) = 4\sqrt{m}\int_0^R \frac{dx}{\sqrt{2E - kx^2}} = 4\sqrt{m}\frac{1}{\sqrt{2E}}\int_0^R \frac{dx}{\sqrt{1 - (\frac{k}{2E})x^2}}$$
$$= 4\sqrt{\frac{m}{2E}}\sqrt{\frac{2E}{k}}\sin^{-1}\left(\sqrt{\frac{k}{2E}}R\right) = 4\sqrt{m/k}\sin^{-1}\left(\sqrt{\frac{k}{2E}}R\right).$$

Therefore

$$T = \lim_{R\to\sqrt{2E/k}} T(R) = 4\sqrt{\frac{m}{k}}\sin^{-1}(1) = 2\pi\sqrt{\frac{m}{k}}.$$

In Exercises 86–89, the ***Laplace transform*** *of a function $f(x)$ is the function $Lf(s)$ of the variable s defined by the improper integral (if it converges):*

$$Lf(s) = \int_0^{\infty} f(x)e^{-sx}\,dx$$

Laplace transforms are widely used in physics and engineering.

86. Show that if $f(x) = C$, where C is a constant, then $Lf(s) = C/s$ for $s > 0$.

SOLUTION If $f(x) = C$, a constant, then the Laplace transform of $f(x)$ is

$$Lf(s) = \int_0^{\infty} Ce^{-sx}\,dx = \lim_{R\to\infty} \frac{-C}{s}e^{-sx}\Big|_0^R = \lim_{R\to\infty} \frac{-C}{s}\left(e^{-sR} - 1\right) = \frac{-C}{s}(0-1) = \frac{C}{s}.$$

87. Show that if $f(x) = \sin\alpha x$, then $Lf(s) = \dfrac{\alpha}{s^2+\alpha^2}$.

SOLUTION If $f(x) = \sin\alpha x$, then the Laplace transform of $f(x)$ is

$$Lf(s) = \int_0^{\infty} e^{-sx}\sin\alpha x\,dx$$

First evaluate the indefinite integral using Integration by Parts, with $u = \sin\alpha x$ and $v' = e^{-sx}$. Then $u' = \alpha\cos\alpha x$, $v = -\frac{1}{s}e^{-sx}$, and

$$\int e^{-sx}\sin\alpha x\,dx = -\frac{1}{s}e^{-sx}\sin\alpha x + \frac{\alpha}{s}\int e^{-sx}\cos\alpha x\,dx.$$

Use Integration by Parts again, with $u = \cos\alpha x$, $v' = e^{-sx}$. Then $u' = -\alpha\sin\alpha x$, $v = -\frac{1}{s}e^{-sx}$, and

$$\int e^{-sx}\cos\alpha x\,dx = -\frac{1}{s}e^{-sx}\cos\alpha x - \frac{\alpha}{s}\int e^{-sx}\sin\alpha x\,dx.$$

Substituting this into the first equation and solving for $\int e^{-sx}\sin\alpha x\,dx$, we get

$$\int e^{-sx}\sin\alpha x\,dx = -\frac{1}{s}e^{-sx}\sin\alpha x - \frac{\alpha}{s^2}e^{-sx}\cos\alpha x - \frac{\alpha^2}{s^2}\int e^{-sx}\sin\alpha x\,dx$$

$$\int e^{-sx}\sin\alpha x\,dx = \frac{-e^{-sx}\left(\frac{1}{s}\sin\alpha x + \frac{\alpha}{s^2}\cos\alpha x\right)}{\left(1+\frac{\alpha^2}{s^2}\right)} = \frac{-e^{-sx}(s\sin\alpha x + \alpha\cos\alpha x)}{s^2+\alpha^2}$$

Thus,

$$\int_0^R e^{-sx}\sin\alpha x\,dx = \frac{1}{s^2+\alpha^2}\left[\frac{s\sin\alpha R + \alpha\cos\alpha R}{-e^{sR}} - \frac{0+\alpha}{-1}\right] = \frac{1}{s^2+\alpha^2}\left[\alpha - \frac{s\sin\alpha R + \alpha\cos\alpha R}{e^{sR}}\right].$$

Finally we take the limit, noting the fact that, for all values of R, $|s\sin\alpha R + \alpha\cos\alpha R| \le s + |\alpha|$

$$Lf(s) = \lim_{R\to\infty} \frac{1}{s^2+\alpha^2}\left[\alpha - \frac{s\sin\alpha R + \alpha\cos\alpha R}{e^{sR}}\right] = \frac{1}{s^2+\alpha^2}(\alpha - 0) = \frac{\alpha}{s^2+\alpha^2}.$$

88. Compute $Lf(s)$, where $f(x) = e^{\alpha x}$ and $s > \alpha$.

SOLUTION If $f(x) = e^{\alpha x}$, where $s > \alpha$, then the Laplace transform of $f(x)$ is

$$Lf(s) = \int_0^{\infty} e^{\alpha x}e^{-sx}\,dx = \int_0^{\infty} e^{-(s-\alpha)x}\,dx = \lim_{R\to\infty} \frac{-1}{s-\alpha}e^{-(s-\alpha)x}\Big|_0^R = \lim_{R\to\infty} \frac{-1}{s-\alpha}\left(e^{-(s-\alpha)R} - 1\right).$$

Because $s > \alpha$, $-(s-\alpha) < 0$, which gives us

$$\lim_{R\to\infty} \frac{1}{s-\alpha}\left(1 - e^{-(s-\alpha)R}\right) = \frac{1}{s-\alpha}(1-0) = \frac{1}{s-\alpha}.$$

The final answer is

$$Lf(s) = \frac{1}{s-\alpha}.$$

89. Compute $Lf(s)$, where $f(x) = \cos \alpha x$ and $s > 0$.

SOLUTION If $f(x) = \cos \alpha x$, then the Laplace transform of $f(x)$ is

$$Lf(x) = \int_0^\infty e^{-sx} \cos \alpha x \, dx$$

First evaluate the indefinite integral using Integration by Parts, with $u = \cos \alpha x$ and $v' - e^{-sx}$. Then $u' = -\alpha \sin \alpha x$, $v = -\frac{1}{s} e^{-sx}$, and

$$\int e^{-sx} \cos \alpha x \, dx = -\frac{1}{s} e^{-sx} \cos \alpha x - \frac{\alpha}{s} \int e^{-sx} \sin \alpha x \, dx.$$

Use Integration by Parts again, with $u = \sin \alpha x \, dx$ and $v' = -e^{-sx}$. Then $u' = \alpha \cos \alpha x$, $v = -\frac{1}{s} e^{-sx}$, and

$$\int e^{-sx} \sin \alpha x \, dx = -\frac{1}{s} e^{-sx} \sin \alpha x + \frac{\alpha}{s} \int e^{-sx} \cos \alpha x \, dx.$$

Substituting this into the first equation and solving for $\int e^{-sx} \cos \alpha x \, dx$, we get

$$\int e^{-sx} \cos \alpha x \, dx = -\frac{1}{s} e^{-sx} \cos \alpha x - \frac{\alpha}{s} \left[-\frac{1}{s} e^{-sx} \sin \alpha x + \frac{\alpha}{s} \int e^{-sx} \cos \alpha \, dx \right]$$

$$= -\frac{1}{s} e^{-sx} \cos \alpha x + \frac{\alpha}{s^2} e^{-sx} \sin \alpha x - \frac{\alpha^2}{s^2} \int e^{-sx} \cos \alpha x \, dx$$

$$\int e^{-sx} \cos \alpha x \, dx = \frac{e^{-sx} \left(\frac{\alpha}{s^2} \sin \alpha x - \frac{1}{s} \cos \alpha x \right)}{1 + \frac{\alpha^2}{s^2}} = \frac{e^{-sx} (\alpha \sin \alpha x - s \cos \alpha x)}{s^2 + \alpha^2}$$

Thus,

$$\int_0^R e^{-sx} \cos \alpha x \, dx = \frac{1}{s^2 + \alpha^2} \left[\frac{\alpha \sin \alpha R - s \cos \alpha R}{e^{sR}} - \frac{0 - s}{1} \right].$$

Finally we take the limit, noting the fact that, for all values of R, $|\alpha \sin \alpha R - s \cos \alpha R| \le |\alpha| + s$

$$Lf(s) = \lim_{R \to \infty} \frac{1}{s^2 + \alpha^2} \left[s + \frac{\alpha \sin \alpha R - s \cos \alpha R}{e^{sR}} \right] = \frac{1}{s^2 + \alpha^2} (s + 0) = \frac{s}{s^2 + \alpha^2}.$$

90. When a radioactive substance decays, the fraction of atoms present at time t is $f(t) = e^{-kt}$, where $k > 0$ is the decay constant. It can be shown that the *average* life of an atom (until it decays) is $A = -\int_0^\infty t f'(t) \, dt$. Use Integration by Parts to show that $A = \int_0^\infty f(t) \, dt$ and compute A. What is the average decay time of radon-222, whose half-life is 3.825 days?

SOLUTION Let $u = t$, $v' = f'(t)$. Then $u' = 1$, $v = f(t)$, and

$$A = -\int_0^\infty t f'(t) \, dt = -t f(t) \Big|_0^\infty + \int_0^\infty f(t) \, dt.$$

Since $f(t) = e^{-kt}$, we have

$$-t f(t) \Big|_0^\infty = \lim_{R \to \infty} -t e^{-kt} \Big|_0^R = \lim_{R \to \infty} -R e^{-Rt} + 0 = \lim_{R \to \infty} \frac{-R}{e^{Rt}} = \lim_{R \to \infty} \frac{-1}{R e^{Rt}} = 0.$$

Here we used L'Hôpital's Rule to compute the limit. Thus

$$A = \int_0^\infty f(t) \, dt = \int_0^\infty e^{-kt} \, dt.$$

Now,

$$\int_0^R e^{-kt} \, dt = -\frac{1}{k} e^{-kt} \Big|_0^R = -\frac{1}{k} \left(e^{-kR} - 1 \right) = \frac{1}{k} \left(1 - e^{-kR} \right),$$

so

$$A = \lim_{R \to \infty} \frac{1}{k} \left(1 - e^{-kR} \right) = \frac{1}{k} (1 - 0) = \frac{1}{k}.$$

Because k has units of $(\text{time})^{-1}$, A does in fact have the appropriate units of time. To find the average decay time of Radon-222, we need to determine the decay constant k, given the half-life of 3.825 days. Recall that

$$k = \frac{\ln 2}{t_n}$$

where t_n is the half-life. Thus,

$$A = \frac{1}{k} = \frac{t_n}{\ln 2} = \frac{3.825}{\ln 2} \approx 5.518 \text{ days.}$$

91. Let $J_n = \int_0^\infty x^n\, e^{-\alpha x}\, dx$, where $n \ge 1$ is an integer and $\alpha > 0$. Prove that

$$J_n = \frac{n}{\alpha} J_{n-1}$$

and $J_0 = 1/\alpha$. Use this to compute J_4. Show that $J_n = n!/\alpha^{n+1}$.

SOLUTION Using Integration by Parts, with $u = x^n$ and $v' = e^{-\alpha x}$, we get $u' = nx^{n-1}$, $v = -\frac{1}{\alpha}e^{-\alpha x}$, and

$$\int x^n e^{-\alpha x}\, dx = -\frac{1}{\alpha}x^n e^{-\alpha x} + \frac{n}{\alpha}\int x^{n-1}e^{-\alpha x}\, dx.$$

Thus,

$$J_n = \int_0^\infty x^n e^{-\alpha x}\, dx = \lim_{R\to\infty}\left(-\frac{1}{\alpha}x^n e^{-\alpha x}\right)\Bigg|_0^R + \frac{n}{\alpha}\int_0^\infty x^{n-1}e^{-\alpha x}\, dx = \lim_{R\to\infty}\frac{-R^n}{\alpha e^{\alpha R}} + 0 + \frac{n}{\alpha}J_{n-1}.$$

Use L'Hôpital's Rule repeatedly to compute the limit:

$$\lim_{R\to\infty}\frac{-R_n}{\alpha e^{\alpha R}} = \lim_{R\to\infty}\frac{-nR^{n-1}}{\alpha^2 e^{\alpha R}} = \lim_{R\to\infty}\frac{-n(n-1)R^{n-2}}{\alpha^3 e^{\alpha R}} = \cdots = \lim_{R\to\infty}\frac{-n(n-1)(n-2)\cdots(3)(2)(1)}{\alpha^{n+1}e^{\alpha R}} = 0.$$

Finally,

$$J_n = 0 + \frac{n}{\alpha}J_{n-1} = \frac{n}{\alpha}J_{n-1}.$$

J_0 can be computed directly:

$$J_0 = \int_0^\infty e^{-\alpha x}\, dx = \lim_{R\to\infty}\int_0^R e^{-\alpha x}\, dx = \lim_{R\to\infty} -\frac{1}{\alpha}e^{-\alpha x}\Bigg|_0^R = \lim_{R\to\infty} -\frac{1}{\alpha}\left(e^{-\alpha R} - 1\right) = -\frac{1}{\alpha}(0-1) = \frac{1}{\alpha}.$$

With this starting point, we can work up to J_4:

$$J_1 = \frac{1}{\alpha}J_0 = \frac{1}{\alpha}\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha^2};$$

$$J_2 = \frac{2}{\alpha}J_1 = \frac{2}{\alpha}\left(\frac{1}{\alpha^2}\right) = \frac{2}{\alpha^3} = \frac{2!}{\alpha^{2+1}};$$

$$J_3 = \frac{3}{\alpha}J_2 = \frac{3}{\alpha}\left(\frac{2}{\alpha^3}\right) = \frac{6}{\alpha^4} = \frac{3!}{\alpha^{3+1}};$$

$$J_4 = \frac{4}{\alpha}J_3 = \frac{4}{\alpha}\left(\frac{6}{\alpha^4}\right) = \frac{24}{\alpha^5} = \frac{4!}{\alpha^{4+1}}.$$

We can use induction to prove the formula for J_n. If

$$J_{n-1} = \frac{(n-1)!}{\alpha^n},$$

then we have

$$J_n = \frac{n}{\alpha}J_{n-1} = \frac{n}{\alpha}\cdot\frac{(n-1)!}{\alpha^n} = \frac{n!}{\alpha^{n+1}}.$$

92. Let $a > 0$ and $n > 1$. Define $f(x) = \dfrac{x^n}{e^{ax}-1}$ for $x \ne 0$ and $f(0) = 0$.

(a) Use L'Hôpital's Rule to show that $f(x)$ is continuous at $x = 0$.

(b) Show that $\int_0^\infty f(x)\, dx$ converges. *Hint:* Show that $f(x) \le 2x^n e^{-ax}$ if x is large enough. Then use the Comparison Test and Exercise 91.

SOLUTION

(a) Using L'Hôpital's Rule, we find

$$\lim_{x\to 0} \frac{x^n}{e^{\alpha x} - 1} = \lim_{x\to 0} \frac{nx^{n-1}}{\alpha e^{\alpha x}} = \frac{0}{\alpha} = 0;$$

thus,

$$\lim_{x\to 0} f(x) = f(0),$$

and $f(x)$ is continuous at $x = 0$.

(b) Since $a > 0$, $\lim_{x\to\infty} e^{ax} = \infty$. Therefore there will be some value of x, say $x = M$, such that, for all $x \ge M$, we'll have $e^{ax} \ge 2$. With this, we have

$$\frac{1}{e^{ax}} \le \frac{1}{2} \quad \text{so} \quad \frac{1}{e^{ax}} + \frac{1}{2} \le 1 \quad \text{and} \quad 1 - \frac{1}{e^{\alpha x}} \ge \frac{1}{2}.$$

Multiply this last inequality through by $e^{\alpha x}$ to obtain

$$e^{\alpha x} - 1 \ge \frac{e^{\alpha x}}{2} \quad \text{so} \quad \frac{1}{e^{\alpha x} - 1} \le \frac{2}{e^{\alpha x}} \quad \text{and} \quad \frac{x^n}{e^{\alpha x} - 1} \le \frac{2x^n}{e^{\alpha x}}.$$

From Exercise 91, we know that

$$\int_0^\infty x^n e^{-\alpha x}\, dx \text{ converges, so } \int_M^\infty 2x^n e^{-\alpha x}\, dx \text{ also converges.}$$

Therefore, by the comparison test,

$$\int_M^\infty \frac{x^n}{e^{\alpha x} - 1}\, dx \text{ also converges.}$$

Now, from part (a), we know that $f(x)$ is continuous on $[0, M]$, so

$$\int_0^M \frac{x^n}{e^{\alpha x} - 1}\, dx$$

exists and is finite. Thus we have shown

$$\int_0^\infty \frac{x^n}{e^{\alpha x} - 1}\, dx = \int_0^M \frac{x^n}{e^{\alpha x} - 1}\, dx + \int_M^\infty \frac{x^n}{e^{\alpha x} - 1}\, dx \text{ converges.}$$

93. According to **Planck's Radiation Law**, the amount of electromagnetic energy with frequency between ν and $\nu + \Delta\nu$ that is radiated by a so-called black body at temperature T is proportional to $F(\nu)\,\Delta\nu$, where

$$F(\nu) = \left(\frac{8\pi h}{c^3}\right) \frac{\nu^3}{e^{h\nu/kT} - 1}$$

where c, h, k are physical constants. Use Exercise 92 to show that the total radiated energy

$$E = \int_0^\infty F(\nu)\, d\nu$$

is finite. To derive his law, Planck introduced the quantum hypothesis in 1900, which marked the birth of quantum mechanics.

SOLUTION The total radiated energy E is given by

$$E = \int_0^\infty F(\nu)\, d\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3}{e^{h\nu/kT} - 1}\, d\nu.$$

Let $\alpha = h/kT$. Then

$$E = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3}{e^{\alpha\nu} - 1}\, d\nu.$$

Because $\alpha > 0$ and $8\pi h/c^3$ is a constant, we know E is finite by Exercise 92.

Further Insights and Challenges

94. Let $I = \int_0^1 x^p \ln x \, dx$.

(a) Show that I diverges for $p = -1$.

(b) Show that if $p \neq -1$, then

$$\int x^p \ln x \, dx = \frac{x^{p+1}}{p+1}\left(\ln x - \frac{1}{p+1}\right) + C$$

(c) Use L'Hôpital's Rule to show that I converges if $p > -1$ and diverges if $p < -1$.

SOLUTION

(a) If $p = -1$, then

$$I = \int_0^1 x^{-1} \ln x \, dx = \int_0^1 \frac{\ln x}{x} \, dx.$$

Let $u = \ln x$, $du = (1/x)\, dx$. Then

$$\int \frac{\ln x}{x} \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C.$$

Thus,

$$\int_R^1 \frac{\ln x}{x} \, dx = \frac{1}{2}(\ln 1)^2 - \frac{1}{2}(\ln R)^2 = -\frac{1}{2}(\ln R)^2,$$

and

$$I = \lim_{R\to 0^+} -\frac{1}{2}(\ln R)^2 = \infty.$$

The integral diverges for $p = -1$.

(b) If $p \neq 1$, then use Integration by Parts, with $u = \ln x$ and $v' = x^p$. Then $u' = 1/x$, $v = x^{p+1}/p + 1$, and

$$\begin{aligned}\int x^p \ln x \, dx &= \frac{x^{p+1}}{p+1}\ln x - \frac{1}{p+1}\int \left(x^{p+1}\right)\left(\frac{1}{x}\right) dx = \frac{x^{p+1}}{p+1}\ln x - \frac{1}{p+1}\int x^p \, dx \\ &= \frac{x^{p+1}}{p+1}\ln x - \frac{1}{p+1}\left(\frac{x^{p+1}}{p+1}\right) + C = \frac{x^{p+1}}{p+1}\left(\ln x - \frac{1}{p+1}\right) + C.\end{aligned}$$

(c) Let $p < -1$. Then

$$\begin{aligned}I &= \lim_{R\to 0^+} \int_R^1 x^p \ln x = \lim_{R\to 0^+}\left[\frac{1}{p+1}\left(\ln 1 - \frac{1}{p+1}\right) - \frac{R^{p+1}}{p+1}\left(\ln R - \frac{1}{p+1}\right)\right] \\ &= \lim_{R\to 0^+}\left(\frac{-1}{(p+1)^2} - \frac{R^{p+1}}{p+1}\ln R + \frac{R^{p+1}}{(p+1)^2}\right).\end{aligned}$$

Since $p < -1$, $p + 1 < 0$, and we have

$$I = \lim_{R\to 0^+}\left(\frac{-1}{(p+1)^2} - \frac{\ln R}{(p+1)R^{-p-1}} + \frac{1}{(p+1)^2 R^{-p-1}}\right) = \infty.$$

The integral diverges for $p < -1$. On the other hand, if $p > -1$, then $p + 1 > 0$, and

$$I = \frac{-1}{(p+1)^2} + \frac{1}{p+1}\lim_{R\to 0+} R^{p+1}\ln R + \frac{1}{(p+1)^2}\lim_{R\to 0^+} R^{p+1} = \frac{-1}{(p+1)^2} + 0 = \frac{-1}{(p+1)^2}.$$

95. Let

$$F(x) = \int_2^x \frac{dt}{\ln t} \qquad \text{and} \qquad G(x) = \frac{x}{\ln x}$$

Verify that L'Hôpital's Rule applies to the limit $L = \lim_{x\to\infty} \frac{F(x)}{G(x)}$ and evaluate L.

SOLUTION Because $\ln t < t$ for $t > 2$, we have $\frac{1}{\ln t} > \frac{1}{t}$ for $t > 2$, and so

$$F(x) = \int_2^x \frac{dt}{\ln t} > \int_2^x \frac{dt}{t} = \ln x - \ln 2$$

Thus, $F(x) \to \infty$ as $x \to \infty$. Moreover, by L'Hôpital's Rule

$$\lim_{x\to\infty} G(x) = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

Thus, $\lim_{x\to\infty} \dfrac{F(x)}{G(x)}$ is of the form ∞/∞, and L'Hôpital's Rule applies. Finally,

$$L = \lim_{x\to\infty} \frac{F(x)}{G(x)} = \lim_{x\to\infty} \frac{\frac{1}{\ln x}}{\frac{\ln x - 1}{(\ln x)^2}} = \lim_{x\to\infty} \frac{\ln x}{\ln x - 1} = \lim_{x\to\infty} \frac{1}{1 - (1/\ln x)} = 1.$$

In Exercises 96–98, an improper integral $I = \int_a^\infty f(x)\,dx$ is called ***absolutely convergent*** *if $\int_a^\infty |f(x)|\,dx$ converges. It can be shown that if I is absolutely convergent, then it is convergent.*

96. Show that $\displaystyle\int_1^\infty \frac{\sin x}{x^2}\,dx$ is absolutely convergent.

SOLUTION For all x, $|\sin x| \le 1$. This implies

$$\left|\frac{\sin x}{x^2}\right| = \frac{|\sin x|}{x^2} \le \frac{1}{x^2}.$$

The integral $\displaystyle\int_1^\infty x^{-2}\,dx$ converges because $p = 2 > 1$. Therefore, by the comparison test,

$$\int_1^\infty \left|\frac{\sin x}{x^2}\right| dx \text{ also converges.}$$

Because the integral

$$\int_1^\infty \frac{\sin x}{x^2}\,dx$$

is absolutely convergent, it is also convergent.

97. Show that $\displaystyle\int_1^\infty e^{-x^2}\cos x\,dx$ is absolutely convergent.

SOLUTION By the result of Exercise 57, we know that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ is convergent. Then $\displaystyle\int_1^\infty e^{-x^2}\,dx$ is also convergent. Because $|\cos x| \le 1$ for all x, we have

$$\left|e^{-x^2}\cos x\right| = |\cos x|\left|e^{-x^2}\right| \le \left|e^{-x^2}\right| = e^{-x^2}.$$

Therefore, by the comparison test, we have

$$\int_1^\infty \left|e^{-x^2}\cos x\right| dx \text{ also converges.}$$

Since $\displaystyle\int_1^\infty e^{-x^2}\cos x\,dx$ converges absolutely, it itself converges.

98. Let $f(x) = \sin x/x$ and $I = \int_0^\infty f(x)\,dx$. We define $f(0) = 1$. Then $f(x)$ is continuous and I is not improper at $x = 0$.

(a) Show that

$$\int_1^R \frac{\sin x}{x}\,dx = -\frac{\cos x}{x}\Big|_1^R - \int_1^R \frac{\cos x}{x^2}\,dx$$

(b) Show that $\int_1^\infty (\cos x/x^2)\,dx$ converges. Conclude that the limit as $R \to \infty$ of the integral in (a) exists and is finite.

(c) Show that I converges.

It is known that $I = \frac{\pi}{2}$. However, I is *not* absolutely convergent. The convergence depends on cancellation, as shown in Figure 12.

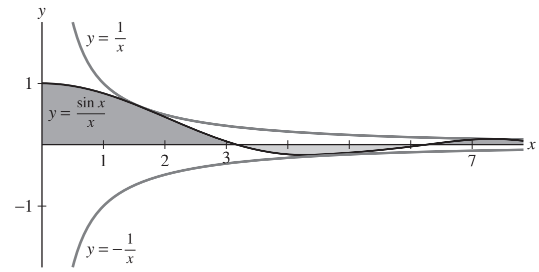


FIGURE 12 Convergence of $\int_1^\infty (\sin x/x)\,dx$ is due to the cancellation arising from the periodic change of sign.

SOLUTION

(a) Use Integration by Parts, with $u = \frac{1}{x}$ and $v' = \sin x$. Then $u' = -1/x^2$, $v = -\cos x$, and we have

$$\int_1^R \frac{\sin x}{x}\,dx = \left.\frac{-\cos x}{x}\right|_1^R - \int_1^R \frac{\cos x}{x^2}\,dx.$$

(b) For all x, $|\cos x| \le 1$, and therefore

$$\left|\frac{\cos x}{x^2}\right| = \frac{|\cos x|}{x^2} \le \frac{1}{x^2}.$$

The integral $\displaystyle\int_1^\infty x^{-2}\,dx$ converges, because $p = 2 > 1$. Therefore, by the comparison test,

$$\int_1^\infty \left|\frac{\cos x}{x^2}\right| dx \text{ also converges.}$$

Because $\displaystyle\int_1^\infty (\cos x/x^2)\,dx$ converges absolutely, it also converges. By this result,

$$\lim_{R\to\infty} \int_1^R \frac{\sin x}{x}\,dx = \lim_{R\to\infty} \left[\frac{-\cos R}{R} + \frac{\cos 1}{1} - \int_1^R \frac{\cos x}{x^2}\,dx\right] = 0 + \frac{\cos 1}{1} - \int_0^\infty \frac{\cos x}{x^2}\,dx = \cos 1 - M,$$

where $M = \displaystyle\int_1^\infty (\cos x/x^2)\,dx$, the existence of which was shown in the argument above. Therefore the integral $\displaystyle\int_1^\infty (\sin x/x)\,dx$ converges to a finite value.

(c) The integral can be split up as follows:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx.$$

The second integral converges by part (b). For the first integral, if we define $f(0) = 1$, then the integrand is continuous on [0, 1], and therefore

$$\int_0^1 \frac{\sin x}{x}\,dx = N$$

where N is some finite value. Thus, we have shown that I converges.

99. The **gamma function**, which plays an important role in advanced applications, is defined for $n \ge 1$ by

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt$$

(a) Show that the integral defining $\Gamma(n)$ converges for $n \ge 1$ (it actually converges for all $n > 0$). *Hint:* Show that $t^{n-1}e^{-t} < t^{-2}$ for t sufficiently large.

(b) Show that $\Gamma(n+1) = n\Gamma(n)$ using Integration by Parts.

(c) Show that $\Gamma(n+1) = n!$ if $n \ge 1$ is an integer. *Hint:* Use (a) repeatedly. Thus, $\Gamma(n)$ provides a way of defining n-factorial when n is not an integer.

SOLUTION
(a) By repeated use of L'Hôpital's Rule, we can compute the following limit:

$$\lim_{t\to\infty} \frac{e^t}{t^{n+1}} = \lim_{t\to\infty} \frac{e^t}{(n+1)t^n} = \cdots = \lim_{t\to\infty} \frac{e^t}{(n+1)!} = \infty.$$

This implies that, for t sufficiently large, we have

$$e^t \geq t^{n+1};$$

therefore

$$\frac{e^t}{t^{n-1}} \geq \frac{t^{n+1}}{t^{n-1}} = t^2 \quad \text{or} \quad t^{n-1}e^{-t} \leq t^{-2}.$$

The integral $\int_1^\infty t^{-2}\,dt$ converges because $p = 2 > 1$. Therefore, by the comparison test,

$$\int_M^\infty t^{n-1}e^{-t}\,dt \text{ also converges,}$$

where M is the value above which the above comparisons hold. Finally, because the function $t^{n-1}e^{-t}$ is continuous for all t, we know that

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt \text{ converges} \quad \text{for all } n \geq 1.$$

(b) Using Integration by Parts, with $u = t^n$ and $v' - e^{-t}$, we have $u' = nt^{n-1}$, $v = -e^{-t}$, and

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}\,dt = -t^n e^{-t}\Big|_0^\infty + n\int_0^\infty t^{n-1}e^{-t}\,dt$$
$$= \lim_{R\to\infty}\left(\frac{-R^n}{e^R} - 0\right) + n\Gamma(n) = 0 + n\Gamma(n) = n\Gamma(n).$$

Here, we've computed the limit as in part (a) with repeated use of L'Hôpital's Rule.
(c) By the result of part (b), we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) = \cdots = n!\,\Gamma(1).$$

If $n = 1$, then

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = \lim_{R\to\infty} -e^{-t}\Big|_0^R = \lim_{R\to\infty}\left(1 - e^{-R}\right) = 1.$$

Thus

$$\Gamma(n+1) = n!\,(1) = n!$$

100. Use the results of Exercise 99 to show that the Laplace transform (see Exercises 86–89 above) of x^n is $\frac{n!}{s^{n+1}}$.

SOLUTION If $f(x) = x^n$, then the Laplace transform of $f(x)$ is

$$Lf(s) = \int_0^\infty x^n e^{-sx}\,dx$$

Let $t = sx$. Then $dt = s\,dx$, and $x^n = t^n/s^n$. This gives us

$$Lf(s) = \int_0^\infty \frac{t^n}{s^n}e^{-t}\frac{dt}{s} = \frac{1}{s^{n+1}}\int_0^\infty t^n e^{-t}\,dt = \frac{1}{s^{n+1}}\Gamma(n+1) = \frac{n!}{s^{n+1}}.$$

7.7 Probability and Integration

Preliminary Questions

1. The function $p(x) = \cos x$ satisfies $\int_{-\pi/2}^{\pi} p(x)\,dx = 1$. Is $p(x)$ a probability density function on $[-\pi/2, \pi]$?

SOLUTION Since $p(x) = \cos x < 0$ for some points in $(-\pi/2, \pi)$, $p(x)$ is not a probability density function.

2. Estimate $P(2 \le X \le 2.1)$ assuming that the probability density function of X satisfies $p(2) = 0.2$.

SOLUTION $P(2 \le X \le 2.1) \approx p(2) \cdot (2.1 - 2) = 0.02$.

3. Which exponential probability density has mean $\mu = \frac{1}{4}$?

SOLUTION $\frac{1}{1/4} e^{-x/(1/4)} = 4e^{-4x}$.

Exercises

In Exercises 1–6, find a constant C such that $p(x)$ is a probability density function on the given interval, and compute the probability indicated.

1. $p(x) = \dfrac{C}{(x+1)^3}$ on $[0, \infty)$; $P(0 \le X \le 1)$

SOLUTION Compute the indefinite integral using the substitution $u = x + 1$, $du = dx$:

$$\int p(x)\,dx = \int \frac{C}{(x+1)^3}\,dx = -\frac{1}{2}C(x+1)^{-2} + K$$

For $p(x)$ to be a probability density function, we must have

$$1 = \int_0^\infty p(x)\,dx = -\frac{1}{2}C \lim_{R\to\infty} (x+1)^{-2}\Big|_0^R = \frac{1}{2}C - \frac{1}{2}C \lim_{R\to\infty}(R+1)^{-2} = \frac{1}{2}C$$

so that $C = 2$, and $p(x) = \frac{2}{(x+1)^3}$. Then using the indefinite integral above,

$$P(0 \le X \le 1) = \int_0^1 \frac{2}{(x+1)^3} = -\frac{1}{2} \cdot 2 \cdot (x+1)^{-2}\Big|_0^1 = -\frac{1}{4} + 1 = \frac{3}{4}$$

2. $p(x) = Cx(4 - x)$ on $[0, 4]$; $P(3 \le X \le 4)$

SOLUTION Compute the indefinite integral:

$$\int p(x)\,dx = C\int x(4-x)\,dx = C\int 4x - x^2\,dx = C\left(2x^2 - \frac{1}{3}x^3\right) + K$$

For $p(x)$ to be a probability density function, we must have

$$1 = \int_0^4 p(x)\,dx = C\left(2x^2 - \frac{1}{3}x^3\right)\Big|_0^4 = C\left(32 - \frac{64}{3}\right) = \frac{32}{3}C$$

so that $C = \frac{3}{32}$ and $p(x) = \frac{3}{32}x(4-x)$. Then using the indefinite integral above,

$$P(3 \le X \le 4) = \int_3^4 p(x)\,dx = \frac{3}{32}\left(2x^2 - \frac{1}{3}x^3\right)\Big|_3^4 = \frac{3}{32}\left(32 - \frac{64}{3} - 18 + 9\right) = \frac{5}{32}$$

3. $p(x) = \dfrac{C}{\sqrt{1-x^2}}$ on $(-1, 1)$; $P\left(-\frac{1}{2} \le X \le \frac{1}{2}\right)$

SOLUTION Compute the indefinite integral:

$$\int p(x)\,dx = C\int \frac{1}{\sqrt{1-x^2}}\,dx = C\sin^{-1}x + K$$

valid for $-1 < x < 1$. For $p(x)$ to be a probability density function, we must have

$$\begin{aligned}1 = \int_{-1}^1 p(x)\,dx = \int_{-1}^0 p(x)\,dx + \int_0^1 p(x)\,dx &= C\left(\lim_{R\to-1^+} \sin^{-1}x\Big|_R^0 + \lim_{R\to1^-}\sin^{-1}x\Big|_0^R\right)\\ &= C\left(\sin^{-1}(0) - \lim_{R\to-1^+}\sin^{-1}(R) + \lim_{R\to1^-}\sin^{-1}R - \sin^{-1}(0)\right)\\ &= C\left(-\sin^{-1}(-1) + \sin^{-1}(1)\right) = \pi C\end{aligned}$$

so that $C = \frac{1}{\pi}$ and $p(x) = \frac{1}{\pi\sqrt{1-x^2}}$. Then using the indefinite integral above,

$$P\left(-\frac{1}{2} \le X \le \frac{1}{2}\right) = \int_{-1/2}^{1/2} p(x)\,dx = \frac{1}{\pi}\sin^{-1}x\Big|_{-1/2}^{1/2} = \frac{1}{\pi}\left(\frac{\pi}{6} - \frac{-\pi}{6}\right) = \frac{1}{3}$$

4. $p(x) = \dfrac{Ce^{-x}}{1+e^{-2x}}$ on $(-\infty, \infty)$; $P(X \le -4)$

SOLUTION Compute the indefinite integral using the substitution $u = e^{-x}$; then $du = -e^{-x}\,dx = -u\,dx$ so that $dx = -\frac{1}{u}\,du$:

$$\int p(x)\,dx = \int \frac{Ce^{-x}}{1+e^{-2x}}\,dx = C\int \frac{u\cdot\left(-\frac{1}{u}\right)}{1+u^2}\,du = -C\int \frac{1}{1+u^2}\,du$$
$$= -C\tan^{-1}u + K = -C\tan^{-1}(e^{-x}) + K = C\tan^{-1}(e^{x}) + K$$

For $p(x)$ to be a probability density function, we must have

$$1 = \int_{-\infty}^{\infty} p(x)\,dx = C\lim_{R\to\infty}\tan^{-1}(e^x)\Big|_{-R}^{R} = C\lim_{R\to\infty}\left(\tan^{-1}(e^R) - \tan^{-1}(e^{-R})\right) = C\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}C$$

so that $C = \frac{2}{\pi}$ and $p(x) = \frac{2e^{-x}}{\pi(1+e^{-2x})}$. Then using the indefinite integral above,

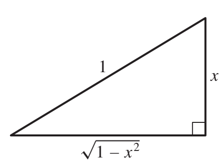
$$P(X \le -4) = \int_{-\infty}^{-4} p(x)\,dx = \lim_{R\to-\infty}\frac{2}{\pi}\tan^{-1}(e^x)\Big|_{R}^{-4} = \frac{2}{\pi}\tan^{-1}(e^{-4}) - \frac{2}{\pi}\lim_{R\to-\infty}\tan^{-1}(e^R)$$
$$= \frac{2}{\pi}\tan^{-1}(e^{-4}) \approx 0.0117$$

5. $p(x) = C\sqrt{1-x^2}$ on $(-1, 1)$; $P\left(-\frac{1}{2} \le X \le 1\right)$

SOLUTION Compute the indefinite integral using the substitution $x = \sin u$, so that $dx = \cos u\,du$:

$$\int p(x)\,dx = C\int\sqrt{1-x^2}\,dx = C\int\sqrt{1-\sin^2 u}\cos u\,du = C\int\cos^2 u\,du$$
$$= C\left(\frac{1}{2}u + \frac{1}{2}\cos u\sin u\right) + K$$

Since $x = \sin u$, we construct the following right triangle:



and we see that $\cos u = \sqrt{1-x^2}$, so that

$$\int p(x)\,dx = \frac{1}{2}C\left(\sin^{-1}x + x\sqrt{1-x^2}\right) + K$$

For $p(x)$ to be a probability density function, we must have

$$1 = \int_{-1}^{1} p(x)\,dx = \frac{1}{2}C\left(\sin^{-1}x + x\sqrt{1-x^2}\right)\Big|_{-1}^{1} = \frac{1}{2}C(\sin^{-1}1 - \sin^{-1}(-1)) = \frac{\pi}{2}C$$

so that $C = \frac{2}{\pi}$ and $p(x) = \frac{2}{\pi}\sqrt{1-x^2}$. Then using the indefinite integral above,

$$P\left(-\frac{1}{2} \le X \le 1\right) = \int_{-1/2}^{1}\frac{2}{\pi}\sqrt{1-x^2}\,dx = \frac{1}{\pi}\left(\sin^{-1}x + x\sqrt{1-x^2}\right)\Big|_{-1/2}^{1}$$
$$= \frac{1}{\pi}\left(\sin^{-1}1 + 0 - \sin^{-1}\left(-\frac{1}{2}\right) - \frac{-1}{2}\sqrt{1-\frac{1}{4}}\right)$$
$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{-\pi}{6} + \frac{\sqrt{3}}{4}\right) = \frac{2}{3} + \frac{\sqrt{3}}{4\pi} \approx 0.804$$

6. $p(x) = Ce^{-x}e^{-e^{-x}}$ on $(-\infty, \infty)$; $P(-4 \le X \le 4)$ This function, called the **Gumbel density**, is used to model extreme events such as floods and earthquakes.

SOLUTION Find the indefinite integral via the substitution $u = -e^{-x}$ so that $du = e^{-x}\,dx$; then

$$\int p(x)\,dx = C\int e^{-x}e^{-e^{-x}}\,dx = C\int e^{u}\,du = Ce^{u} = Ce^{-e^{-x}} + K$$

For $p(x)$ to be a probability density function, we must have

$$1 = \int_{-\infty}^{\infty} p(x)\,dx = C\lim_{R\to\infty} e^{-e^{-x}}\Big|_{-R}^{R} = C\lim_{R\to\infty}\left(e^{-e^{-R}} - e^{-e^{R}}\right) = C$$

since $e^{-R} \to 0$ so that the first term approaches $e^{0} = 1$, and $e^{R} \to \infty$ so that the second term approaches $e^{-\infty} = 0$. Thus $C = 1$ and $p(x) = e^{-x}e^{-e^{-x}}$. Then using the indefinite integral above,

$$P(-4 \le X \le 4) = e^{-e^{-4}} - e^{-e^{4}} \approx 0.982$$

7. Verify that $p(x) = 3x^{-4}$ is a probability density function on $[1, \infty)$ and calculate its mean value.

SOLUTION We have

$$\int_{1}^{\infty} 3x^{-4}\,dx = \lim_{R\to\infty}\left(-x^{-3}\right)\Big|_{1}^{R} = \lim_{R\to\infty}\left(-\frac{1}{R^3}\right) + 1 = 1$$

so that $p(x)$ is a probability density function on $[1, \infty)$. Its mean value is

$$\int_{1}^{\infty} x\cdot 3x^{-4}\,dx = \int_{1}^{\infty} 3x^{-3}\,dx = -\frac{3}{2}x^{-2}\Big|_{1}^{R} = \lim_{R\to\infty}\left(-\frac{3}{2R^2}\right) + \frac{3}{2} = \frac{3}{2}$$

8. Show that the density function $p(x) = \dfrac{2}{\pi(x^2+1)}$ on $[0, \infty)$ has infinite mean.

SOLUTION To verify that $p(x)$ is a probability density function, note that

$$\int_{0}^{\infty} \frac{2}{\pi}\frac{1}{x^2+1}\,dx = \frac{2}{\pi}\lim_{R\to\infty}\tan^{-1}x\Big|_{0}^{R} = \frac{2}{\pi}\left(\frac{\pi}{2} - 0\right) = 1$$

Its average value is (using the substitution $u = x^2 + 1$, $du = 2x\,dx$):

$$\frac{2}{\pi}\int_{0}^{\infty}\frac{x}{x^2+1}\,dx = \frac{1}{\pi}\int_{0}^{\infty}\frac{1}{u}\,du$$

The indefinite integral is $\ln u$, so the definite integral approaches $\infty - (-\infty) = \infty$, so this integral diverges and the mean is infinite.

9. Verify that $p(t) = \frac{1}{50}e^{-t/50}$ satisfies the condition $\int_0^{\infty} p(t)\,dt = 1$.

SOLUTION Use the substitution $u = \frac{t}{50}$, so that $du = \frac{1}{50}\,dt$. Then

$$\int_{0}^{\infty} p(t)\,dt = \int_{0}^{\infty}\frac{1}{50}e^{-t/50}\,dt = \int_{0}^{\infty} e^{-u}\,du = \lim_{R\to\infty}(-e^{-u})\Big|_{0}^{R} = \lim_{R\to\infty} 1 - e^{-R} = 1$$

10. Verify that for all $r > 0$, the exponential density function $p(t) = \frac{1}{r}e^{-t/r}$ satisfies the condition $\int_0^{\infty} p(t)\,dt = 1$.

SOLUTION This is similar to the preceding problem. Use the substitution $u = \frac{t}{r}$, so that $du = \frac{1}{r}\,dt$. Then

$$\int_{0}^{\infty} p(t)\,dt = \int_{0}^{\infty}\frac{1}{r}e^{-t/r}\,dt = \int_{0}^{\infty} e^{-u}\,du = \lim_{R\to\infty}(e^{-u})\Big|_{0}^{R} = \lim_{R\to\infty} 1 - e^{-R} = 1$$

11. The life X (in hours) of a battery in constant use is a random variable with exponential density. What is the probability that the battery will last more than 12 hours if the average life is 8 hours?

SOLUTION If the average life is 8 hours, then the mean of the exponential distribution is 8, so that the distribution is

$$p(x) = \frac{1}{8}e^{-x/8}$$

The probability that the battery will last more than 12 hours is given by (using the substitution $u = x/8$, so that $du = 1/8\,dx$ and $x = 12$ corresponds to $u = 3/2$)

$$P(X \geq 12) = \int_{12}^{\infty} p(x)\,dx = \int_{12}^{\infty} \frac{1}{8}e^{-x/8}\,dx = \int_{3/2}^{\infty} e^{-u}\,du = \lim_{R\to\infty}(-e^{-u})\Big|_{3/2}^{R}$$
$$= e^{-3/2} - \lim_{R\to\infty} e^{-R} = e^{-3/2} \approx 0.223$$

12. The time between incoming phone calls at a call center is a random variable with exponential density. There is a 50% probability of waiting 20 seconds or more between calls. What is the average time between calls?

SOLUTION The distribution is exponential, so $p(x) = \frac{1}{r}e^{-x/r}$. Since there is a 50% probability of waiting 20 seconds or more between calls, this means that

$$\int_{20}^{\infty} \frac{1}{r}e^{-x/r}\,dx = \frac{1}{2}$$

But

$$\int_{20}^{\infty} \frac{1}{r}e^{-x/r}\,dx = e^{-x/r}\Big|_{20}^{\infty} = e^{-20/r}$$

Thus $\frac{1}{2} = e^{-20/r}$, so that $-\frac{20}{r} = \ln\frac{1}{2} = -\ln 2$; it follows that $r = \frac{20}{\ln 2}$, which is the mean value.

13. The distance r between the electron and the nucleus in a hydrogen atom (in its lowest energy state) is a random variable with probability density $p(r) = 4a_0^{-3}r^2e^{-2r/a_0}$ for $r \geq 0$, where a_0 is the Bohr radius (Figure 7). Calculate the probability P that the electron is within one Bohr radius of the nucleus. The value of a_0 is approximately 5.29×10^{-11} m, but this value is not needed to compute P.

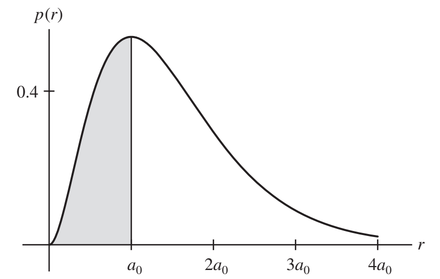


FIGURE 7 Probability density function $p(r) = 4a_0^{-3}r^2e^{-2r/a_0}$.

SOLUTION The probability P is the area of the shaded region in Figure 7. To calculate p, use the substitution $u = 2r/a_0$:

$$P = \int_0^{a_0} p(r)\,dr = \frac{4}{a_0^3}\int_0^{a_0} r^2e^{-2r/a_0}\,dr = \left(\frac{4}{a_0^3}\right)\left(\frac{a_0^3}{8}\right)\int_0^2 u^2e^{-u}\,du$$

The constant in front simplifies to $\frac{1}{2}$ and the formula in the margin gives us

$$P = \frac{1}{2}\int_0^2 u^2e^{-u}\,du = \frac{1}{2}\left(-(u^2+2u+2)e^{-u}\right)\Big|_0^2 = \frac{1}{2}\left(2 - 10e^{-2}\right) \approx 0.32$$

Thus, the electron within a distance a_0 of the nucleus with probability 0.32.

14. Show that the distance r between the electron and the nucleus in Exercise 13 has mean $\mu = 3a_0/2$.

SOLUTION The mean of the distribution is

$$\mu = \int_0^{\infty} rp(r)\,dr = \int_0^{\infty} r \cdot 4a_0^{-3}r^2e^{-2r/a_0}\,dr = \frac{4}{a_0^3}\int_0^{\infty} r^3e^{-2r/a_0}\,dr$$

To calculate this integral, use as before the substitution $x = 2r/a_0$ to get

$$\mu = \frac{4}{a_0^3}\cdot\frac{a_0^3}{8}\cdot\frac{a_0}{2}\int_0^{\infty} x^3e^{-x}\,dx = \frac{a_0}{4}\int_0^{\infty} x^3e^{-x}\,dx$$

To calculate this integral, we use integration by parts, with $u = x^3$, $v' = e^{-x}$, so that $u' = 3x^2$ and $v = -e^{-x}$; then

$$\mu = \frac{a_0}{4}\left(-x^3e^{-x}\Big|_0^{\infty} + 3\int_0^{\infty} x^2e^{-x}\,dx\right)$$

The first term is evaluated as follows, using L'Hôpital's Rule multiple times:

$$-x^3e^{-x}\Big|_0^{\infty} = \lim_{R\to\infty}\left(-x^3e^{-x}\right)\Big|_0^R = \lim_{R\to\infty}\left(-\frac{R^3}{e^R}\right)$$
$$= \lim_{R\to\infty}\left(-\frac{3R^2}{e^R}\right) = \lim_{R\to\infty}\left(-\frac{6R}{e^R}\right) = \lim_{R\to\infty}\left(-\frac{6}{e^R}\right) = 0$$

The second term, by the marginal note in the previous problem, is

$$\int_0^{\infty} x^2e^{-x}\,dx = \lim_{R\to\infty}\left((-u^2+2u+2)e^{-u}\right)\Big|_0^R = \lim_{R\to\infty}\left(2-\frac{-R^2+2R+2}{e^R}\right) = 2$$

using L'Hôpital's Rule as in the previous formulas. Thus, finally,

$$\mu = \frac{a_0}{4}(0+3\cdot 2) = \frac{3}{2}a_0$$

In Exercises 15–21, $F(z)$ denotes the cumulative normal distribution function. Refer to a calculator, computer algebra system, or online resource to obtain values of $F(z)$.

15. Express the area of region A in Figure 8 in terms of $F(z)$ and compute its value.

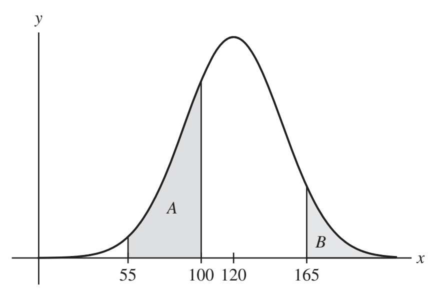


FIGURE 8 Normal density function with $\mu = 120$ and $\sigma = 30$.

SOLUTION The area of region A is $P(55 \le X \le 100)$. By Theorem 1, we have

$$P(55 \le X \le 100) = F\left(\frac{100-120}{30}\right) - F\left(\frac{55-120}{30}\right) = F\left(-\frac{2}{3}\right) - F\left(-\frac{13}{6}\right) \approx 0.237$$

16. Show that the area of region B in Figure 8 is equal to $1 - F(1.5)$ and compute its value. Verify numerically that this area is also equal to $F(-1.5)$ and explain why graphically.

SOLUTION The area of region B is $P(X \ge 165)$, and $P(X \ge 165) + P(X \le 165) = 1$. But by Theorem 1,

$$P(X \le 165) = F\left(\frac{165-120}{30}\right) = F(1.5)$$

so that

$$P(X \ge 165) = 1 - P(X \le 165) = 1 - F(1.5) \approx 0.0668$$

Using a computer algebra system, we also get $F(-1.5) \approx 0.0668$. Graphically, since the density function $p(x)$ is symmetric around $x = 120$, we see that the area to the right of $x = 165$ is equal to the area to the left of $x = 120 - (165 - 120) = 75$; this area is

$$F\left(\frac{75-120}{30}\right) = F\left(\frac{-45}{30}\right) = F(-1.5)$$

17. Assume X has a standard normal distribution ($\mu = 0, \sigma = 1$). Find:

(a) $P(X \le 1.2)$ **(b)** $P(X \ge -0.4)$

SOLUTION

(a) $P(X \le 1.2) = F(1.2) \approx 0.8849$

(b) $P(X \ge -0.4) = 1 - P(X \le -0.4) = 1 - F(-0.4) \approx 1 - 0.3446 \approx 0.6554$

18. Evaluate numerically: $\dfrac{1}{3\sqrt{2\pi}}\displaystyle\int_{14.5}^{\infty} e^{-(z-10)^2/18}\,dz$.

SOLUTION This is the area to the right of 14.5 under the cumulative distribution function for a normal distribution with $\mu = 10$ and $\sigma = 3$. In terms of the standard normal cumulative distribution function $F(z)$, this is

$$P(X \geq 14.5) = 1 - P(X \leq 14.5) = 1 - F\left(\frac{14.5 - 10}{3}\right) = 1 - F(1.5) \approx 0.0668$$

19. Use a graph to show that $F(-z) = 1 - F(z)$ for all z. Then show that if $p(x)$ is a normal density function with mean μ and standard deviation σ, then for all $r \geq 0$,

$$P(\mu - r\sigma \leq X \leq \mu + r\sigma) = 2F(r) - 1$$

SOLUTION Consider the graph of the standard normal density function in Figure 5. This graph is symmetric around the y-axis, so that the area under the curve from z to ∞, which is $1 - F(z)$, is equal to the area under the curve from $-\infty$ to $-z$, which is $F(-z)$. Thus $1 - F(z) = F(-z)$. Now, if $p(x)$ is a normal density function with mean μ and standard deviation σ, then for $r \geq 0$ (so that the range $\mu - r\sigma \leq X \leq \mu + r\sigma$ is nonempty),

$$\begin{aligned} P(\mu - r\sigma \leq X \leq \mu + r\sigma) &= F\left(\frac{\mu + r\sigma - \mu}{\sigma}\right) - F\left(\frac{\mu - r\sigma - \mu}{\sigma}\right) \\ &= F(r) - F(-r) = F(r) - (1 - F(r)) = 2F(r) - 1 \end{aligned}$$

20. The average September rainfall in Erie, Pennsylvania, is a random variable X with mean $\mu = 102$ mm. Assume that the amount of rainfall is normally distributed with standard deviation $\sigma = 48$.

(a) Express $P(128 \leq X \leq 150)$ in terms of $F(z)$ and compute its value numerically.

(b) Let P be the probability that September rainfall will be at least 120 mm. Express P as an integral of an appropriate density function and compute its value numerically.

SOLUTION

(a)

$$P(128 \leq X \leq 150) = F\left(\frac{150 - 102}{48}\right) - F\left(\frac{128 - 102}{48}\right) = F(1) - F\left(\frac{13}{24}\right) \approx 0.135$$

(b) The cumulative density function associated with X is

$$f(z) = \frac{1}{48\sqrt{2\pi}}\int_{-\infty}^{z} e^{-(x-102)^2/(2\cdot 48^2)}\,dx$$

To compute the value numerically, we use the standard normal cumulative distribution $F(z)$. Recall that $P(X \geq 120) = 1 - P(X \leq 120)$, and that

$$P(X \leq 120) = F\left(\frac{120 - 102}{48}\right) = F\left(\frac{3}{8}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{3/8} e^{-x^2/2}\,dx \approx 0.646$$

so that $P(X \geq 120) \approx 1 - 0.646 \approx 0.354$.

21. A bottling company produces bottles of fruit juice that are filled, on average, with 32 ounces of juice. Due to random fluctuations in the machinery, the actual volume of juice is normally distributed with a standard deviation of 0.4 ounce. Let P be the probability of a bottle having less than 31 ounces. Express P as an integral of an appropriate density function and compute its value numerically.

SOLUTION The associated cumulative distribution function is

$$f(z) = \frac{1}{0.4\sqrt{2\pi}}\int_{-\infty}^{z} e^{-(x-32)^2/(2\cdot 0.4^2)}\,dx$$

To compute the value numerically, we use the standard normal cumulative distribution function $F(z)$:

$$P(X \leq 31) = F\left(\frac{31 - 32}{0.4}\right) = F\left(-\frac{5}{2}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-5/2} e^{-x^2/2}\,dx \approx 0.0062$$

22. According to **Maxwell's Distribution Law**, in a gas of molecular mass m, the speed v of a molecule in a gas at temperature T (kelvins) is a random variable with density

$$p(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/(2kT)} \qquad (v \geq 0)$$

where k is Boltzmann's constant. Show that the average molecular speed is equal to $(8kT/\pi m)^{1/2}$. The average speed of oxygen molecules at room temperature is around 450 m/s.

SOLUTION The average speed $\bar{v}$ is given by

$$\bar{v} = \int_0^\infty vp(v)\,dv = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty v^3 e^{-mv^2/2kT}\,dv.$$

Let $\alpha = -m/2kT$. We'll first compute the indefinite integral

$$\int v^3 e^{\alpha v^2}\,dv.$$

Using Integration by Parts, let $u = v^2$, $v' = ve^{\alpha v^2}$. Then $u' = 2v$ and $v = \frac{1}{2\alpha}e^{\alpha v^2}$. This gives us

$$\int v^3 e^{\alpha v^2}\,dv = \frac{1}{2\alpha}v^2 e^{\alpha v^2} - \frac{1}{\alpha}\int ve^{\alpha v^2}\,dv.$$

To compute the remaining integral, let $w = \alpha v^2$, $dw = 2\alpha v\,dv$. The result is

$$\int v^3 e^{\alpha v^2}\,dv = \frac{1}{2\alpha}v^2 e^{\alpha v^2} - \frac{1}{2\alpha^2}e^{\alpha v^2} + C.$$

Thus,

$$\int_0^R vp(v)\,dv = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\left[\frac{e^{\alpha v^2}}{2\alpha}\left(v^2 - \frac{1}{\alpha}\right)\right]_0^R = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{1}{2\alpha}\left[e^{\alpha R^2}\left(R^2 - \frac{1}{\alpha}\right) + \frac{1}{\alpha}\right],$$

and

$$\bar{v} = \lim_{R\to\infty} 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{1}{2\alpha}\left[e^{\alpha R^2}\left(R^2 - \frac{1}{\alpha}\right) + \frac{1}{\alpha}\right] = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{1}{2\alpha}\left[\lim_{R\to\infty} e^{\alpha R^2}\left(R^2 - \frac{1}{\alpha}\right) + \frac{1}{\alpha}\right].$$

Use L'Hôpital's Rule to compute the limit, recalling that $\alpha = -m/2kT < 0$:

$$\lim_{R\to\infty} e^{\alpha R^2}\left(R^2 - \frac{1}{\alpha}\right) = \lim_{R\to\infty}\frac{R^2 - \frac{1}{\alpha}}{e^{-\alpha R^2}} = \lim_{R\to\infty}\frac{2R}{-2\alpha Re^{-\alpha R^2}} = \lim_{R\to\infty}\frac{-1}{\alpha e^{-\alpha R^2}} = 0.$$

Thus

$$\bar{v} = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{1}{2\alpha}\left(0 + \frac{1}{\alpha}\right) = \frac{2\pi}{\alpha^2}\left(\frac{m}{2\pi kT}\right)^{3/2} = 2\pi\left(-\frac{2kT}{m}\right)^2\left(\frac{m}{2\pi kT}\right)\sqrt{\frac{m}{2\pi kT}}$$
$$= \frac{4kT}{m}\sqrt{\frac{m}{2\pi kT}} = \sqrt{\frac{8kT}{\pi m}}.$$

*In Exercises 23–26, calculate μ and σ, where σ is the **standard deviation**, defined by*

$$\sigma^2 = \int_{-\infty}^\infty (x-\mu)^2\,p(x)\,dx$$

The smaller the value of σ, the more tightly clustered are the values of the random variable X about the mean μ.

23. $p(x) = \dfrac{5}{2x^{7/2}}$ on $[1,\infty)$

SOLUTION The mean is

$$\int_1^\infty xp(x)\,dx = \int_1^\infty \frac{5}{2}x^{-5/2}\,dx = -\frac{5}{3}x^{-3/2}\Big|_1^\infty = \frac{5}{3}$$

and

$$\sigma^2 = \int_1^\infty (x-\mu)^2 p(x)\,dx = \int_1^\infty (x^2 - 2\mu x + \mu^2)\frac{5}{2}x^{-7/2}\,dx$$
$$= \frac{5}{2}\int_1^\infty x^{-3/2} - 2\mu x^{-5/2} + \mu^2 x^{-7/2}\,dx = \frac{5}{2}\left(-2x^{-1/2} + \frac{4}{3}\mu x^{-3/2} - \frac{2}{5}\mu^2 x^{-5/2}\right)\Big|_1^\infty$$
$$= \frac{5}{2}\left(2 - \frac{4}{3}\mu + \frac{2}{5}\mu^2\right) = \frac{5}{2}\left(2 - \frac{4}{3}\cdot\frac{5}{3} + \frac{2}{5}\cdot\frac{25}{9}\right) = \frac{20}{9}$$

24. $p(x) = \dfrac{1}{\pi\sqrt{1-x^2}}$ on $(-1, 1)$

SOLUTION Use the substituion $u = 1 - x^2$ so that $du = -2x\,dx$. The mean is

$$\mu = \int_{-1}^{1} \frac{x}{\pi\sqrt{1-x^2}}\,dx = -\frac{1}{2\pi}\int_{x=-1}^{1} \frac{-2x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2\pi}\int_{x=-1}^{1} \frac{1}{\sqrt{u}}\,du$$
$$= -\frac{1}{\pi}\sqrt{u}\Big|_{x=-1}^{x=1} = -\frac{1}{\pi}\sqrt{1-x^2}\Big|_{-1}^{1} = 0$$

To compute the standard deviation, use the substitution $x = \sin u$, $dx = \cos u\,du$; then $x = -1$ corresponds to $u = -\pi/2$ and $x = 1$ to $u = \pi/2$:

$$\sigma^2 = \int_{-1}^{1} (x-\mu)^2 p(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\sin^2 u}{\sqrt{1-\sin^2 u}}\cos u\,du$$
$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\sin^2 u}{\cos u}\cos u\,du = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin^2 u\,du = \frac{1}{2\pi}(u - \cos u \sin u)\Big|_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{2\pi}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = \frac{1}{2}$$

25. $p(x) = \dfrac{1}{3}e^{-x/3}$ on $[0, \infty)$

SOLUTION This is an exponential density function with mean $\mu = 3$. The standard deviation is

$$\sigma^2 = \frac{1}{3}\int_0^{\infty} (x-3)^2 e^{-x/3}\,dx = \frac{1}{3}\int_0^{\infty} \left(x^2e^{-x/3} - 6xe^{-x/3} + 9e^{-x/3}\right)dx$$
$$= \frac{1}{3}\int_0^{\infty} x^2e^{-x/3}\,dx - 2\int_0^{\infty} xe^{-x/3}\,dx + 3\int_0^{\infty} e^{-x/3}\,dx$$

We tackle the third integral first:

$$\int_0^{\infty} e^{-x/3}\,dx = -3e^{-x/3}\Big|_0^{\infty} = 3$$

For the second integral, use integration by parts with $u = x$, $v' = e^{-x/3}$ so that $u' = 1$ and $v = -3e^{-x/3}$. Then

$$\int_0^{\infty} xe^{-x/3}\,dx = -3xe^{-x/3}\Big|_0^{\infty} + 3\int_0^{\infty} e^{-x/3}\,dx = 0 + 3 \cdot 3 = 9$$

Finally, the first integral is solved using integration by parts with $u = x^2$, $v' = e^{-x/3}$ so that $u' = 2x$ and $v = -3e^{-x/3}$; then

$$\int_0^{\infty} x^2e^{-x/3}\,dx = -3x^2e^{-x/3}\Big|_0^{\infty} + 6\int_0^{\infty} xe^{-x/3}\,dx = 0 + 6 \cdot 9 = 54$$

and, finally,

$$\sigma^2 = \frac{1}{3}\int_0^{\infty} x^2e^{-x/3}\,dx - 2\int_0^{\infty} xe^{-x/3}\,dx + 3\int_0^{\infty} e^{-x/3}\,dx$$
$$= \frac{1}{3}\cdot 54 - 2\cdot 9 + 3\cdot 3 = 9$$

26. $p(x) = \dfrac{1}{r}e^{-x/r}$ on $[0, \infty)$, where $r > 0$

SOLUTION This is similar to the previous problem. We have an exponential density function with mean $\mu = r$. The standard deviation is

$$\sigma^2 = \frac{1}{r}\int_0^{\infty} (x-r)^2 e^{-x/r}\,dx = \frac{1}{r}\int_0^{\infty} \left(x^2e^{-x/r} - 2rxe^{-x/r} + r^2e^{-x/r}\right)dx$$
$$= \frac{1}{r}\int_0^{\infty} x^2e^{-x/r}\,dx - 2\int_0^{\infty} xe^{-x/r}\,dx + r\int_0^{\infty} e^{-x/r}\,dx$$

We tackle the third integral first:

$$\int_0^\infty e^{-x/r}\,dx = -re^{-x/r}\Big|_0^\infty = r$$

For the second integral, use integration by parts with $u = x$, $v' = e^{-x/r}$ so that $u' = 1$ and $v = -re^{-x/r}$. Then

$$\int_0^\infty xe^{-x/r}\,dx = -rxe^{-x/r}\Big|_0^\infty + r\int_0^\infty e^{-x/r}\,dx = 0 + r \cdot r = r^2$$

Finally, the first integral is solved using integration by parts with $u = x^2$, $v' = e^{-x/r}$ so that $u' = 2x$ and $v = -re^{-x/r}$; then

$$\int_0^\infty x^2e^{-x/r}\,dx = -rx^2e^{-x/r}\Big|_0^\infty + 2r\int_0^\infty xe^{-x/r}\,dx = 0 + 2r \cdot r^2 = 2r^3$$

and, finally,

$$\begin{aligned}\sigma^2 &= \frac{1}{r}\int_0^\infty x^2e^{-x/r}\,dx - 2\int_0^\infty xe^{-x/3}\,dx + r\int_0^\infty e^{-x/3}\,dx \\ &= \frac{1}{r}\cdot 2r^3 - 2\cdot r^2 + r\cdot r = r^2\end{aligned}$$

Further Insights and Challenges

27. The time to decay of an atom in a radioactive substance is a random variable X. The law of radioactive decay states that if N atoms are present at time $t = 0$, then $Nf(t)$ atoms will be present at time t, where $f(t) = e^{-kt}$ ($k > 0$ is the decay constant). Explain the following statements:

(a) The fraction of atoms that decay in a small time interval $[t, t + \Delta t]$ is approximately $-f'(t)\Delta t$.

(b) The probability density function of X is $-f'(t)$.

(c) The average time to decay is $1/k$.

SOLUTION

(a) The number of atoms that decay in the interval $[t, t + \Delta t]$ is just $f(t) - f(t + \Delta t)$; the statement simply says that $f(t) - f(t + \Delta t) \approx -f'(t)\Delta t$, which is the same as saying that

$$f'(t) \approx \frac{f(t) - f(t+\Delta t)}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

which is true by the definition of the derivative. Intuitively, since $f'(t)$ is the instantaneous rate of decay, we would expect that over a short interval, the number of atoms decaying is proportional to both $f'(t)$ and the size of the interval.

(b) The probability density function is defined by the property in (a): the probability that X lies in a small interval $[t, t + \Delta t]$ is approximately $p(t)\Delta t$, so that $p(t) = -f'(t)$.

(c) The average time to decay is the mean of the distribution, which is

$$\mu = \int_0^\infty t\cdot(-f'(t))\,dt = -\int_0^\infty tf'(t)\,dt$$

We compute this integral using integration by parts, with $u = t$, $v' = f'(t)$. Then $u' = 1$, $v = f(t)$, and

$$\mu = -\int_0^\infty tf'(t)\,dt = -tf(t)\Big|_0^\infty + \int_0^\infty f(t)\,dt.$$

Since $f(t) = e^{-kt}$, we have

$$-tf(t)\Big|_0^\infty = \lim_{R\to\infty} -te^{-kt}\Big|_0^R = \lim_{R\to\infty} -Re^{-Rt} + 0 = \lim_{R\to\infty}\frac{-R}{e^{Rt}} = \lim_{R\to\infty}\frac{-1}{Re^{Rt}} = 0.$$

Here we used L'Hôpital's Rule to compute the limit. Thus

$$\mu = \int_0^\infty f(t)\,dt = \int_0^\infty e^{-kt}\,dt.$$

Now,

$$\int_0^R e^{-kt}\,dt = -\frac{1}{k}e^{-kt}\Big|_0^R = -\frac{1}{k}\left(e^{-kR} - 1\right) = \frac{1}{k}\left(1 - e^{-kR}\right),$$

so

$$\mu = \lim_{R\to\infty} \frac{1}{k}\left(1 - e^{-kR}\right) = \frac{1}{k}(1-0) = \frac{1}{k}.$$

Because k has units of $(\text{time})^{-1}$, μ does in fact have the appropriate units of time.

28. The half-life of radon-222, is 3.825 days. Use Exercise 27 to compute:

(a) The average time to decay of a radon-222 atom.

(b) The probability that a given atom will decay in the next 24 hours.

SOLUTION

(a) The average decay time is just the mean, μ; to determine it, we must determine the decay constant k, given the half-life of 3.825 days. Recall that

$$k = \frac{\ln 2}{t_n}$$

where t_n is the half-life. Thus,

$$\mu = \frac{1}{k} = \frac{t_n}{\ln 2} = \frac{3.825}{\ln 2} \approx 5.518 \text{ days.}$$

(b) The probability that a particular atom will decay in the next 24 hours is the area under the probability density function between $t = 0$ and $t = 1$ (note that t is measured in days). Since $f(t) = e^{-kt}$, the probability density function is $-ke^{-kt}$; from part (a), $k \approx 0.1812$, so the required probability is

$$\int_0^1 (-f'(t))\,dt = f(0) - f(1) = 1 - e^{-0.1812} \approx 0.1657$$

7.8 Numerical Integration

Preliminary Questions

1. What are T_1 and T_2 for a function on [0, 2] such that $f(0) = 3$, $f(1) = 4$, and $f(2) = 3$?

SOLUTION Using the given function values

$$T_1 = \frac{1}{2}(2)(3+3) = 6 \quad \text{and} \quad T_2 = \frac{1}{2}(1)(3+8+3) = 7.$$

2. For which graph in Figure 16 will T_N *overestimate* the integral? What about M_N?

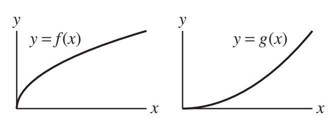


FIGURE 16

SOLUTION T_N overestimates the value of the integral when the integrand is concave up; thus, T_N will overestimate the integral of $y = g(x)$. On the other hand, M_N overestimates the value of the integral when the integrand is concave down; thus, M_N will overestimate the integral of $y = f(x)$.

3. How large is the error when the Trapezoidal Rule is applied to a linear function? Explain graphically.

SOLUTION The Trapezoidal Rule integrates linear functions exactly, so the error will be zero.

4. What is the maximum possible error if T_4 is used to approximate

$$\int_0^3 f(x)\,dx$$

where $|f''(x)| \le 2$ for all x.

SOLUTION The maximum possible error in T_4 is

$$\max |f''(x)| \frac{(b-a)^3}{12n^2} \le \frac{2(3-0)^3}{12(4)^2} = \frac{9}{32}.$$

5. What are the two graphical interpretations of the Midpoint Rule?

SOLUTION The two graphical interpretations of the Midpoint Rule are the sum of the areas of the midpoint rectangles and the sum of the areas of the tangential trapezoids.

Exercises

In Exercises 1–12, calculate T_N and M_N for the value of N indicated.

1. $\int_0^2 x^2\,dx, \quad N = 4$

SOLUTION Let $f(x) = x^2$. We divide [0, 2] into 4 subintervals of width

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

with endpoints 0, 0.5, 1, 1.5, 2, and midpoints 0.25, 0.75, 1.25, 1.75. With this data, we get

$$T_4 = \frac{1}{2}\cdot\frac{1}{2}\left(0^2 + 2(0.5)^2 + 2(1)^2 + 2(1.5)^2 + 2^2\right) = 2.75;\ \text{and}$$

$$M_4 = \frac{1}{2}\left(0.25^2 + 0.75^2 + 1.25^2 + 1.75^2\right) = 2.625.$$

2. $\int_0^4 \sqrt{x}\,dx, \quad N = 4$

SOLUTION Let $f(x) = \sqrt{x}$. We divide [0, 4] into 4 subintervals of width

$$\Delta x = \frac{4-0}{4} = 1$$

with endpoints 0, 1, 2, 3, 4, and midpoints 0.5, 1.5, 2.5, 3.5. With this data, we get

$$T_4 = \frac{1}{2}\cdot 1\cdot\left(\sqrt{0} + 2\sqrt{1} + 2\sqrt{2} + 2\sqrt{3} + \sqrt{4}\right) \approx 5.14626;\ \text{and}$$

$$M_4 = 1\cdot\left(\sqrt{0.5} + \sqrt{1.5} + \sqrt{2.5} + \sqrt{3.5}\right) \approx 5.38382.$$

3. $\int_1^4 x^3\,dx, \quad N = 6$

SOLUTION Let $f(x) = x^3$. We divide [1, 4] into 6 subintervals of width

$$\Delta x = \frac{4-1}{6} = \frac{1}{2}$$

with endpoints 1, 1.5, 2, 2.5, 3, 3.5, 4, and midpoints 1.25, 1.75, 2.25, 2.75, 3.25, 3.75. With this data, we get

$$T_6 = \frac{1}{2}\left(\frac{1}{2}\right)\left(1^3 + 2(1.5)^3 + 2(2)^3 + 2(2.5)^3 + 2(3)^3 + 2(3.5)^3 + 4^3\right) = 64.6875;\ \text{and}$$

$$M_6 = \frac{1}{2}\left(1.25^3 + 1.75^3 + 2.25^3 + 2.75^3 + 3.25^3 + 3.75^3\right) = 63.28125.$$

4. $\int_1^2 \sqrt{x^4+1}\,dx, \quad N = 5$

SOLUTION We divide [1, 2] into 5 subintervals of width

$$\Delta x = \frac{2-1}{5} = \frac{1}{5} = 0.2$$

with endpoints 1, 1.2, 1.4, 1.6, 1.8, 2, and midpoints 1.1, 1.3, 1.5, 1.7, 1.9. With this data, we have

$$T_5 = \frac{1}{2}\cdot\frac{1}{5}\left(\sqrt{1^4+1} + 2\sqrt{1.2^4+1} + 2\sqrt{1.4^4+1} + 2\sqrt{1.6^4+1} + 2\sqrt{1.8^4+1} + \sqrt{2^2+1}\right) \approx 2.57228$$

$$M_5 = \frac{1}{5}\left(\sqrt{1.1^4+1} + \sqrt{1.3^4+1} + \sqrt{1.5^4+1} + \sqrt{1.7^4+1} + \sqrt{1.9^4+1}\right) \approx 2.55994$$

5. $\int_1^4 \frac{dx}{x}, \quad N = 6$

SOLUTION Let $f(x) = 1/x$. We divide [1, 4] into 6 subintervals of width

$$\Delta x = \frac{4-1}{6} = \frac{1}{2}$$

with endpoints 1, 1.5, 2, 2.5, 3, 3.5, 4, and midpoints 1.25, 1.75, 2.25, 2.75, 3.25, 3.75. With this data, we get

$$T_6 = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{1} + \frac{2}{1.5} + \frac{2}{2} + \frac{2}{2.5} + \frac{2}{3} + \frac{2}{3.5} + \frac{1}{4}\right) \approx 1.40536; \text{ and}$$

$$M_6 = \frac{1}{2}\left(\frac{1}{1.25} + \frac{1}{1.75} + \frac{1}{2.25} + \frac{1}{2.75} + \frac{1}{3.25} + \frac{1}{3.75}\right) \approx 1.37693.$$

6. $\displaystyle\int_{-2}^{-1} \frac{dx}{x}, \quad N = 5$

SOLUTION Let $f(x) = 1/x$. We divide $[-2, -1]$ into 5 subintervals of width

$$\Delta x = \frac{-1 - (-2)}{5} = \frac{1}{5} = 0.2$$

with endpoints $-2, -1.8, -1.6, -1.4, -1.2, -1$, and midpoints $-1.9, -1.7, -1.5, -1.3, -1.1$. With this data, we get

$$T_5 = \frac{1}{2}\left(\frac{1}{5}\right)\left(\frac{1}{-2} + \frac{2}{-1.8} + \frac{2}{-1.6} + \frac{2}{-1.4} + \frac{2}{-1.2} + \frac{1}{-1}\right) \approx -0.695635; \text{ and}$$

$$M_5 = \frac{1}{5}\left(\frac{1}{-1.9} + \frac{1}{-1.7} + \frac{1}{-1.5} + \frac{1}{-1.3} + \frac{1}{-1.1}\right) \approx -0.691908.$$

7. $\displaystyle\int_{0}^{\pi/2} \sqrt{\sin x}\, dx, \quad N = 6$

SOLUTION Let $f(x) = \sqrt{\sin x}$. We divide $[0, \pi/2]$ into 6 subintervals of width

$$\Delta x = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

with endpoints

$$0, \frac{\pi}{12}, \frac{2\pi}{12}, \ldots, \frac{6\pi}{12} = \frac{\pi}{2},$$

and midpoints

$$\frac{\pi}{24}, \frac{3\pi}{24}, \ldots, \frac{11\pi}{24}.$$

With this data, we get

$$T_6 = \frac{1}{2}\left(\frac{\pi}{12}\right)\left(\sqrt{\sin(0)} + 2\sqrt{\sin(\pi/12)} + \cdots + \sqrt{\sin(6\pi/12)}\right) \approx 1.17029; \text{ and}$$

$$M_6 = \frac{\pi}{12}\left(\sqrt{\sin(\pi/24)} + \sqrt{\sin(3\pi/24)} + \cdots + \sqrt{\sin(11\pi/24)}\right) \approx 1.20630.$$

8. $\displaystyle\int_{0}^{\pi/4} \sec x\, dx, \quad N = 6$

SOLUTION Let $f(x) = \sec x$. We divide $[0, \pi/4]$ into 6 subintervals of width

$$\Delta x = \frac{\frac{\pi}{4} - 0}{6} = \frac{\pi}{24}$$

with endpoints

$$0, \frac{\pi}{24}, \frac{2\pi}{24}, \ldots, \frac{6\pi}{24} = \frac{\pi}{4},$$

and midpoints

$$\frac{\pi}{48}, \frac{3\pi}{48}, \ldots, \frac{11\pi}{48}.$$

With this data, we get

$$T_6 = \frac{1}{2}\left(\frac{\pi}{24}\right)\left(\sec(0) + 2\sec(\pi/24) + 2\sec(2\pi/24) + \cdots + \sec(6\pi/24)\right) \approx 0.883387; \text{ and}$$

$$M_6 = \frac{\pi}{24}\left(\sec(\pi/48) + \sec(3\pi/48) + \sec(5\pi/48) + \cdots + \sec(11\pi/48)\right) \approx 0.880369.$$

9. $\int_1^2 \ln x\, dx, \quad N = 5$

SOLUTION Let $f(x) = \ln x$. We divide [1, 2] into 5 subintervals of width

$$\Delta x = \frac{2-1}{5} = \frac{1}{5} = 0.2$$

with endpoints 1, 1.2, 1.4, 1.6, 1.8, 2, and midpoints 1.1, 1.3, 1.5, 1.7, 1.9. With this data, we get

$$T_5 = \frac{1}{2}\left(\frac{1}{5}\right)\left(\ln 1 + 2\ln 1.2 + 2\ln 1.4 + 2\ln 1.6 + 2\ln 1.8 + \ln 2\right) \approx 0.384632; \text{ and}$$

$$M_5 = \frac{1}{5}\left(\ln 1.1 + \ln 1.3 + \ln 1.5 + \ln 1.7 + \ln 1.9\right) \approx 0.387124.$$

10. $\int_2^3 \frac{dx}{\ln x}, \quad N = 5$

SOLUTION Let $f(x) = 1/\ln x$. We divide [2, 3] into 5 subintervals of width

$$\Delta x = \frac{3-2}{5} = \frac{1}{5} = 0.2$$

with endpoints 2, 2.2, 2.4, 2.6, 2.8, 3, and midpoints 2.1, 2.3, 2.5, 2.7, 2.9. With this data, we get

$$T_5 = \frac{1}{2}\left(\frac{1}{5}\right)\left(\frac{1}{\ln 2} + \frac{2}{\ln 2.2} + \frac{2}{\ln 2.4} + \frac{2}{\ln 2.6} + \frac{2}{\ln 2.8} + \frac{1}{\ln 3}\right) \approx 1.12096; \text{ and}$$

$$M_5 = \frac{1}{5}\left(\frac{1}{\ln 2.1} + \frac{1}{\ln 2.3} + \frac{1}{\ln 2.5} + \frac{1}{\ln 2.7} + \frac{1}{\ln 2.9}\right) \approx 1.11716.$$

11. $\int_0^1 e^{-x^2}\, dx, \quad N = 5$

SOLUTION Let $f(x) = e^{-x^2}$. We divide [0, 1] into 5 subintervals of width

$$\Delta x = \frac{1-0}{5} = \frac{1}{5} = 0.2$$

with endpoints

$$0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$$

and midpoints

$$\frac{1}{10}, \frac{3}{10}, \frac{5}{10}, \frac{7}{10}, \frac{9}{10}.$$

With this data, we get

$$T_5 = \frac{1}{2}\left(\frac{1}{5}\right)\left(e^{-0^2} + 2e^{-(1/5)^2} + 2e^{-(2/5)^2} + 2e^{-(3/5)^2} + 2e^{-(4/5)^2} + e^{-1^2}\right) \approx 0.74437; \text{ and}$$

$$M_5 = \frac{1}{5}\left(e^{-(1/10)^2} + e^{-(3/10)^2} + e^{-(5/10)^2} + e^{-(7/10)^2} + e^{-(9/10)^2}\right) \approx 0.74805.$$

12. $\int_{-2}^1 e^{x^2}\, dx, \quad N = 6$

SOLUTION Let $f(x) = e^{x^2}$. We divide $[-2, 1]$ into 6 subintervals of width

$$\Delta x = \frac{1-(-2)}{6} = \frac{3}{6} = \frac{1}{2} = 0.5$$

with endpoints $-2, -1.5, -1, -0.5, 0, 0.5, 1$, and midpoints $-1.75, -1.25, -0.75, -0.25, 0.25, 0.75$. With this data, we get

$$T_6 = \frac{1}{2}\left(\frac{1}{2}\right)\left(e^{(-2)^2} + 2e^{(-1.5)^2} + 2e^{(-1)^2} + 2e^{(-0.5)^2} + 2e^{0^2} + 2e^{(0.5)^2} + e^{1^2}\right) \approx 22.2161; \text{ and}$$

$$M_6 = \frac{1}{2}\left(e^{(-1.75)^2} + e^{(-1.25)^2} + e^{(-0.75)^2} + e^{(-0.25)^2} + e^{(0.25)^2} + e^{(0.75)^2}\right) \approx 15.8954.$$

In Exercises 13–22, calculate S_N given by Simpson's Rule for the value of N indicated.

13. $\int_0^4 \sqrt{x}\,dx, \quad N = 4$

SOLUTION Let $f(x) = \sqrt{x}$. We divide [0, 4] into 4 subintervals of width

$$\Delta x = \frac{4-0}{4} = 1$$

with endpoints 0, 1, 2, 3, 4. With this data, we get

$$S_4 = \frac{1}{3}(1)\left(\sqrt{0} + 4\sqrt{1} + 2\sqrt{2} + 4\sqrt{3} + \sqrt{4}\right) \approx 5.25221.$$

14. $\int_3^5 (9 - x^2)\,dx, \quad N = 4$

SOLUTION Let $f(x) = 9 - x^2$. We divide [3, 5] into 4 subintervals of length

$$\Delta x = \frac{5-3}{4} = \frac{2}{4} = \frac{1}{2} = 0.5$$

with endpoints 3, 3.5, 4, 4.5, 5. With this data, we get

$$S_4 = \frac{1}{3}\left(\frac{1}{2}\right)\left[(9-3^2) + 4(9-3.5^2) + 2(9-4^2) + 4(9-4.5^2) + (9-5^2)\right] \approx -14.6667.$$

15. $\int_0^3 \frac{dx}{x^4+1}, \quad N = 6$

SOLUTION Let $f(x) = 1/(x^4+1)$. We divide [0, 3] into 6 subintervals of length

$$\Delta x = \frac{3-0}{6} = \frac{1}{2} = 0.5$$

with endpoints 0, 0.5, 1, 1.5, 2, 2.5, 3. With this data, we get

$$S_6 = \frac{1}{3}\left(\frac{1}{2}\right)\left[\frac{1}{0^4+1} + \frac{4}{0.5^4+1} + \frac{2}{1^4+1} + \frac{4}{1.5^4+1} + \frac{2}{2^4+1} + \frac{4}{2.5^4+1} + \frac{1}{3^4+1}\right] \approx 1.10903.$$

16. $\int_0^1 \cos(x^2)\,dx, \quad N = 6$

SOLUTION Let $f(x) = \cos(x^2)$. We divide [0, 1] into 6 subintervals of length

$$\Delta x = \frac{1-0}{6} = \frac{1}{6}$$

with endpoints $0, \frac{1}{6}, \frac{2}{6}, \dots, \frac{6}{6} = 1$. With this data, we get

$$S_6 = \frac{1}{3}\left(\frac{1}{6}\right)\left[\cos\left(0^2\right) + 4\cos\left(\left(\frac{1}{6}\right)^2\right) + 2\cos\left(\left(\frac{2}{6}\right)^2\right) + \cdots + 4\cos\left(\left(\frac{5}{6}\right)^2\right) + \cos\left(1^2\right)\right] \approx 0.904523.$$

17. $\int_0^1 e^{-x^2}\,dx, \quad N = 4$

SOLUTION Let $f(x) = e^{-x^2}$. We divide [0, 1] into 4 subintervals of length

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

with endpoints $0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = 1$. With this data, we get

$$S_4 = \frac{1}{3}\left(\frac{1}{4}\right)\left[e^{-0^2} + 4e^{-(1/4)^2} + 2e^{-(2/4)^2} + 4e^{-(3/4)^2} + e^{-(1)^2}\right] \approx 0.746855.$$

18. $\int_1^2 e^{-x}\,dx, \quad N = 6$

SOLUTION Let $f(x) = e^{-x}$. We divide [1, 2] into 6 subintervals of width

$$\Delta x = \frac{2-1}{6} = \frac{1}{6}$$

with endpoints $1, \frac{7}{6}, \frac{8}{6}, \frac{9}{6}, \ldots, \frac{12}{6} = 2$. With this data, we get

$$S_6 = \frac{1}{3}\left(\frac{1}{6}\right)\left[e^{-1} + 4e^{-7/6} + 2e^{-8/6} + 4e^{-9/6} + 2e^{-10/6} + 4e^{-11/6} + e^{-12/6}\right] \approx 0.232545.$$

19. $\int_1^4 \ln x\, dx, \quad N = 8$

SOLUTION Let $f(x) = \ln x$. We divide [1, 4] into 8 subintervals of length

$$\Delta x = \frac{4-1}{8} = \frac{3}{8} = 0.375$$

with endpoints 1, 1.375, 1.75, 2.125, 2.5, 2.875, 3.25, 3.625, 4. With this data, we get

$$S_8 = \frac{1}{3}\left(\frac{3}{8}\right)\left[\ln 1 + 4\ln(1.375) + 2\ln(1.75) + \cdots + 4\ln(3.625) + \ln 4\right] \approx 2.54499.$$

20. $\int_2^4 \sqrt{x^4+1}\, dx, \quad N = 8$

SOLUTION Let $f(x) = \sqrt{x^4+1}$. We divide [2, 4] into 8 subintervals of width

$$\Delta x = \frac{4-2}{8} = \frac{2}{8} = \frac{1}{4} = 0.25$$

with endpoints 2, 2.25, 2.5, 2.75, 3, 3.25, 3.5, 3.75, 4. With this data, we get

$$S_8 = \frac{1}{3}\left(\frac{1}{4}\right)\left[\sqrt{2^4+1} + 4\sqrt{(2.25)^4+1} + 2\sqrt{(2.5)^4+1} + \cdots + 4\sqrt{(3.75)^4+1} + \sqrt{4^4+1}\right] \approx 18.7909.$$

21. $\int_0^{\pi/4} \tan\theta\, d\theta, \quad N = 10$

SOLUTION Let $f(\theta) = \tan\theta$. We divide $[0, \frac{\pi}{4}]$ into 10 subintervals of width

$$\Delta\theta = \frac{\frac{\pi}{4} - 0}{10} = \frac{\pi}{40}$$

with endpoints $0, \frac{\pi}{40}, \frac{2\pi}{40}, \frac{3\pi}{40}, \ldots, \frac{10\pi}{40} = \frac{\pi}{4}$. With this data, we get

$$S_{10} = \frac{1}{3}\left(\frac{\pi}{40}\right)\left[\tan(0) + 4\tan\left(\frac{\pi}{40}\right) + 2\tan\left(\frac{2\pi}{40}\right) + \cdots + 4\tan\left(\frac{9\pi}{40}\right) + \tan\left(\frac{10\pi}{40}\right)\right] \approx 0.346576.$$

22. $\int_0^2 (x^2+1)^{-1/3}\, dx, \quad N = 10$

SOLUTION Let $f(x) = (x^2+1)^{-1/3}$. We divide [0, 2] into 10 subintervals of width

$$\Delta x = \frac{2-0}{10} = \frac{1}{5} = 0.2$$

with endpoints 0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2. With this data, we get

$$S_{10} = \frac{1}{3}\cdot\frac{1}{5}\left[(0^2+1)^{-1/3} + 4(0.2^2+1)^{-1/3} + 2(0.4^2+1)^{-1/3} + \cdots + 4(1.8^2+1)^{-1/3} + (2^2+1)^{-1/3}\right] \approx 1.598005$$

In Exercises 23–26, calculate the approximation to the volume of the solid obtained by rotating the graph around the given axis.

23. $y = \cos x; \quad \left[0, \frac{\pi}{2}\right]; \quad x\text{-axis}; \quad M_8$

SOLUTION Using the disk method, the volume is given by

$$V = \int_0^{\pi/2} \pi r^2\, dx = \pi\int_0^{\pi/2} (\cos x)^2\, dx$$

which can be estimated as

$$\pi \int_0^{\pi/2} (\cos x)^2\, dx \approx \pi[M_8].$$

Let $f(x) = \cos^2 x$. We divide $[0, \pi/2]$ into 8 subintervals of length

$$\Delta x = \frac{\frac{\pi}{2} - 0}{8} = \frac{\pi}{16}$$

with midpoints

$$\frac{\pi}{32}, \frac{3\pi}{32}, \frac{5\pi}{32}, \dots, \frac{15\pi}{32}.$$

With this data, we get

$$V \approx \pi[M_8] = \pi\left[\Delta x(y_1 + y_2 + \cdots + y_8)\right] = \frac{\pi^2}{16}\left[\cos^2\left(\frac{\pi}{32}\right) + \cos^2\left(\frac{3\pi}{32}\right) + \cdots + \cos^2\left(\frac{15\pi}{32}\right)\right] \approx 2.46740.$$

24. $y = \cos x$; $\left[0, \frac{\pi}{2}\right]$; y-axis; S_8

SOLUTION Using the cylindrical shell method, the volume is given by

$$V = \int_0^{\pi/2} 2\pi r h\, dx = 2\pi \int_0^{\pi/2} x \cos x\, dx$$

where the radius of the cylinder is $r = x$ and the height is $h = \cos x$. This can be approximated as

$$V = 2\pi \int_0^{\pi/2} x \cos x\, dx \approx 2\pi\left[\, S_8 \,\right],$$

where $f(x) = x \cos x$. We divide $[0, \pi/2]$ into 8 subintervals of length

$$\Delta x = \frac{\frac{\pi}{2} - 0}{8} = \frac{\pi}{16}$$

with endpoints

$$0, \frac{\pi}{16}, \frac{2\pi}{16}, \dots, \frac{8\pi}{16}.$$

With this data, we get

$$V \approx 2\pi[S_8] = 2\pi\left[\frac{1}{3} \cdot \frac{\pi}{16}(y_0 + 4y_1 + 2y_2 + \cdots + 4y_7 + y_8)\right]$$

$$= \frac{\pi^2}{24}\left[0(\cos 0) + 4\frac{\pi}{16}\left(\cos\frac{\pi}{16}\right) + \cdots + \frac{8\pi}{16}\left(\cos\frac{8\pi}{16}\right)\right] \approx 3.58666.$$

25. $y = e^{-x^2}$; $[0, 1]$; x-axis; T_8

SOLUTION Using the disk method, the volume is given by

$$V = \int_0^1 \pi r^2\, dx = \pi \int_0^1 \left(e^{-x^2}\right)^2 dx = \pi \int_0^1 e^{-2x^2}\, dx.$$

We can use the approximation

$$V = \pi \int_0^1 e^{-2x^2}\, dx \approx \pi[T_8],$$

where $f(x) = e^{-2x^2}$. Divide $[0, 1]$ into 8 subintervals of length

$$\Delta x = \frac{1 - 0}{8} = \frac{1}{8},$$

with endpoints

$$0, \frac{1}{8}, \frac{2}{8}, \dots, 1.$$

With this data, we get

$$V \approx \pi[T_8] = \pi\left[\frac{1}{2} \cdot \frac{1}{8}\left(e^{-2(0^2)} + 2e^{-2(1/8)^2} + \cdots + 2e^{-2(7/8)^2} + e^{-2(1)^2}\right)\right] \approx 1.87691.$$

26. $y = e^{-x^2}$; [0, 1]; y-axis; S_8

SOLUTION Using the cylindrical shell method, the volume is given by

$$V = \int_0^1 2\pi r h\, dx = 2\pi \int_0^1 x e^{-x^2}\, dx$$

where $r = x$ and $h = e^{-x^2}$. We can use the approximation

$$V = 2\pi \int_0^1 x e^{-x^2}\, dx \approx 2\pi [S_8],$$

where $f(x) = xe^{-x^2}$. Divide [0, 1] into 8 subintervals of length

$$\Delta x = \frac{1-0}{8} = \frac{1}{8},$$

with endpoints

$$0,\ \frac{1}{8},\ \frac{2}{8}, \dots, 1.$$

With this data, we get

$$V \approx 2\pi [S_8] = 2\pi \left(\frac{1}{3}\right)\left(\frac{1}{8}\right)\left[(0)e^{-(0^2)} + 4\left(\frac{1}{8}\right)e^{-(1/8)^2} + \cdots + 4\left(\frac{7}{8}\right)e^{-(7/8)^2} + e^{-1^2}\right] \approx 1.98595.$$

27. An airplane's velocity is recorded at 5-min intervals during a 1-hour period with the following results, in miles per hour:

550, 575, 600, 580, 610, 640, 625,
595, 590, 620, 640, 640, 630

Use Simpson's Rule to estimate the distance traveled during the hour.

SOLUTION The distance traveled is equal to the integral $\int_0^1 v(t)\, dt$, where t is in hours. Since 5 minutes is $1/12$ of an hour, we have $\Delta t = 1/12$. Simpson's Rule gives us

$$S_{12} = \frac{1}{3} \cdot \frac{1}{12}\left[550 + 4 \cdot 575 + 2 \cdot 600 + 4 \cdot 580 + 2 \cdot 610 + \cdots + 4 \cdot 640 + 630\right] \approx 608.611.$$

The distance traveled during the hour is approximately 608.6 miles.

28. Use Simpson's Rule to determine the average temperature in a museum over a 3-hour period, if the temperatures (in degrees Celsius), recorded at 15-min intervals, are

21, 21.3, 21.5, 21.8, 21.6, 21.2, 20.8,
20.6, 20.9, 21.2, 21.1, 21.3, 21.2

SOLUTION If $T(t)$ represents the temperature at time t, then the average temperature T_{ave} from $t = 0$ to $t = 3$ hours is given by

$$T_{\text{ave}} = \frac{1}{3-0}\int_0^3 T(t)\, dt.$$

To use Simpson's Rule to approximate this, let $\Delta t = 1/4$ (15 minute intervals). Then we have

$$T_{\text{ave}} = \frac{1}{3}[S_{12}] = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{4}\left[21 + 4 \cdot 21.3 + 2 \cdot 21.5 + \cdots + 4 \cdot 21.3 + 21.2\right] \approx 21.2111.$$

The average temperature is approximately 21.2° C.

29. **Tsunami Arrival Times** Scientists estimate the arrival times of tsunamis (seismic ocean waves) based on the point of origin P and ocean depths. The speed s of a tsunami in miles per hour is approximately $s = \sqrt{15d}$, where d is the ocean depth in feet.

(a) Let $f(x)$ be the ocean depth x miles from P (in the direction of the coast). Argue using Riemann sums that the time T required for the tsunami to travel M miles toward the coast is

$$T = \int_0^M \frac{dx}{\sqrt{15 f(x)}}$$

(b) Use Simpson's Rule to estimate T if $M = 1000$ and the ocean depths (in feet), measured at 100-mile intervals starting from P, are

13,000, 11,500, 10,500, 9000, 8500,
7000, 6000, 4400, 3800, 3200, 2000

SOLUTION

(a) At a given distance from shore, say, x_i, the speed of the tsunami in mph is $s = \sqrt{15f(x_i)}$. If we assume the speed s is constant over a small interval Δx, then the time to cover that interval at that speed is

$$t_i = \frac{\text{distance}}{\text{speed}} = \frac{\Delta x}{\sqrt{15f(x_i)}}.$$

Now divide the interval $[0, M]$ into N subintervals of length Δx. The total time T is given by

$$T = \sum_{i=1}^{N} t_i = \sum_{i=1}^{N} \frac{\Delta x}{\sqrt{15f(x_i)}}.$$

Taking the limit as $N \to \infty$, we get

$$T = \int_0^M \frac{dx}{\sqrt{15f(x)}}.$$

(b) We have $\Delta x = 100$. Simpson's Rule gives us

$$S_{10} = \frac{1}{3} \cdot 100 \left[\frac{1}{\sqrt{15(13{,}000)}} + \frac{4}{\sqrt{15(11{,}500)}} + \cdots + \frac{1}{\sqrt{15(2000)}} \right] \approx 3.347.$$

It will take the tsunami about 3 hours and 21 minutes to reach shore.

30. Use S_8 to estimate $\int_0^{\pi/2} \frac{\sin x}{x}\, dx$, taking the value of $\frac{\sin x}{x}$ at $x = 0$ to be 1.

SOLUTION Divide $[0, \pi/2]$ into 8 subintervals of length

$$\Delta x = \frac{\frac{\pi}{2} - 0}{8} = \frac{\pi}{16}$$

with endpoints

$$0, \frac{\pi}{16}, \frac{2\pi}{16}, \ldots, \frac{8\pi}{16}.$$

Taking the value of $(\sin x)/x$ at $x = 0$ to be 1, we get

$$S_8 = \frac{1}{3}\left(\frac{\pi}{16}\right)\left[1 + 4\frac{\sin(\pi/16)}{\pi/16} + 2\frac{\sin(2\pi/16)}{2\pi/16} + \cdots + \frac{\sin(\pi/2)}{\pi/2}\right] \approx 1.37076.$$

31. Calculate T_6 for the integral $I = \int_0^2 x^3\, dx$.

(a) Is T_6 too large or too small? Explain graphically.

(b) Show that $K_2 = |f''(2)|$ may be used in the error bound and find a bound for the error.

(c) Evaluate I and check that the actual error is less than the bound computed in (b).

SOLUTION Let $f(x) = x^3$. Divide $[0, 2]$ into 6 subintervals of length $\Delta x = \frac{2-0}{6} = \frac{1}{3}$ with endpoints $0, \frac{1}{3}, \frac{2}{3}, \ldots, 2$. With this data, we get

$$T_6 = \frac{1}{2} \cdot \frac{1}{3}\left[0^3 + 2\left(\frac{1}{3}\right)^3 + 2\left(\frac{2}{3}\right)^3 + 2\left(\frac{3}{3}\right)^3 + 2\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + (1)2^3\right] \approx 4.11111.$$

(a) Since x^3 is concave up on $[0, 2]$, T_6 is too large.

(b) We have $f'(x) = 3x^2$ and $f''(x) = 6x$. Since $|f''(x)| = |6x|$ is *increasing* on $[0, 2]$, its maximum value occurs at $x = 2$ and we may take $K_2 = |f''(2)| = 12$. Then

$$\text{Error}(T_6) \le \frac{K_2(b-a)^3}{12N^2} = \frac{12(2-0)^3}{12(6)^2} = \frac{2}{9} \approx 0.22222.$$

(c) The exact value is

$$\int_0^2 x^3\,dx = \frac{1}{4}x^4\Big|_0^2 = \frac{1}{4}(16-0) = 4.$$

We can use this to compute the actual error:

$$\text{Error}(T_6) = |T_6 - 4| \approx |4.11111 - 4| \approx 0.11111.$$

Since $0.11111 < 0.22222$, the actual error is indeed less than the maximum possible error.

32. Calculate M_4 for the integral $I = \int_0^1 x\sin(x^2)\,dx$.

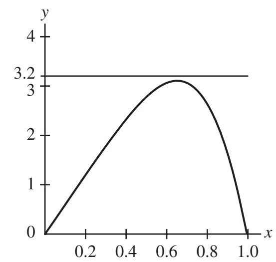
(a) GU Use a plot of $f''(x)$ to show that $K_2 = 3.2$ may be used in the error bound and find a bound for the error.

(b) CAS Evaluate I numerically and check that the actual error is less than the bound computed in (a).

SOLUTION Let $f(x) = x\sin(x^2)$. Divide [0, 1] into 4 subintervals of length $\Delta x = \frac{1-0}{4} = \frac{1}{4} = 0.25$, with endpoint 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1 and midpoints $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$. With this data, we get

$$M_4 = \frac{1}{4}\left[\frac{1}{8}\sin((1/8)^2) + \frac{3}{8}\sin((3/8)^2) + \frac{5}{8}\sin((5/8)^2) + \frac{7}{8}\sin((7/8)^2)\right] \approx 0.224714$$

(a) Consider the following plot of $f''(x) = 6x\cos(x^2) - 4x^3\sin(x^2)$:



From this figure, it is clear that $f''(x)$ is bounded above (in absolute value) by 3.2, so we can choose $K_2 = 3.2$ in the error bound formula. With this choice, the bound for the error in the M_4 approximation is

$$\text{Error}(M_4) \le K_2 \cdot \frac{(b-a)^3}{24N^2} = 3.2 \cdot \frac{(1-0)^3}{24 \cdot 4^2} = \frac{3.2}{384} \approx 0.008333 \approx 8.333 \times 10^{-3}$$

(b) Using a computer algebra system, $I \approx 0.2298488$, so the actual error is

$$\approx 0.2298488 - 0.224714 = 0.005135 < 0.008333$$

In Exercises 33–36, state whether T_N or M_N underestimates or overestimates the integral and find a bound for the error (but do not calculate T_N or M_N).

33. $\int_1^4 \frac{1}{x}\,dx, \quad T_{10}$

SOLUTION Let $f(x) = \frac{1}{x}$. Then $f'(x) = \frac{-1}{x^2}$ and $f''(x) = \frac{2}{x^3} > 0$ on [1, 4], so $f(x)$ is concave up, and T_{10} overestimates the integral. Since $|f''(x)| = |\frac{2}{x^3}|$ has its maximum value on [1, 4] at $x = 1$, we can take $K_2 = \frac{2}{1^3} = 2$, and

$$\text{Error}(T_{10}) \le \frac{K_2(4-1)^3}{12N^2} = \frac{2(3)^3}{12(10)^2} = 0.045.$$

34. $\int_0^2 e^{-x/4}\,dx, \quad T_{20}$

SOLUTION Let $f(x) = e^{-x/4}$. Then $f'(x) = -(1/4)e^{-x/4}$ and

$$f''(x) = \frac{1}{16}e^{-x/4} > 0$$

on [0, 2], so $f(x)$ is concave up, and T_{20} overestimates the integral. Since $|f''(x)| = |(1/16)e^{-x/4}|$ has its maximum value on [0, 2] at $x = 0$, we can take $K_2 = |(1/16)e^0| = 1/16$, and

$$\text{Error}(T_{20}) \le \frac{K_2(2-0)^3}{12N^2} = \frac{\frac{1}{16}(2)^3}{12(20)^2} = 1.04167 \times 10^{-4}.$$

35. $\int_1^4 \ln x\, dx, \quad M_{10}$

SOLUTION Let $f(x) = \ln x$. Then $f'(x) = 1/x$ and

$$f''(x) = -\frac{1}{x^2} < 0$$

on $[1, 4]$, so $f(x)$ is concave down, and M_{10} overestimates the integral. Since $|f''(x)| = |-1/x^2|$ has its maximum value on $[1, 4]$ at $x = 1$, we can take $K_2 = |-1/1^2| = 1$, and

$$\text{Error}(M_{10}) \le \frac{K_2(4-1)^3}{24N^2} = \frac{(1)(3)^3}{24(10)^2} = 0.01125.$$

36. $\int_0^{\pi/4} \cos x, \quad M_{20}$

SOLUTION Let $f(x) = \cos x$. Then $f'(x) = -\sin x$ and $f''(x) = -\cos x < 0$ on $[0, \pi/4]$, so $f(x)$ is concave down, and M_{20} overestimates the integral. Since $|f''(x)| = |-\cos x|$ has its maximum value on $[0, \pi/4]$ at $x = 0$, we can take $K_2 = |-\cos(0)| = 1$, and

$$\text{Error}(M_{20}) \le \frac{K_2(\pi/4-0)^3}{24N^2} = \frac{(1)(\pi/4)^3}{24(20)^2} = 5.04659 \times 10^{-5}.$$

CAS *In Exercises 37–40, use the error bound to find a value of N for which Error(T_N) $\le 10^{-6}$. If you have a computer algebra system, calculate the corresponding approximation and confirm that the error satisfies the required bound.*

37. $\int_0^1 x^4\, dx$

SOLUTION Let $f(x) = x^4$. Then $f'(x) = 4x^3$ and $|f''(x)| = |12x^2|$, which has its maximum value on $[0, 1]$ at $x = 1$, so we can take $K_2 = |12(1)^2| = 12$. Then we have

$$\text{Error}(T_N) \le \frac{K_2(1-0)^3}{12N^2} = \frac{12}{12N^2} = \frac{1}{N^2}.$$

To ensure that the error is at most 10^{-6}, we must choose N such that

$$\frac{1}{N^2} \le \frac{1}{10^6}.$$

This gives $N^2 \ge 10^6$ or $N \ge 10^3$. Thus let $N = 1000$. The exact value of the integral is

$$\int_0^1 x^4\, dx = \left.\frac{x^5}{5}\right|_0^1 = \frac{1}{5} = 0.2.$$

Using a CAS, we find that

$$T_{1000} \approx 0.2000003333.$$

The actual error is approximately $|0.2000003333 - 0.2| \approx 3.333 \times 10^{-7}$, and is indeed less than 10^{-6}.

38. $\int_0^3 (5x^4 - x^5)\, dx$

SOLUTION Let $f(x) = 5x^4 - x^5$. Then $f'(x) = 20x^3 - 5x^4$ and $f''(x) = 60x^2 - 20x^3$. A plot reveals that $f''(x) \ge 0$ on $[0, 3]$; it achieves its maximum value where its derivative is zero, which is where $120x - 60x^2 = 0$, so $x = 2$. $|f''(2)| = |60 \cdot 2^2 - 20 \cdot 2^3| = 80$, so we may take $K_2 = 80$ in the error bound approximation. Then we have

$$\text{Error}(T_N) \le \frac{K_2(3-0)^3}{12N^2} = \frac{180}{N^2}$$

To ensure that the error is at most 10^{-6}, we must choose N such that

$$\frac{180}{N^2} \le 10^{-6}, \qquad \text{or} \quad N^2 \ge 180 \times 10^6 = 1.8 \times 10^8$$

Thus $N \ge \sqrt{1.8} \times 10^4 \approx 1.34164 \times 10^4$, so let $N = 13{,}417$. Using a computer algebra system, we get

$$T_{13417} \approx 121.5000006000$$

The true value of the integral is

$$I = \int_0^3 \left(5x^4 - x^5\right)\,dx = \left(x^5 - \frac{1}{6}x^6\right)\Big|_0^3 = 121.5$$

so that $T_{13417} - I \approx 0.0000006 = 6 \times 10^{-7} < 10^{-6}$.

39. $\displaystyle\int_2^5 \frac{1}{x}\,dx$

SOLUTION Let $f(x) = 1/x$. Then $f'(x) = -1/x^2$ and $|f''(x)| = |2/x^3|$, which has its maximum value on $[2, 5]$ at $x = 2$, so we can take $K_2 = |2/2^3| = 1/4$. Then we have

$$\text{Error}(T_N) \le \frac{K_2(5-2)^3}{12N^2} = \frac{(1/4)3^3}{12N^2} = \frac{9}{16N^2}.$$

To ensure that the error is at most 10^{-6}, we must choose N such that

$$\frac{9}{16N^2} \le \frac{1}{10^6}.$$

This gives us

$$N^2 \ge \frac{9 \cdot 10^6}{16} \Rightarrow N \ge \sqrt{\frac{9 \cdot 10^6}{16}} = 750.$$

Thus let $N = 750$. The exact value of the integral is

$$\int_2^5 \frac{1}{x}\,dx = \ln 5 - \ln 2 \approx 0.9162907314.$$

Using a CAS, we find that

$$T_{750} \approx 0.9162910119.$$

The error is approximately

$$|0.9162907314 - 0.9162910119| \approx 2.805 \times 10^{-7}$$

and is indeed less than 10^{-6}.

40. $\displaystyle\int_0^3 e^{-x}\,dx$

SOLUTION Let $f(x) = e^{-x}$. Then $f'(x) = -e^{-x}$ and $|f''(x)| = |e^{-x}| = e^{-x}$, which has its maximum value on $[0, 3]$ at $x = 0$, so we can take $K_2 = e^0 = 1$. Then we have

$$\text{Error}(T_N) \le \frac{K_2(3-0)^3}{12N^2} = \frac{(1)3^3}{12N^2} = \frac{9}{4N^2}.$$

To ensure that the error is at most 10^{-6}, we must choose N such that

$$\frac{9}{4N^2} \le \frac{1}{10^6}.$$

This gives us

$$N^2 \ge \frac{9 \cdot 10^6}{4} \Rightarrow N \ge \sqrt{\frac{9 \cdot 10^6}{4}} = 1500.$$

Thus let $N = 1500$. The exact value of the integral is

$$\int_0^3 e^{-x}\,dx = \left(-e^{-3}\right) - \left(-e^{-0}\right) = 1 - e^{-3} \approx 0.9502129316.$$

Using a CAS, we find that

$$T_{1500} \approx 0.9502132468.$$

The error is approximately

$$|0.9502129316 - 0.9502132468| \approx 3.152 \times 10^{-7}$$

and is indeed less than 10^{-6}.

41. Compute the error bound for the approximations T_{10} and M_{10} to $\int_0^3 (x^3+1)^{-1/2}\,dx$, using Figure 17 to determine a value of K_2. Then find a value of N such that the error in M_N is at most 10^{-6}.

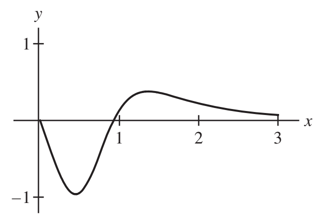


FIGURE 17 Graph of $f''(x)$, where $f(x) = (x^3+1)^{-1/2}$.

SOLUTION Clearly, in the range $0 \le x \le 3$, we have $|f''(x)| \le 1$, so we may choose $K_2 = 1$. Then

$$\text{Error}(T_{10}) \le \frac{K_2(3-0)^3}{12N^2} = \frac{27}{12 \cdot 10^2} = \frac{27}{1200} = 0.0225$$

$$\text{Error}(M_{10}) \le \frac{K_2(3-0)^3}{24N^2} = \frac{27}{24 \cdot 10^2} = \frac{27}{2400} = 0.01125$$

In order for the error in M_N to be at most 10^{-6}, we must have

$$\text{Error}(M_N) \le \frac{K_2(3-0)^3}{24N^2} = \frac{9}{8N^2} \le 10^{-6}$$

so that $8N^2 \ge 9 \times 10^6$ and $N^2 \ge 1{,}125{,}000$. Thus we must choose $N \ge \sqrt{1{,}125{,}000} \approx 1060.7$, so that $N = 1061$.

42. **(a)** Compute S_6 for the integral $I = \displaystyle\int_0^1 e^{-2x}\,dx$.

(b) Show that $K_4 = 16$ may be used in the error bound and compute the error bound.

(c) Evaluate I and check that the actual error is less than the bound for the error computed in (b).

SOLUTION

(a) Let $f(x) = e^{-2x}$. We divide [0, 1] into six subintervals of length $\Delta x = (1-0)/6 = 1/6$, with endpoints $0, 1/6, \ldots, 5/6, 1$. With this data, we get

$$S_6 = \frac{1}{3} \cdot \frac{1}{6}\left[e^{-2(0)} + 4e^{-2(1/6)} + 2e^{-2(2/6)} + \cdots + e^{-2(1)}\right] \approx 0.432361.$$

(b) Taking derivatives, we get

$$f'(x) = -2e^{-2x}, \;\; f''(x) = 4e^{-2x}, \;\; f^{(3)}(x) = -8e^{-2x}, \;\; f^{(4)}(x) = 16e^{-2x}.$$

Since $|f^{(4)}(x)| = |16e^{-2x}|$ assumes its maximum value on [0, 1] at $x = 0$, we can set $K_4 = |16e^0| = 16$. Then we have

$$\text{Error}(S_6) \le \frac{K_4(1-0)^5}{180N^4} = \frac{16}{180 \cdot 6^4} \approx 6.86 \times 10^{-5}.$$

(c) The exact value of the integral is

$$\int_0^1 e^{-2x}\,dx = \left.\frac{e^{-2x}}{-2}\right|_0^1 = \frac{1-e^{-2}}{2} \approx 0.432332.$$

The actual error is

$$\text{Error}(S_6) \approx |0.432361 - 0.432332| \approx 2.9 \times 10^{-5}.$$

The error is indeed less than the maximum possible error.

43. Calculate S_8 for $\int_1^5 \ln x\,dx$ and calculate the error bound. Then find a value of N such that S_N has an error of at most 10^{-6}.

SOLUTION Let $f(x) = \ln x$. We divide [1, 5] into eight subintervals of length $\Delta x = (5-1)/8 = 0.5$, with endpoints $1, 1.5, 2, \ldots, 5$. With this data, we get

$$S_8 = \frac{1}{3} \cdot \frac{1}{2}\left[\ln 1 + 4\ln 1.5 + 2\ln 2 + \cdots + 4\ln 4.5 + \ln 5\right] \approx 4.046655.$$

To find the maximum possible error, we first take derivatives:

$$f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f^{(3)}(x) = \frac{2}{x^3},\ f^{(4)}(x) = -\frac{6}{x^4}.$$

Since $|f^{(4)}(x)| = |-6x^{-4}| = 6x^{-4}$, assumes its maximum value on $[1, 5]$ at $x = 1$, we can set $K_4 = 6(1)^{-4} = 6$. Then we have

$$\text{Error}(S_8) \le \frac{K_4(5-1)^5}{180N^4} = \frac{6 \cdot 4^5}{180 \cdot 8^4} \approx 0.0083333.$$

To ensure that S_N has error at most 10^{-6}, we must find N such that

$$\frac{6 \cdot 4^5}{180N^4} \le \frac{1}{10^6}.$$

This gives us

$$N^4 \ge \frac{6 \cdot 4^5 \cdot 10^6}{180} \Rightarrow N \ge \left(\frac{6 \cdot 4^5 \cdot 10^6}{180}\right)^{1/4} \approx 76.435.$$

Thus let $N = 78$ (remember that N must be even when using Simpson's Rule).

44. Find a bound for the error in the approximation S_{10} to $\int_0^3 e^{-x^2}\,dx$ (use Figure 18 to determine a value of K_4). Then find a value of N such that S_N has an error of at most 10^{-6}.

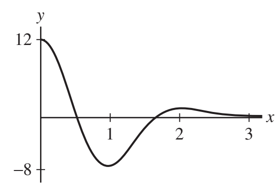


FIGURE 18 Graph of $f^{(4)}(x)$, where $f(x) = e^{-x^2}$.

SOLUTION From the graph, we see that $|f^{(4)}(x)| \le 12$, so we set $K_4 = 12$. This gives us

$$\text{Error}(S_{10}) \le \frac{K_4(3-0)^5}{180N^4} = \frac{12 \cdot 3^5}{180 \cdot 10^4} = 0.00162.$$

To ensure that S_N has error at most 10^{-6}, we must find N such that

$$\frac{12 \cdot 3^5}{180 \cdot N^4} \le \frac{1}{10^6}.$$

This gives us

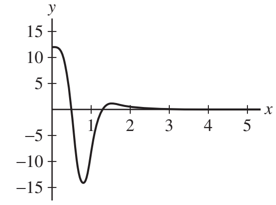
$$N^4 \ge \frac{12 \cdot 3^5 \cdot 10^6}{180} \Rightarrow N \ge \left(\frac{12 \cdot 3^5 \cdot 10^6}{180}\right)^{1/4} \approx 63.44.$$

Thus let $N = 64$.

45. CAS Use a computer algebra system to compute and graph $f^{(4)}(x)$ for $f(x) = \sqrt{1 + x^4}$ and find a bound for the error in the approximation S_{40} to $\displaystyle\int_0^5 f(x)\,dx$.

SOLUTION From the graph of $f^{(4)}(x)$ shown below, we see that $|f^{(4)}(x)| \le 15$ on $[0, 5]$. Therefore we set $K_4 = 15$. Now we have

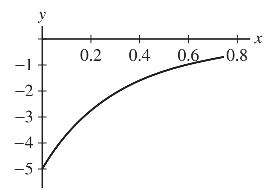
$$\text{Error}(S_{40}) \le \frac{15(5-0)^5}{180(40)^4} = \frac{5}{49152} \approx 1.017 \times 10^{-4}.$$



46. CAS Use a computer algebra system to compute and graph $f^{(4)}(x)$ for $f(x) = \tan x - \sec x$ and find a bound for the error in the approximation S_{40} to $\int_0^{\pi/4} f(x)\,dx$.

SOLUTION From the graph of $f^{(4)}(x)$ shown below, we see that $|f^{(x)}(x)| \le 5$ on $[0, \pi/4]$. Therefore we set $K_4 = 5$. Now we have

$$\text{Error}(S_{40}) \le \frac{5(\pi/4 - 0)^5}{180(40)^4} \approx 3.243 \times 10^{-9}.$$



In Exercises 47–50, use the error bound to find a value of N for which Error(S_N) $\le 10^{-9}$.

47. $\int_1^6 x^{4/3}\,dx$

SOLUTION Let $f(x) = x^{4/3}$. We start by taking derivatives:

$$f'(x) = \frac{4}{3}x^{1/3}$$
$$f''(x) = \frac{4}{9}x^{-2/3}$$
$$f'''(x) = -\frac{8}{27}x^{-5/3}$$
$$f^{(4)}(x) = \frac{40}{81}x^{-8/3}$$

For $x \ge 1$, $f^{(4)}(x)$ is a decreasing function of x, so it takes its maximum value on $[1, 6]$ at $x = 1$. That maximum value is $\frac{40}{81}$, which is quite close to (but smaller than) $\frac{1}{2}$. For simplicity, we take $K_4 = \frac{1}{2}$. Then

$$\text{Error}(S_N) \le \frac{K_4(b-a)^5}{180N^4} = \frac{(6-1)^5}{2 \cdot 180 \cdot N^4} = \frac{5^5}{360N^4} = \frac{625}{72N^4} \le 10^{-9}$$

Thus $72N^4 \ge 625 \times 10^9$, so that

$$N \ge \left(\frac{625 \times 10^9}{72}\right)^{1/4} \approx 305.24$$

so we can take $N = 306$.

48. $\int_0^4 xe^x\,dx$

SOLUTION Let $f(x) = xe^x$. To find K_4, we first take derivatives:

$$f'(x) = xe^x + e^x$$
$$f''(x) = xe^x + 2e^x$$
$$f^{(3)}(x) = xe^x + 3e^x$$
$$f^{(4)}(x) = xe^x + 4e^x.$$

On the interval $[0, 4]$,

$$|f^{(4)}(x)| = |xe^x + 4e^x| \le |4e^4 + 4e^4| = 8e^4.$$

Therefore we set $K_4 = 8e^4$, and we have

$$\text{Error}(S_N) \le \frac{K_4(4-0)^5}{180N^4} = \frac{8e^4 \cdot 4^5}{180N^4}.$$

To ensure that S_N has error at most 10^{-9}, we must find N such that

$$\frac{8e^4 \cdot 4^5}{180N^4} \le \frac{1}{10^9}.$$

This gives us

$$N^4 \ge \frac{8e^4 \cdot 4^5 \cdot 10^9}{180} \Rightarrow N \ge \left(\frac{8e^4 \cdot 4^5 \cdot 10^9}{180}\right)^{1/4} \approx 1255.52.$$

Thus let $N = 1256$.

49. $\int_0^1 e^{x^2}\,dx$

SOLUTION Let $f(x) = e^{x^2}$. To find K_4, we first take derivatives:

$$\begin{aligned} f'(x) &= 2xe^{x^2} \\ f''(x) &= 4x^2e^{x^2} + 2e^{x^2} \\ f^{(3)}(x) &= 8x^3e^{x^2} + 12xe^{x^2} \\ f^{(4)}(x) &= 16x^4e^{x^2} + 48x^2e^{x^2} + 12e^{x^2}. \end{aligned}$$

On the interval [0, 1], $|f^{(4)}(x)|$ assumes its maximum value at $x = 1$. Therefore we set

$$K_4 = |f^{(4)}(1)| = 16e + 48e + 12e = 76e.$$

Now we have

$$\text{Error}(S_N) \le \frac{K_4(1-0)^5}{180N^4} = \frac{76e}{180N^4}.$$

To ensure that S_N has error at most 10^{-9}, we must find N such that

$$\frac{76e}{180N^4} \le \frac{1}{10^9}.$$

This gives us

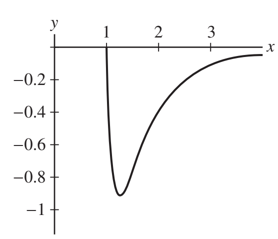
$$N^4 \ge \frac{76e \cdot 10^9}{180} \Rightarrow N \ge \left(\frac{76e \cdot 10^9}{180}\right)^{1/4} \approx 184.06.$$

Thus we let $N = 186$ (remember that N must be even when using Simpson's Rule).

50. $\int_1^4 \sin(\ln x)\,dx$

SOLUTION Let $f(x) = \sin(\ln x)$. To find K_4, we first take derivatives:

$$\begin{aligned} f'(x) &= \frac{\cos(\ln x)}{x} \\ f''(x) &= \frac{-\sin(\ln x) - \cos(\ln x)}{x^2} \\ f^{(3)}(x) &= \frac{\cos(\ln x) + 3\sin(\ln x)}{x^3} \\ f^{(4)}(x) &= \frac{-10\sin(\ln x)}{x^4} \end{aligned}$$



From the graph of $y = f^{(4)}(x)$ shown above, we can see that on the interval $[1, 4]$, $|f^{(4)}(x)| \le 1$. Therefore we set $K_4 = 1$. Now we have

$$\text{Error}(S_N) \le \frac{(1)(4-1)^5}{180N^4} = \frac{3^5}{180N^4}.$$

To ensure that S_N has error at most 10^{-9}, we must find N such that

$$\frac{3^5}{180N^4} \le \frac{1}{10^9}.$$

This gives us

$$N^4 \ge \frac{3^5 \cdot 10^9}{180} \Rightarrow N \ge \left(\frac{3^5 \cdot 10^9}{180}\right)^{1/4} \approx 191.7.$$

Thus we let $N = 192$.

51. CAS Show that $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$ [use Eq. (3) in Section 5.7].

(a) Use a computer algebra system to graph $f^{(4)}(x)$ for $f(x) = (1+x^2)^{-1}$ and find its maximum on $[0, 1]$.

(b) Find a value of N such that S_N approximates the integral with an error of at most 10^{-6}. Calculate the corresponding approximation and confirm that you have computed $\frac{\pi}{4}$ to at least four places.

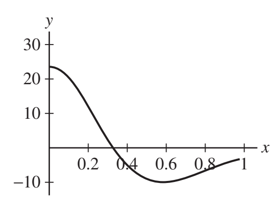
SOLUTION Recall from Section 3.9 that

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}.$$

So then

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}.$$

(a) From the graph of $f^{(4)}(x)$ shown below, we can see that the maximum value of $|f^{(4)}(x)|$ on the interval $[0, 1]$ is 24.



(b) From part (a), we set $K_4 = 24$. Then we have

$$\text{Error}(S_N) \le \frac{24(1-0)^5}{180N^4} = \frac{2}{15N^4}.$$

To ensure that S_N has error at most 10^{-6}, we must find N such that

$$\frac{2}{15N^4} \le \frac{1}{10^6}.$$

This gives us

$$N^4 \ge \frac{2 \cdot 10^6}{15} \Rightarrow N \ge \left(\frac{2 \cdot 10^6}{15}\right)^{1/4} \approx 19.1.$$

Thus let $N = 20$. To compute S_{20}, let $\Delta x = (1-0)/20 = 0.05$. The endpoints of $[0, 1]$ are $0,\ 0.05,\ \ldots,\ 1$. With this data, we get

$$S_{20} = \frac{1}{3}\left(\frac{1}{20}\right)\left[\frac{1}{1+0^2} + \frac{4}{1+(0.05)^2} + \frac{2}{1+(0.1)^2} + \cdots + \frac{1}{1+1^2}\right] \approx 0.785398163242.$$

The actual error is

$$|0.785398163242 - \pi/4| = |0.785398163242 - 0.785398163397| = 1.55 \times 10^{-10}.$$

52. Let $J = \int_0^\infty e^{-x^2}\,dx$ and $J_N = \int_0^N e^{-x^2}\,dx$. Although e^{-x^2} has no elementary antiderivative, it is known that $J = \sqrt{\pi}/2$. Let T_N be the Nth trapezoidal approximation to J_N. Calculate T_4 and show that T_4 approximates J to three decimal places.

SOLUTION T_4 is the 4$^{\text{th}}$ trapezoidal approximation to $J_4 = \int_0^4 e^{-x^2}\,dx$. We divide the interval [0, 4] into four subintervals, with endpoints 0, 1, 2, 3, and 4. Then

$$T_4 = \frac{1}{2}\cdot 1\left[e^{-0^2} + 2e^{-1^2} + 2e^{-2^2} + 2e^{-3^2} + e^{-4^2}\right] \approx 0.8863185$$

We have

$$T_4 - J \approx 0.8863185 - \frac{\sqrt{\pi}}{2} \approx 0.8863185 - 0.8862269 \approx 0.0000916$$

53. Let $f(x) = \sin(x^2)$ and $I = \int_0^1 f(x)\,dx$.

(a) Check that $f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$. Then show that $|f''(x)| \le 6$ for $x \in [0, 1]$. *Hint:* Note that $|2\cos(x^2)| \le 2$ and $|4x^2\sin(x^2)| \le 4$ for $x \in [0, 1]$.

(b) Show that Error(M_N) is at most $\dfrac{1}{4N^2}$.

(c) Find an N such that $|I - M_N| \le 10^{-3}$.

SOLUTION

(a) Taking derivatives, we get

$$f'(x) = 2x\cos(x^2)$$
$$f''(x) = 2x(-\sin(x^2)\cdot 2x) + 2\cos(x^2) = 2\cos(x^2) - 4x^2\sin(x^2).$$

On the interval [0, 1],

$$|f''(x)| = |2\cos(x^2) - 4x^2\sin(x^2)| \le |2\cos(x^2)| + |4x^2\sin(x^2)| \le 2 + 4 = 6.$$

(b) Using $K_2 = 6$, we get

$$\text{Error}(M_N) \le \frac{K_2(1-0)^3}{24N^2} = \frac{6}{24N^2} = \frac{1}{4N^2}.$$

(c) To ensure that M_N has error at most 10^{-3}, we must find N such that

$$\frac{1}{4N^2} \le \frac{1}{10^3}.$$

This gives us

$$N^2 \ge \frac{10^3}{4} = 250 \Rightarrow N \ge \sqrt{250} \approx 15.81.$$

Thus let $N = 16$.

54. CAS The error bound for M_N is proportional to $1/N^2$, so the error bound decreases by $\frac{1}{4}$ if N is increased to $2N$. Compute the actual error in M_N for $\int_0^\pi \sin x\,dx$ for $N = 4$, 8, 16, 32, and 64. Does the actual error seem to decrease by $\frac{1}{4}$ as N is doubled?

SOLUTION The exact value of the integral is

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -(-1) - (1) = 2.$$

To compute M_4, we have $\Delta x = (\pi - 0)/4 = \pi/4$, and midpoints $\pi/8,\ 3\pi/8,\ 5\pi/8,\ 7\pi/8$. With this data, we get

$$M_4 = \frac{\pi}{4}\left[\sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right) + \sin\left(\frac{7\pi}{8}\right)\right] \approx 2.052344.$$

The values for M_8, M_{16}, M_{32}, and M_{64} are computed similarly:

$$M_8 = \frac{\pi}{8}\left[\sin\left(\frac{\pi}{16}\right) + \sin\left(\frac{3\pi}{16}\right) + \cdots + \sin\left(\frac{15\pi}{16}\right)\right] \approx 2.012909;$$

$$M_{16} = \frac{\pi}{16}\left[\sin\left(\frac{\pi}{32}\right) + \sin\left(\frac{3\pi}{32}\right) + \cdots + \sin\left(\frac{31\pi}{32}\right)\right] \approx 2.0032164;$$

$$M_{32} = \frac{\pi}{32}\left[\sin\left(\frac{\pi}{64}\right) + \sin\left(\frac{3\pi}{64}\right) + \cdots + \sin\left(\frac{63\pi}{64}\right)\right] \approx 2.00080342;$$

$$M_{64} = \frac{\pi}{64}\left[\sin\left(\frac{\pi}{128}\right) + \sin\left(\frac{3\pi}{128}\right) + \cdots + \sin\left(\frac{127\pi}{128}\right)\right] \approx 2.00020081.$$

Now we can compute the actual errors for each N:

$$\text{Error}(M_4) = |2 - 2.052344| = 0.052344$$

$$\text{Error}(M_8) = |2 - 2.012909| = 0.012909$$

$$\text{Error}(M_{16}) = |2 - 2.0032164| = 0.0032164$$

$$\text{Error}(M_{32}) = |2 - 2.00080342| = 0.00080342$$

$$\text{Error}(M_{64}) = |2 - 2.00020081| = 0.00020081$$

The actual error does in fact decrease by about 1/4 each time N is doubled.

55. CAS Observe that the error bound for T_N (which has 12 in the denominator) is twice as large as the error bound for M_N (which has 24 in the denominator). Compute the actual error in T_N for $\int_0^\pi \sin x\,dx$ for $N = 4, 8, 16, 32$, and 64 and compare with the calculations of Exercise 54. Does the actual error in T_N seem to be roughly twice as large as the error in M_N in this case?

SOLUTION The exact value of the integral is

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -(-1) - (1) = 2.$$

To compute T_4, we have $\Delta x = (\pi - 0)/4 = \pi/4$, and endpoints $0,\ \pi/4,\ 2\pi/4,\ 3\pi/4,\ \pi$. With this data, we get

$$T_4 = \frac{1}{2}\cdot\frac{\pi}{4}\left[\sin(0) + 2\sin\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{2\pi}{4}\right) + 2\sin\left(\frac{3\pi}{4}\right) + \sin(\pi)\right] \approx 1.896119.$$

The values for T_8, T_{16}, T_{32}, and T_{64} are computed similarly:

$$T_8 = \frac{1}{2}\cdot\frac{\pi}{8}\left[\sin(0) + 2\sin\left(\frac{\pi}{8}\right) + 2\sin\left(\frac{2\pi}{8}\right) + \cdots + 2\sin\left(\frac{7\pi}{8}\right) + \sin(\pi)\right] \approx 1.974232;$$

$$T_{16} = \frac{1}{2}\cdot\frac{\pi}{16}\left[\sin(0) + 2\sin\left(\frac{\pi}{16}\right) + 2\sin\left(\frac{2\pi}{16}\right) + \cdots + 2\sin\left(\frac{15\pi}{16}\right) + \sin(\pi)\right] \approx 1.993570;$$

$$T_{32} = \frac{1}{2}\cdot\frac{\pi}{32}\left[\sin(0) + 2\sin\left(\frac{\pi}{32}\right) + 2\sin\left(\frac{2\pi}{32}\right) + \cdots + 2\sin\left(\frac{31\pi}{32}\right) + \sin(\pi)\right] \approx 1.998393;$$

$$T_{64} = \frac{1}{2}\cdot\frac{\pi}{64}\left[\sin(0) + 2\sin\left(\frac{\pi}{64}\right) + 2\sin\left(\frac{2\pi}{64}\right) + \cdots + 2\sin\left(\frac{63\pi}{64}\right) + \sin(\pi)\right] \approx 1.999598.$$

Now we can compute the actual errors for each N:

$$\text{Error}(T_4) = |2 - 1.896119| = 0.103881$$

$$\text{Error}(T_8) = |2 - 1.974232| = 0.025768$$

$$\text{Error}(T_{16}) = |2 - 1.993570| = 0.006430$$

$$\text{Error}(T_{32}) = |2 - 1.998393| = 0.001607$$

$$\text{Error}(T_{64}) = |2 - 1.999598| = 0.000402$$

Comparing these results with the calculations of Exercise 54, we see that the actual error in T_N is in fact about twice as large as the error in M_N.

56. CAS Explain why the error bound for S_N decreases by $\frac{1}{16}$ if N is increased to $2N$. Compute the actual error in S_N for $\int_0^\pi \sin x\,dx$ for $N = 4, 8, 16, 32$, and 64. Does the actual error seem to decrease by $\frac{1}{16}$ as N is doubled?

SOLUTION If we plug in $2N$ for N in the formula for the error bound for S_N, we get

$$\frac{K_4(b-a)^5}{180(2N)^4} = \frac{K_4(b-a)^5}{180 \cdot 2^4 \cdot N^4} = \frac{1}{16}\left(\frac{K_4(b-a)^5}{180N^4}\right).$$

Thus we see that, since N is raised to the fourth power in the denominator, the Error Bound for S_N decreases by $1/16$ if N is increased to $2N$. The exact value of the integral is

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -(-1) - (1) = 2.$$

To compute S_4, we have $\Delta x = (\pi - 0)/4 = \pi/4$, and endpoints $0,\ \pi/4,\ 2\pi/4,\ 3\pi/4,\ \pi$. With this data, we get

$$S_4 = \frac{1}{3}\cdot\frac{\pi}{4}\left[\sin(0) + 4\sin\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{2\pi}{4}\right) + 4\sin\left(\frac{3\pi}{4}\right) + \sin(\pi)\right] \approx 2.004560.$$

The values for S_8, S_{16}, S_{32}, and S_{64} are computed similarly:

$$S_8 = \frac{1}{3}\cdot\frac{\pi}{8}\left[\sin(0) + 4\sin\left(\frac{\pi}{8}\right) + 2\sin\left(\frac{2\pi}{8}\right) + \cdots + 4\sin\left(\frac{7\pi}{8}\right) + \sin(\pi)\right] \approx 2.0002692;$$

$$S_{16} = \frac{1}{3}\cdot\frac{\pi}{16}\left[\sin(0) + 4\sin\left(\frac{\pi}{16}\right) + 2\sin\left(\frac{2\pi}{16}\right) + \cdots + 4\sin\left(\frac{15\pi}{16}\right) + \sin(\pi)\right] \approx 2.00001659;$$

$$S_{32} = \frac{1}{3}\cdot\frac{\pi}{32}\left[\sin(0) + 4\sin\left(\frac{\pi}{32}\right) + 2\sin\left(\frac{2\pi}{32}\right) + \cdots + 4\sin\left(\frac{31\pi}{32}\right) + \sin(\pi)\right] \approx 2.000001033;$$

$$S_{64} = \frac{1}{3}\cdot\frac{\pi}{64}\left[\sin(0) + 4\sin\left(\frac{\pi}{64}\right) + 2\sin\left(\frac{2\pi}{64}\right) + \cdots + 4\sin\left(\frac{63\pi}{64}\right) + \sin(\pi)\right] \approx 2.00000006453.$$

Now we can compute the actual errors for each N:

$$\text{Error}(S_4) = |2 - 2.004560| = 0.004560$$

$$\text{Error}(S_8) = |2 - 2.0002692| = 2.692 \times 10^{-4}$$

$$\text{Error}(S_{16}) = |2 - 2.00001659| = 1.659 \times 10^{-5}$$

$$\text{Error}(S_{32}) = |2 - 2.000001033| = 1.033 \times 10^{-6}$$

$$\text{Error}(S_{64}) = |2 - 2.00000006453| = 6.453 \times 10^{-8}$$

The actual error does in fact decrease by about $1/16$ each time N is doubled. For example, $0.004560/16 = 2.85 \times 10^{-4}$, which is roughly the same as 2.692×10^{-4}.

57. Verify that S_2 yields the exact value of $\displaystyle\int_0^1 (x - x^3)\,dx$.

SOLUTION Let $f(x) = x - x^3$. Clearly $f^{(4)}(x) = 0$, so we may take $K_4 = 0$ in the error bound estimate for S_2. Then

$$\text{Error}(S_2) \le \frac{K_4(1-0)^5}{180 \cdot 2^4} = 0 \cdot \frac{1}{2880} = 0$$

so that S_2 yields the exact value of the integral.

58. Verify that S_2 yields the exact value of $\displaystyle\int_a^b (x - x^3)\,dx$ for all $a < b$.

SOLUTION Let $f(x) = x - x^3$. Clearly $f^{(4)}(x) = 0$, so we may take $K_4 = 0$ in the error bound estimate for S_2. Then

$$\text{Error}(S_2) \le \frac{K_4(b-a)^5}{180 \cdot 2^4} = 0 \cdot \frac{(b-a)^5}{2880} = 0$$

so that S_2 yields the exact value of the integral.

Further Insights and Challenges

59. Show that if $f(x) = rx + s$ is a linear function (r, s constants), then $T_N = \int_a^b f(x)\,dx$ for all N and all endpoints a, b.

SOLUTION First, note that

$$\int_a^b (rx+s)\,dx = \frac{r(b^2-a^2)}{2} + s(b-a).$$

Now,

$$T_N(rx+s) = \frac{b-a}{2N}\left[f(a) + 2\sum_{i=1}^{N-1} f(x_i) + f(b)\right] = \frac{r(b-a)}{2N}\left[a + 2\sum_{i=1}^{N-1} a + 2\frac{b-a}{N}\sum_{i=1}^{N-1} i + b\right] + s\frac{b-a}{2N}(2N)$$

$$= \frac{r(b-a)}{2N}\left[(2N-1)a + 2\frac{b-a}{N}\frac{(N-1)N}{2} + b\right] + s(b-a) = \frac{r(b^2-a^2)}{2} + s(b-a).$$

60. Show that if $f(x) = px^2 + qx + r$ is a quadratic polynomial, then $S_2 = \int_a^b f(x)\,dx$. In other words, show that

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\big(y_0 + 4y_1 + y_2\big)$$

where $y_0 = f(a)$, $y_1 = f\left(\frac{a+b}{2}\right)$, and $y_2 = f(b)$. *Hint:* Show this first for $f(x) = 1, x, x^2$ and use linearity.

SOLUTION For S_2, $\Delta x = (b-a)/2$, and the endpoints are a, $(a+b)/2$, b. Following the hint, let $f(x) = 1$. In this case,

$$S_2(1) = \frac{1}{3}\left(\frac{b-a}{2}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = \frac{b-a}{6}(1 + 4(1) + 1) = \frac{b-a}{6}(6)$$

$$= b - a = \int_a^b 1\,dx.$$

If $f(x) = x$, then

$$S_2(x) = \frac{1}{3}\left(\frac{b-a}{2}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = \frac{b-a}{6}\left(a + 4\left(\frac{a+b}{2}\right) + b\right) = \frac{b-a}{6}\left(\frac{6a+6b}{2}\right)$$

$$= \frac{b^2-a^2}{2} = \int_a^b x\,dx;$$

and if $f(x) = x^2$, then

$$S_2(x^2) = \frac{1}{3}\left(\frac{b-a}{2}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = \frac{b-a}{6}\left(a^2 + 4\left(\frac{a+b}{2}\right)^2 + b^2\right)$$

$$= \frac{b-a}{6}\left(a^2 + (a^2 + 2ab + b^2) + b^2\right) = \frac{b-a}{6}(2)(a^2 + ab + b^2) = \frac{b^3-a^3}{3} = \int_a^b x^2\,dx.$$

Now we use linearity:

$$\int_a^b (px^2 + qx + r)\,dx = p\int_a^b x^2\,dx + q\int_a^b x\,dx + r\int_a^b dx$$

$$= pS_2(x^2) + qS_2(x) + rS_2(1) = S_2(pa^2 + qa + r).$$

61. For N even, divide $[a, b]$ into N subintervals of width $\Delta x = \frac{b-a}{N}$. Set $x_j = a + j\,\Delta x$, $y_j = f(x_j)$, and

$$S_2^{2j} = \frac{b-a}{3N}(y_{2j} + 4y_{2j+1} + y_{2j+2})$$

(a) Show that S_N is the sum of the approximations on the intervals $[x_{2j}, x_{2j+2}]$—that is, $S_N = S_2^0 + S_2^2 + \cdots + S_2^{N-2}$.

(b) By Exercise 60, $S_2^{2j} = \int_{x_{2j}}^{x_{2j+2}} f(x)\,dx$ if $f(x)$ is a quadratic polynomial. Use (a) to show that S_N is exact *for all N* if $f(x)$ is a quadratic polynomial.

SOLUTION

(a) This result follows because the even-numbered interior endpoints overlap:

$$\sum_{i=0}^{(N-2)/2} S_2^{2j} = \frac{b-a}{6}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots\right]$$

$$= \frac{b-a}{6}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{N-1} + y_N\right] = S_N.$$

(b) If $f(x)$ is a quadratic polynomial, then by part (a) we have

$$S_N = S_2^0 + S_2^2 + \cdots + S_2^{N-2} = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{N-2}}^{x_N} f(x)\,dx = \int_a^b f(x)\,dx.$$

62. Show that S_2 also gives the exact value for $\int_a^b x^3\,dx$ and conclude, as in Exercise 61, that S_N is exact for all cubic polynomials. Show by counterexample that S_2 is not exact for integrals of x^4.

SOLUTION Let $f(x) = x^3$. Then $\Delta x = (b-a)/2$ and the endpoints are a, $(a+b)/2$, b. With this data, we get

$$S_2(x^3) = \frac{1}{3}\left(\frac{b-a}{2}\right)\left[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right] = \frac{b-a}{6}\left[a^3 + \frac{1}{2}(a^3 + 3a^2b + 3ab^2 + b^3) + b^3\right]$$

$$= \frac{b-a}{6}\left(\frac{3}{2}\right)[a^3 + a^2b + ab^2 + b^3] = \frac{1}{4}(b-a)(a^3 + a^2b + ab^2 + b^3) = \frac{b^4 - a^4}{4} = \int_a^b x^3\,dx.$$

By linearity, and using the result from Exercise 60, we have that

$$\int_a^b (sx^3 + px^2 + qx + r)\,dx = s\int_a^b x^3\,dx + \int_a^b (px^2 + qx + r)\,dx$$

$$= s\left(S_2(x^3)\right) + S_2(px^2 + qx + r)$$

$$= S_2(sx^3 + px^2 + qx + r).$$

For N even, we can now follow the procedure of Exercise 61; that is, divide $[a, b]$ into N subintervals and on each subinterval compute S_2. Then, for any cubic polynomial $f(x)$, we have

$$\int_a^b f(x)\,dx = \int_a^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{N-2}}^{b} f(x)\,dx = S_2^0 + S_2^2 + \cdots + S_2^{N-2} = S_N.$$

However, S_2 is not exact for polynomials of degree 4. For example,

$$\int_0^1 x^4\,dx = \frac{1}{5}$$

but

$$S_2 = \frac{1}{3}\left(\frac{1}{2}\right)\left[0^5 + 4(0.5)^5 + 1^5\right] = \frac{1}{6}\left(\frac{33}{32}\right) = \frac{11}{64} \neq \frac{1}{5}.$$

63. Use the error bound for S_N to obtain another proof that Simpson's Rule is exact for all cubic polynomials.

SOLUTION Let $f(x) = ax^3 + bx^2 + cx + d$, with $a \neq 0$, be any cubic polynomial. Then, $f^{(4)}(x) = 0$, so we can take $K_4 = 0$. This yields

$$\text{Error}(S_N) \leq \frac{0}{180N^4} = 0.$$

In other words, S_N is exact for all cubic polynomials for all N.

64. **Sometimes, Simpson's Rule Performs Poorly** Calculate M_{10} and S_{10} for the integral $\int_0^1 \sqrt{1-x^2}\,dx$, whose value we know to be $\frac{\pi}{4}$ (one-quarter of the area of the unit circle).

(a) We usually expect S_N to be more accurate than M_N. Which of M_{10} and S_{10} is more accurate in this case?

(b) How do you explain the result of part (a)? *Hint:* The error bounds are not valid because $|f''(x)|$ and $|f^{(4)}(x)|$ tend to ∞ as $x \to 1$, but $|f^{(4)}(x)|$ goes to infinity faster.

SOLUTION Let $f(x) = \sqrt{1-x^2}$. Divide [0, 1] into 10 subintervals of length $\Delta x = (1-0)/10 = 0.1$ Then we have

$$M_{10} = \frac{1}{10}\left[\sqrt{1-(0.05)^2} + \sqrt{1-(0.15)^2} + \cdots + \sqrt{1-(0.95)^2}\right] \approx 0.788103;$$

$$S_{10} = \frac{1}{3}\left(\frac{1}{10}\right)\left[\sqrt{1-0^2} + 4\sqrt{1-(0.1)^2} + 2\sqrt{1-(0.2)^2} + \cdots + \sqrt{1-1^2}\right] \approx 0.781752.$$

(a) Since $\pi/4 = 0.785389$, we have

$$\text{Error}(M_{10}) = 0.0027;$$

$$\text{Error}(S_{10}) = 0.00365.$$

Thus, M_{10} is more accurate.

(b) These results can be explained by looking at the derivatives:

$$f'(x) = \frac{-x}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{-1}{(1-x^2)^{3/2}}$$

$$f^{(3)}(x) = \frac{-3x}{(1-x^2)^{5/2}}$$

$$f^{(4)}(x) = \frac{-3(x^2+1)}{(1-x^2)^{7/2}}$$

Both $|f''(x)|$ and $|f^{(4)}(x)|$ tend to ∞ as $x \to 1$, but $|f^{(4)}(x)|$ tends to ∞ faster due to the $7/2$ exponent in the denominator.

CHAPTER REVIEW EXERCISES

1. Match the integrals (a)–(e) with their antiderivatives (i)–(v) on the basis of the general form (do not evaluate the integrals).

(a) $\displaystyle\int \frac{x\,dx}{x^2-4}$

(b) $\displaystyle\int \frac{(2x+9)\,dx}{x^2+4}$

(c) $\displaystyle\int \sin^3 x\cos^2 x\,dx$

(d) $\displaystyle\int \frac{dx}{x\sqrt{16x^2-1}}$

(e) $\displaystyle\int \frac{16\,dx}{x(x-4)^2}$

(i) $\sec^{-1} 4x + C$

(ii) $\log|x| - \log|x-4| - \dfrac{4}{x-4} + C$

(iii) $\dfrac{1}{30}(3\cos^5 x - 3\cos^3 x\sin^2 x - 7\cos^3 x) + C$

(iv) $\dfrac{9}{2}\tan^{-1}\dfrac{x}{2} + \ln(x^2+4) + C$

(v) $\sqrt{x^2-4} + C$

SOLUTION

(a) $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-4}}$

Since x is a constant multiple of the derivative of x^2-4, the substitution method implies that the integral is a constant multiple of $\int \frac{du}{\sqrt{u}}$ where $u = x^2-4$, that is a constant multiple of $\sqrt{u} = \sqrt{x^2-4}$. It corresponds to the function in **(v)**.

(b) $\displaystyle\int \frac{(2x+9)\,dx}{x^2+4}$

The part $\int \frac{2x}{x^2+4}\,dx$ corresponds to $\ln(x^2+4)$ in **(iv)** and the part $\int \frac{9}{x^2+4}\,dx$ corresponds to $\frac{9}{2}\tan^{-1}\frac{x}{2}$. Hence the integral corresponds to the function in **(iv)**.

(c) $\displaystyle\int \sin^3 x\cos^2 x\,dx$

The reduction formula for $\int \sin^m x\cos^n x\,dx$ shows that this integral is equal to a sum of constant multiples of products in the form $\cos^i x\sin^j x$ as in **(iii)**.

(d) $\displaystyle\int \frac{dx}{x\sqrt{16x^2-1}}$

Since $\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x + C$, we expect the integral $\int \frac{dx}{x\sqrt{16x^2-1}}$ to be equal to the function in **(i)**.

(e) $\displaystyle\int \frac{16\,dx}{x(x-4)^2}$

The partial fraction decomposition of the integrand has the form:

$$\frac{A}{x} + \frac{B}{x-4} + \frac{C}{(x-4)^2}$$

The term $\frac{A}{x}$ contributes the function $A \ln |x|$ to the integral, the term $\frac{B}{x-4}$ contributes $B \ln |x-4|$ and the term $\frac{C}{(x-4)^2}$ contributes $-\frac{C}{x-4}$. Therefore, we expect the integral to be equal to the function in **(ii)**.

2. Evaluate $\displaystyle\int \frac{x\,dx}{x+2}$ in two ways: using substitution and using the Method of Partial Fractions.

SOLUTION Using substitution, write $u = x + 2$; then $du = dx$ and

$$\int \frac{x}{x+2}\,dx = \int \frac{u-2}{u}\,du = \int 1\,du - 2\int \frac{1}{u}\,du = u - 2\ln|u| + C_1$$
$$= x + 2 - 2\ln|x+2| + C_1 = x - 2\ln|x+2| + C$$

Using partial fractions, first do long division to get

$$\frac{x}{x+2} = 1 - \frac{2}{x+2}$$

Then

$$\int \frac{x}{x+2}\,dx = \int \left(1 - \frac{2}{x+2}\right)\,dx = \int 1\,dx - 2\int \frac{1}{x+2}\,dx = x - 2\ln|x+2| + C$$

In Exercises 3–12, evaluate using the suggested method.

3. $\displaystyle\int \cos^3\theta \sin^8\theta\,d\theta$ [write $\cos^3\theta$ as $\cos\theta(1-\sin^2\theta)$]

SOLUTION We use the identity $\cos^2\theta = 1 - \sin^2\theta$ to rewrite the integral:

$$\int \cos^3\theta \sin^8\theta\,d\theta = \int \cos^2\theta \sin^8\theta \cos\theta\,d\theta = \int \left(1 - \sin^2\theta\right)\sin^8\theta\cos\theta\,d\theta.$$

Now, we use the substitution $u = \sin\theta$, $du = \cos\theta\,d\theta$:

$$\int \cos^3\theta \sin^8\theta\,d\theta = \int \left(1-u^2\right)u^8\,du = \int \left(u^8 - u^{10}\right)\,du = \frac{u^9}{9} - \frac{u^{11}}{11} + C = \frac{\sin^9\theta}{9} - \frac{\sin^{11}\theta}{11} + C.$$

4. $\displaystyle\int xe^{-12x}\,dx$ (Integration by Parts)

SOLUTION We use Integration by Parts with $u = x$ and $v' = e^{-12x}$. Then $u' = 1$, $v = -\frac{1}{12}e^{-12x}$, and we obtain:

$$\int xe^{-12x}\,dx = -\frac{xe^{-12x}}{12} + \int \frac{1}{12}e^{-12x}\,dx = -\frac{xe^{-12x}}{12} - \frac{1}{144}e^{-12x} + C = -\frac{e^{-12x}}{144}(12x+1) + C.$$

5. $\displaystyle\int \sec^3\theta \tan^4\theta\,d\theta$ (trigonometric identity, reduction formula)

SOLUTION We use the identity $1 + \tan^2\theta = \sec^2\theta$ to write $\tan^4\theta = \left(\sec^2\theta - 1\right)^2$ and to rewrite the integral as

$$\int \sec^3\theta \tan^4\theta\,d\theta \int \sec^3\theta\left(1 - \sec^2\theta\right)^2 d\theta = \int \sec^3\theta\left(1 - 2\sec^2\theta + \sec^4\theta\right)\,d\theta$$
$$= \int \sec^7\theta\,d\theta - 2\int \sec^5\theta\,d\theta + \int \sec^3\theta\,d\theta.$$

Now we use the reduction formula

$$\int \sec^m\theta\, d\theta = \frac{\tan\theta \sec^{m-2}\theta}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2}\theta\, d\theta.$$

We have

$$\int \sec^5\theta\, d\theta = \frac{\tan\theta\sec^3\theta}{4} + \frac{3}{4}\int \sec^3\theta\, d\theta + C,$$

and

$$\int \sec^7\theta\, d\theta = \frac{\tan\theta\sec^5\theta}{6} + \frac{5}{6}\int \sec^5\theta\, d\theta = \frac{\tan\theta\sec^5\theta}{6} + \frac{5}{6}\left(\frac{\tan\theta\sec^3\theta}{4} + \frac{3}{4}\int \sec^3\theta\, d\theta\right) + C$$

$$= \frac{\tan\theta\sec^5\theta}{6} + \frac{5}{24}\tan\theta\sec^3\theta + \frac{5}{8}\int \sec^3\theta\, d\theta + C.$$

Therefore,

$$\int \sec^3\theta\tan^4\theta\, d\theta = \left(\frac{\tan\theta\sec^5\theta}{6} + \frac{5}{24}\tan\theta\sec^3\theta + \frac{5}{8}\int \sec^3\theta\, d\theta\right)$$

$$- 2\left(\frac{\tan\theta\sec^3\theta}{4} + \frac{3}{4}\int \sec^3\theta\, d\theta\right) + \int \sec^3\theta\, d\theta$$

$$= \frac{\tan\theta\sec^5\theta}{6} - \frac{7\tan\theta\sec^3\theta}{24} + \frac{1}{8}\int \sec^3\theta\, d\theta.$$

We again use the reduction formula to compute

$$\int \sec^3\theta\, d\theta = \frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\, d\theta = \frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$

Finally,

$$\int \sec^3\theta\tan^4\theta\, d\theta = \frac{\tan\theta\sec^5\theta}{6} - \frac{7\tan\theta\sec^3\theta}{24} + \frac{\tan\theta\sec\theta}{16} + \frac{1}{16}\ln|\sec\theta + \tan\theta| + C.$$

6. $\displaystyle\int \frac{4x+4}{(x-5)(x+3)}\, dx$ (partial fractions)

SOLUTION The following partial fraction decomposition takes the form

$$\frac{4x+4}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}.$$

Clearing denominators gives us

$$4x + 4 = A(x+3) + B(x-5).$$

Setting $x = 5$ then yields $A = 3$, while setting $x = -3$ yields $B = 1$. Hence,

$$\int \frac{4x+4}{(x-5)(x+3)}\, dx = \int \frac{3}{x-5}\, dx + \int \frac{1}{x+3}\, dx = 3\ln|x-5| + \ln|x+3| + C.$$

7. $\displaystyle\int \frac{dx}{x(x^2-1)^{3/2}}\, dx$ (trigonometric substitution)

SOLUTION Substitute $x = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$. Then,

$$\left(x^2-1\right)^{3/2} = \left(\sec^2\theta - 1\right)^{3/2} = \left(\tan^2\theta\right)^{3/2} = \tan^3\theta,$$

and

$$\int \frac{dx}{x\left(x^2-1\right)^{3/2}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\tan^3\theta} = \int \frac{d\theta}{\tan^2\theta} = \int \cot^2\theta\, d\theta.$$

Using a reduction formula we find that:

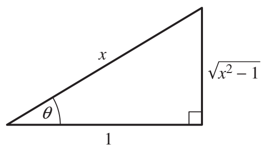
$$\int \cot^2\theta\, d\theta = -\cot\theta - \theta + C$$

so

$$\int \frac{dx}{x\left(x^2-1\right)^{3/2}} = -\cot\theta - \theta + C.$$

We now must return to the original variable x. We use the relation $x = \sec\theta$ and the figure to obtain:

$$\int \frac{dx}{x\left(x^2-1\right)^{3/2}} = -\frac{1}{\sqrt{x^2-1}} - \sec^{-1}x + C.$$

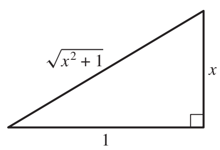


8. $\int (1+x^2)^{-3/2}dx$ (trigonometric substitution)

SOLUTION Use the substitution $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$. Then

$$\int (1+x^2)^{-3/2}\,dx = \int (1+\tan^2\theta)^{-3/2}\sec^2\theta\, d\theta = \int (\sec^2\theta)^{-3/2}\sec^2\theta\, d\theta = \int \frac{1}{\sec\theta}\, d\theta$$
$$= \int \cos\theta\, d\theta = \sin\theta + C$$

Since $x = \tan\theta$, draw the following right triangle:



From the figure, we see that $\sin\theta = \frac{x}{\sqrt{x^2+1}}$, so that

$$\int (1+x^2)^{-3/2}\,dx = x(1+x^2)^{-1/2} + C$$

9. $\int \frac{dx}{x^{3/2}+x^{1/2}}$ (substitution)

SOLUTION Let $t = x^{1/2}$. Then $dt = \frac{1}{2}x^{-1/2}\,dx$ or $dx = 2x^{1/2}\,dt = 2t\,dt$. Therefore,

$$\int \frac{dx}{x^{3/2}+x^{1/2}} = \int \frac{2t\,dt}{t^3+t} = \int \frac{2\,dt}{t^2+1} = 2\tan^{-1}t + C = 2\tan^{-1}\sqrt{x} + C.$$

10. $\int \frac{dx}{x+x^{-1}}$ (rewrite integrand)

SOLUTION We rewrite the integrand as follows:

$$\int \frac{dx}{x+x^{-1}} = \int \frac{x\,dx}{x^2+1}.$$

Now, we substitute $u = x^2+1$. Then $du = 2x\,dx$ and

$$\int \frac{dx}{x+x^{-1}} = \int \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(1+x^2\right) + C.$$

11. $\int x^{-2}\tan^{-1}x\,dx$ (Integration by Parts)

SOLUTION We use Integration by Parts with $u = \tan^{-1}x$ and $v' = x^{-2}$. Then $u' = \frac{1}{1+x^2}$, $v = -x^{-1}$ and

$$\int x^{-2}\tan^{-1}x\,dx = -\frac{\tan^{-1}x}{x} + \int \frac{dx}{x\left(1+x^2\right)}.$$

For the remaining integral, the partial fraction decomposition takes the form

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}.$$

Clearing denominators gives us

$$1 = A(1+x^2) + (Bx+C)x.$$

Setting $x = 0$ then yields $A = 1$. Next, equating the x^2-coefficients gives

$$0 = A + B \qquad \text{so} \qquad B = -1,$$

while equating x-coefficients gives $C = 0$. Hence,

$$\frac{1}{x\left(1+x^2\right)} = \frac{1}{x} - \frac{x}{1+x^2},$$

and

$$\int \frac{dx}{x(1+x^2)} = \int \frac{1}{x}\,dx - \int \frac{x\,dx}{1+x^2} = \ln|x| - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

Therefore,

$$\int x^{-2}\tan^{-1}x\,dx = -\frac{\tan^{-1}x}{x} + \ln|x| - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

12. $\displaystyle\int \frac{dx}{x^2+4x-5}$ (complete the square, substitution, partial fractions)

SOLUTION The partial fraction decomposition takes the form

$$\frac{1}{x^2+4x-5} = \frac{A}{x-1} + \frac{B}{x+5}.$$

Clearing denominators gives us

$$1 = A(x+5) + B(x-1).$$

Setting $x = 1$ then yields $A = \frac{1}{6}$, while setting $x = -5$ yields $B = -\frac{1}{6}$. Therefore,

$$\int \frac{dx}{x^2+4x-5} = \frac{1}{6}\int \frac{dx}{x-1} - \frac{1}{6}\int \frac{dx}{x+5} = \frac{1}{6}\ln|x-1| - \frac{1}{6}\ln|x+5| + C = \frac{1}{6}\ln\left|\frac{x-1}{x+5}\right| + C.$$

In Exercises 13–64, evaluate using the appropriate method or combination of methods.

13. $\displaystyle\int_0^1 x^2 e^{4x}\,dx$

SOLUTION We evaluate the indefinite integral using Integration by Parts with $u = x^2$ and $v' = e^{4x}$. Then $u' = 2x$, $v = \frac{1}{4}e^{4x}$ and

$$\int x^2 e^{4x}\,dx = \frac{x^2}{4}e^{4x} - \frac{1}{2}\int x e^{4x}\,dx.$$

We compute the resulting integral using Integration by Parts again, this time with $u = x$ and $v' = e^{4x}$. Then $u' = 1$, $v = \frac{1}{4}e^{4x}$ and

$$\int x e^{4x}\,dx = x \cdot \frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x}\,dx = \frac{x}{4}e^{4x} - \frac{1}{16}e^{4x} + C.$$

Therefore,

$$\int x^2 e^{4x}\,dx = \frac{x^2}{4}e^{4x} - \frac{1}{2}\left(\frac{x}{4}e^{4x} - \frac{1}{16}e^{4x}\right) + C = \frac{e^{4x}}{32}\left(8x^2 - 4x + 1\right) + C.$$

Finally,

$$\int_0^1 x^2 e^{4x}\,dx = \left(\frac{e^{4x}}{32}\left(8x^2 - 4x + 1\right)\right)\Big|_0^1 = \frac{e^4}{32}(8-4+1) - \frac{1}{32}(1) = \frac{5e^4 - 1}{32}$$

14. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

SOLUTION Substitute $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$. Then

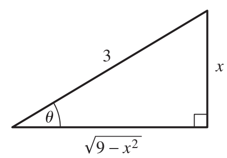
$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\left(1-\sin^2\theta\right)} = \sqrt{9\cos^2\theta} = 3\cos\theta,$$

and

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{3\cos\theta} = 9\int \sin^2\theta\,d\theta$$
$$= 9\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C = \frac{9\theta}{2} - \frac{9\sin\theta\cos\theta}{2} + C.$$

We now must return to the original variable x. Since $x = 3\sin\theta$, we have $t = \sin^{-1}\frac{x}{3}$. Using the figure we obtain

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C.$$



15. $\displaystyle\int \cos^9 6\theta \sin^3 6\theta\,d\theta$

SOLUTION We use the identity $\sin^2 6\theta = 1 - \cos^2 6\theta$ to rewrite the integral:

$$\int \cos^9 6\theta \sin^3 6\theta\,d\theta = \int \cos^9 6\theta \sin^2 6\theta \sin 6\theta\,d\theta = \int \cos^9 6\theta\left(1-\cos^2 6\theta\right)\sin 6\theta\,d\theta.$$

Now, we use the substitution $u = \cos 6\theta$, $du = -6\sin 6\theta\,d\theta$:

$$\int \cos^9 6\theta \sin^3 6\theta\,d\theta = \int u^9\left(1-u^2\right)\left(-\frac{du}{6}\right) = -\frac{1}{6}\int\left(u^9 - u^{11}\right)du$$
$$= -\frac{1}{6}\left(\frac{u^{10}}{10} - \frac{u^{12}}{12}\right) + C = \frac{\cos^{12} 6\theta}{72} - \frac{\cos^{10} 6\theta}{60} + C.$$

16. $\displaystyle\int \sec^2\theta \tan^4\theta\,d\theta$

SOLUTION We substitute $u = \tan\theta$, $du = \sec^2\theta\,d\theta$ to obtain

$$\int \sec^2\theta\tan^4\theta\,d\theta = \int u^4\,du = \frac{u^5}{5} + C = \frac{\tan^5\theta}{5} + C.$$

17. $\displaystyle\int \frac{(6x+4)\,dx}{x^2-1}$

SOLUTION The partial fraction decomposition takes the form

$$\frac{6x+4}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Clearing the denominators gives us

$$6x + 4 = A(x+1) + B(x-1).$$

Setting $x = 1$ then yields $A = 5$, while setting $x = -1$ yields $B = 1$. Hence,

$$\int \frac{(6x+4)dx}{x^2-1} = \int \frac{5}{x-1}\,dx + \int \frac{1}{x+1}\,dx = 5\ln|x-1| + \ln|x+1| + C.$$

18. $\displaystyle\int_4^9 \frac{dt}{(t^2-1)^2}$

SOLUTION First evaluate the indefinite integral. Substitute $t = \sin\theta$, $dt = \cos\theta\, d\theta$. Then

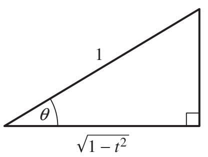
$$\left(t^2-1\right)^2 = \left(1-t^2\right)^2 = \left(1-\sin^2\theta\right)^2 = \left(\cos^2\theta\right)^2 = \cos^4\theta,$$

and

$$\int \frac{dt}{\left(t^2-1\right)^2} = \int \frac{\cos\theta\, d\theta}{\cos^4\theta} = \int \frac{d\theta}{\cos^3\theta} = \int \sec^3\theta\, d\theta.$$

We use a reduction formula to compute the resulting integral:

$$\int \frac{dt}{\left(t^2-1\right)^2} = \int \sec^3\theta\, d\theta = \frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\, d\theta = \frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$



We now must return to the original variable t. Using the relation $t = \sin\theta$ and the accompanying figure,

$$\begin{aligned}\int \frac{dt}{\left(t^2-1\right)^2} &= \frac{1}{2}\cdot\frac{t}{\sqrt{1-t^2}}\cdot\frac{1}{\sqrt{1-t^2}} + \frac{1}{2}\ln\left|\frac{1}{\sqrt{1-t^2}} + \frac{t}{\sqrt{1-t^2}}\right| + C \\ &= \frac{1}{2}\left(\frac{t}{1-t^2} + \ln\left|\frac{1+t}{\sqrt{1-t^2}}\right|\right) + C = \frac{1}{2}\left(\frac{t}{1-t^2} + \ln\left|\sqrt{\frac{1+t}{1-t}}\right|\right) + C \\ &= \frac{1}{2}\frac{t}{1-t^2} + \frac{1}{4}\ln\left|\frac{1+t}{1-t}\right| + C\end{aligned}$$

Finally,

$$\begin{aligned}\int_4^9 \frac{dt}{(t^2-1)^2} &= \left(\frac{1}{2}\frac{t}{1-t^2} + \frac{1}{4}\ln\left|\frac{1+t}{1-t}\right|\right)\Bigg|_4^9 \\ &= \frac{1}{2}\cdot\frac{9}{-80} + \frac{1}{4}\ln\frac{10}{8} - \frac{1}{2}\cdot\frac{4}{-15} - \frac{1}{4}\ln\frac{5}{3} = -\frac{9}{160} + \frac{2}{15} + \frac{1}{4}\left(\ln\frac{5}{4} - \ln\frac{5}{3}\right) \\ &= \frac{37}{480} + \frac{1}{4}\ln\frac{3}{4} = \frac{37}{480} + \frac{1}{4}\ln 3 - \frac{1}{2}\ln 2\end{aligned}$$

19. $\displaystyle\int \frac{d\theta}{\cos^4\theta}$

SOLUTION We use the identity $1 + \tan^2\theta = \sec^2\theta$ to rewrite the integral:

$$\int \frac{d\theta}{\cos^4\theta} = \int \sec^4\theta\, d\theta = \int \left(1+\tan^2\theta\right)\sec^2\theta\, d\theta.$$

Now, we substitute $u = \tan\theta$. Then, $du = \sec^2\theta\, d\theta$ and

$$\int \frac{d\theta}{\cos^4\theta} = \int \left(1+u^2\right) du = u + \frac{u^3}{3} + C = \frac{\tan^3\theta}{3} + \tan\theta + C.$$

20. $\displaystyle\int \sin 2\theta \sin^2\theta\, d\theta$

SOLUTION We use the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$ to rewrite the integral:

$$\int \sin 2\theta \sin^2\theta\, d\theta = \int 2\sin\theta\cos\theta\sin^2\theta\, d\theta = \int 2\sin^3\theta\cos\theta\, d\theta.$$

Now, we substitute $u = \sin\theta$. Then $du = \cos\theta\, d\theta$ and

$$\int \sin 2\theta\sin^2\theta\, d\theta = 2\int u^3\, du = \frac{u^4}{2} + C = \frac{\sin^4\theta}{2} + C.$$

21. $\int_0^1 \ln(4-2x)\,dx$

SOLUTION Note that $\ln(4-2x) = \ln(2(2-x)) = \ln 2 + \ln(2-x)$. Use integration by parts to integrate $\ln(2-x)$, with $u = \ln(2-x)$, $v' = 1$, so that $u' = -\frac{1}{2-x}$ and $v = x$. Then

$$I = \int_0^1 \ln(4-2x)\,dx = \int_0^1 \ln 2\,dx + \int_0^1 \ln(2-x)\,dx = \ln 2 + (x\ln(2-x))\Big|_0^1 + \int_0^1 \frac{x}{2-x}\,dx$$

Now use long division on the remaining integral, and the substitution $u = 2-x$:

$$\begin{aligned} I &= \ln 2 + (x\ln(2-x))\Big|_0^1 + \int_0^1 \left(-1 + \frac{2}{2-x}\right) dx \\ &= \ln 2 + 1\ln 1 - \int_0^1 1\,dx + 2\int_0^1 \frac{1}{2-x}\,dx = \ln 2 - 1 - 2\int_2^1 \frac{1}{u}\,du \\ &= \ln 2 - 1 - 2\ln u\Big|_2^1 = \ln 2 - 1 + 2\ln 2 = 3\ln 2 - 1 \end{aligned}$$

22. $\int (\ln(x+1))^2\,dx$

SOLUTION First, substitute $w = x+1$, $dw = dx$. Then

$$\int (\ln(x+1))^2\,dx = \int (\ln w)^2\,dw.$$

Now, we use Integration by Parts with $u = (\ln w)^2$ and $v' = 1$. We find $u' = 2\frac{\ln w}{w}$, $v = w$, and

$$\int (\ln w)^2\,dw = w(\ln w)^2 - 2\int \ln w\,dw.$$

We use Integration by Parts again, this time with $u = \ln w$ and $v' = 1$. We find $u' = \frac{1}{w}$, $v = w$, and

$$\int \ln w\,dx = w\ln w - \int dw = w\ln w - w + C.$$

Thus,

$$\int (\ln w)^2\,dw = w(\ln w)^2 - 2w\ln w + 2w + C,$$

and

$$\int (\ln(x+1))^2\,dx = (x+1)\,[\ln(x+1)]^2 - 2(x+1)\ln(x+1) + 2(x+1) + C.$$

23. $\int \sin^5\theta\,d\theta$

SOLUTION We use the trigonometric identity $\sin^2\theta = 1 - \cos^2\theta$ to rewrite the integral:

$$\int \sin^5\theta\,d\theta = \int \sin^4\theta\sin\theta\,d\theta = \int \left(1-\cos^2\theta\right)^2 \sin\theta\,d\theta.$$

Now, we substitute $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and

$$\begin{aligned} \int \sin^5\theta\,d\theta &= \int \left(1-u^2\right)^2(-du) = -\int \left(1 - 2u^2 + u^4\right) du \\ &= -\left(u - \frac{2}{3}u^3 + \frac{u^5}{5}\right) + C = -\frac{\cos^5\theta}{5} + \frac{2\cos^3\theta}{3} - \cos\theta + C. \end{aligned}$$

24. $\int \cos^4(9x-2)\,dx$

SOLUTION We substitute $u = 9x-2$, $du = 9\,dx$ and then use a reduction formula to evaluate the resulting integral. We obtain:

$$\int \cos^4(9x-2)\,dx = \frac{1}{9}\int \cos^4 u\,du = \frac{1}{9}\left(\frac{\cos^3 u\sin u}{4} + \frac{3}{4}\int \cos^2 u\,du\right)$$

$$= \frac{\cos^3 u \sin u}{36} + \frac{1}{12}\int \cos^2 u\, du = \frac{\cos^3 u \sin u}{36} + \frac{1}{12}\left(\frac{u}{2} + \frac{\sin 2u}{4}\right) + C$$

$$= \frac{\cos^3(9x-2)\sin(9x-2)}{36} + \frac{9x-2}{24} + \frac{\sin(18x-4)}{48} + C.$$

25. $\displaystyle\int_0^{\pi/4} \sin 3x \cos 5x\, dx$

SOLUTION First compute the indefinite integral, using the trigonometric identity:

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right).$$

For $\alpha = 3x$ and $\beta = 5x$ we get:

$$\sin 3x \cos 5x = \frac{1}{2}\left(\sin 8x + \sin(-2x)\right) = \frac{1}{2}(\sin 8x - \sin 2x).$$

Hence,

$$\int \sin 3x \cos 5x\, dx = \frac{1}{2}\int \sin 8x\, dx - \frac{1}{2}\int \sin 2x\, dx = -\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + C.$$

Then

$$\int_0^{\pi/4} \sin 3x \cos 5x\, dx = \left(\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x\right)\Big|_0^{\pi/4} = \frac{1}{4}\cos\frac{\pi}{2} - \frac{1}{16}\cos 2\pi - \frac{1}{4}\cos 0 + \frac{1}{16}\cos 0 = -\frac{1}{4}$$

26. $\displaystyle\int \sin 2x \sec^2 x\, dx$

SOLUTION We use the trigonometric identity $\sin 2x = 2\cos x \sin x$ to rewrite the integrand:

$$\sin 2x \sec^2 x = 2\sin x \cos x \sec^2 x = \frac{2\sin x \cos x}{\cos^2 x} = \frac{2\sin x}{\cos x} = 2\tan x.$$

Hence,

$$\int \sin 2x \sec^2 x\, dx = \int 2\tan x\, dx = 2\ln|\sec x| + C.$$

27. $\displaystyle\int \sqrt{\tan x}\, \sec^2 x\, dx$

SOLUTION We substitute $u = \tan x$. Then $du = \sec^2 x\, dx$ and we obtain:

$$\int \sqrt{\tan x}\, \sec^2 x\, dx = \int \sqrt{u}\, du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C.$$

28. $\displaystyle\int (\sec x + \tan x)^2\, dx$

SOLUTION We rewrite the integrand as

$$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x = 2\sec x \tan x + 2\sec^2 x - 1.$$

Therefore,

$$\int (\sec x + \tan x)^2\, dx = 2\int \sec x \tan x\, dx + 2\int \sec^2 x\, dx - \int dx = 2\sec x + 2\tan x - x + C.$$

29. $\displaystyle\int \sin^5\theta \cos^3\theta\, d\theta$

SOLUTION We use the identity $\cos^2\theta = 1 - \sin^2\theta$ to rewrite the integral:

$$\int \sin^5\theta \cos^3\theta\, d\theta = \int \sin^5\theta \cos^2\theta \cos\theta\, d\theta = \int \sin^5\theta\left(1 - \sin^2\theta\right)\cos\theta\, d\theta.$$

Now, we use the substitution $u = \sin\theta$, $du = \cos\theta\, d\theta$:

$$\int \sin^5\theta \cos^3\theta\, d\theta = \int u^5\left(1 - u^2\right) du = \int \left(u^5 - u^7\right) du = \frac{u^6}{6} - \frac{u^8}{8} + C = \frac{\sin^6\theta}{6} - \frac{\sin^8\theta}{8} + C.$$

30. $\int \cot^3 x \csc x \, dx$

SOLUTION Use the identity $\cot^2 x = \csc^2 x - 1$ to write

$$\int \cot^3 x \csc x \, dx = \int \left(\csc^2 x - 1\right) \csc x \cot x \, dx.$$

Now use the substitution $u = \csc x$, $du = -\csc x \cot x \, dx$:

$$\int \cot^3 x \csc x \, dx = -\int \left(u^2 - 1\right) du = \int \left(1 - u^2\right) du = u - \frac{1}{3}u^3 + C = \csc x - \frac{1}{3}\csc^3 x + C.$$

31. $\int \cot^2 x \csc^2 x \, dx$

SOLUTION Use the substitution $u = \cot x$, $du = -\csc^2 x \, dx$:

$$\int \cot^2 x \csc^2 x \, dx = -\int \cot^2 x \left(-\csc^2 x \, dx\right) = -\int u^2 \, du = -\frac{1}{3}u^3 + C = -\frac{1}{3}\cot^3 x + C.$$

32. $\int_{\pi/2}^{\pi} \cot^2 \frac{\theta}{2} \, d\theta$

SOLUTION To compute the indefinite integral, substitute $u = \frac{\theta}{2}$. Then $du = \frac{1}{2}\, d\theta$ and

$$\int \cot^2 \frac{\theta}{2} \, d\theta = 2 \int \cot^2 u \, du.$$

Now, we use a reduction formula to compute

$$\int \cot^2 \frac{\theta}{2} \, d\theta = 2 \int \cot^2 u \, du = 2(-\cot u - u) + C = -2 \cot \frac{\theta}{2} - \theta + C.$$

Then

$$\int_{\pi/2}^{\pi} \cot^2 \frac{\theta}{2} \, d\theta = \left(-2 \cot \frac{\theta}{2} - \theta\right)\bigg|_{\pi/2}^{\pi} = -2 \cot \frac{\pi}{2} - \pi + 2 \cot \frac{\pi}{4} + \frac{\pi}{2} = 0 - \pi + 2 + \frac{\pi}{2} = 2 - \frac{\pi}{2}$$

33. $\int_{\pi/4}^{\pi/2} \cot^2 x \csc^3 x \, dx$

SOLUTION To compute the indefinite integral, use the identity $\cot^2 x = \csc^2 x - 1$ to write

$$\int \cot^2 x \csc^3 x \, dx = \int \left(\csc^2 x - 1\right) \csc^3 x \, dx = \int \csc^5 x \, dx - \int \csc^3 x \, dx.$$

Now use the reduction formula for $\csc^m x$:

$$\begin{aligned}
\int \cot^2 x \csc^3 x \, dx &= \left(-\frac{1}{4}\cot x \csc^3 x + \frac{3}{4}\int \csc^3 x \, dx\right) - \int \csc^3 x \, dx \\
&= -\frac{1}{4}\cot x \csc^3 x - \frac{1}{4}\int \csc^3 x \, dx \\
&= -\frac{1}{4}\cot x \csc^3 x - \frac{1}{4}\left(-\frac{1}{2}\cot x \csc x + \frac{1}{2}\int \csc x \, dx\right) \\
&= -\frac{1}{4}\cot x \csc^3 x + \frac{1}{8}\cot x \csc x - \frac{1}{8}\ln|\csc x - \cot x| + C.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \cos^2 x \csc^3 x \, dx &= \left(-\frac{1}{4}\cot x \csc^3 x + \frac{1}{8}\cot x \csc x - \frac{1}{8}\ln|\csc x - \cot x|\right)\bigg|_{\pi/4}^{\pi/2} \\
&= -\frac{1}{4}\cot\frac{\pi}{2}\csc^3\frac{\pi}{2} + \frac{1}{8}\cot\frac{\pi}{2}\csc\frac{\pi}{2} - \frac{1}{8}\ln\left|\csc\frac{\pi}{2} - \cot\frac{\pi}{2}\right| \\
&\quad + \frac{1}{4}\cot\frac{\pi}{4}\csc^3\frac{\pi}{4} - \frac{1}{8}\cot\frac{\pi}{4}\csc\frac{\pi}{4} + \frac{1}{8}\ln\left|\csc\frac{\pi}{4} - \cot\frac{\pi}{4}\right| \\
&= 0 + 0 - \frac{1}{8}\ln|1 - 0| + \frac{1}{4}\cdot 1 \cdot (\sqrt{2})^3 - \frac{1}{8}\cdot 1 \cdot \sqrt{2} + \frac{1}{8}\ln\left|\sqrt{2} - 1\right| \\
&= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8} + \frac{1}{8}\ln(\sqrt{2} - 1) = \frac{3}{8}\sqrt{2} + \frac{1}{8}\ln(\sqrt{2} - 1)
\end{aligned}$$

34. $\displaystyle\int_4^6 \frac{dt}{(t-3)(t+4)}$

SOLUTION The partial fraction decomposition takes the form

$$\frac{1}{(t-3)(t+4)} = \frac{A}{t-3} + \frac{B}{t+4}.$$

Clearing denominators gives us

$$1 = A(t+4) + B(t-3) = (A+B)t + 4A - 3B.$$

Setting $t = 3$ then yields $A = \frac{1}{7}$, while setting $t = -4$ yields $B = -\frac{1}{7}$. Hence,

$$\int_4^6 \frac{dt}{(t-3)(t+4)} = \frac{1}{7}\int_4^6 \frac{dt}{t-3} - \frac{1}{7}\int_4^6 \frac{dt}{t+4} = \left(\frac{1}{7}\ln|t-3| - \frac{1}{7}\ln|t+4|\right)\Big|_4^6$$
$$= \left(\frac{1}{7}\ln\left|\frac{t-3}{t+4}\right|\right)\Big|_4^6 = \frac{1}{7}\left(\ln\frac{3}{10} - \ln\frac{1}{8}\right) = \frac{1}{7}\ln\frac{12}{5}$$

35. $\displaystyle\int \frac{dt}{(t-3)^2(t+4)}$

SOLUTION The partial fraction decomposition has the form

$$\frac{1}{(t-3)^2(t+4)} = \frac{A}{t+4} + \frac{B}{t-3} + \frac{C}{(t-3)^2}.$$

Clearing denominators gives us

$$1 = A(t-3)^2 + B(t-3)(t+4) + C(t+4).$$

Setting $t = 3$ then yields $C = \frac{1}{7}$, while setting $t = -4$ yields $A = \frac{1}{49}$. Lastly, setting $t = 0$ yields

$$1 = 9A - 12B + 4C \qquad \text{or} \qquad B = -\frac{1}{49}.$$

Hence,

$$\int \frac{dt}{(t-3)^2(t+4)} = \frac{1}{49}\int \frac{dt}{t+4} - \frac{1}{49}\int \frac{dt}{t-3} + \frac{1}{7}\int \frac{dt}{(t-3)^2}$$
$$= \frac{1}{49}\ln|t+4| - \frac{1}{49}\ln|t-3| + \frac{1}{7}\cdot\frac{-1}{t-3} + C = \frac{1}{49}\ln\left|\frac{t+4}{t-3}\right| - \frac{1}{7}\cdot\frac{1}{t-3} + C.$$

36. $\displaystyle\int \sqrt{x^2+9}\,dx$

SOLUTION Substitute $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$. Then

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = \sqrt{9\left(\tan^2\theta + 1\right)} = 3\sqrt{\sec^2\theta} = 3\sec\theta,$$

and

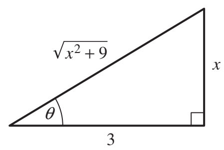
$$\int \sqrt{x^2+9}\,dx = \int 3\sec\theta \cdot 3\sec^2\theta\,d\theta = 9\int \sec^3\theta\,d\theta.$$

We use a reduction formula to compute the resulting integral:

$$\int \sqrt{x^2+9}\,dx = 9\int \sec^3\theta\,d\theta = 9\left(\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\,d\theta\right) = \frac{9\tan\theta\sec\theta}{2} + \frac{9}{2}\ln|\sec\theta + \tan\theta| + C.$$

We now return to the original variable x. Since $x = 3\tan\theta$, we have $\theta = \tan^{-1}\frac{x}{3}$. We also use the figure to obtain:

$$\int \sqrt{x^2+9}\,dx = \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{x^2+9}}{3} + \frac{9}{2}\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C = \frac{x\sqrt{x^2+9}}{2} + \frac{9}{2}\ln\left|\frac{x+\sqrt{x^2+9}}{3}\right| + C.$$



37. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}$

SOLUTION Substitute $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\, d\theta$. Then

$$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4\left(\sec^2\theta - 1\right)} = \sqrt{4\tan^2\theta} = 2\tan\theta,$$

and

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{2\sec\theta \cdot 2\tan\theta} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C.$$

Now, return to the original variable x. Since $x = 2\sec\theta$, we have $\sec\theta = \frac{x}{2}$ or $\theta = \sec^{-1}\frac{x}{2}$. Thus,

$$\int \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2}\sec^{-1}\frac{x}{2} + C.$$

38. $\displaystyle\int_8^{27} \frac{dx}{x + x^{2/3}}$

SOLUTION We rewrite the integrand:

$$\int_8^{27} \frac{dx}{x + x^{2/3}} = \int_8^{27} \frac{dx}{x^{2/3}\left(x^{1/3}+1\right)} = \int_8^{27} t\frac{x^{-2/3}\, dx}{1 + x^{1/3}}.$$

Now, use the substitution $u = 1 + x^{1/3}$, $du = \frac{1}{3}x^{-2/3}\, dx$. $x = 8$ corresponds to $u = 3$, and $x = 27$ corresponds to $u = 4$. Then

$$\int_8^{27} \frac{dx}{x + x^{2/3}} = \int_8^{27} \frac{x^{-2/3}\, dx}{1 + x^{1/3}} = 3\int_3^4 \frac{du}{u} = 3\left(\ln|u|\right)\Big|_3^4 = 3(\ln 4 - \ln 3)$$

39. $\displaystyle\int \frac{dx}{x^{3/2} + ax^{1/2}}$

SOLUTION Let $u = x^{1/2}$ or $x = u^2$. Then $dx = 2u\, du$ and

$$\int \frac{dx}{x^{3/2} + ax^{1/2}} = \int \frac{2u\, du}{u^3 + au} = 2\int \frac{du}{u^2 + a}.$$

If $a > 0$, then

$$\int \frac{dx}{x^{3/2} + ax^{1/2}} = 2\int \frac{du}{u^2 + a} = \frac{2}{\sqrt{a}}\tan^{-1}\left(\frac{u}{\sqrt{a}}\right) + C = \frac{2}{\sqrt{a}}\tan^{-1}\sqrt{\frac{x}{a}} + C.$$

If $a = 0$, then

$$\int \frac{dx}{x^{3/2}} = -\frac{2}{\sqrt{x}} + C.$$

Finally, if $a < 0$, then

$$\int \frac{du}{u^2 + a} = \int \frac{du}{u^2 - \left(\sqrt{-a}\right)^2},$$

and the partial fraction decomposition takes the form

$$\frac{1}{u^2 - \left(\sqrt{-a}\right)^2} = \frac{A}{u - \sqrt{-a}} + \frac{B}{u + \sqrt{-a}}.$$

Clearing denominators gives us

$$1 = A(u + \sqrt{-a}) + B(u - \sqrt{-a}).$$

Setting $u = \sqrt{-a}$ then yields $A = \frac{1}{2\sqrt{-a}}$, while setting $u = -\sqrt{-a}$ yields $B = -\frac{1}{2\sqrt{-a}}$. Hence,

$$\begin{aligned}\int \frac{dx}{x^{3/2} + ax^{1/2}} &= 2\int \frac{du}{u^2 + a} = \frac{1}{\sqrt{-a}}\int \frac{du}{u - \sqrt{-a}} - \frac{1}{\sqrt{-a}}\int \frac{du}{u + \sqrt{-a}} \\ &= \frac{1}{\sqrt{-a}}\ln|u - \sqrt{-a}| - \frac{1}{\sqrt{-a}}\ln\left|u + \sqrt{-a}\right| + C \\ &= \frac{1}{\sqrt{-a}}\ln\left|\frac{u - \sqrt{-a}}{u + \sqrt{-a}}\right| + C = \frac{1}{\sqrt{-a}}\ln\left|\frac{\sqrt{x} - \sqrt{-a}}{\sqrt{x} + \sqrt{-a}}\right| + C.\end{aligned}$$

In summary,

$$\int \frac{dx}{x^{3/2}+ax^{1/2}} = \begin{cases} \frac{2}{\sqrt{a}}\tan^{-1}\sqrt{\frac{x}{a}}+C & a>0 \\ \frac{1}{\sqrt{-a}}\ln\left|\frac{\sqrt{x}-\sqrt{-a}}{\sqrt{x}+\sqrt{-a}}\right|+C & a<0 \\ -\frac{2}{\sqrt{x}}+C & a=0 \end{cases}$$

40. $\int \frac{dx}{(x-b)^2+4}$

SOLUTION Substitute $u = x-b$, $du = dx$. Then

$$\int \frac{dx}{(x-b)^2+4} = \int \frac{du}{u^2+4} = \frac{1}{2}\tan^{-1}\frac{u}{2}+C = \frac{1}{2}\tan^{-1}\left(\frac{x-b}{2}\right)+C.$$

41. $\int \frac{(x^2-x)\,dx}{(x+2)^3}$

SOLUTION The partial fraction decomposition has the form

$$\frac{x^2-x}{(x+2)^3} = \frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3}.$$

Clearing denominators gives us

$$x^2-x = A(x+2)^2+B(x+2)+C.$$

Setting $x=-2$ then yields $C=6$. Equating x^2-coefficients gives us $A=1$, and equating x-coefficients yields $4A+B=-1$, or $B=-5$. Thus,

$$\int \frac{x^2-x}{(x+2)^3}\,dx = \int \frac{dx}{x+2}+\int \frac{-5\,dx}{(x+2)^2}+\int \frac{6\,dx}{(x+2)^3} = \ln|x+2|+\frac{5}{x+2}-\frac{3}{(x+2)^2}+C.$$

42. $\int \frac{(7x^2+x)\,dx}{(x-2)(2x+1)(x+1)}$

SOLUTION The partial fraction decomposition has the form

$$\frac{7x^2+x}{(x-2)(2x+1)(x+1)} = \frac{A}{x-2}+\frac{B}{2x+1}+\frac{C}{x+1}.$$

Clearing denominators gives us

$$7x^2+x = A(2x+1)(x+1)+B(x-2)(x+1)+C(x-2)(2x+1).$$

Setting $x=2$ then yields $A=2$, while setting $x=-\frac{1}{2}$ yields $B=-1$, and setting $x=-1$ yields $C=2$. Hence,

$$\int \frac{7x^2+x}{(x-2)(2x+1)(x+1)}\,dx = 2\int \frac{dx}{x-2}-\int \frac{dx}{2x+1}+2\int \frac{dx}{x+1}$$

$$= 2\ln|x-2|-\frac{1}{2}\ln|2x+1|+2\ln|x+1|+C.$$

43. $\int \frac{16\,dx}{(x-2)^2(x^2+4)}$

SOLUTION The partial fraction decomposition has the form

$$\frac{16}{(x-2)^2\left(x^2+4\right)} = \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{Cx+D}{x^2+4}.$$

Clearing denominators gives us

$$16 = A(x-2)\left(x^2+4\right)+B\left(x^2+4\right)+(Cx+D)(x-2)^2.$$

Setting $x=2$ then yields $B=2$. With $B=2$,

$$16 = A\left(x^3-2x^2+4x-8\right)+2\left(x^2+4\right)+Cx^3+(D-4C)x^2+(4C-4D)x+4D$$

$$16 = (A+C)x^3+(-2A+2+D-4C)\,x^2+(4A+4C-4D)x+(-8A+8+4D)$$

Equating coefficients of like powers of x now gives us the system of equations

$$\begin{aligned} A + C &= 0 \\ -2A - 4C + D + 2 &= 0 \\ 4A + 4C - 4D &= 0 \\ -8A + 4D + 8 &= 1 \end{aligned}$$

whose solution is

$$A = -1,\ C = 1,\ D = 0.$$

Thus,

$$\begin{aligned} \int \frac{dx}{(x-2)^2\left(x^2+4\right)} &= -\int \frac{dx}{x-2} + 2\int \frac{dx}{(x-2)^2} + \int \frac{x}{x^2+4}\,dx \\ &= -\ln|x-2| - 2\frac{1}{x-2} + \frac{1}{2}\ln\left(x^2+4\right) + C. \end{aligned}$$

44. $\displaystyle\int \frac{dx}{(x^2+25)^2}$

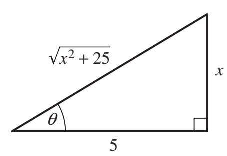
SOLUTION Use the trigonometric substitution $x = 5\tan\theta$, $dx = 5\sec^2\theta\,d\theta$,

$$x^2 + 25 = (5\tan\theta)^2 + 25 = 25\left(\tan^2\theta + 1\right) = 25\sec^2\theta.$$

Then,

$$\begin{aligned} \int \frac{dx}{\left(x^2+25\right)^2} &= \int \frac{5\sec^2\theta\,d\theta}{\left(25\sec^2\theta\right)^2} = \int \frac{d\theta}{125\sec^2\theta} = \frac{1}{125}\int \cos^2\theta\,d\theta \\ &= \frac{1}{125}\left(\frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\theta\right) + C = \frac{1}{250}(\cos\theta\sin\theta + \theta) + C. \end{aligned}$$

To return to the original variable x we use the relation $x = 5\tan\theta$ and the accompanying figure.



Thus,

$$\int \frac{dx}{\left(x^2+25\right)^2} = \frac{1}{250}\left(\frac{5}{\sqrt{x^2+25}} \cdot \frac{x}{\sqrt{x^2+25}} + \tan^{-1}\left(\frac{x}{5}\right)\right) + C = \frac{1}{50}\frac{x}{x^2+25} + \frac{1}{250}\tan^{-1}\left(\frac{x}{5}\right) + C.$$

45. $\displaystyle\int \frac{dx}{x^2+8x+25}$

SOLUTION Complete the square to rewrite the denominator as

$$x^2 + 8x + 25 = (x+4)^2 + 9.$$

Now, let $u = x + 4$, $du = dx$. Then,

$$\int \frac{dx}{x^2+8x+25} = \int \frac{du}{u^2+9} = \frac{1}{3}\tan^{-1}\frac{u}{3} + C = \frac{1}{3}\tan^{-1}\left(\frac{x+4}{3}\right) + C.$$

46. $\displaystyle\int \frac{dx}{x^2+8x+4}$

SOLUTION Use the method of partial fractions. To facilitate the computations we first complete the square in the denominator:

$$\frac{1}{x^2+8x+4} = \frac{1}{(x+4)^2 - 12}.$$

Now we substitute $t = x + 4$. Then $dt = dx$ and

$$\int \frac{dx}{x^2 + 8x + 4} = \int \frac{dt}{t^2 - 12} = \int \frac{dt}{\left(t - 2\sqrt{3}\right)\left(t + 2\sqrt{3}\right)}.$$

We use the following partial fraction decomposition of the integrand:

$$\frac{1}{\left(t - 2\sqrt{3}\right)\left(t + 2\sqrt{3}\right)} = \frac{A}{t - 2\sqrt{3}} + \frac{B}{t + 2\sqrt{3}}.$$

Clearing denominators gives us

$$1 = A\left(t + 2\sqrt{3}\right) + B\left(t - 2\sqrt{3}\right).$$

Setting $t = 2\sqrt{3}$ then yields $A = \frac{1}{4\sqrt{3}}$, while setting $t = -2\sqrt{3}$ yields $B = -\frac{1}{4\sqrt{3}}$. Hence,

$$\int \frac{dx}{x^2 + 8x + 4} = \frac{1}{4\sqrt{3}} \int \frac{dt}{t - 2\sqrt{3}} - \frac{1}{4\sqrt{3}} \int \frac{dt}{t + 2\sqrt{3}} = \frac{1}{4\sqrt{3}} \ln|t - 2\sqrt{3}| - \frac{1}{4\sqrt{3}} \ln|t + 2\sqrt{3}| + C$$

$$= \frac{1}{4\sqrt{3}} \ln\left|\frac{t - 2\sqrt{3}}{t + 2\sqrt{3}}\right| + C = \frac{1}{4\sqrt{3}} \ln\left|\frac{x + 42\sqrt{3}}{x + 4 + 2\sqrt{3}}\right| + C.$$

47. $\int \frac{(x^2 - x)\,dx}{(x + 2)^3}$

SOLUTION The partial fraction decomposition has the form

$$\frac{x^2 - x}{(x + 2)^3} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2} + \frac{C}{(x + 2)^3}.$$

Clearing denominators gives us

$$x^2 - x = A(x + 2)^2 + B(x + 2) + C.$$

Setting $x = -2$ then yields $C = 6$. Equating x^2-coefficients gives us $A = 1$, and equating x-coefficients yields $4A + B = -1$, or $B = -5$. Thus,

$$\int \frac{x^2 - x}{(x + 2)^3}\,dx = \int \frac{dx}{x + 2} + \int \frac{-5\,dx}{(x + 2)^2} + \int \frac{6\,dx}{(x + 2)^3} = \ln|x + 2| + \frac{5}{x + 2} - \frac{3}{(x + 2)^2} + C.$$

48. $\int_0^1 t^2\sqrt{1 - t^2}\,dt$

SOLUTION First compute the indefinite integral by using the substitution $t = \sin\theta$, $dt = \cos\theta\,d\theta$. We have

$$\sqrt{1 - t^2} = \sqrt{1 - \sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta,$$

and

$$\int t^2\sqrt{1 - t^2}\,dt = \int \sin^2\theta\cos\theta\cos\theta\,d\theta = \int \sin^2\theta\cos^2\theta\,d\theta$$

$$= \int (1 - \cos^2\theta)\cos^2\theta\,d\theta = \int \cos^2\theta\,d\theta - \int \cos^4\theta\,d\theta$$

$$= \int \cos^2\theta\,d\theta - \left(\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\int \cos^2\theta\,d\theta\right)$$

$$= -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{4}\int \cos^2\theta\,d\theta$$

$$= -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{4}\left(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right) + C$$

$$= -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{8}\cos\theta\sin\theta + \frac{1}{8}\theta + C.$$

Now, return to the original variable t. Since $t = \sin\theta$, $\cos\theta = \sqrt{1 - t^2}$ and

$$\int t^2\sqrt{1 - t^2}\,dt = -\frac{t(1 - t^2)^{3/2}}{4} + \frac{t\sqrt{1 - t^2}}{8} + \frac{\sin^{-1} t}{8} + C = \frac{t^3\sqrt{1 - t^2}}{4} + \frac{\sin^{-1} t}{8} - \frac{t\sqrt{1 - t^2}}{8} + C.$$

Then

$$\int_0^1 t^2\sqrt{1-t^2}\,dt = \left(\frac{t^3\sqrt{1-t^2}}{4} + \frac{\sin^{-1} t}{8} - \frac{t\sqrt{1-t^2}}{8}\right)\Bigg|_0^1$$

$$= 0 + \frac{1}{8}\sin^{-1} 1 - 0 - 0 + \frac{1}{8}\sin^{-1} 0 + 0 = \frac{\sin^{-1} 1}{8} = \frac{\pi}{16}$$

49. $\displaystyle\int \frac{dx}{x^4\sqrt{x^2+4}}$

SOLUTION Substitute $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$. Then

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta+4} = \sqrt{4\left(\tan^2\theta+1\right)} = 2\sqrt{\sec^2\theta} = 2\sec\theta,$$

and

$$\int \frac{dx}{x^4\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\,d\theta}{16\tan^4\theta\cdot 2\sec\theta} = \int \frac{\sec\theta\,d\theta}{16\tan^4\theta}.$$

We have

$$\frac{\sec\theta}{\tan^4\theta} = \frac{\cos^3\theta}{\sin^4\theta}.$$

Hence,

$$\int \frac{dx}{x^4\sqrt{x^2+4}} = \frac{1}{16}\int \frac{\cos^3\theta\,d\theta}{\sin^4\theta} = \frac{1}{16}\int \frac{\cos^2\theta\cos\theta\,d\theta}{\sin^4\theta} = \frac{1}{16}\int \frac{\left(1-\sin^2\theta\right)\cos\theta\,d\theta}{\sin^4\theta}.$$

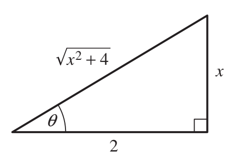
Now substitute $u = \sin\theta$ and $du = \cos\theta\,d\theta$ to obtain

$$\int \frac{dx}{x^4\sqrt{x^2+4}} = \frac{1}{16}\int \frac{1-u^2}{u^4}\,du = \frac{1}{16}\int \left(u^{-4}-u^{-2}\right)du = -\frac{1}{48u^3} + \frac{1}{16}\frac{1}{u} + C$$

$$= -\frac{1}{48}\cdot\frac{1}{\sin^3\theta} + \frac{1}{16}\frac{1}{\sin\theta} + C = -\frac{1}{48}\csc^3\theta + \frac{1}{16}\csc\theta + C.$$

Finally, return to the original to the original variable x using the relation $x = 2\tan\theta$ and the figure below.

$$\int \frac{dx}{x^4\sqrt{x^2+4}} = -\frac{1}{48}\left(\frac{\sqrt{x^2+4}}{x}\right)^3 + \frac{1}{16}\frac{\sqrt{x^2+4}}{x} + C = -\frac{\left(x^2+4\right)^{3/2}}{48x^3} + \frac{\sqrt{x^2+4}}{16x} + C.$$



50. $\displaystyle\int \frac{dx}{(x^2+5)^{3/2}}$

SOLUTION Substitute $x = \sqrt{5}\tan\theta$. Then $dx = \sqrt{5}\sec^2\theta\,d\theta$,

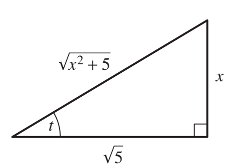
$$x^2+5 = 5\tan^2\theta + 5 = 5(\tan^2\theta+1) = 5\sec^2\theta,$$

and

$$\int \frac{dx}{\left(x^2+5\right)^{3/2}} = \frac{1}{5}\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{5}\int \cos\theta\,d\theta = \frac{1}{5}\sin\theta + C.$$

We now return to the original variable x using the relation $x = \sqrt{5}\tan\theta$ and the figure below. Thus,

$$\int \frac{dx}{\left(x^2+5\right)^{3/2}} = \frac{1}{5}\cdot\frac{x}{\sqrt{x^2+5}} + C.$$



51. $\int (x+1)e^{4-3x}\,dx$

SOLUTION We compute the integral using Integration by Parts with $u = x+1$ and $v' = e^{4-3x}$. Then $u' = 1$, $v = -\frac{1}{3}e^{4-3x}$ and

$$\int (x+1)e^{4-3x}\,dx = -\frac{1}{3}(x+1)e^{4-3x} + \frac{1}{3}\int e^{4-3x}\,dx = -\frac{1}{3}(x+1)e^{4-3x} + \frac{1}{3}\cdot\left(-\frac{1}{3}\right)e^{4-3x} + C$$
$$= -\frac{1}{9}e^{4-3x}(3x+4) + C.$$

52. $\int x^{-2}\tan^{-1}x\,dx$

SOLUTION We use Integration by Parts with $u = \tan^{-1}x$ and $v' = x^{-2}$. Then $u' = \frac{1}{1+x^2}$, $v = -x^{-1}$ and

$$\int x^{-2}\tan^{-1}x\,dx = -\frac{\tan^{-1}x}{x} + \int \frac{dx}{x\left(1+x^2\right)}.$$

For the remaining integral, the partial fraction decomposition takes the form

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}.$$

Clearing denominators gives us

$$1 = A(1+x^2) + (Bx+C)x.$$

Setting $x = 0$ then yields $A = 1$. Next, equating the x^2-coefficients gives

$$0 = A + B \qquad \text{so} \qquad B = -1,$$

while equating x-coefficients gives $C = 0$. Hence,

$$\frac{1}{x\left(1+x^2\right)} = \frac{1}{x} - \frac{x}{1+x^2},$$

and

$$\int \frac{dx}{x\left(1+x^2\right)} = \int \frac{1}{x}\,dx - \int \frac{x\,dx}{1+x^2} = \ln|x| - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

Therefore,

$$\int x^{-2}\tan^{-1}x\,dx = -\frac{\tan^{-1}x}{x} + \ln|x| - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

53. $\int x^3\cos(x^2)\,dx$

SOLUTION Substitute $t = x^2$, $dt = 2x\,dx$. Then

$$\int x^3\cos\left(x^2\right)\,dx = \frac{1}{2}\int t\cos t\,dt.$$

We compute the resulting integral using Integration by Parts with $u = t$ and $v' = \cos t$. Then $u' = 1$, $v = \sin t$ and

$$\int t\cos t\,dt = t\sin t - \int \sin t\,dt = t\sin t + \cos t + C.$$

Thus,

$$\int x^3\cos\left(x^2\right)\,dx = \frac{1}{2}x^2\sin x^2 + \frac{1}{2}\cos x^2 + C.$$

54. $\int x^2(\ln x)^2\,dx$

SOLUTION We use Integration by Parts with $u = (\ln x)^2$ and $v' = x^2$. Then $u' = \frac{2\ln x}{x}$, $v = \frac{x^3}{3}$ and

$$\int x^2(\ln x)^2dx = \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\int x^2\ln x\,dx.$$

To calculate the resulting integral, we again use Integration by Parts, this time with $u = \ln x$ and $v' = x^2$. Then, $u' = \frac{1}{x}$, $v = \frac{x^3}{3}$, and

$$\int x^2 \ln x\, dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2\, dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C.$$

Finally,

$$\int x^2 (\ln x)^2 dx = \frac{x^3}{3} (\ln x)^2 - \frac{2}{3} \left(\frac{x^3}{3} \ln x - \frac{x^3}{9} \right) + C = \frac{x^3}{3} \left((\ln x)^2 - \frac{2}{3} \ln x + \frac{2}{9} \right) + C.$$

55. $\int x \tanh^{-1} x\, dx$

SOLUTION We use Integration by Parts with $u = \tanh^{-1} x$ and $v' = x$. Then $u' = \frac{1}{1-x^2}$, $v = \frac{x^2}{2}$ and

$$\int x \tanh^{-1} x\, dx = \frac{x^2}{2} \tanh^{-1} x - \frac{1}{2} \int \frac{x^2}{1 - x^2}\, dx.$$

Now

$$\frac{x^2}{1 - x^2} = \frac{x^2 - 1 + 1}{1 - x^2} = -1 + \frac{1}{1 - x^2},$$

and the partial fraction decomposition for the remaining fraction takes the form

$$\frac{1}{1 - x^2} = \frac{A}{1 - x} + \frac{B}{1 + x}.$$

Clearing denominators gives us

$$1 = A(1 + x) + B(1 - x).$$

Setting $x = 1$ then yields $A = \frac{1}{2}$, while setting $x = -1$ yields $B = \frac{1}{2}$. Thus,

$$\int \frac{x^2}{1 - x^2} = -\int dx + \frac{1}{2} \int \frac{1}{1 - x}\, dx + \frac{1}{2} \int \frac{1}{1 + x}\, dx$$
$$= -x - \frac{1}{2} \ln |1 - x| + \frac{1}{2} \ln |1 + x| + C = -x + \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| + C.$$

Therefore,

$$\int x \tanh^{-1} x\, dx = \frac{x^2}{2} \tanh^{-1} x - \frac{1}{2} \left(-x + \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| \right) + C = \frac{x^2}{2} \tanh^{-1} x + \frac{x}{2} - \frac{1}{4} \ln \left| \frac{1 + x}{1 - x} \right| + C.$$

56. $\int \frac{\tan^{-1} t\, dt}{1 + t^2}$

SOLUTION Substitute $u = \tan^{-1} t$. Then, $du = \frac{dt}{1+t^2}$ and

$$\int \frac{\tan^{-1} t\, dt}{1 + t^2} = \int u\, du = \frac{1}{2} u^2 + C = \frac{1}{2} \left(\tan^{-1} t \right)^2 + C.$$

57. $\int \ln(x^2 + 9)\, dx$

SOLUTION We compute the integral using Integration by Parts with $u = \ln \left(x^2 + 9 \right)$ and $v' = 1$. Then $u' = \frac{2x}{x^2+9}$, $v = x$, and

$$\int \ln \left(x^2 + 9 \right)\, dx = x \ln \left(x^2 + 9 \right) - \int \frac{2x^2}{x^2 + 9}\, dx.$$

To compute this integral we write:

$$\frac{x^2}{x^2 + 9} = \frac{\left(x^2 + 9 \right) - 9}{x^2 + 9} = 1 - \frac{9}{x^2 + 9};$$

hence,

$$\int \frac{x^2}{x^2+9}\,dx = \int 1\,dx - 9\int \frac{dx}{x^2+9} = x - 3\tan^{-1}\frac{x}{3} + C.$$

Therefore,

$$\int \ln\left(x^2+9\right)\,dx = x\ln\left(x^2+9\right) - 2x + 6\tan^{-1}\left(\frac{x}{3}\right) + C.$$

58. $\int (\sin x)(\cosh x)\,dx$

SOLUTION We compute the integral using Integration by Parts with $u = \sin x$ and $v' = \cosh x$. Then $u' = \cos x$, $v = \sinh x$ and

$$\int \sin x \cosh x\,dx = \sin x \sinh x - \int \cos x \sinh x\,dx.$$

We compute the resulting integral using Integration by Parts, this time with $u = \cos x$ and $v' = \sinh x$. Then $u' = -\sin x$, $v = \cosh x$ and

$$\int \cos x \sinh x\,dx = \cos x \cosh x + \int \sin x \cosh x\,dx.$$

Therefore,

$$\int \sin x \cosh x\,dx = \sin x \sinh x - \cos x \cosh x - \int \sin x \cosh x\,dx.$$

Solving for $\int (\sin x)(\cosh x)\,dx$, we find

$$2\int \sin x \cosh x\,dx = \sin x \sinh x - \cos x \cosh x + C$$

$$\int \sin x \cosh x\,dx = \frac{1}{2}\sin x \sinh x - \frac{1}{2}\cos x \cosh x + C$$

59. $\int_0^1 \cosh 2t\,dt$

SOLUTION $\int_0^1 \cosh 2t\,dt = \frac{1}{2}\sinh 2t\Big|_0^1 = \frac{1}{2}\sinh 2.$

60. $\int \sinh^3 x \cosh x\,dx$

SOLUTION Let $u = \sinh x$. Then $du = \cosh x\,dx$ and

$$\int \sinh^3 x \cosh x\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\sinh^4 x + C.$$

61. $\int \coth^2(1-4t)\,dt$

SOLUTION $\int \coth^2(1-4t)\,dt = \int \left(1 + \operatorname{csch}^2(1-4t)\right)\,dt = t + \frac{1}{4}\coth(1-4t) + C.$

62. $\int_{-0.3}^{0.3} \frac{dx}{1-x^2}$

SOLUTION $\int_{-0.3}^{0.3} \frac{dx}{1-x^2} = \tanh^{-1} x\Big|_{-0.3}^{0.3} = 2\tanh^{-1}(0.3).$

63. $\int_0^{3\sqrt{3}/2} \frac{dx}{\sqrt{9-x^2}}$

SOLUTION $\int_0^{3\sqrt{3}/2} \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3}\Big|_0^{3\sqrt{3}/2} = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}.$

64. $\displaystyle\int \frac{\sqrt{x^2+1}\,dx}{x^2}$

SOLUTION Let $x = \sinh t$. Then $dx = \cosh t\,dt$ and

$$\int \frac{\sqrt{x^2+1}\,dx}{x^2} = \int \frac{\cosh^2 t}{\sinh^2 t}\,dt = \int \coth^2 t\,dt = \int (1+\operatorname{csch}^2 t)\,dt = t - \coth t + C$$

$$= \sinh^{-1} x - \frac{\sqrt{x^2+1}}{x} + C.$$

65. Use the substitution $u = \tanh t$ to evaluate $\displaystyle\int \frac{dt}{\cosh^2 t + \sinh^2 t}$.

SOLUTION Let $u = \tanh t$. Then $du = \operatorname{sech}^2 t\,dt$ and

$$\int \frac{dt}{\cosh^2 t + \sinh^2 t} = \int \frac{\operatorname{sech}^2 t}{1+\tanh^2 t}\,dt = \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1}(\tanh x) + C.$$

66. Find the volume obtained by rotating the region enclosed by $y = \ln x$ and $y = (\ln x)^2$ about the y-axis.

SOLUTION The curves meet at $(1, 0)$ and at $(e, 1)$. We compute the volume of the solid using the method of cylindrical shells:

$$V = \int_1^e 2\pi x \cdot (\ln x - (\ln x)^2)\,dx = 2\pi \int_1^e x \ln x\,dx - 2\pi \int_0^1 x(\ln x)^2\,dx$$

For the second integral, use integration by parts, with $u = (\ln x)^2$ and $v' = x$, so that $u' = \frac{2\ln x}{x}$ and $v = \frac{1}{2}x^2$. Then

$$V = 2\pi \int_1^e x \ln x\,dx - 2\pi \left(\frac{1}{2}x^2(\ln x)^2\Big|_1^e - \int_1^e x \ln x\,dx\right) = -\pi e^2 + 4\pi \int_1^e x \ln x\,dx$$

Again apply integration by parts, with $u = \ln x$ and $v' = x$, so that $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$. Then

$$V = -\pi e^2 + 4\pi \int_1^e x \ln x\,dx = -\pi e^2 + 4\pi \left(\frac{1}{2}x^2 \ln x\Big|_1^e - \frac{1}{2}\int_1^e x\,dx\right) = -\pi e^2 + 4\pi \left(\frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}\right) = \pi$$

67. Let $I_n = \displaystyle\int \frac{x^n\,dx}{x^2+1}$.

(a) Prove that $I_n = \dfrac{x^{n-1}}{n-1} - I_{n-2}$.

(b) Use (a) to calculate I_n for $0 \le n \le 5$.

(c) Show that, in general,

$$I_{2n+1} = \frac{x^{2n}}{2n} - \frac{x^{2n-2}}{2n-2} + \cdots + (-1)^{n-1}\frac{x^2}{2} + (-1)^n \frac{1}{2}\ln(x^2+1) + C$$

$$I_{2n} = \frac{x^{2n-1}}{2n-1} - \frac{x^{2n-3}}{2n-3} + \cdots + (-1)^{n-1}x + (-1)^n \tan^{-1} x + C$$

SOLUTION

(a) $\displaystyle I_n = \int \frac{x^n}{x^2+1}\,dx = \int \frac{x^{n-2}(x^2+1-1)}{x^2+1}\,dx = \int x^{n-2}\,dx - \int \frac{x^{n-2}}{x^2+1}\,dx = \frac{x^{n-1}}{n-1} - I_{n-2}.$

(b) First compute I_0 and I_1 directly:

$$I_0 = \int \frac{x^0\,dx}{x^2+1} = \int \frac{dx}{x^2+1} = \tan^{-1}x + C \quad \text{and} \quad I_1 = \int \frac{x\,dx}{x^2+1} = \frac{1}{2}\ln\left(x^2+1\right) + C.$$

We now use the equality obtained in part (a) to compute I_2, I_3, I_4 and I_5:

$$I_2 = \frac{x^{2-1}}{2-1} - I_{2-2} = x - I_0 = x - \tan^{-1}x + C;$$

$$I_3 = \frac{x^{3-1}}{3-1} - I_{3-2} = \frac{x^2}{2} - I_1 = \frac{x^2}{2} - \frac{1}{2}\ln\left(x^2+1\right) + C;$$

$$I_4 = \frac{x^{4-1}}{4-1} - I_{4-2} = \frac{x^3}{3} - I_2 = \frac{x^3}{3} - \left(x - \tan^{-1}x\right) + C = \frac{x^3}{3} - x + \tan^{-1}x + C;$$

$$I_5 = \frac{x^{5-1}}{5-1} - I_{5-2} = \frac{x^4}{4} - I_3 = \frac{x^4}{4} - \left(\frac{x^2}{2} - \frac{1}{2}\ln\left(x^2+1\right)\right) + C = \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\ln\left(x^2+1\right) + C.$$

(c) We prove the two identities using mathematical induction. We first prove that for $n \geq 1$:

$$I_{2n+1} = \frac{x^{2n}}{2n} - \frac{x^{2n-2}}{2n-2} + \cdots + (-1)^n \cdot \frac{1}{2} \ln\left(x^2+1\right) + C.$$

We verify the equality for $n = 1$. Setting $n = 1$, we find

$$I_3 = \frac{x^2}{2} + (-1)^1 \cdot \frac{1}{2} \ln\left(x^2+1\right) + C = \frac{x^2}{2} - \frac{1}{2} \ln\left(x^2+1\right) + C,$$

which agrees with the value obtained in part (b). We now assume that for $n = k$:

$$I_{2k+1} = \frac{x^{2k}}{2k} - \frac{x^{2k-2}}{2k-2} + \cdots + (-1)^k \cdot \frac{1}{2} \ln\left(x^2+1\right) + C.$$

We use this assumption to prove the equality for $n = k + 1$. By part (a) and the induction hypothesis

$$\begin{aligned} I_{2k+3} = \frac{x^{2k+2}}{2k+2} - I_{2k+1} &= \frac{x^{2k+2}}{2k+2} - \frac{x^{2k}}{2k} + \frac{x^{2k-2}}{2k-2} - \cdots - (-1)^k \cdot \frac{1}{2} \ln\left(x^2+1\right) + C \\ &= \frac{x^{2k+2}}{2k+2} - \frac{x^{2k}}{2k} + \cdots + (-1)^{k+1} \cdot \frac{1}{2} \ln\left(x^2+1\right) + C \end{aligned}$$

as required. We now prove the second identity for $n \geq 1$:

$$I_{2n} = \frac{x^{2n-1}}{2n-1} - \frac{x^{2n-3}}{2n-3} + \cdots + (-1)^n \tan^{-1} x + C.$$

We verify this equality for $n = 1$:

$$I_2 = x - \tan^{-1} x + C,$$

which agrees with the value obtained in part (b). We now assume that for $n = k$

$$I_{2k} = \frac{x^{2k-1}}{2k-1} - \frac{x^{2k-3}}{2k-3} + \cdots + (-1)^k \tan^{-1} x + C.$$

We use this assumption to prove the equality for $n = k + 1$. By part (a) and the induction hypothesis

$$\begin{aligned} I_{2k+2} = \frac{x^{2k+1}}{2k+1} - I_{2k} &= \frac{x^{2k+1}}{2k+1} - \frac{x^{2k-1}}{2k-1} + \frac{x^{2k-3}}{2k-3} - \cdots - (-1)^k \cdot \tan^{-1} x + C \\ &= \frac{x^{2k+1}}{2k+1} - \frac{x^{2k-1}}{2k-1} + \cdots + (-1)^{k+1} \cdot \tan^{-1} x + C \end{aligned}$$

as required.

68. Let $J_n = \int x^n e^{-x^2/2}\,dx$.

(a) Show that $J_1 = -e^{-x^2/2}$.

(b) Prove that $J_n = -x^{n-1} e^{-x^2/2} + (n-1) J_{n-2}$.

(c) Use (a) and (b) to compute J_3 and J_5.

SOLUTION

(a) Let $u = -\frac{x^2}{2}$. Then $du = -x\,dx$ and

$$J_1 = \int x e^{-x^2/2}\,dx = -\int e^u\,du = -e^u + C = -e^{-x^2/2} + C.$$

(b) Using Integration by Parts with $u = x^{n-1}$ and $v' = x e^{-x^2/2}$, we find

$$J_n = -x^{n-1} e^{-x^2/2} + (n-1) \int x^{n-2} e^{-x^2/2}\,dx = -x^{n-1} e^{-x^2/2} + (n-1) J_{n-2}.$$

(c) Using the results from parts (a) and (b),

$$\begin{aligned} J_3 &= -x^{3-1} e^{-x^2/2} + (3-1) J_{3-2} = -x^2 e^{-x^2/2} + 2J_1 \\ &= -x^2 e^{-x^2/2} - 2e^{-x^2/2} + C = -e^{-x^2/2}(x^2+2) + C \end{aligned}$$

and then

$$J_5 = -x^{5-1}e^{-x^2/2} + (5-1)J_{5-2} = -x^4e^{-x^2/2} + 4J_3$$
$$= -x^4e^{-x^2/2} - 4e^{-x^2/2}(x^2+2) + C = -e^{-x^2/2}(x^4 + 4x^2 + 8) + C.$$

69. Compute $p(X \le 1)$, where X is a continuous random variable with probability density $p(x) = \dfrac{1}{\pi(x^2+1)}$.

SOLUTION

$$P(X \le 1) = \int_{-\infty}^{1} p(x)\,dx = \frac{1}{\pi}\int_{-\infty}^{1} \frac{1}{x^2+1}\,dx = \frac{1}{\pi}\tan^{-1}x\Big|_{-\infty}^{1} = \frac{1}{\pi}\cdot\left(\frac{\pi}{4} - \frac{-\pi}{2}\right) = \frac{3}{4}$$

70. Show that $p(x) = \frac{1}{4}e^{-x/2} + \frac{1}{6}e^{-x/3}$ is a probability density on $[0,\infty)$ and find its mean.

SOLUTION To show that $p(x)$ is a probability density, we must show that its integral over $[0,\infty)$ is 1:

$$\int_0^\infty p(x)\,dx = \int_0^\infty \left(\frac{1}{4}e^{-x/2} + \frac{1}{6}e^{-x/3}\right)dx = \left(-\frac{1}{2}e^{-x/2} - \frac{1}{2}e^{-x/3}\right)\Big|_0^\infty = 0 + 0 + \frac{1}{2} + \frac{1}{2} = 1$$

The mean of $p(x)$ is

$$\mu = \int_0^\infty xp(x)\,dx = \int_0^\infty \left(\frac{1}{4}xe^{-x/2} + \frac{1}{6}xe^{-x/3}\right)dx$$

Now, for a positive constant a, using integration by parts with $u = x$, $v' = e^{-x/a}$, we have $u' = 1$, $v = -ae^{-x/a}$, and

$$\int_0^\infty xe^{-x/a}\,dx = -axe^{-x/a}\Big|_0^\infty + a\int_0^\infty e^{-x/a}\,dx = -a^2\left(e^{-x/a}\right)\Big|_0^\infty = a^2$$

so that

$$\mu = \frac{1}{4}\int_0^\infty xe^{-x/2}\,dx + \frac{1}{6}\int_0^\infty xe^{-x/3}\,dx = \frac{1}{4}\cdot 4 + \frac{1}{6}\cdot 9 = \frac{5}{2}$$

71. Find a constant C such that $p(x) = Cx^3e^{-x^2}$ is a probability density and compute $p(0 \le X \le 1)$.

SOLUTION We first find the indefinite integral of $p(x)$ using integration by parts, with $u = x^2$, $v' = xe^{-x^2}$, so that $u' = 2x$ and $v = -\frac{1}{2}e^{-x^2}$:

$$\int Cx^3e^{-x^2}\,dx = C\left(-\frac{1}{2}x^2e^{-x^2} + \int xe^{-x^2}\,dx\right) = C\left(-\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2}\right) = -\frac{C}{2}e^{-x^2}(x^2+1)$$

To determine the constant C, the value of the integral on the interval $[0,\infty)$ must be 1:

$$1 = \int_0^\infty Cx^3e^{-x^2}\,dx = -\frac{C}{2}e^{-x/2}(x^2+1)\Big|_0^\infty = -\frac{C}{2}\left(\lim_{R\to\infty}\frac{x^2+1}{e^{x/2}} - 1\right) = \frac{C}{2}$$

so that $C = 2$. Then

$$P(0 \le X \le 1) = \int_0^1 2x^3e^{-x^2}\,dx = -e^{-x^2}(x^2+1)\Big|_0^1 = 1 - 2e^{-1} \approx 0.13212$$

72. The interval between patient arrivals in an emergency room is a random variable with exponential density function $p(t) = 0.125e^{-0.125t}$ (t in minutes). What is the average time between patient arrivals? What is the probability of two patients arriving within 3 minutes of each other?

SOLUTION The mean of the distribution is (using integration by parts with $u = t$, $v' = 0.125e^{-0.125t}$):

$$\int_0^\infty tp(t)\,dt = \int_0^\infty 0.125te^{-0.125t}\,dt = te^{-0.125t}\Big|_0^\infty + \int_0^\infty e^{-0.125t}\,dt = -8e^{-0.125t}\Big|_0^\infty = 8$$

Since the distribution gives the waiting time between arrivals, it follows that the probability of two patients arriving within 3 minutes of each other is

$$\int_0^3 p(t)\,dt = \int_0^3 0.125e^{-0.125t}\,dt = -e^{-0.125t}\Big|_0^3 = 1 - e^{-0.375} \approx 1 - 0.68729 \approx 0.31271$$

73. Calculate the following probabilities, assuming that X is normally distributed with mean $\mu = 40$ and $\sigma = 5$.

(a) $p(X \geq 45)$ **(b)** $p(0 \leq X \leq 40)$

SOLUTION Let F be the standard normal cumulative distribution function. Then by Theorem 1 in Section 7.7,

(a)

$$p(X \geq 45) = 1 - p(X \leq 45) = 1 - F\left(\frac{45-40}{5}\right) = 1 - F(1) \approx 1 - 0.8413 \approx 0.1587$$

(b)

$$p(0 \leq X \leq 40) = p(X \leq 40) - p(X \leq 0) = F\left(\frac{40-40}{5}\right) - F\left(\frac{0-40}{5}\right)$$
$$= F(0) - F(-8) = \frac{1}{2} - F(-8) \approx \frac{1}{2} - 0 = \frac{1}{2}$$

Note that $p(X \leq 40)$ is exactly $\frac{1}{2}$ since 40 is the mean. Also, since -8 is so far to the left in the standard normal distribution, the probability of its occurrence is quite small (approximately 8×10^{-11}).

74. According to kinetic theory, the molecules of ordinary matter are in constant random motion. The energy E of a molecule is a random variable with density function $p(E) = \frac{1}{kT}e^{-E/(kT)}$, where T is the temperature (in kelvins) and k is Boltzmann's constant. Compute the *mean* kinetic energy $\overline{E}$ in terms of k and T.

SOLUTION By definition,

$$\int_0^\infty Ee^{-E/kT}\,dE = \lim_{R\to\infty}\int_0^R Ee^{-E/kT}\,dE.$$

We compute the definite integral using Integration by Parts with $u = E$, $v' = e^{-E/kT}$. Then $u' = 1$, $v = -kTe^{-E/kT}$ and

$$\int_0^R Ee^{-E/kT}\,dE = -kTe^{-E/kT}E\Big|_{E=0}^R + \int_0^R kTe^{-E/kT}\,dE = -kTe^{-R/kT}R - (kT)^2e^{-E/kT}\Big|_{E=0}^R$$
$$= -kTRe^{-R/kT} - \left(k^2T^2e^{-R/kT} - k^2T^2e^0\right) = k^2T^2 - kTRe^{-R/kT} - k^2T^2e^{-R/kT}.$$

We now let $R \to \infty$, obtaining:

$$\int_0^\infty Ee^{-E/RT}\,dE = \lim_{R\to\infty}\int_0^R Ee^{-E/RT}\,dE = \lim_{R\to\infty}\left(k^2T^2 - kTRe^{-R/kT} - k^2T^2e^{-R/kT}\right)$$
$$= k^2T^2 - kT\lim_{R\to\infty}Re^{-R/kT} - 0 = k^2T^2 - kT\lim_{R\to\infty}Re^{-R/kT}.$$

We compute the remaining limit using L'Hôpital's Rule:

$$\lim_{R\to\infty}Re^{-R/kT} = \lim_{R\to\infty}\frac{R}{e^{R/kT}} = \lim_{R\to\infty}\frac{\frac{dR}{dR}}{\frac{d}{dR}\left(e^{R/kT}\right)} = \lim_{R\to\infty}\frac{1}{\frac{1}{kT}e^{R/kT}} = 0.$$

Thus,

$$\int_0^\infty Ee^{-E/RT}\,dE = k^2T^2,$$

and

$$\overline{E} = \frac{1}{kT}\int_0^\infty Ee^{-E/kT}\,dE = \frac{1}{kT}\cdot k^2T^2 = kT.$$

In Exercises 75–84, determine whether the improper integral converges and, if so, evaluate it.

75. $\displaystyle\int_0^\infty \frac{dx}{(x+2)^2}$

SOLUTION

$$\int_0^\infty \frac{dx}{(x+2)^2} = \lim_{R\to\infty}\int_0^R \frac{dx}{(x+2)^2} = \lim_{R\to\infty} -\frac{1}{x+2}\Big|_0^R$$
$$= \lim_{R\to\infty}\left(-\frac{1}{R+2} + \frac{1}{0+2}\right) = \lim_{R\to\infty}\left(-\frac{1}{R+2} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}.$$

76. $\displaystyle\int_4^\infty \frac{dx}{x^{2/3}}$

SOLUTION The integral $\displaystyle\int_a^\infty \frac{dx}{x^p}$ $(a > 0)$ converges if $p > 1$ and diverges if $p \le 1$. Here, $p = \frac{2}{3} < 1$, hence the integral diverges.

77. $\displaystyle\int_0^4 \frac{dx}{x^{2/3}}$

SOLUTION

$$\int_0^4 \frac{dx}{x^{2/3}} = \lim_{R\to 0+} \int_R^4 \frac{dx}{x^{2/3}} = \lim_{R\to 0+} 3x^{1/3}\Big|_R^4 = \lim_{R\to 0+} \left(3\cdot 4^{1/3} - 3\cdot R^{1/3}\right) = 3\sqrt[3]{4}.$$

78. $\displaystyle\int_9^\infty \frac{dx}{x^{12/5}}$

SOLUTION

$$\int_9^\infty \frac{dx}{x^{12/5}} = \lim_{R\to\infty} \int_9^R \frac{dx}{x^{12/5}} = \lim_{R\to\infty} -\frac{5}{7}x^{-7/5}\Big|_9^R = \lim_{R\to\infty} \left(-\frac{5}{7}R^{-7/5} + \frac{5}{7}\cdot 9^{-7/5}\right)$$
$$= 0 + \frac{5}{7}\cdot 9^{-7/5} = \frac{5}{7\cdot 9\cdot 9^{2/5}} = \frac{5}{63\cdot 9^{2/5}}.$$

79. $\displaystyle\int_{-\infty}^0 \frac{dx}{x^2+1}$

SOLUTION

$$\int_{-\infty}^0 \frac{dx}{x^2+1} = \lim_{R\to-\infty} \int_R^0 \frac{dx}{x^2+1} = \lim_{R\to-\infty} \tan^{-1}x\Big|_R^0 = \lim_{R\to-\infty} \left(\tan^{-1}0 - \tan^{-1}R\right)$$
$$= \lim_{R\to-\infty} \left(-\tan^{-1}R\right) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

80. $\displaystyle\int_{-\infty}^9 e^{4x}\,dx$

SOLUTION

$$\int_{-\infty}^9 e^{4x}\,dx = \lim_{R\to-\infty} \int_R^9 e^{4x}\,dx = \lim_{R\to-\infty} \frac{1}{4}e^{4x}\Big|_R^9 = \lim_{R\to-\infty} \frac{1}{4}e^{36} - \frac{1}{4}e^{4R} = \frac{e^{36}}{4}.$$

81. $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$

SOLUTION

$$\int_0^{\pi/2} \cot\theta\,d\theta = \lim_{R\to 0+} \int_R^{\pi/2} \cot\theta\,d\theta = \lim_{R\to 0+} \ln|\sin\theta|\Big|_R^{\pi/2} = \lim_{R\to 0+} \left(\ln\left(\sin\frac{\pi}{2}\right) - \ln(\sin R)\right)$$
$$= \lim_{R\to 0+} (\ln 1 - \ln(\sin R)) = \lim_{R\to 0+} \ln\left(\frac{1}{\sin R}\right) = \infty.$$

We conclude that the improper integral diverges.

82. $\displaystyle\int_1^\infty \frac{dx}{(x+2)(2x+3)}$

SOLUTION First, evaluate the indefinite integral. The following partial fraction decomposition has the form

$$\frac{1}{(x+2)(2x+3)} = -\frac{1}{x+2} + \frac{2}{2x+3}.$$

Clearing denominators gives us

$$1 = A(2x+3) + B(x+2).$$

Setting $x = -2$ then yields $A = -1$, while setting $x = -\frac{3}{2}$ yields $B = 2$. Hence,

$$\int \frac{dx}{(x+2)(2x+3)} = -\int \frac{dx}{x+2} + 2\int \frac{dx}{2x+3} = -\ln|x+2| + \ln|2x+3| + C = \ln\left|\frac{2x+3}{x+2}\right| + C.$$

Now, for $R > 1$,

$$\int_1^R \frac{dx}{(x+2)(2x+3)} = \ln\left|\frac{2x+3}{x+2}\right|\Bigg|_1^R = \ln\frac{2R+3}{R+2} - \ln\frac{5}{3},$$

and

$$\int_1^\infty \frac{dx}{(x+2)(2x+3)} = \lim_{R\to\infty}\left(\ln\frac{2R+3}{R+2}\right) - \ln\frac{5}{3} = \ln 2 + \ln\frac{3}{5} = \ln\frac{6}{5}.$$

83. $\int_0^\infty (5+x)^{-1/3}\,dx$

SOLUTION

$$\int_0^\infty (5+x)^{-1/3}\,dx = \lim_{R\to\infty}\int_0^R (5+x)^{-1/3}\,dx = \lim_{R\to\infty}\frac{3}{2}(5+x)^{2/3}\Big|_0^R$$
$$= \lim_{R\to\infty}\left(\frac{3}{2}(5+R)^{2/3} - \frac{3}{2}5^{2/3}\right) = \infty.$$

We conclude that the improper integral diverges.

84. $\int_2^5 (5-x)^{-1/3}\,dx$

SOLUTION

$$\int_2^5 (5-x)^{-1/3}\,dx = \lim_{R\to 5-}\int_2^R (5-x)^{-1/3}\,dx = \lim_{R\to 5-} -\frac{3}{2}(5-x)^{2/3}\Big|_2^R$$
$$= \lim_{R\to 5-} -\frac{3}{2}\left((5-R)^{2/3} - 3^{2/3}\right) = -\frac{3}{2}\left(0 - 3^{2/3}\right) = \frac{3^{5/3}}{2}.$$

In Exercises 85–90, use the Comparison Test to determine whether the improper integral converges or diverges.

85. $\int_8^\infty \frac{dx}{x^2-4}$

SOLUTION For $x \ge 8$, $\frac{1}{2}x^2 \ge 4$, so that

$$-\frac{1}{2}x^2 \le -4$$
$$\frac{1}{2}x^2 \le x^2 - 4$$

and

$$\frac{1}{x^2-4} \le \frac{2}{x^2}.$$

Now, $\int_1^\infty \frac{dx}{x^2}$ converges, so $\int_8^\infty \frac{2}{x^2}\,dx$ also converges. Therefore, by the comparison test,

$$\int_8^\infty \frac{dx}{x^2-4} \text{ converges.}$$

86. $\int_8^\infty (\sin^2 x)e^{-x}\,dx$

SOLUTION The following inequality holds for all x,

$$0 \le \left(\sin^2 x\right)e^{-x} \le e^{-x}.$$

We use direct computation to show that the improper integral of e^{-x} over the interval $[8, \infty)$ converges:

$$\int_8^\infty e^{-x}\,dx = \lim_{R\to\infty}\int_8^R e^{-x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_8^R = \lim_{R\to\infty}\left(-e^{-R}+e^{-8}\right) = 0+e^{-8} = e^{-8}.$$

Therefore, by the Comparison Test, the improper integral $\int_8^\infty (\sin^2 x)e^{-x}\,dx$ also converges.

87. $\int_3^\infty \frac{dx}{x^4+\cos^2 x}$

SOLUTION For $x \ge 1$, we have

$$\frac{1}{x^4+\cos^2 x} \le \frac{1}{x^4}.$$

Since $\int_1^\infty \frac{dx}{x^4}$ converges, the Comparison Test guarantees that $\int_1^\infty \frac{dx}{x^4+\cos^2 x}$ also converges. The integral $\int_1^3 \frac{dx}{x^4+\cos^2 x}$ has a finite value (notice that $x^4+\cos^2 x \ne 0$) hence we conclude that the integral $\int_3^\infty \frac{dx}{x^4+\cos^2 x}$ also converges.

88. $\int_1^\infty \frac{dx}{x^{1/3}+x^{2/3}}$

SOLUTION If $x \ge 1$, then $x^{1/3} \ge 1$; therefore,

$$x^{1/3}+x^{2/3} = x^{1/3}\left(1+x^{1/3}\right) \le x^{1/3}\left(x^{1/3}+x^{1/3}\right) = x^{1/3}\cdot 2x^{1/3} = 2x^{2/3}.$$

Hence,

$$\frac{1}{x^{1/3}+x^{2/3}} \ge \frac{1}{2x^{2/3}}.$$

The integral $\int_1^\infty \frac{dx}{x^{2/3}}$ diverges; hence $\int_1^\infty \frac{dx}{2x^{2/3}}$ also diverges. Therefore, by the Comparison Test, the improper integral $\int_1^\infty \frac{dx}{x^{1/3}+x^{2/3}}$ also diverges.

89. $\int_0^1 \frac{dx}{x^{1/3}+x^{2/3}}$

SOLUTION For $0 \le x \le 1$,

$$x^{1/3}+x^{2/3} \ge x^{1/3} \quad \text{so} \quad \frac{1}{x^{1/3}+x^{2/3}} \le \frac{1}{x^{1/3}}.$$

Now, $\int_0^1 x^{-1/3}\,dx$ converges. Therefore, by the Comparison Test, the improper integral $\int_0^1 \frac{dx}{x^{1/3}+x^{2/3}}$ also converges.

90. $\int_0^\infty e^{-x^3}\,dx$

SOLUTION For $x > 1$, $e^x \ge x$; hence $e^{x^3} \ge x^3$, therefore $0 \le e^{-x^3} \le x^{-3}$. Since $\int_1^\infty \frac{dx}{x^3}$ converges, the integral $\int_1^\infty e^{-x^3}\,dx$ also converges by the Comparison Test. We write

$$\int_0^\infty e^{-x^3}\,dx = \int_0^1 e^{-x^3}\,dx + \int_1^\infty e^{-x^3}\,dx.$$

The first integral on the right hand side has a finite value and the second integral converges. We conclude that the integral $\int_0^\infty e^{-x^3}\,dx$ converges.

91. Calculate the volume of the infinite solid obtained by rotating the region under $y = (x^2+1)^{-2}$ for $0 \le x < \infty$ about the y-axis.

SOLUTION Using the Shell Method, the volume of the infinite solid obtained by rotating the region under the graph of $y = \left(x^2+1\right)^{-2}$ over the interval $[0, \infty)$ about the y-axis is

$$V = 2\pi\int_0^\infty \frac{x}{\left(x^2+1\right)^2}\,dx.$$

Now,

$$\int_0^\infty \frac{x}{\left(x^2+1\right)^2}\,dx = \lim_{R\to\infty}\int_0^R \frac{x\,dx}{\left(x^2+1\right)^2}$$

We substitute $t = x^2 + 1$, $dt = 2x\,dx$. The new limits of integration are $t = 1$ and $t = R^2 + 1$. Thus,

$$\int_0^R \frac{x\,dx}{\left(x^2+1\right)^2} = \int_1^{R^2+1} \frac{\frac{1}{2}\,dt}{t^2} = -\frac{1}{2t}\bigg|_1^{R^2+1} = \frac{1}{2}\left(1 - \frac{1}{R^2+1}\right).$$

Taking the limit as $R \to \infty$ yields:

$$\int_0^\infty \frac{x\,dx}{\left(x^2+1\right)^2} = \lim_{R\to\infty}\frac{1}{2}\left(1 - \frac{1}{R^2+1}\right) = \frac{1}{2}(1-0) = \frac{1}{2}.$$

Therefore,

$$V = 2\pi \cdot \frac{1}{2} = \pi.$$

92. Let R be the region under the graph of $y = (x+1)^{-1}$ for $0 \le x < \infty$. Which of the following quantities is finite?

(a) The area of R

(b) The volume of the solid obtained by rotating R about the x-axis

(c) The volume of the solid obtained by rotating R about the y-axis

SOLUTION

(a) The area of R is

$$\int_0^\infty \frac{dx}{x+1} = \lim_{R\to\infty}\int_0^R \frac{dx}{x+1} = \lim_{R\to\infty} \ln|x+1|\bigg|_0^R = \lim_{R\to\infty}\left(\ln(R+1) - \ln 1\right) = \infty.$$

Hence, the area of R is not finite.

(b) Using the Disk Method, the volume of the solid obtained by rotating R about the x-axis is

$$\pi\int_0^\infty \frac{dx}{(x+1)^2} = \pi\lim_{R\to\infty}\int_0^R \frac{dx}{(x+1)^2} = \pi\lim_{R\to\infty} -\frac{1}{x+1}\bigg|_0^R = \pi\lim_{R\to\infty}\left(-\frac{1}{R+1} + 1\right) = \pi.$$

Hence, the volume of the solid obtained by rotating R about the x-axis is finite.

(c) Using the Shell Method, the volume of the solid obtained by rotating R about the y-axis is

$$2\pi\int_0^\infty \frac{x}{x+1}\,dx = 2\pi\lim_{R\to\infty}\int_0^R \frac{x\,dx}{x+1}.$$

Now,

$$\int_0^R \frac{x\,dx}{x+1} = \int_0^R \frac{(x+1)-1}{x+1}\,dx = \int_0^R\left(1 - \frac{1}{x+1}\right)dx = (x - \ln(x+1))\bigg|_0^R$$
$$= R - (\ln(R+1) - \ln 1) = R - \ln(R+1).$$

Thus,

$$2\pi\lim_{R\to\infty}\int_0^R \frac{x\,dx}{x+1} = 2\pi\lim_{R\to\infty}(R - \ln(R+1)) = 2\pi\lim_{R\to\infty} R\left(1 - \frac{\ln(R+1)}{R}\right) = \infty.$$

Hence, the volume of the solid obtained by rotating R about the y-axis is not finite.

93. Show that $\int_0^\infty x^n e^{-x^2}\,dx$ converges for all $n > 0$. *Hint:* First observe that $x^n e^{-x^2} < x^n e^{-x}$ for $x > 1$. Then show that $x^n e^{-x} < x^{-2}$ for x sufficiently large.

SOLUTION For $x > 1$, $x^2 > x$; hence $e^{x^2} > e^x$, and $0 < e^{-x^2} < e^{-x}$. Therefore, for $x > 1$ the following inequality holds:

$$x^{n+2}e^{-x^2} < x^{n+2}e^{-x}.$$

Now, using L'Hôpital's Rule $n+2$ times, we find

$$\lim_{x\to\infty} x^{n+2}e^{-x} = \lim_{x\to\infty} \frac{x^{n+2}}{e^x} = \lim_{x\to\infty} \frac{(n+2)x^{n+1}}{e^x} = \lim_{x\to\infty} \frac{(n+2)(n+1)x^n}{e^x}$$
$$= \cdots = \lim_{x\to\infty} \frac{(n+2)!}{e^x} = 0.$$

Therefore,

$$\lim_{x\to\infty} x^{n+2}e^{-x^2} = 0$$

by the Squeeze Theorem, and there exists a number $R > 1$ such that, for all $x > R$:

$$x^{n+2}e^{-x^2} < 1 \quad \text{or} \quad x^n e^{-x^2} < x^{-2}.$$

Finally, write

$$\int_0^\infty x^n e^{-x^2}\,dx = \int_0^R x^n e^{-x^2}\,dx + \int_R^\infty x^n e^{-x^2}\,dx.$$

The first integral on the right-hand side has finite value since the integrand is a continuous function. The second integral converges since on the interval of integration, $x^n e^{-x^2} < x^{-2}$ and we know that $\int_R^\infty x^{-2}\,dx = \int_R^\infty \frac{dx}{x^2}$ converges. We conclude that the integral $\int_0^\infty x^n e^{-x^2}\,dx$ converges.

94. Compute the Laplace transform $Lf(s)$ of the function $f(x) = x$ for $s > 0$. See Exercises 86–89 in Section 7.6 for the definition of $Lf(s)$.

SOLUTION The Laplace transform of $f(x) = x$ is the following integral:

$$L(x)(s) = \int_0^\infty xe^{-sx}\,dx = \lim_{R\to\infty} \int_0^R xe^{-sx}\,dx.$$

We compute the definite integral using Integration by Parts with $u = x$ and $v' = e^{-sx}$. Then $u' = 1$, $v = -\frac{1}{s}e^{-sx}$ and

$$\int_0^R xe^{-sx}\,dx = -\frac{1}{s}xe^{-sx}\Big|_0^R + \int_0^R \frac{1}{s}e^{-sx}\,dx = \left(-\frac{1}{s}Re^{-sR} - \frac{1}{s^2}e^{-sx}\right)\Big|_0^R$$
$$= -\frac{1}{s}Re^{-sR} - \frac{1}{s^2}\left(e^{-sR} - e^0\right) = \frac{1}{s^2} - \frac{1}{s^2}e^{-sR} - \frac{1}{s}Re^{-sR}.$$

Therefore,

$$L(x)(s) = \lim_{R\to\infty}\left(\frac{1}{s^2} - \frac{1}{s^2}e^{-sR} - \frac{1}{s}Re^{-sR}\right) = \frac{1}{s^2} - \frac{1}{s^2}\lim_{R\to\infty} e^{-sR} - \frac{1}{s}\lim_{R\to\infty} Re^{-sR}.$$

Since $s > 0$, we have $\lim_{R\to\infty} e^{-sR} = 0$. Also by L'Hôpital's Rule:

$$\lim_{R\to\infty} Re^{-sR} = \lim_{R\to\infty} \frac{R}{e^{sR}} = \lim_{R\to\infty} \frac{1}{se^{sR}} = 0.$$

Finally,

$$L(x)(s) = \frac{1}{s^2} - 0 - 0 = \frac{1}{s^2}.$$

95. Compute the Laplace transform $Lf(s)$ of the function $f(x) = x^2 e^{\alpha x}$ for $s > \alpha$.

SOLUTION The Laplace transform is the following integral:

$$L\left(x^2 e^{\alpha x}\right)(s) = \int_0^\infty x^2 e^{\alpha x} e^{-sx}\,dx = \int_0^\infty x^2 e^{(\alpha - s)x}\,dx = \lim_{R\to\infty} \int_0^R x^2 e^{(\alpha - s)x}\,dx.$$

We compute the definite integral using Integration by Parts with $u = x^2$, $v' = e^{(\alpha-s)x}$. Then $u' = 2x$, $v = \frac{1}{\alpha - s}e^{(\alpha-s)x}$ and

$$\int_0^R x^2 e^{(\alpha-s)x}\,dx = \frac{1}{\alpha - s}x^2 e^{(\alpha-s)x}\Big|_{x=0}^R - \int_0^R 2x \cdot \frac{1}{\alpha - s}e^{(\alpha-s)x}\,dx$$
$$= \frac{1}{\alpha - s}R^2 e^{(\alpha-s)R} - \frac{2}{\alpha - s}\int_0^R xe^{(\alpha-s)x}\,dx.$$

We compute the resulting integral using Integration by Parts again, this time with $u = x$ and $v' = e^{(\alpha-s)x}$. Then $u' = 1$, $v = \frac{1}{\alpha-s}e^{(\alpha-s)x}$ and

$$\int_0^R xe^{(\alpha-s)x}\,dx = x\cdot\frac{1}{\alpha-s}e^{(\alpha-s)x}\bigg|_{x=0}^{R} - \frac{1}{\alpha-s}\int_0^R e^{(\alpha-s)x}\,dx = \left(\frac{x}{\alpha-s}e^{(\alpha-s)x} - \frac{1}{(\alpha-s)^2}e^{(\alpha-s)x}\right)\bigg|_{x=0}^{R}$$
$$= \frac{R}{\alpha-s}e^{(\alpha-s)R} - \frac{1}{(\alpha-s)^2}\left(e^{(\alpha-s)R} - e^0\right) = \frac{1}{(\alpha-s)^2} - \frac{1}{(\alpha-s)^2}e^{(\alpha-s)R} + \frac{R}{\alpha-s}e^{(\alpha-s)R}.$$

Thus,

$$\int_0^R x^2e^{(\alpha-s)x}\,dx = \frac{1}{\alpha-s}R^2e^{(\alpha-s)R} - \frac{2}{\alpha-s}\left(\frac{1}{(\alpha-s)^2} - \frac{1}{(\alpha-s)^2}e^{(\alpha-s)R} + \frac{R}{\alpha-s}e^{(\alpha-s)R}\right)$$
$$= \frac{1}{\alpha-s}R^2e^{(\alpha-s)R} - \frac{2}{(\alpha-s)^3} + \frac{2}{(\alpha-s)^3}e^{(\alpha-s)R} - \frac{2R}{(\alpha-s)^2}e^{(\alpha-s)R},$$

and

$$L\left(x^2e^{\alpha x}\right)(s) = \frac{2}{(s-\alpha)^3} - \frac{1}{s-\alpha}\lim_{R\to\infty}R^2e^{-(s-\alpha)R} - \frac{2}{(s-\alpha)^3}\lim_{R\to\infty}e^{-(s-\alpha)R} - \frac{2}{(s-\alpha)^2}\lim_{R\to\infty}Re^{-(s-\alpha)R}.$$

Now, since $s > \alpha$, $\lim\limits_{R\to\infty} e^{-(s-\alpha)R} = 0$. We use L'Hôpital's Rule to compute the other two limits:

$$\lim_{R\to\infty}Re^{-(s-\alpha)R} = \lim_{R\to\infty}\frac{R}{e^{(s-\alpha)R}} = \lim_{R\to\infty}\frac{1}{(s-\alpha)e^{(s-\alpha)R}} = 0;$$
$$\lim_{R\to\infty}R^2e^{-(s-\alpha)R} = \lim_{R\to\infty}\frac{R^2}{e^{(s-\alpha)R}} = \lim_{R\to\infty}\frac{2R}{(s-\alpha)e^{(s-\alpha)R}} = \lim_{R\to\infty}\frac{2}{(s-\alpha)^2e^{(s-\alpha)R}} = 0.$$

Finally,

$$L\left(x^2e^{\alpha x}\right)(s) = \frac{2}{(s-\alpha)^3} - 0 - 0 - 0 = \frac{2}{(s-\alpha)^3}.$$

96. Estimate $\int_2^5 f(x)\,dx$ by computing T_2, M_3, T_6, and S_6 for a function $f(x)$ taking on the values in the following table:

x	2	2.5	3	3.5	4	4.5	5
$f(x)$	$\frac{1}{2}$	2	1	0	$-\frac{3}{2}$	-4	-2

SOLUTION To calculate T_2, divide [2, 5] into two subintervals of length $\Delta x = \frac{3}{2}$ with endpoints $x_0 = 2$, $x_1 = 3.5$, $x_2 = 5$. Then

$$T_2 = \frac{1}{2}\cdot\frac{3}{2}\left(f(2) + 2f(3.5) + f(5)\right) = 0.75\left(\frac{1}{2} + 2\cdot 0 + (-2)\right) = -\frac{9}{8}.$$

To calculate M_3, divide [2, 5] into three subintervals of length $\Delta x = 1$ with midpoints $c_1 = 2.5$, $c_2 = 3.5$, $c_3 = 4.5$. Then

$$M_3 = 1\cdot\left(f(2.5) + f(3.5) + f(4.5)\right) = 2 + 0 - 4 = -2.$$

To calculate T_6, divide [2, 5] into 6 subintervals of length $\frac{5-2}{6} = \frac{1}{2}$ with endpoints $x_0 = 2$, $x_1 = 2.5$, $x_2 = 3$, $x_3 = 3.5$, $x_4 = 4$, $x_5 = 4.5$, $x_6 = 5$. Then

$$T_6 = \frac{1}{2}\cdot\frac{1}{2}\left(f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + 2f(4) + 2f(4.5) + f(5)\right)$$
$$= \frac{1}{4}\left(\frac{1}{2} + 2\cdot 2 + 2\cdot 1 + 2\cdot 0 + 2\cdot\left(-\frac{3}{2}\right) + 2(-4) + (-2)\right) = -\frac{13}{8}.$$

Finally, to calculate S_6, divide [2, 5] into 6 subintervals of length $\Delta x = \frac{5-2}{6} = \frac{1}{2}$ with endpoints $x_0 = 2$, $x_1 = 2.5$, $x_2 = 3$, $x_3 = 3.5$, $x_4 = 4$, $x_5 = 4.5$, $x_6 = 5$. Then

$$S_6 = \frac{1}{3}\cdot\frac{1}{2}\left(f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + 2f(4) + 4f(4.5) + f(5)\right)$$
$$= \frac{1}{6}\left(\frac{1}{2} + 4\cdot 2 + 2\cdot 1 + 4\cdot 0 + 2\cdot\left(-\frac{3}{2}\right) + 4(-4) + (-2)\right) = -\frac{7}{4}.$$

97. State whether the approximation M_N or T_N is larger or smaller than the integral.

(a) $\int_0^{\pi} \sin x\, dx$

(b) $\int_{\pi}^{2\pi} \sin x\, dx$

(c) $\int_1^8 \frac{dx}{x^2}$

(d) $\int_2^5 \ln x\, dx$

SOLUTION

(a) Because $f(x) = \sin x$ is concave down on the interval $[0, \pi]$,

$$T_N \le \int_0^{\pi} \sin x\, dx \le M_N;$$

that is, T_N is smaller and M_N is larger than the integral.

(b) On the interval $[\pi, 2\pi]$, the function $f(x) = \sin x$ is concave up, therefore

$$M_N \le \int_{\pi}^{2\pi} \sin x\, dx \le T_N;$$

that is, M_N is smaller and T_N is larger than the integral.

(c) The function $f(x) = \frac{1}{x^2}$ is concave up on the interval [1, 8]; therefore,

$$M_N \le \int_1^8 \frac{dx}{x^2} \le T_N;$$

that is, M_N is smaller and T_N is larger than the integral.

(d) The integrand $y = \ln x$ is concave down on the interval [2, 5]; hence,

$$T_N \le \int_2^5 \ln x\, dx \le M_N;$$

that is, T_N is smaller and M_N is larger than the integral.

98. The rainfall rate (in inches per hour) was measured hourly during a 10-hour thunderstorm with the following results:

0, 0.41, 0.49, 0.32, 0.3, 0.23,
0.09, 0.08, 0.05, 0.11, 0.12

Use Simpson's Rule to estimate the total rainfall during the 10-hour period.

SOLUTION We have 10 subintervals of length $\Delta x = 1$. Thus, the total rainfall during the 10-hour period is approximately

$$\begin{aligned} S_{10} &= \frac{1}{3} \cdot 1[0 + 4 \cdot 0.41 + 2 \cdot 0.49 + 4 \cdot 0.32 + 2 \cdot 0.3 + 4 \cdot 0.23 + 2 \cdot 0.09 + 4 \cdot 0.08 + 2 \cdot 0.05 \\ &\qquad + 4 \cdot 0.11 + 0.12] \\ &= 2.19 \text{ inches.} \end{aligned}$$

In Exercises 99–104, compute the given approximation to the integral.

99. $\int_0^1 e^{-x^2}\, dx, \quad M_5$

SOLUTION Divide the interval [0, 1] into 5 subintervals of length $\Delta x = \frac{1-0}{5} = \frac{1}{5}$, with midpoints $c_1 = \frac{1}{10}$, $c_2 = \frac{3}{10}$, $c_3 = \frac{1}{2}$, $c_4 = \frac{7}{10}$, and $c_5 = \frac{9}{10}$. Then

$$\begin{aligned} M_5 &= \Delta x \left[f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{7}{10}\right) + f\left(\frac{9}{10}\right) \right] \\ &= \frac{1}{5}\left[e^{-(1/10)^2} + e^{-(3/10)^2} + e^{-(1/2)^2} + e^{-(7/10)^2} + e^{-(9/10)^2} \right] = 0.748053. \end{aligned}$$

100. $\int_2^4 \sqrt{6t^3+1}\,dt, \quad T_3$

SOLUTION Divide the interval [2, 4] into 3 subintervals of length $\Delta x = \frac{4-2}{3} = \frac{2}{3}$, with endpoints 2, $\frac{8}{3}$, $\frac{10}{3}$, 4. Then,

$$T_3 = \frac{1}{2}\Delta x\left(f(2) + 2f\left(\frac{8}{3}\right) + 2f\left(\frac{10}{3}\right) + f(4)\right)$$
$$= \frac{1}{2}\cdot\frac{2}{3}\left(\sqrt{6\cdot 2^3+1} + 2\sqrt{6\cdot\left(\frac{8}{3}\right)^3+1} + 2\sqrt{6\cdot\left(\frac{10}{3}\right)^3+1} + \sqrt{6\cdot 4^3+1}\right) = 25.976514.$$

101. $\int_{\pi/4}^{\pi/2} \sqrt{\sin\theta}\,d\theta, \quad M_4$

SOLUTION Divide the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ into 4 subintervals of length $\Delta x = \frac{\frac{\pi}{2}-\frac{\pi}{4}}{4} = \frac{\pi}{16}$ with midpoints $\frac{9\pi}{32}$, $\frac{11\pi}{32}$, $\frac{13\pi}{32}$, and $\frac{15\pi}{32}$. Then

$$M_4 = \Delta x\left(f\left(\frac{9\pi}{32}\right) + f\left(\frac{11\pi}{32}\right) + f\left(\frac{13\pi}{32}\right) + f\left(\frac{15\pi}{32}\right)\right)$$
$$= \frac{\pi}{16}\left(\sqrt{\sin\frac{9\pi}{32}} + \sqrt{\sin\frac{11\pi}{32}} + \sqrt{\sin\frac{13\pi}{32}} + \sqrt{\sin\frac{15\pi}{32}}\right) = 0.744978.$$

102. $\int_1^4 \frac{dx}{x^3+1}, \quad T_6$

SOLUTION Divide the interval [1, 4] into 6 subintervals of length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$ with endpoints 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, 4. Then

$$T_6 = \frac{1}{2}\Delta x\left(f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4)\right)$$
$$= \frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{1^3+1} + 2\frac{1}{\left(\frac{3}{2}\right)^3+1} + 2\frac{1}{2^3+1} + 2\frac{1}{\left(\frac{5}{2}\right)^3+1} + 2\frac{1}{3^3+1} + 2\frac{1}{(\frac{7}{2})^2+1} + \frac{1}{4^3+1}\right) = 0.358016.$$

103. $\int_0^1 e^{-x^2}\,dx, \quad S_4$

SOLUTION Divide the interval [0, 1] into 4 subintervals of length $\Delta x = \frac{1}{4}$ with endpoints 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1. Then

$$S_6 = \frac{1}{3}\Delta x\left(f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right)$$
$$= \frac{1}{3}\cdot\frac{1}{4}\left(e^{-0^2} + 4e^{-(1/4)^2} + 2e^{-(1/2)^2} + 4e^{-(3/4)^2} + e^{-1^2}\right) = 0.746855.$$

104. $\int_5^9 \cos(x^2)\,dx, \quad S_8$

SOLUTION Divide the interval [5, 9] into 8 subintervals of length $\Delta x = \frac{9-5}{8} = \frac{1}{2}$ with endpoints 5, $\frac{11}{2}$, 6, $\frac{13}{2}$, 7, $\frac{15}{2}$, 8, $\frac{17}{2}$, 9. Then

$$S_8 = \frac{1}{3}\Delta x\left(f(5) + 4f\left(\frac{11}{2}\right) + 2f(6) + 4f\left(\frac{13}{2}\right) + 2f(7) + 4f\left(\frac{15}{2}\right) + 2f(8) + 4f\left(\frac{17}{2}\right) + f(9)\right)$$
$$= \frac{1}{3}\cdot\frac{1}{2}\left(\cos(5^2) + 4\cos(5.5^2) + 2\cos(6^2) + 4\cos(6.5^2)\right.$$
$$\left. + 2\cos(7^2) + 4\cos(7.5^2) + 2\cos(8^2) + 4\cos(8.5^2) + \cos(9^2)\right)$$
$$= 0.608711.$$

105. The following table gives the area $A(h)$ of a horizontal cross section of a pond at depth h. Use the Trapezoidal Rule to estimate the volume V of the pond (Figure 1).

h (ft)	$A(h)$ (acres)	h (ft)	$A(h)$ (acres)
0	2.8	10	0.8
2	2.4	12	0.6
4	1.8	14	0.2
6	1.5	16	0.1
8	1.2	18	0

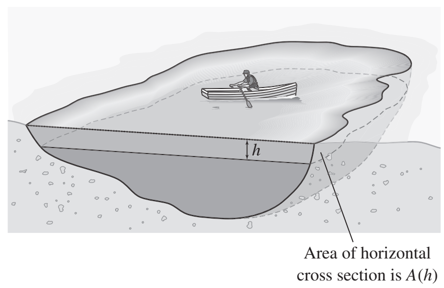


FIGURE 1

SOLUTION The volume of the pond is the following integral:

$$V = \int_0^{18} A(h)dh$$

We approximate the integral using the trapezoidal approximation T_9. The interval of depth [0, 18] is divided to 9 subintervals of length $\Delta x = 2$ with endpoints 0, 2, 4, 6, 8, 10, 12, 14, 16, 18. Thus,

$$\begin{aligned} V \approx T_9 &= \frac{1}{2} \cdot 2(2.8 + 2 \cdot 2.4 + 2 \cdot 1.8 + 2 \cdot 1.5 + 2 \cdot 1.2 + 2 \cdot 0.8 + 2 \cdot 0.6 + 2 \cdot 0.2 + 2 \cdot 0.1 + 0) \\ &= 20 \text{ acre} \cdot \text{ft} = 871{,}200 \text{ ft}^3, \end{aligned}$$

where we have used the fact that 1 acre = 43,560 ft^2.

106. Suppose that the second derivative of the function $A(h)$ in Exercise 105 satisfies $|A''(h)| \le 1.5$. Use the error bound to find the maximum possible error in your estimate of the volume V of the pond.

SOLUTION The Error Bound for the Trapezoidal Rule states that

$$\text{Error } (T_N) \le \frac{K_2(b-a)^3}{12N^2},$$

where K_2 is a number such that $\left|f''(x)\right| \le K_2$ for all $x \in [a, b]$. We estimated the volume of the pond by T_9; hence $N = 9$. The interval of depth is [0, 18] hence $b - a = 18 - 0 = 18$. Since $\left|A''(h)\right| \le 1.5$ acres/ft^2 we may take $K_2 = 1.5$, to find that the error cannot exceed

$$\frac{K_2(b-a)^3}{12N^2} = \frac{1.5 \cdot 18^3}{12 \cdot 9^2} = 9 \text{ acre} \cdot \text{ft} = 392{,}040 \text{ ft}^3,$$

where we have used the fact that 1 acre = 43,560 ft^2.

107. Find a bound for the error $\left|M_{16} - \int_1^3 x^3\,dx\right|$.

SOLUTION The Error Bound for the Midpoint Rule states that

$$\left|M_N - \int_a^b f(x)\,dx\right| \le \frac{K_2(b-a)^3}{24N^2},$$

where K_2 is a number such that $\left|f''(x)\right| \le K_2$ for all $x \in [1, 3]$. Here $b - a = 3 - 1 = 2$ and $N = 16$. Therefore,

$$\left|M_{16} - \int_1^3 x^3\,dx\right| \le \frac{K_2 \cdot 2^3}{24 \cdot 16^2} = \frac{K_2}{768}.$$

To find K_2, we differentiate $f(x) = x^3$ twice:

$$f'(x) = 3x^2 \qquad \text{and} \qquad f''(x) = 6x.$$

On the interval [1, 3] we have $\left|f''(x)\right| = 6x \le 6 \cdot 3 = 18$; hence, we may take $K_2 = 18$. Thus,

$$\left| M_{16} - \int_1^3 x^3\,dx \right| \le \frac{18}{768} = \frac{3}{128} = 0.0234375.$$

108. GU Let $f(x) = \sin(x^3)$. Find a bound for the error

$$\left| T_{24} - \int_0^{\pi/2} f(x)\,dx \right|$$

Hint: Find a bound K_2 for $|f''(x)|$ by plotting $f''(x)$ with a graphing utility.

SOLUTION Using the error bound for T_{24} we obtain:

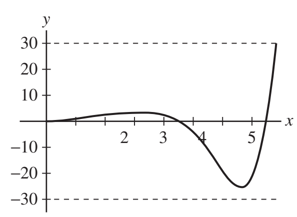
$$\left| T_{24} - \int_0^{\pi/2} f(x)\,dx \right| \le \frac{K_2\left(\frac{\pi}{2} - 0\right)^3}{12 \cdot 24^2} = \frac{K_2\pi^3}{55,296},$$

where K_2 is a number such that $\left|f''(x)\right| < k_2$ for all $x \in \left[0, \frac{\pi}{2}\right]$. We compute the first and second derivative of $f(x) = \sin(x^3)$:

$$f'(x) = 3x^2\cos(x^3)$$
$$f''(x) = 6x\cos(x^3) + 3x^2 \cdot 3x^2\left(-\sin(x^3)\right) = 6x\cos(x^3) - 9x^4\sin(x^3)$$

The graph of $f''(x) = 6x\cos(x^3) - 9x^4\sin(x^3)$ on the interval $\left[0, \frac{\pi}{2}\right]$ shows that $\left|f''(x)\right| \le 30$ on this interval. We may choose $K_2 = 30$ and find

$$\left| T_{24} - \int_0^{\pi/2} f(x)\,dx \right| \le \frac{30\pi^3}{55,296} = \frac{5\pi^3}{9216} = 0.0168220.$$



109. Find a value of N such that

$$\left| M_N - \int_0^{\pi/4} \tan x\,dx \right| \le 10^{-4}$$

SOLUTION To use the Error Bound we must find the second derivative of $f(x) = \tan x$. We differentiate f twice to obtain:

$$f'(x) = \sec^2 x$$
$$f''(x) = 2\sec x\tan x = \frac{2\sin x}{\cos^2 x}$$

For $0 \le x \le \frac{\pi}{4}$, we have $\sin x \le \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos x \ge \frac{1}{\sqrt{2}}$ or $\cos^2 x \ge \frac{1}{2}$. Therefore, for $0 \le x \le \frac{\pi}{4}$ we have:

$$f''(x) = \frac{2\sin x}{\cos^2 x} \le \frac{2 \cdot \frac{1}{\sqrt{2}}}{\frac{1}{2}} = 2\sqrt{2}.$$

Using the Error Bound with $b = \frac{\pi}{4}$, $a = 0$ and $K_2 = 2\sqrt{2}$ we have:

$$\left| M_N - \int_0^{\pi/4} \tan x\,dx \right| \le \frac{2\sqrt{2} \cdot \left(\frac{\pi}{4} - 0\right)^3}{24N^2} = \frac{\pi^3\sqrt{2}}{768N^2}.$$

We must choose a value of N such that:

$$\frac{\pi^3\sqrt{2}}{768N^2} \le 10^{-4}$$

$$N^2 \ge \frac{10^4 \cdot \sqrt{2}\pi^3}{768}$$

$$N \ge 23.9$$

The smallest integer that is needed to obtain the required precision is $N = 24$.

110. Find a value of N such that S_N approximates $\int_2^5 x^{-1/4}\,dx$ with an error of at most 10^{-2} (but do not calculate S_N).

SOLUTION To use the error bound we must find the fourth derivative $f^{(4)}(x)$. We differentiate $f(x) = x^{-1/4}$ four times to obtain:

$$f'(x) = -\frac{1}{4}x^{-5/4},\ f''(x) = \frac{5}{16}x^{-9/4},\ f'''(x) = -\frac{45}{64}x^{-13/4},\ f^{(4)}(x) = \frac{585}{256}x^{-17/4}.$$

For $2 \le x \le 5$ we have:

$$\left|f^{(4)}(x)\right| = \frac{585}{256x^{17/4}} \le \frac{585}{256 \cdot 2^{17/4}} = 0.120099.$$

Using the error bound with $b = 5$, $a = 2$ and $K_4 = 0.120099$ we have:

$$\text{Error}\,(S_N) \le \frac{0.120099(5-2)^5}{180N^4} = \frac{0.162134}{N^4}.$$

We must choose a value of N such that:

$$\frac{0.162134}{N^4} \le 10^{-2}$$

$$N^4 \ge 16.2134$$

$$N \ge 2.00664$$

The smallest even value of N that is needed to obtain the required precision is $N = 4$.