6 APPLICATIONS OF THE INTEGRAL

6.1 Area Between Two Curves

Preliminary Questions

1. What is the area interpretation of $\int_a^b \big(f(x) - g(x)\big)\,dx$ if $f(x) \ge g(x)$?

SOLUTION Because $f(x) \ge g(x)$, $\int_a^b (f(x) - g(x))\,dx$ represents the area of the region bounded between the graphs of $y = f(x)$ and $y = g(x)$, bounded on the left by the vertical line $x = a$ and on the right by the vertical line $x = b$.

2. Is $\int_a^b \big(f(x) - g(x)\big)\,dx$ still equal to the area between the graphs of f and g if $f(x) \ge 0$ but $g(x) \le 0$?

SOLUTION Yes. Since $f(x) \ge 0$ and $g(x) \le 0$, it follows that $f(x) - g(x) \ge 0$.

3. Suppose that $f(x) \ge g(x)$ on [0, 3] and $g(x) \ge f(x)$ on [3, 5]. Express the area between the graphs over [0, 5] as a sum of integrals.

SOLUTION Remember that to calculate an area between two curves, one must subtract the equation for the lower curve from the equation for the upper curve. Over the interval [0, 3], $y = f(x)$ is the upper curve. On the other hand, over the interval [3, 5], $y = g(x)$ is the upper curve. The area between the graphs over the interval [0, 5] is therefore given by

$$\int_0^3 (f(x) - g(x))\,dx + \int_3^5 (g(x) - f(x))\,dx.$$

4. Suppose that the graph of $x = f(y)$ lies to the left of the y-axis. Is $\int_a^b f(y)\,dy$ positive or negative?

SOLUTION If the graph of $x = f(y)$ lies to the left of the y-axis, then for each value of y, the corresponding value of x is less than zero. Hence, the value of $\int_a^b f(y)\,dy$ is negative.

Exercises

1. Find the area of the region between $y = 3x^2 + 12$ and $y = 4x + 4$ over $[-3, 3]$ (Figure 9).

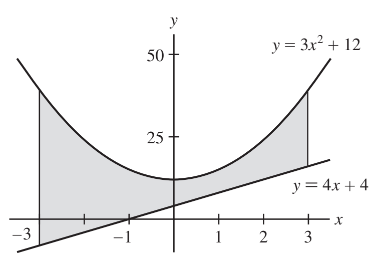


FIGURE 9

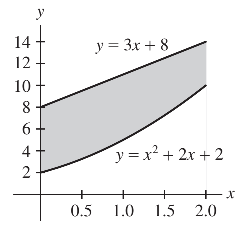
SOLUTION As the graph of $y = 3x^2 + 12$ lies above the graph of $y = 4x + 4$ over the interval $[-3, 3]$, the area between the graphs is

$$\int_{-3}^{3} \Big((3x^2 + 12) - (4x + 4)\Big)\,dx = \int_{-3}^{3} (3x^2 - 4x + 8)\,dx = \Big(x^3 - 2x^2 + 8x\Big)\Big|_{-3}^{3} = 102.$$

2. Find the area of the region between the graphs of $f(x) = 3x + 8$ and $g(x) = x^2 + 2x + 2$ over [0, 2].

SOLUTION From the diagram below, we see that the graph of $f(x) = 3x + 8$ lies above the graph of $g(x) = x^2 + 2x + 2$ over the interval [0, 2]. Thus, the area between the graphs is

$$\int_0^2 \Big[(3x + 8) - \Big(x^2 + 2x + 2\Big)\Big]\,dx = \int_0^2 \Big(-x^2 + x + 6\Big)\,dx = \Big(-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x\Big)\Big|_0^2 = \frac{34}{3}.$$



3. Find the area of the region enclosed by the graphs of $f(x) = x^2 + 2$ and $g(x) = 2x + 5$ (Figure 10).

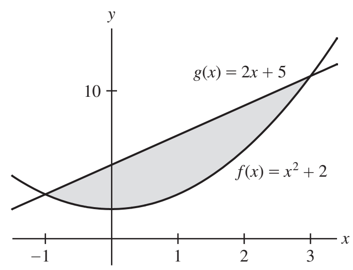


FIGURE 10

SOLUTION From the figure, we see that the graph of $g(x) = 2x + 5$ lies above the graph of $f(x) = x^2 + 2$ over the interval $[-1, 3]$. Thus, the area between the graphs is

$$\int_{-1}^{3} \left[(2x+5) - \left(x^2+2\right)\right] dx = \int_{-1}^{3} \left(-x^2 + 2x + 3\right) dx$$
$$= \left(-\frac{1}{3}x^3 + x^2 + 3x\right)\Big|_{-1}^{3}$$
$$= 9 - \left(-\frac{5}{3}\right) = \frac{32}{3}.$$

4. Find the area of the region enclosed by the graphs of $f(x) = x^3 - 10x$ and $g(x) = 6x$ (Figure 11).

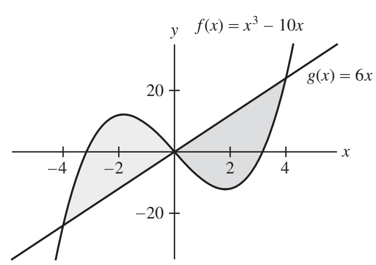


FIGURE 11

SOLUTION From the figure, we see that the graph of $f(x) = x^3 - 10x$ lies above the graph of $g(x) = 6x$ over the interval $[-4, 0]$, while the graph of $g(x) = 6x$ lies above the graph of $f(x) = x^3 - 10x$ over the interval $[0, 4]$. Thus, the area enclosed by the two graphs is

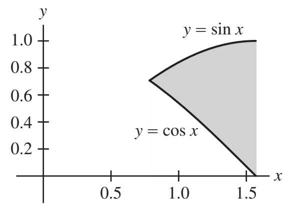
$$A = \int_{-4}^{0} \left(x^3 - 10x - 6x\right) dx + \int_{0}^{4} \left(6x - (x^3 - 10x)\right) dx$$
$$= \int_{-4}^{0} (x^3 - 16x)\, dx + \int_{0}^{4} (16x - x^3)\, dx$$
$$= \left(\frac{1}{4}x^4 - 8x^2\right)\Big|_{-4}^{0} + \left(8x^2 - \frac{1}{4}x^4\right)\Big|_{0}^{4}$$
$$= 64 + 64 = 128.$$

In Exercises 5 and 6, sketch the region between $y = \sin x$ and $y = \cos x$ over the interval and find its area.

5. $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

SOLUTION Over the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$, the graph of $y = \cos x$ lies below that of $y = \sin x$ (see the sketch below). Hence, the area between the two curves is

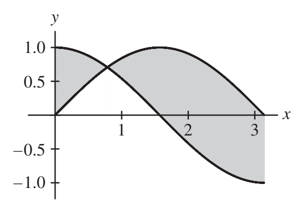
$$\int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx = \left.(-\cos x - \sin x)\right|_{\pi/4}^{\pi/2} = (0 - 1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = \sqrt{2} - 1.$$



6. $[0, \pi]$

SOLUTION Over the interval $[0, \frac{\pi}{4}]$, the graph of $y = \sin x$ lies below that of $y = \cos x$, while over the interval $[\frac{\pi}{4}, \pi]$, the orientation of the graphs is reversed (see the sketch below). The area between the graphs over $[0, \pi]$ is then

$$\begin{aligned}
&\int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx \\
&= \left.(\sin x + \cos x)\right|_0^{\pi/4} + \left.(-\cos x - \sin x)\right|_{\pi/4}^{\pi} \\
&= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 + 1) + (1 - 0) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 2\sqrt{2}.
\end{aligned}$$



In Exercises 7 and 8, let $f(x) = 20 + x - x^2$ and $g(x) = x^2 - 5x$.

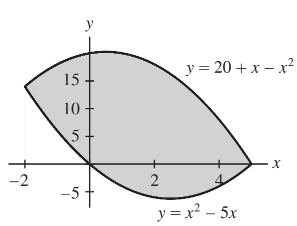
7. Sketch the region enclosed by the graphs of $f(x)$ and $g(x)$ and compute its area.

SOLUTION Setting $f(x) = g(x)$ gives $20 + x - x^2 = x^2 - 5x$, which simplifies to

$$0 = 2x^2 - 6x - 20 = 2(x - 5)(x + 2).$$

Thus, the curves intersect at $x = -2$ and $x = 5$. With $y = 20 + x - x^2$ being the upper curve (see the sketch below), the area between the two curves is

$$\int_{-2}^{5} \left((20 + x - x^2) - (x^2 - 5x)\right) dx = \int_{-2}^{5} \left(20 + 6x - 2x^2\right) dx = \left.\left(20x + 3x^2 - \frac{2}{3}x^3\right)\right|_{-2}^{5} = \frac{343}{3}.$$



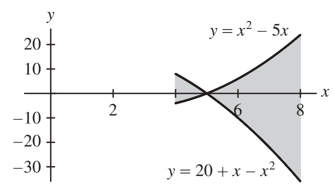
8. Sketch the region between the graphs of $f(x)$ and $g(x)$ over [4, 8] and compute its area as a sum of two integrals.

SOLUTION Setting $f(x) = g(x)$ gives $20 + x - x^2 = x^2 - 5x$, which simplifies to

$$0 = 2x^2 - 6x - 20 = 2(x-5)(x+2).$$

Thus, the curves intersect at $x = -2$ and $x = 5$. Over the interval [4, 5], $y = 20 + x - x^2$ is the upper curve but over the interval [5, 8], $y = x^2 - 5x$ is the upper curve (see the sketch below). The area between the two curves over the interval [4, 8] is then

$$\int_4^5 \left((20 + x - x^2) - (x^2 - 5x)\right) dx + \int_5^8 \left((x^2 - 5x) - (20 + x - x^2)\right) dx$$
$$= \int_4^5 \left(-2x^2 + 6x + 20\right) dx + \int_5^8 \left(2x^2 - 6x - 20\right) dx$$
$$= \left(-\frac{2}{3}x^2 + 3x^2 + 20x\right)\Big|_4^5 + \left(\frac{2}{3}x^3 - 3x^2 - 20x\right)\Big|_5^8 = \frac{19}{3} + 81 = \frac{262}{3}.$$



9. Find the area between $y = e^x$ and $y = e^{2x}$ over [0, 1].

SOLUTION As the graph of $y = e^{2x}$ lies above the graph of $y = e^x$ over the interval [0, 1], the area between the graphs is

$$\int_0^1 (e^{2x} - e^x)\, dx = \left(\frac{1}{2}e^{2x} - e^x\right)\Big|_0^1 = \frac{1}{2}e^2 - e - \left(\frac{1}{2} - 1\right) = \frac{1}{2}e^2 - e + \frac{1}{2}.$$

10. Find the area of the region bounded by $y = e^x$ and $y = 12 - e^x$ and the y-axis.

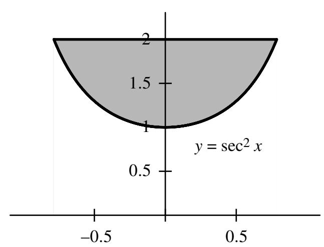
SOLUTION The two graphs intersect when $e^x = 12 - e^x$, or when $x = \ln 6$. As the graph of $y = 12 - e^x$ lies above the graph of $y = e^x$ over the interval [0, ln 6], the area between the graphs is

$$\int_0^{ln 6} \left(12 - e^x - e^x\right) dx = (12x - 2e^x)\Big|_0^{\ln 6} = 12\ln 6 - 12 - (0 - 2) = 12\ln 6 - 10.$$

11. Sketch the region bounded by the line $y = 2$ and the graph of $y = \sec^2 x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and find its area.

SOLUTION A sketch of the region bounded by $y = \sec^2 x$ and $y = 2$ is shown below. Note the region extends from $x = -\frac{\pi}{4}$ on the left to $x = \frac{\pi}{4}$ on the right. As the graph of $y = 2$ lies above the graph of $y = \sec^2 x$, the area between the graphs is

$$\int_{-\pi/4}^{\pi/4} (2 - \sec^2 x)\, dx = (2x - \tan x)\Big|_{-\pi/4}^{\pi/4} = \left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right) = \pi - 2.$$



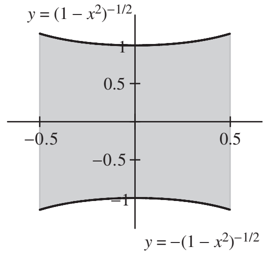
12. Sketch the region bounded by

$$y = \frac{1}{\sqrt{1 - x^2}} \quad \text{and} \quad y = -\frac{1}{\sqrt{1 - x^2}}$$

for $-\frac{1}{2} \le x \le \frac{1}{2}$ and find its area.

SOLUTION A sketch of the region bounded by $y = \dfrac{1}{\sqrt{1-x^2}}$ and $y = -\dfrac{1}{\sqrt{1-x^2}}$ for $-\frac{1}{2} \le x \le \frac{1}{2}$ is shown below. As the graph of $y = \dfrac{1}{\sqrt{1-x^2}}$ lies above the graph of $y = -\dfrac{1}{\sqrt{1-x^2}}$, the area between the graphs is

$$\int_{-1/2}^{1/2} \left[\frac{1}{\sqrt{1-x^2}} - \left(-\frac{1}{\sqrt{1-x^2}} \right) \right] dx = 2\sin^{-1} x \Big|_{-1/2}^{1/2} = 2\left[\frac{\pi}{6} - \left(-\frac{\pi}{6} \right) \right] = \frac{2\pi}{3}.$$



In Exercises 13–16, find the area of the shaded region in Figures 12–15.

13.

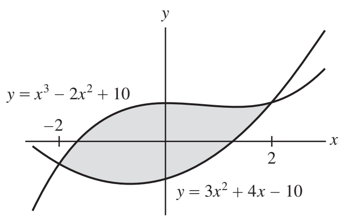


FIGURE 12

SOLUTION As the graph of $y = x^3 - 2x^2 + 10$ lies above the graph of $y = 3x^2 + 4x - 10$, the area of the shaded region is

$$\int_{-2}^{2} \left((x^3 - 2x^2 + 10) - (3x^2 + 4x - 10) \right) dx = \int_{-2}^{2} \left(x^3 - 5x^2 - 4x + 20 \right) dx$$

$$= \left(\frac{1}{4}x^4 - \frac{5}{3}x^3 - 2x^2 + 20x \right) \Big|_{-2}^{2} = \frac{160}{3}.$$

14.

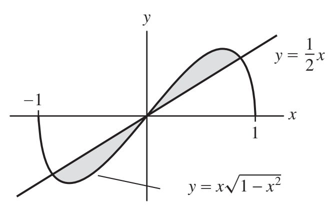


FIGURE 13

SOLUTION Setting $\frac{1}{2}x = x\sqrt{1-x^2}$ yields $x = 0$ or $\frac{1}{2} = \sqrt{1-x^2}$, so that $x = \pm\frac{\sqrt{3}}{2}$. Over the interval $[-\frac{\sqrt{3}}{2}, 0]$, $y = \frac{1}{2}x$ is the upper curve but over the interval $[0, \frac{\sqrt{3}}{2}]$, $y = x\sqrt{1-x^2}$ is the upper curve. The area of the shaded region is then

$$\int_{-\sqrt{3}/2}^{0} \left(\frac{1}{2}x - x\sqrt{1-x^2} \right) dx + \int_{0}^{\sqrt{3}/2} \left(x\sqrt{1-x^2} - \frac{1}{2}x \right) dx$$

$$= \left(\frac{1}{4}x^2 + \frac{1}{3}(1-x^2)^{3/2} \right) \Big|_{-\sqrt{3}/2}^{0} + \left(-\frac{1}{3}(1-x^2)^{3/2} - \frac{1}{4}x^2 \right) \Big|_{0}^{\sqrt{3}/2} = \frac{5}{48} + \frac{5}{48} = \frac{5}{24}.$$

15.

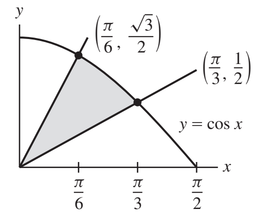


FIGURE 14

SOLUTION The line on the top-left has equation $y = \frac{3\sqrt{3}}{\pi}x$, and the line on the bottom-right has equation $y = \frac{3}{2\pi}x$. Thus, the area to the left of $x = \frac{\pi}{6}$ is

$$\int_0^{\pi/6} \left(\frac{3\sqrt{3}}{\pi}x - \frac{3}{2\pi}x\right) dx = \left(\frac{3\sqrt{3}}{2\pi}x^2 - \frac{3}{4\pi}x^2\right)\bigg|_0^{\pi/6} = \frac{3\sqrt{3}}{2\pi}\frac{\pi^2}{36} - \frac{3}{4\pi}\frac{\pi^2}{36} = \frac{(2\sqrt{3}-1)\pi}{48}.$$

The area to the right of $x = \frac{\pi}{6}$ is

$$\int_{\pi/6}^{\pi/3} \left(\cos x - \frac{3}{2\pi}x\right) dx = \left(\sin x - \frac{3}{4\pi}x^2\right)\bigg|_{\pi/6}^{\pi/3} = \frac{8\sqrt{3} - 8 - \pi}{16}.$$

The entire area is then

$$\frac{(2\sqrt{3}-1)\pi}{48} + \frac{8\sqrt{3}-8-\pi}{16} = \frac{12\sqrt{3} - 12 + (\sqrt{3}-2)\pi}{24}.$$

16.

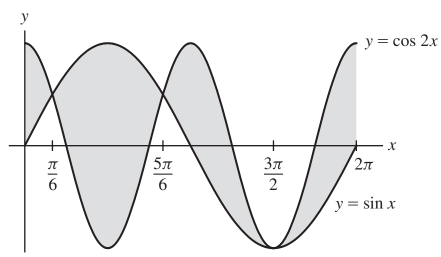


FIGURE 15

SOLUTION Over the interval $[0, \pi/6]$, the graph of $y = \cos 2x$ lies above the graph of $y = \sin x$. The orientation of the two graphs reverses over $[\pi/6, 5\pi/6]$ and reverses again over $[5\pi/6, 2\pi]$. Thus, the area between the two graphs is given by

$$A = \int_0^{\pi/6} (\cos 2x - \sin x)\, dx + \int_{\pi/6}^{5\pi/6} (\sin x - \cos 2x)\, dx + \int_{5\pi/6}^{2\pi} (\cos 2x - \sin x)\, dx.$$

Carrying out the integration and evaluation, we find

$$A = \left(\frac{1}{2}\sin 2x + \cos x\right)\bigg|_0^{\pi/6} + \left(-\cos x - \frac{1}{2}\sin 2x\right)\bigg|_{\pi/6}^{5\pi/6} + \left(\frac{1}{2}\sin 2x + \cos x\right)\bigg|_{5\pi/6}^{2\pi}$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} - 1 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} - \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right) + 1 - \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2}\right)$$

$$= 3\sqrt{3}.$$

In Exercises 17 and 18, find the area between the graphs of $x = \sin y$ and $x = 1 - \cos y$ over the given interval (Figure 16).

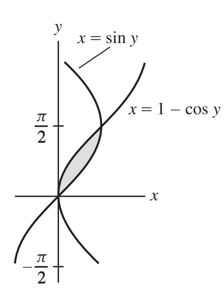


FIGURE 16

17. $0 \le y \le \dfrac{\pi}{2}$

SOLUTION As shown in the figure, the graph on the right is $x = \sin y$ and the graph on the left is $x = 1 - \cos y$. Therefore, the area between the two curves is given by

$$\int_0^{\pi/2} (\sin y - (1 - \cos y))\, dy = \left(-\cos y - y + \sin y\right)\Big|_0^{\pi/2} = \left(-\frac{\pi}{2} + 1\right) - (-1) = 2 - \frac{\pi}{2}.$$

18. $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

SOLUTION The shaded region in the figure shows the area between the graphs from $y = 0$ to $y = \frac{\pi}{2}$. It is bounded on the right by $x = \sin y$ and on the left by $x = 1 - \cos y$. Therefore, the area between the graphs from $y = 0$ to $y = \frac{\pi}{2}$ is

$$\int_0^{\pi/2} (\sin y - (1 - \cos y))\, dy = \left(-\cos y - y + \sin y\right)\Big|_0^{\pi/2} = \left(-\frac{\pi}{2} + 1\right) - (-1) = 2 - \frac{\pi}{2}.$$

The graphs cross at $y = 0$. Since $x = 1 - \cos y$ lies to the right of $x = \sin y$ on the interval $[-\frac{\pi}{2}, 0]$ along the y-axis, the area between the graphs from $y = -\frac{\pi}{2}$ to $y = 0$ is

$$\int_{-\pi/2}^{0} ((1 - \cos y) - \sin y)\, dy = \left(y - \sin y + \cos y\right)\Big|_{-\pi/2}^{0} = 1 - \left(-\frac{\pi}{2} + 1\right) = \frac{\pi}{2}.$$

The total area between the graphs from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$ is the sum

$$\int_0^{\pi/2} (\sin y - (1 - \cos y))\, dy + \int_{-\pi/2}^{0} ((1 - \cos y) - \sin y)\, dy = 2 - \frac{\pi}{2} + \frac{\pi}{2} = 2.$$

19. Find the area of the region lying to the right of $x = y^2 + 4y - 22$ and to the left of $x = 3y + 8$.

SOLUTION Setting $y^2 + 4y - 22 = 3y + 8$ yields

$$0 = y^2 + y - 30 = (y + 6)(y - 5),$$

so the two curves intersect at $y = -6$ and $y = 5$. The area in question is then given by

$$\int_{-6}^{5} \left((3y + 8) - (y^2 + 4y - 22)\right) dy = \int_{-6}^{5} \left(-y^2 - y + 30\right) dy = \left(-\frac{y^3}{3} - \frac{y^2}{2} + 30y\right)\Bigg|_{-6}^{5} = \frac{1331}{6}.$$

20. Find the area of the region lying to the right of $x = y^2 - 5$ and to the left of $x = 3 - y^2$.

SOLUTION Setting $y^2 + 5 = 3 - y^2$ yields $2y^2 = 8$ or $y = \pm 2$. The area of the region enclosed by the two graphs is then

$$\int_{-2}^{2} \left((3 - y^2) - (y^2 + 5)\right) dy = \int_{-2}^{2} \left(8 - 2y^2\right) dy = \left(8y - \frac{2}{3}y^3\right)\Bigg|_{-2}^{2} = \frac{64}{3}.$$

21. Figure 17 shows the region enclosed by $x = y^3 - 26y + 10$ and $x = 40 - 6y^2 - y^3$. Match the equations with the curves and compute the area of the region.

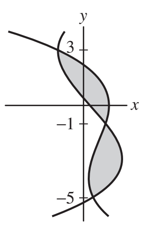


FIGURE 17

SOLUTION Substituting $y = 0$ into the equations for both curves indicates that the graph of $x = y^3 - 26y + 10$ passes through the point $(10, 0)$ while the graph of $x = 40 - 6y^2 - y^3$ passes through the point $(40, 0)$. Therefore, over the

y-interval $[-1, 3]$, the graph of $x = 40 - 6y^2 - y^3$ lies to the right of the graph of $x = y^3 - 26y + 10$. The orientation of the two graphs is reversed over the y-interval $[-5, -1]$. Hence, the area of the shaded region is

$$\int_{-5}^{-1} \left((y^3 - 26y + 10) - (40 - 6y^2 - y^3)\right) dy + \int_{-1}^{3} \left((40 - 6y^2 - y^3) - (y^3 - 26y + 10)\right) dy$$

$$= \int_{-5}^{-1} \left(2y^3 + 6y^2 - 26y - 30\right) dy + \int_{-1}^{3} \left(-2y^3 - 6y^2 + 26y + 30\right) dy$$

$$= \left(\frac{1}{2}y^4 + 2y^3 - 13y^2 - 30y\right)\bigg|_{-5}^{-1} + \left(-\frac{1}{2}y^4 - 2y^3 + 13y^2 + 30y\right)\bigg|_{-1}^{3} = 256.$$

22. Figure 18 shows the region enclosed by $y = x^3 - 6x$ and $y = 8 - 3x^2$. Match the equations with the curves and compute the area of the region.

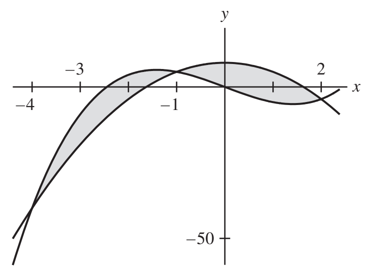


FIGURE 18 Region between $y = x^3 - 6x$ and $y = 8 - 3x^2$.

SOLUTION Setting $x^3 - 6x = 8 - 3x^2$ yields $(x + 1)(x + 4)(x - 2) = 0$, so the two curves intersect at $x = -4$, $x = -1$ and $x = 2$. Over the interval $[-4, -1]$, $y = x^3 - 6x$ is the upper curve, while $y = 8 - 3x^2$ is the upper curve over the interval $[-1, 2]$. The area of the region enclosed by the two curves is then

$$\int_{-4}^{-1} \left((x^3 - 6x) - (8 - 3x^2)\right) dx + \int_{-1}^{2} \left((8 - 3x^2) - (x^3 - 6x)\right) dx$$

$$= \left(\frac{1}{4}x^4 - 3x^2 - 8x + x^3\right)\bigg|_{-4}^{-1} + \left(8x - x^3 - \frac{1}{4}x^4 + 3x^2\right)\bigg|_{-1}^{2} = \frac{81}{4} + \frac{81}{4} = \frac{81}{2}.$$

In Exercises 23 and 24, find the area enclosed by the graphs in two ways: by integrating along the x-axis and by integrating along the y-axis.

23. $x = 9 - y^2, \quad x = 5$

SOLUTION Along the y-axis, we have points of intersection at $y = \pm 2$. Therefore, the area enclosed by the two curves is

$$\int_{-2}^{2} \left(9 - y^2 - 5\right) dy = \int_{-2}^{2} \left(4 - y^2\right) dy = \left(4y - \frac{1}{3}y^3\right)\bigg|_{-2}^{2} = \frac{32}{3}.$$

Along the x-axis, we have integration limits of $x = 5$ and $x = 9$. Therefore, the area enclosed by the two curves is

$$\int_{5}^{9} 2\sqrt{9 - x}\, dx = -\frac{4}{3}(9 - x)^{3/2}\bigg|_{5}^{9} = 0 - \left(-\frac{32}{3}\right) = \frac{32}{3}.$$

24. The *semicubical parabola* $y^2 = x^3$ and the line $x = 1$.

SOLUTION Since $y^2 = x^3$, it follows that $x \ge 0$ since $y^2 \ge 0$. Therefore, $y = \pm x^{3/2}$, and the area of the region enclosed by the semicubical parabola and $x = 1$ is

$$\int_{0}^{1} \left(x^{3/2} - (-x^{3/2})\right) dx = \int_{0}^{1} 2x^{3/2}\, dx = \frac{4}{5}x^{5/2}\bigg|_{0}^{1} = \frac{4}{5}.$$

Along the y-axis, we have integration limits of $y = \pm 1$. Therefore, the area enclosed by the two curves is

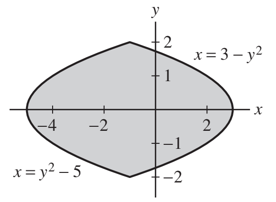
$$\int_{-1}^{1} \left(1 - y^{2/3}\right) dy = \left(y - \frac{3}{5}y^{5/3}\right)\bigg|_{-1}^{1} = \left(1 - \frac{3}{5}\right) - \left(-1 + \frac{3}{5}\right) = \frac{4}{5}.$$

In Exercises 25 and 26, find the area of the region using the method (integration along either the x- or the y-axis) that requires you to evaluate just one integral.

25. Region between $y^2 = x + 5$ and $y^2 = 3 - x$

SOLUTION From the figure below, we see that integration along the x-axis would require two integrals, but integration along the y-axis requires only one integral. Setting $y^2 - 5 = 3 - y^2$ yields points of intersection at $y = \pm 2$. Thus, the area is given by

$$\int_{-2}^{2} \left((3 - y^2) - (y^2 + 5) \right) dy = \int_{-2}^{2} \left(8 - 2y^2 \right) dy = \left(8y - \frac{2}{3}y^3 \right) \Big|_{-2}^{2} = \frac{64}{3}.$$



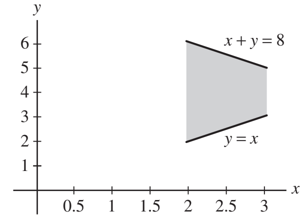
26. Region between $y = x$ and $x + y = 8$ over $[2, 3]$

SOLUTION From the figure below, we see that integration along the y-axis would require three integrals, but integration along the x-axis requires only one integral. The area of the region is then

$$\int_{2}^{3} \left((8 - x) - x \right) dx = (8x - x^2) \Big|_{2}^{3} = (24 - 9) - (16 - 4) = 3.$$

As a check, the area of a trapezoid is given by

$$\frac{h}{2}(b_1 + b_2) = \frac{1}{2}(4 + 2) = 3.$$

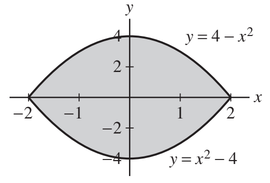


In Exercises 27–44, sketch the region enclosed by the curves and compute its area as an integral along the x- or y-axis.

27. $y = 4 - x^2, \quad y = x^2 - 4$

SOLUTION Setting $4 - x^2 = x^2 - 4$ yields $2x^2 = 8$ or $x^2 = 4$. Thus, the curves $y = 4 - x^2$ and $y = x^2 - 4$ intersect at $x = \pm 2$. From the figure below, we see that $y = 4 - x^2$ lies above $y = x^2 - 4$ over the interval $[-2, 2]$; hence, the area of the region enclosed by the curves is

$$\int_{-2}^{2} \left((4 - x^2) - (x^2 - 4) \right) dx = \int_{-2}^{2} (8 - 2x^2)\, dx = \left(8x - \frac{2}{3}x^3 \right) \Big|_{-2}^{2} = \frac{64}{3}.$$



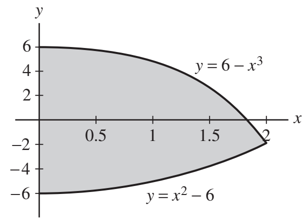
28. $y = x^2 - 6, \quad y = 6 - x^3, \quad y\text{-axis}$

SOLUTION Setting $x^2 - 6 = 6 - x^3$ yields

$$0 = x^3 + x^2 - 12 = (x - 2)(x^2 + 3x + 6),$$

so the curves $y = x^2 - 6$ and $y = 6 - x^3$ intersect at $x = 2$. Using the graph shown below, we see that $y = 6 - x^3$ lies above $y = x^2 - 6$ over the interval [0, 2]; hence, the area of the region enclosed by these curves and the y-axis is

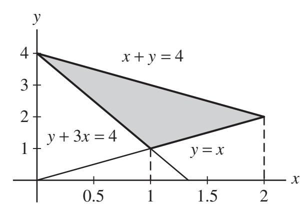
$$\int_0^2 \left((6 - x^3) - (x^2 - 6)\right) dx = \int_0^2 (-x^3 - x^2 + 12)\, dx = \left(-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 12x\right)\Big|_0^2 = \frac{52}{3}.$$



29. $x + y = 4, \quad x - y = 0, \quad y + 3x = 4$

SOLUTION From the graph below, we see that the top of the region enclosed by the three lines is always bounded by $x + y = 4$. On the other hand, the bottom of the region is bounded by $y + 3x = 4$ for $0 \le x \le 1$ and by $x - y = 0$ for $1 \le x \le 2$. The total area of the region is then

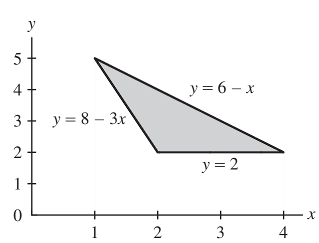
$$\int_0^1 ((4 - x) - (4 - 3x))\, dx + \int_1^2 ((4 - x) - x)\, dx = \int_0^1 2x\, dx + \int_1^2 (4 - 2x)\, dx$$
$$= x^2\Big|_0^1 + (4x - x^2)\Big|_1^2 = 1 + (8 - 4) - (4 - 1) = 2.$$



30. $y = 8 - 3x, \quad y = 6 - x, \quad y = 2$

SOLUTION From the figure below, we see that the graph of $y = 6 - x$ lies to the right of the graph of $y = 8 - 3x$, so integration in y is most appropriate for this problem. Setting $8 - 3x = 6 - x$ yields $x = 1$, so the y-coordinate of the point of intersection between $y = 8 - 3x$ and $y = 6 - x$ is 5. The area bounded by the three given curves is thus

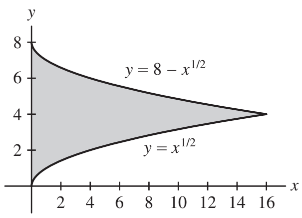
$$\begin{aligned} A &= \int_2^5 \left((6 - y) - \frac{1}{3}(8 - y)\right) dy \\ &= \int_2^5 \left(\frac{10}{3} - \frac{2}{3}y\right) dy \\ &= \left(\frac{10}{3}y - \frac{1}{3}y^2\right)\Big|_2^5 \\ &= \left(\frac{50}{3} - \frac{25}{3}\right) - \left(\frac{20}{3} - \frac{4}{3}\right) \\ &= 3. \end{aligned}$$



31. $y = 8 - \sqrt{x}, \quad y = \sqrt{x}, \quad x = 0$

SOLUTION Setting $8 - \sqrt{x} = \sqrt{x}$ yields $\sqrt{x} = 4$ or $x = 16$. Using the graph shown below, we see that $y = 8 - \sqrt{x}$ lies above $y = \sqrt{x}$ over the interval [0, 16]. The area of the region enclosed by these two curves and the y-axis is then

$$\int_0^{16} \left(8 - \sqrt{x} - \sqrt{x}\right) dx = \int_0^{16} \left(8 - 2\sqrt{x}\right) dx = \left(8x - \frac{4}{3}x^{3/2}\right)\Big|_0^{16} = \frac{128}{3}.$$



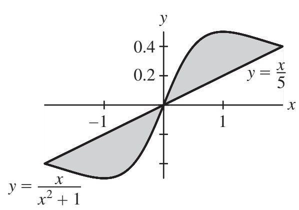
32. $y = \dfrac{x}{x^2+1}, \quad y = \dfrac{x}{5}$

SOLUTION Setting

$$\frac{x}{x^2+1} = \frac{x}{5} \quad \text{yields} \quad x = -2, 0, 2.$$

From the figure below, we see that the graph of $y = x/5$ lies above the graph of $y = x/(x^2+1)$ over $[-2, 0]$ and that the orientation is reversed over $[0, 2]$. Thus,

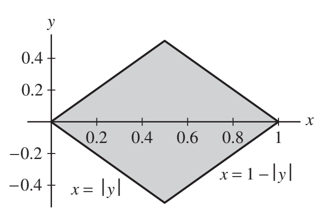
$$\begin{aligned} A &= \int_{-2}^{0} \left(\frac{x}{5} - \frac{x}{x^2+1}\right) dx + \int_0^2 \left(\frac{x}{x^2+1} - \frac{x}{5}\right) dx \\ &= \left(\frac{x^2}{10} - \frac{1}{2}\ln(x^2+1)\right)\Big|_{-2}^{0} + \left(\frac{1}{2}\ln(x^2+1) - \frac{x^2}{10}\right)\Big|_0^2 \\ &= \left(0 - \frac{2}{5} + \frac{1}{2}\ln 5\right) + \left(\frac{1}{2}\ln 5 - \frac{2}{5} - 0\right) \\ &= \ln 5 - \frac{4}{5}. \end{aligned}$$



33. $x = |y|, \quad x = 1 - |y|$

SOLUTION From the graph below, we see that the region enclosed by the curves $x = |y|$ and $x = 1 - |y|$ is symmetric with respect to the x-axis. We can therefore determine the total area by doubling the area in the first quadrant. For $y > 0$, setting $y = 1 - y$ yields $y = \frac{1}{2}$ as the point of intersection. Moreover, $x = 1 - |y| = 1 - y$ lies to the right of $x = |y| = y$, so the total area of the region is

$$2\int_0^{1/2} \left((1-y) - y\right) dy = 2\left(y - y^2\right)\Big|_0^{1/2} = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}.$$



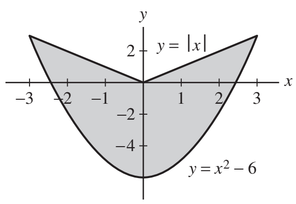
34. $y = |x|, \quad y = x^2 - 6$

SOLUTION From the graph below, we see that the region enclosed by the curves $y = |x|$ and $y = x^2 - 6$ is symmetric with respect to the y-axis. We can therefore determine the total area of the region by doubling the area of the portion of the region to the right of the y-axis. For $x > 0$, setting $x = x^2 - 6$ yields

$$0 = x^2 - x - 6 = (x-3)(x+2),$$

so the curves intersect at $x = 3$. Moreover, on the interval $[0, 3]$, $y = |x| = x$ lies above $y = x^2 - 6$. Therefore, the area of the region enclosed by the two curves is

$$2\int_0^3 \left(x - (x^2 - 6)\right) dx = 2\left(\frac{1}{2}x^2 - \frac{1}{3}x^3 + 6x\right)\Big|_0^3 = 2\left(\frac{9}{2} - 9 + 18\right) = 27.$$



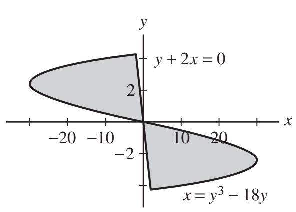
35. $x = y^3 - 18y, \quad y + 2x = 0$

SOLUTION Setting $y^3 - 18y = -\frac{y}{2}$ yields

$$0 = y^3 - \frac{35}{2}y = y\left(y^2 - \frac{35}{2}\right),$$

so the points of intersection occur at $y = 0$ and $y = \pm\frac{\sqrt{70}}{2}$. From the graph below, we see that both curves are symmetric with respect to the origin. It follows that the portion of the region enclosed by the curves in the second quadrant is identical to the region enclosed in the fourth quadrant. We can therefore determine the total area enclosed by the two curves by doubling the area enclosed in the second quadrant. In the second quadrant, $y + 2x = 0$ lies to the right of $x = y^3 - 18y$, so the total area enclosed by the two curves is

$$2\int_0^{\sqrt{70}/2} \left(-\frac{y}{2} - (y^3 - 18y)\right) dy = 2\left(\frac{35}{4}y^2 - \frac{1}{4}y^4\right)\Big|_0^{\sqrt{70}/2} = 2\left(\frac{1225}{8} - \frac{1225}{16}\right) = \frac{1225}{8}.$$



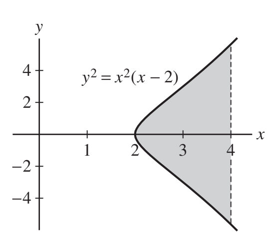
36. $y = x\sqrt{x-2}, \quad y = -x\sqrt{x-2}, \quad x = 4$

SOLUTION Note that $y = x\sqrt{x-2}$ and $y = -x\sqrt{x-2}$ are the upper and lower branches, respectively, of the curve $y^2 = x^2(x-2)$. The area enclosed by this curve and the vertical line $x = 4$ is

$$\int_2^4 \left(x\sqrt{x-2} - (-x\sqrt{x-2})\right) dx = \int_2^4 2x\sqrt{x-2}\, dx.$$

Substitute $u = x - 2$. Then $du = dx$, $x = u + 2$ and

$$\int_2^4 2x\sqrt{x-2}\, dx = \int_0^2 2(u+2)\sqrt{u}\, du = \int_0^2 \left(2u^{3/2} + 4u^{1/2}\right) du = \left(\frac{4}{5}u^{5/2} + \frac{8}{3}u^{3/2}\right)\Big|_0^2 = \frac{128\sqrt{2}}{15}.$$



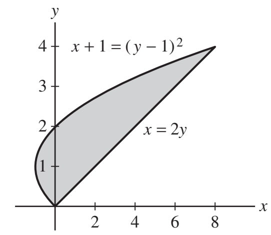
37. $x = 2y, \quad x + 1 = (y-1)^2$

SOLUTION Setting $2y = (y-1)^2 - 1$ yields

$$0 = y^2 - 4y = y(y-4),$$

so the two curves intersect at $y = 0$ and at $y = 4$. From the graph below, we see that $x = 2y$ lies to the right of $x + 1 = (y-1)^2$ over the interval $[0, 4]$ along the y-axis. Thus, the area of the region enclosed by the two curves is

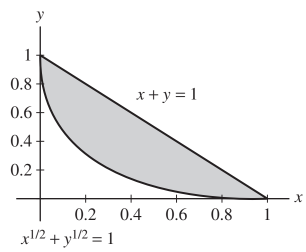
$$\int_0^4 \left(2y - ((y-1)^2 - 1)\right) dy = \int_0^4 \left(4y - y^2\right) dy = \left(2y^2 - \frac{1}{3}y^3\right)\Big|_0^4 = \frac{32}{3}.$$



38. $x + y = 1, \quad x^{1/2} + y^{1/2} = 1$

SOLUTION From the graph below, we see that the two curves intersect at $x = 0$ and at $x = 1$ and that $x + y = 1$ lies above $x^{1/2} + y^{1/2} = 1$. The area of the region enclosed by the two curves is then

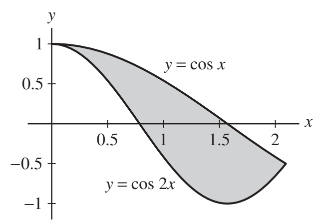
$$\int_0^1 \left((1-x) - (1-\sqrt{x})^2\right) dx = \int_0^1 \left(-2x + 2\sqrt{x}\right) dx = \left(-x^2 + \frac{4}{3}x^{3/2}\right)\Big|_0^1 = \frac{1}{3}.$$



39. $y = \cos x, \quad y = \cos 2x, \quad x = 0, \quad x = \dfrac{2\pi}{3}$

SOLUTION From the graph below, we see that $y = \cos x$ lies above $y = \cos 2x$ over the interval $[0, \frac{2\pi}{3}]$. The area of the region enclosed by the two curves is therefore

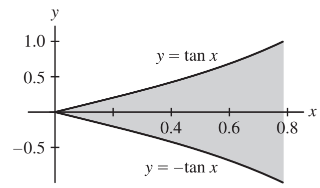
$$\int_0^{2\pi/3} (\cos x - \cos 2x)\, dx = \left(\sin x - \frac{1}{2}\sin 2x\right)\Big|_0^{2\pi/3} = \frac{3\sqrt{3}}{4}.$$



40. $y = \tan x, \quad y = -\tan x, \quad x = \dfrac{\pi}{4}$

SOLUTION Because the graph of $y = \tan x$ lies above the graph of $y = -\tan x$ over the interval $[0, \pi/4]$, the area bounded by the two curves is

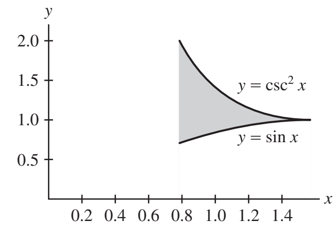
$$\begin{aligned} A &= \int_0^{\pi/4} (\tan x - (-\tan x))\, dx = 2\int_0^{\pi/4} \tan x\, dx \\ &= 2\ln|\sec x|\Big|_0^{\pi/4} \\ &= 2\ln 2 - 2\ln 1 = 2\ln 2. \end{aligned}$$



41. $y = \sin x, \quad y = \csc^2 x, \quad x = \dfrac{\pi}{4}$

SOLUTION Over the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$, $y = \csc^2 x$ lies above $y = \sin x$. The area of the region enclosed by the two curves is then

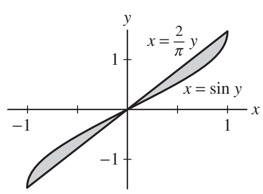
$$\int_{\pi/4}^{\pi/2} \left(\csc^2 x - \sin x\right) dx = \left(-\cot x + \cos x\right)\Big|_{\pi/4}^{\pi/2} = (0 - 0) - \left(-1 + \frac{\sqrt{2}}{2}\right) = 1 - \frac{\sqrt{2}}{2}.$$



42. $x = \sin y, \quad x = \dfrac{2}{\pi} y$

SOLUTION Here, integration along the y-axis will require less work than integration along the x-axis. The curves intersect when $\frac{2y}{\pi} = \sin y$ or when $y = 0, \pm\frac{\pi}{2}$. From the graph below, we see that both curves are symmetric with respect to the origin. It follows that the portion of the region enclosed by the curves in the first quadrant is identical to the region enclosed in the third quadrant. We can therefore determine the total area enclosed by the two curves by doubling the area enclosed in the first quadrant. In the first quadrant, $x = \sin y$ lies to the right of $x = \frac{2y}{\pi}$, so the total area enclosed by the two curves is

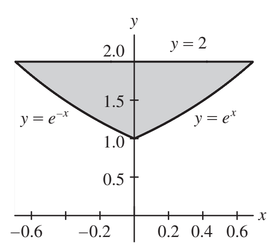
$$2\int_0^{\pi/2} \left(\sin y - \frac{2}{\pi} y\right) dy = 2\left(-\cos y - \frac{1}{\pi} y^2\right)\Big|_0^{\pi/2} = 2\left[\left(0 - \frac{\pi}{4}\right) - (-1 - 0)\right] = 2 - \frac{\pi}{2}.$$



43. $y = e^x, \quad y = e^{-x}, \quad y = 2$

SOLUTION From the figure below, we see that integration in y would be most appropriate - unfortunately, we have not yet learned how to integrate $\ln y$. Consequently, we will calculate the area using two integrals in x:

$$\begin{aligned} A &= \int_{-\ln 2}^{0} (2 - e^{-x})\, dx + \int_0^{\ln 2} (2 - e^x)\, dx \\ &= \left(2x + e^{-x}\right)\Big|_{-\ln 2}^{0} + \left(2x - e^x\right)\Big|_0^{\ln 2} \\ &= 1 - (-2\ln 2 + 2) + (2\ln 2 - 2) - (-1) = 4\ln 2 - 2. \end{aligned}$$



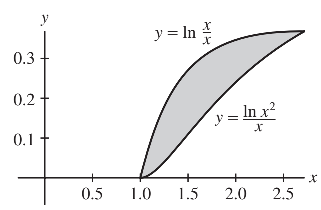
44. $y = \dfrac{\ln x}{x}, \quad y = \dfrac{(\ln x)^2}{x}$

SOLUTION Setting

$$\frac{\ln x}{x} = \frac{(\ln x)^2}{2} \quad \text{yields} \quad x = 1, e.$$

From the figure below, we see that the graph of $y = \ln x/x$ lies above the graph of $y = (\ln x)^2/x$ over the interval $[1, e]$. Thus, the area between the two curves is

$$A = \int_1^e \left(\frac{\ln x}{x} - \frac{(\ln x)^2}{x}\right) dx$$
$$= \left(\frac{1}{2}(\ln x)^2 - \frac{1}{3}(\ln x)^3\right)\Big|_1^e$$
$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$



45. CAS Plot

$$y = \frac{x}{\sqrt{x^2+1}} \quad \text{and} \quad y = (x-1)^2$$

on the same set of axes. Use a computer algebra system to find the points of intersection numerically and compute the area between the curves.

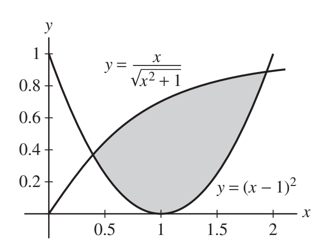
SOLUTION Using a computer algebra system, we find that the curves

$$y = \frac{x}{\sqrt{x^2+1}} \qquad \text{and} \qquad y = (x-1)^2$$

intersect at $x = 0.3943285581$ and at $x = 1.942944418$. From the graph below, we see that $y = \frac{x}{\sqrt{x^2+1}}$ lies above $y = (x-1)^2$, so the area of the region enclosed by the two curves is

$$\int_{0.3943285581}^{1.942944418} \left(\frac{x}{\sqrt{x^2+1}} - (x-1)^2\right) dx = 0.7567130951$$

The value of the definite integral was also obtained using a computer algebra system.



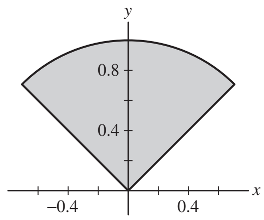
46. Sketch a region whose area is represented by

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left(\sqrt{1-x^2} - |x|\right) dx$$

and evaluate using geometry.

SOLUTION Matching the integrand $\sqrt{1-x^2} - |x|$ with the $y_{\text{TOP}} - y_{\text{BOT}}$ template for calculating area, we see that the region in question is bounded along the top by the curve $y = \sqrt{1-x^2}$ (the upper half of the unit circle) and is bounded along the bottom by the curve $y = |x|$. Hence, the region is $\frac{1}{4}$ of the unit circle (see the figure below). The area of the region must then be

$$\frac{1}{4}\pi(1)^2 = \frac{\pi}{4}.$$



47. Athletes 1 and 2 run along a straight track with velocities $v_1(t)$ and $v_2(t)$ (in m/s) as shown in Figure 19.

(a) Which of the following is represented by the area of the shaded region over [0, 10]?

i. The distance between athletes 1 and 2 at time $t = 10$ s.

ii. The difference in the distance traveled by the athletes over the time interval [0, 10].

(b) Does Figure 19 give us enough information to determine who is ahead at time $t = 10$ s?

(c) If the athletes begin at the same time and place, who is ahead at $t = 10$ s? At $t = 25$ s?

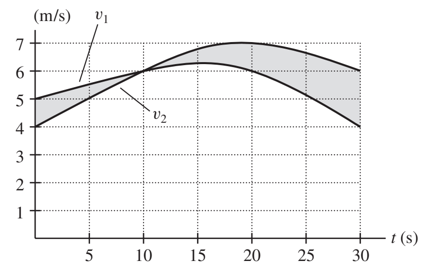


FIGURE 19

SOLUTION

(a) The area of the shaded region over [0, 10] represents **(ii)**: the difference in the distance traveled by the athletes over the time interval [0, 10].

(b) No, Figure 19 does not give us enough information to determine who is ahead at time $t = 10$ s. We would additionally need to know the relative position of the runners at $t = 0$ s.

(c) If the athletes begin at the same time and place, then athlete 1 is ahead at $t = 10$ s because the velocity graph for athlete 1 lies above the velocity graph for athlete 2 over the interval [0, 10]. Over the interval [10, 25], the velocity graph for athlete 2 lies above the velocity graph for athlete 1 and appears to have a larger area than the area between the graphs over [0, 10]. Thus, it appears that athlete 2 is ahead at $t = 25$ s.

48. Express the area (not signed) of the shaded region in Figure 20 as a sum of three integrals involving $f(x)$ and $g(x)$.

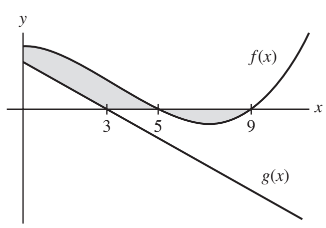


FIGURE 20

SOLUTION Because either the curve bounding the top of the region or the curve bounding the bottom of the region or both change at $x = 3$ and at $x = 5$, the area is calculated using three integrals. Specifically, the area is

$$\int_0^3 (f(x) - g(x))\,dx + \int_3^5 (f(x) - 0)\,dx + \int_5^9 (0 - f(x))\,dx$$

$$= \int_0^3 (f(x) - g(x))\,dx + \int_3^5 f(x)\,dx - \int_5^9 f(x)\,dx.$$

49. Find the area enclosed by the curves $y = c - x^2$ and $y = x^2 - c$ as a function of c. Find the value of c for which this area is equal to 1.

SOLUTION The curves intersect at $x = \pm\sqrt{c}$, with $y = c - x^2$ above $y = x^2 - c$ over the interval $[-\sqrt{c}, \sqrt{c}]$. The area of the region enclosed by the two curves is then

$$\int_{-\sqrt{c}}^{\sqrt{c}} \left(c - x^2) - (x^2 - c)\right) dx = \int_{-\sqrt{c}}^{\sqrt{c}} \left(2c - 2x^2\right) dx = \left(2cx - \frac{2}{3}x^3\right)\Big|_{-\sqrt{c}}^{\sqrt{c}} = \frac{8}{3}c^{3/2}.$$

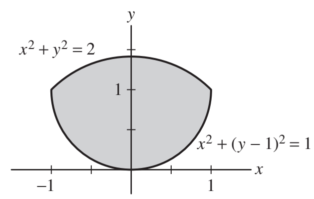
In order for the area to equal 1, we must have $\frac{8}{3}c^{3/2} = 1$, which gives

$$c = \frac{9^{1/3}}{4} \approx 0.520021.$$

50. Set up (but do not evaluate) an integral that expresses the area between the circles $x^2 + y^2 = 2$ and $x^2 + (y - 1)^2 = 1$.

SOLUTION Setting $2 - y^2 = 1 - (y - 1)^2$ yields $y = 1$. The two circles therefore intersect at the points $(1, 1)$ and $(-1, 1)$. From the graph below, we see that over the interval $[-1, 1]$, the upper half of the circle $x^2 + y^2 = 2$ lies above the lower half of the circle $x^2 + (y - 1)^2 = 1$. The area enclosed by the two circles is therefore given by the integral

$$\int_{-1}^{1} \left(\sqrt{2 - x^2} - (1 - \sqrt{1 - x^2})\right) dx.$$



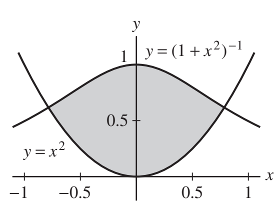
51. Set up (but do not evaluate) an integral that expresses the area between the graphs of $y = (1 + x^2)^{-1}$ and $y = x^2$.

SOLUTION Setting $(1 + x^2)^{-1} = x^2$ yields $x^4 + x^2 - 1 = 0$. This is a quadratic equation in the variable x^2. By the quadratic formula,

$$x^2 = \frac{-1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

As x^2 must be nonnegative, we discard $\frac{-1-\sqrt{5}}{2}$. Finally, we find the two curves intersect at $x = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}$. From the graph below, we see that $y = (1 + x^2)^{-1}$ lies above $y = x^2$. The area enclosed by the two curves is then

$$\int_{-\sqrt{\frac{-1+\sqrt{5}}{2}}}^{\sqrt{\frac{-1+\sqrt{5}}{2}}} \left((1 + x^2)^{-1} - x^2\right) dx.$$

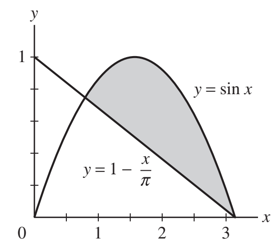


52. CAS Find a numerical approximation to the area above $y = 1 - (x/\pi)$ and below $y = \sin x$ (find the points of intersection numerically).

SOLUTION The region in question is shown in the figure below. Using a computer algebra system, we find that $y = 1 - x/\pi$ and $y = \sin x$ intersect on the left at $x = 0.8278585215$. Analytically, we determine the two curves intersect on the right at $x = \pi$. The area above $y = 1 - x/\pi$ and below $y = \sin x$ is then

$$\int_{0.8278585215}^{\pi} \left(\sin x - \left(1 - \frac{x}{\pi}\right)\right) dx = 0.8244398727,$$

where the definite integral was evaluated using a computer algebra system.

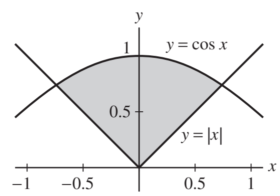


53. CAS Find a numerical approximation to the area above $y = |x|$ and below $y = \cos x$.

SOLUTION The region in question is shown in the figure below. We see that the region is symmetric with respect to the y-axis, so we can determine the total area of the region by doubling the area of the portion in the first quadrant. Using a computer algebra system, we find that $y = \cos x$ and $y = |x|$ intersect at $x = 0.7390851332$. The area of the region between the two curves is then

$$2\int_0^{0.7390851332} (\cos x - x)\, dx = 0.8009772242,$$

where the definite integral was evaluated using a computer algebra system.

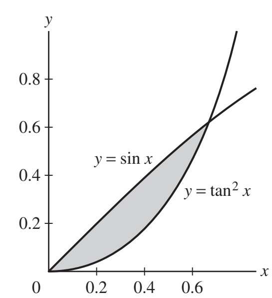


54. CAS Use a computer algebra system to find a numerical approximation to the number c (besides zero) in $\left[0, \frac{\pi}{2}\right]$, where the curves $y = \sin x$ and $y = \tan^2 x$ intersect. Then find the area enclosed by the graphs over $[0, c]$.

SOLUTION The region in question is shown in the figure below. Using a computer algebra system, we find that $y = \sin x$ and $y = \tan^2 x$ intersect at $x = 0.6662394325$. The area of the region enclosed by the two curves is then

$$\int_0^{0.6662394325} \left(\sin x - \tan^2 x\right) dx = 0.09393667698,$$

where the definite integral was evaluated using a computer algebra system.



55. The back of Jon's guitar (Figure 21) is 19 inches long. Jon measured the width at 1-in. intervals, beginning and ending $\frac{1}{2}$ in. from the ends, obtaining the results

$$6,\ 9,\ 10.25,\ 10.75,\ 10.75,\ 10.25,\ 9.75,\ 9.5,\ 10,\ 11.25,$$

$$12.75,\ 13.75,\ 14.25,\ 14.5,\ 14.5,\ 14,\ 13.25,\ 11.25,\ 9$$

Use the midpoint rule to estimate the area of the back.

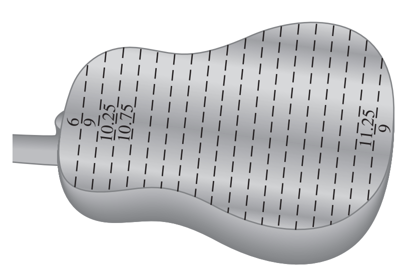


FIGURE 21 Back of guitar.

SOLUTION Note that the measurements were taken at the midpoint of each one-inch section of the guitar. For example, in the 0 to 1 inch section, the midpoint would be at $\frac{1}{2}$ inch, and thus the approximate area of the first rectangle would be $1 \cdot 6$ inches^2. An approximation for the entire area is then

$$A = 1(6 + 9 + 10.25 + 10.75 + 10.75 + 10.25 + 9.75 + 9.5 + 10 + 11.25 \\ + 12.75 + 13.75 + 14.25 + 14.5 + 14.5 + 14 + 13.25 + 11.25 + 9) \\ = 214.75 \text{ in}^2.$$

56. Referring to Figure 1 at the beginning of this section, estimate the projected number of additional joules produced in the years 2009–2030 as a result of government stimulus spending in 2009–2010. *Note:* One watt is equal to one joule per second, and one gigawatt is 10^9 watts.

SOLUTION We make some rough estimates of the areas depicted in Figure 1. From 2009 through 2012, the area between the curves is roughly a right triangle with a base of 3 and a height of 40; from 2012 through 2020, the area is roughly an 8 by 40 rectangle. Finally, from 2020 through 2030, the area is roughly a trapezoid with height 10 and bases 40 and 27. Thus, additional energy produced is approximately

$$\frac{1}{2}(3)(40) + 8(40) + \frac{1}{2}(10)(40 + 27) = 715 \text{ gigawatt-years.}$$

Because 1 gigawatt is equal to 10^9 joules per second and 1 year (assuming 365 days) is equal to 31536000 seconds, the additional joules produced in the years 2009–2030 as a result of government stimulus spending in 2009–2010 is approximately 2.25×10^{19}.

Exercises 57 and 58 use the notation and results of Exercises 49–51 of Section 3.4. For a given country, $F(r)$ is the fraction of total income that goes to the bottom rth fraction of households. The graph of $y = F(r)$ is called the Lorenz curve.

57. Let A be the area between $y = r$ and $y = F(r)$ over the interval [0, 1] (Figure 22). The **Gini index** is the ratio $G = A/B$, where B is the area under $y = r$ over [0, 1].

(a) Show that $G = 2\int_0^1 (r - F(r))\,dr$.

(b) Calculate G if

$$F(r) = \begin{cases} \frac{1}{3}r & \text{for } 0 \le r \le \frac{1}{2} \\ \frac{5}{3}r - \frac{2}{3} & \text{for } \frac{1}{2} \le r \le 1 \end{cases}$$

(c) The Gini index is a measure of income distribution, with a lower value indicating a more equal distribution. Calculate G if $F(r) = r$ (in this case, all households have the same income by Exercise 51(b) of Section 3.4).

(d) What is G if all of the income goes to one household? *Hint:* In this extreme case, $F(r) = 0$ for $0 \le r < 1$.

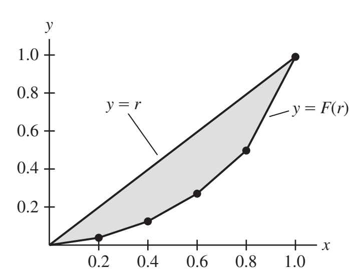


FIGURE 22 Lorenz Curve for U.S. in 2001.

SOLUTION

(a) Because the graph of $y = r$ lies above the graph of $y = F$¨ in Figure 22,

$$A = \int_0^1 (r - F(r))\,dr.$$

Moreover,

$$B = \int_0^1 r\,dr = \frac{1}{2}r^2\bigg|_0^1 = \frac{1}{2}.$$

Thus,

$$G = \frac{A}{B} = 2\int_0^1 (r - F(r))\,dr.$$

(b) With the given $F(r)$,

$$G = 2\int_0^{1/2}\left(r - \frac{1}{3}r\right)dr + 2\int_{1/2}^{1}\left(r - \left(\frac{5}{3}r - \frac{2}{3}\right)\right)dr$$
$$= \frac{4}{3}\int_0^{1/2} r\,dr - \frac{4}{3}\int_{1/2}^{1}(r-1)\,dr$$
$$= \frac{2}{3}r^2\Big|_0^{1/2} - \frac{4}{3}\left(\frac{1}{2}r^2 - r\right)\Big|_{1/2}^{1}$$
$$= \frac{1}{6} - \frac{4}{3}\left(-\frac{1}{2}\right) + \frac{4}{3}\left(-\frac{3}{8}\right) = \frac{1}{3}.$$

(c) If $F(r) = r$, then

$$G = 2\int_0^1 (r - r)\,dr = 0.$$

(d) If $F(r) = 0$ for $0 \le r < 1$, then

$$G = 2\int_0^1 (r - 0)\,dr = 2\left(\frac{1}{2}r^2\right)\Big|_0^1 = 2\left(\frac{1}{2}\right) = 1.$$

58. Calculate the Gini index of the United States in the year 2001 from the Lorenz curve in Figure 22, which consists of segments joining the data points in the following table.

r	0	0.2	0.4	0.6	0.8	1
$F(r)$	0	0.035	0.123	0.269	0.499	1

SOLUTION From part (a) of the previous exercise,

$$G = 2\int_0^1 (r - F(r))\,dr = 1 - 2\int_0^1 F(r)\,dr.$$

Because $F(r)$ consists of segments joining the data points in the given table, the area under the graph of $y = F(r)$ consists of a triangle and four trapezoids. The area is

$$\frac{1}{2}(0.2)(0.035) + \frac{1}{2}(0.2)(0.035 + 0.123) + \frac{1}{2}(0.2)(0.123 + 0.269) + \frac{1}{2}(0.2)(0.269 + 0.499) + \frac{1}{2}(0.2)(0.499 + 1)$$

or 0.2852. Finally,

$$G = 1 - 2(0.2852) = 0.4296.$$

Further Insights and Challenges

59. Find the line $y = mx$ that divides the area under the curve $y = x(1 - x)$ over [0, 1] into two regions of equal area.

SOLUTION First note that

$$\int_0^1 x(1-x)\,dx = \int_0^1 \left(x - x^2\right)dx = \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{6}.$$

Now, the line $y = mx$ and the curve $y = x(1 - x)$ intersect when $mx = x(1 - x)$, or at $x = 0$ and at $x = 1 - m$. The area of the region enclosed by the two curves is then

$$\int_0^{1-m} (x(1-x) - mx)\,dx = \int_0^{1-m}\left((1-m)x - x^2\right)dx = \left((1-m)\frac{x^2}{2} - \frac{1}{3}x^3\right)\Big|_0^{1-m} = \frac{1}{6}(1-m)^3.$$

To have $\frac{1}{6}(1-m)^3 = \frac{1}{2}\cdot\frac{1}{6}$ requires

$$m = 1 - \left(\frac{1}{2}\right)^{1/3} \approx 0.206299.$$

60. CAS Let c be the number such that the area under $y = \sin x$ over $[0, \pi]$ is divided in half by the line $y = cx$ (Figure 23). Find an equation for c and solve this equation *numerically* using a computer algebra system.

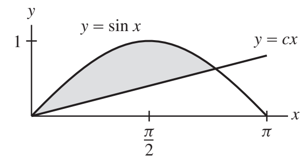


FIGURE 23

SOLUTION First note that

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = 2.$$

Now, let $y = cx$ and $y = \sin x$ intersect at $x = a$. Then $ca = \sin a$, which gives $c = \frac{\sin a}{a}$ and $y = cx = \frac{\sin a}{a}x$. Then

$$\int_0^a \left(\sin x - \frac{\sin a}{a}x\right) dx = \left(-\cos x - \frac{\sin a}{2a}x^2\right)\Big|_0^a = 1 - \cos a - \frac{a\sin a}{2}.$$

We need

$$1 - \cos a - \frac{a\sin a}{2} = \frac{1}{2}(2) = 1,$$

which gives $a = 2.458714176$ and finally

$$c = \frac{\sin a}{a} = 0.2566498570.$$

61. Explain geometrically (without calculation):

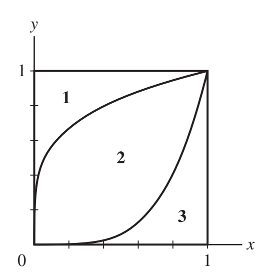
$$\int_0^1 x^n\,dx + \int_0^1 x^{1/n}\,dx = 1 \qquad (\text{for } n > 0)$$

SOLUTION Let A_1 denote the area of region 1 in the figure below. Define A_2 and A_3 similarly. It is clear from the figure that

$$A_1 + A_2 + A_3 = 1.$$

Now, note that x^n and $x^{1/n}$ are inverses of each other. Therefore, the graphs of $y = x^n$ and $y = x^{1/n}$ are symmetric about the line $y = x$, so regions 1 and 3 are also symmetric about $y = x$. This guarantees that $A_1 = A_3$. Finally,

$$\int_0^1 x^n\,dx + \int_0^1 x^{1/n}\,dx = A_3 + (A_2 + A_3) = A_1 + A_2 + A_3 = 1.$$



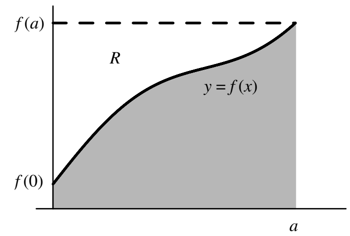
62. Let $f(x)$ be an increasing function with inverse $g(x)$. Explain geometrically:

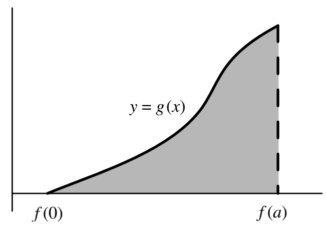
$$\int_0^a f(x)\,dx + \int_{f(0)}^{f(a)} g(x)\,dx = af(a)$$

SOLUTION The region whose area is represented by $\int_0^a f(x)\,dx$ is shown as the shaded portion of the graph below on the left, and the region whose area is represented by $\int_{f(0)}^{f(a)} g(x)\,dx$ is shown as the shaded portion of the graph below on the right. Because f and g are inverse functions, the graph of $y = f(x)$ is obtained by reflecting the graph of $y = g(x)$

through the line $y = x$. It then follows that if we were to reflect the shaded region in the graph below on the right through the line $y = x$, the reflected region would coincide exactly with the region R in the graph below on the left. Thus

$$\int_0^a f(x)\,dx + \int_{f(0)}^{f(a)} g(x)\,dx = \text{ area of a rectangle with width } a \text{ and height } f(a) = af(a).$$





6.2 Setting Up Integrals: Volume, Density, Average Value

Preliminary Questions

1. What is the average value of $f(x)$ on [0, 4] if the area between the graph of $f(x)$ and the x-axis is equal to 12?

SOLUTION Assuming that $f(x) \geq 0$ over the interval [1, 4], the fact that the area between the graph of f and the x-axis is equal to 9 indicates that $\int_1^4 f(x)\,dx = 9$. The average value of f over the interval [1, 4] is then

$$\frac{\int_1^4 f(x)\,dx}{4-1} = \frac{9}{3} = 3.$$

2. Find the volume of a solid extending from $y = 2$ to $y = 5$ if every cross section has area $A(y) = 5$.

SOLUTION Because the cross-sectional area of the solid is constant, the volume is simply the cross-sectional area times the length, or $5 \times 3 = 15$.

3. What is the definition of flow rate?

SOLUTION The flow rate of a fluid is the volume of fluid that passes through a cross-sectional area at a given point per unit time.

4. Which assumption about fluid velocity did we use to compute the flow rate as an integral?

SOLUTION To express flow rate as an integral, we assumed that the fluid velocity depended only on the radial distance from the center of the tube.

5. The average value of $f(x)$ on [1, 4] is 5. Find $\int_1^4 f(x)\,dx$.

SOLUTION

$$\int_1^4 f(x)\,dx = \text{average value on } [1,4] \times \text{ length of } [1,4]$$
$$= 5 \times 3 = 15.$$

Exercises

1. Let V be the volume of a pyramid of height 20 whose base is a square of side 8.

(a) Use similar triangles as in Example 1 to find the area of the horizontal cross section at a height y.

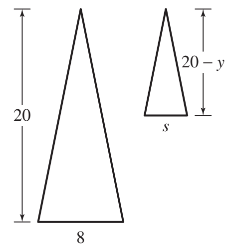
(b) Calculate V by integrating the cross-sectional area.

SOLUTION

(a) We can use similar triangles to determine the side length, s, of the square cross section at height y. Using the diagram below, we find

$$\frac{8}{20} = \frac{s}{20-y} \qquad \text{or} \qquad s = \frac{2}{5}(20-y).$$

The area of the cross section at height y is then given by $\frac{4}{25}(20-y)^2$.



(b) The volume of the pyramid is

$$\int_0^{20} \frac{4}{25}(20-y)^2\,dy = -\frac{4}{75}(20-y)^3\Big|_0^{20} = \frac{1280}{3}.$$

2. Let V be the volume of a right circular cone of height 10 whose base is a circle of radius 4 [Figure 17(A)].

(a) Use similar triangles to find the area of a horizontal cross section at a height y.

(b) Calculate V by integrating the cross-sectional area.

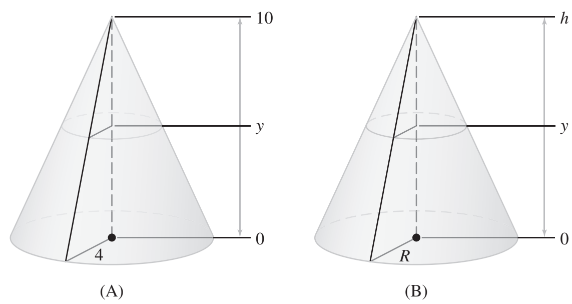


FIGURE 17 Right circular cones.

SOLUTION

(a) If r is the radius at height y (see Figure 17), then

$$\frac{10}{4} = \frac{10-y}{r}$$

from similar triangles, which implies that $r = 4 - \frac{2}{5}y$. The area of the cross-section at height y is then

$$A = \pi\left(4 - \frac{2}{5}y\right)^2.$$

(b) The volume of the cone is

$$V = \int_0^{10} \pi\left(4 - \frac{2}{5}y\right)^2 dy = -\frac{5\pi}{6}\left(4 - \frac{2}{5}y\right)^3\Big|_0^{10} = \frac{160\pi}{3}.$$

3. Use the method of Exercise 2 to find the formula for the volume of a right circular cone of height h whose base is a circle of radius R [Figure 17(B)].

SOLUTION

(a) From similar triangles (see Figure 17),

$$\frac{h}{h-y} = \frac{R}{r_0},$$

where r_0 is the radius of the cone at a height of y. Thus, $r_0 = R - \frac{Ry}{h}$.

(b) The volume of the cone is

$$\pi\int_0^h \left(R - \frac{Ry}{h}\right)^2 dy = \frac{-h\pi}{R}\,\frac{\left(R - \frac{Ry}{h}\right)^3}{3}\Bigg|_0^h = \frac{h\pi}{R}\,\frac{R^3}{3} = \frac{\pi R^2 h}{3}.$$

4. Calculate the volume of the ramp in Figure 18 in three ways by integrating the area of the cross sections:

(a) Perpendicular to the x-axis (rectangles).

(b) Perpendicular to the y-axis (triangles).

(c) Perpendicular to the z-axis (rectangles).

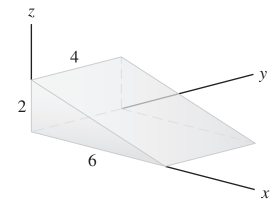


FIGURE 18 Ramp of length 6, width 4, and height 2.

SOLUTION

(a) Cross sections perpendicular to the x-axis are rectangles of width 4 and height $2 - \frac{1}{3}x$. The volume of the ramp is then

$$\int_0^6 4\left(-\frac{1}{3}x + 2\right) dx = \left(-\frac{2}{3}x^2 + 8x\right)\Big|_0^6 = 24.$$

(b) Cross sections perpendicular to the y-axis are right triangles with legs of length 2 and 6. The volume of the ramp is then

$$\int_0^4 \left(\frac{1}{2}\cdot 2 \cdot 6\right) dy = (6y)\Big|_0^4 = 24.$$

(c) Cross sections perpendicular to the z-axis are rectangles of length $6 - 3z$ and width 4. The volume of the ramp is then

$$\int_0^2 4\,(-3(z-2))\; dz = (-6z^2 + 24z)\Big|_0^2 = 24.$$

5. Find the volume of liquid needed to fill a sphere of radius R to height h (Figure 19).

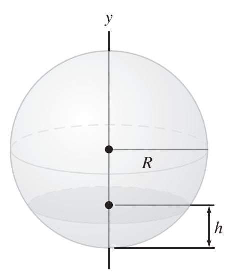


FIGURE 19 Sphere filled with liquid to height h.

SOLUTION The radius r at any height y is given by $r = \sqrt{R^2 - (R-y)^2}$. Thus, the volume of the filled portion of the sphere is

$$\pi \int_0^h r^2\, dy = \pi \int_0^h \left(R^2 - (R-y)^2\right) dy = \pi \int_0^h (2Ry - y^2)\, dy = \pi\left(Ry^2 - \frac{y^3}{3}\right)\Big|_0^h = \pi\left(Rh^2 - \frac{h^3}{3}\right).$$

6. Find the volume of the wedge in Figure 20(A) by integrating the area of vertical cross sections.

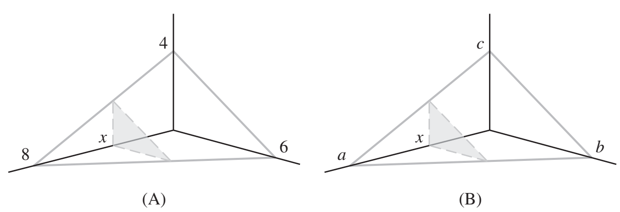


FIGURE 20

SOLUTION Cross sections of the wedge taken perpendicular to the x-axis are right triangles. Using similar triangles, we find the base and the height of the cross sections to be $\frac{3}{4}(8-x)$ and $\frac{1}{2}(8-x)$, respectively. The volume of the wedge is then

$$\frac{3}{16}\int_0^8 (8-x)^2\, dx = \frac{3}{16}\int_0^8 \left(64 - 16x + x^2\right) dx = \frac{3}{16}\left(64x - 8x^2 + \frac{1}{3}x^3\right)\Big|_0^8 = 32.$$

7. Derive a formula for the volume of the wedge in Figure 20(B) in terms of the constants a, b, and c.

SOLUTION The line from c to a is given by the equation $(z/c) + (x/a) = 1$ and the line from b to a is given by $(y/b) + (x/a) = 1$. The cross sections perpendicular to the x-axis are right triangles with height $c(1 - x/a)$ and base $b(1 - x/a)$. Thus we have

$$\int_0^a \frac{1}{2}bc\,(1 - x/a)^2\,dx = -\frac{1}{6}abc\left(1 - \frac{x}{a}\right)^3\Big|_0^a = \frac{1}{6}abc.$$

8. Let B be the solid whose base is the unit circle $x^2 + y^2 = 1$ and whose vertical cross sections perpendicular to the x-axis are equilateral triangles. Show that the vertical cross sections have area $A(x) = \sqrt{3}(1 - x^2)$ and compute the volume of B.

SOLUTION At the arbitrary location x, the side of the equilateral triangle cross section that lies in the base of the solid extends from the top half of the unit circle (with $y = \sqrt{1 - x^2}$) to the bottom half (with $y = -\sqrt{1 - x^2}$). The equilateral triangle therefore has sides of length $s = 2\sqrt{1 - x^2}$ and an area of

$$A(x) = \frac{s^2\sqrt{3}}{4} = \sqrt{3}(1 - x^2).$$

Finally, the volume of the solid is

$$\sqrt{3}\int_{-1}^{1}\left(1 - x^2\right)dx = \sqrt{3}\left(x - \frac{1}{3}x^3\right)\Big|_{-1}^{1} = \frac{4\sqrt{3}}{3}.$$

In Exercises 9–14, find the volume of the solid with the given base and cross sections.

9. The base is the unit circle $x^2 + y^2 = 1$, and the cross sections perpendicular to the x-axis are triangles whose height and base are equal.

SOLUTION At each location x, the side of the triangular cross section that lies in the base of the solid extends from the top half of the unit circle (with $y = \sqrt{1 - x^2}$) to the bottom half (with $y = -\sqrt{1 - x^2}$). The triangle therefore has base and height equal to $2\sqrt{1 - x^2}$ and area $2(1 - x^2)$. The volume of the solid is then

$$\int_{-1}^{1} 2(1 - x^2)\,dx = 2\left(x - \frac{1}{3}x^3\right)\Big|_{-1}^{1} = \frac{8}{3}.$$

10. The base is the triangle enclosed by $x + y = 1$, the x-axis, and the y-axis. The cross sections perpendicular to the y-axis are semicircles.

SOLUTION The diameter of the semicircle lies in the base of the solid and thus has length $1 - y$ for each y. The area of the semicircle is then

$$\frac{1}{2}\pi\left(\frac{1 - y}{2}\right)^2 = \frac{1}{8}\pi(1 - y)^2.$$

Finally, the volume of the solid is

$$\frac{\pi}{8}\int_0^1 (1 - y)^2\,dy = \frac{\pi}{8}\int_0^1 (1 - 2y + y^2)\,dy = \frac{\pi}{8}\left(y - y^2 + \frac{1}{3}y^3\right)\Big|_0^1 = \frac{\pi}{24}.$$

11. The base is the semicircle $y = \sqrt{9 - x^2}$, where $-3 \le x \le 3$. The cross sections perpendicular to the x-axis are squares.

SOLUTION For each x, the base of the square cross section extends from the semicircle $y = \sqrt{9 - x^2}$ to the x-axis. The square therefore has a base with length $\sqrt{9 - x^2}$ and an area of $\left(\sqrt{9 - x^2}\right)^2 = 9 - x^2$. The volume of the solid is then

$$\int_{-3}^{3}\left(9 - x^2\right)dx = \left(9x - \frac{1}{3}x^3\right)\Big|_{-3}^{3} = 36.$$

12. The base is a square, one of whose sides is the interval $[0, \ell]$ along the x-axis. The cross sections perpendicular to the x-axis are rectangles of height $f(x) = x^2$.

SOLUTION For each x, the rectangular cross section has base ℓ and height x^2. The cross-sectional area is then ℓx^2, and the volume of the solid is

$$\int_0^{\ell}\left(\ell x^2\right)dx = \left(\frac{1}{3}\ell x^3\right)\Big|_0^{\ell} = \frac{1}{3}\ell^4.$$

13. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are squares.

SOLUTION At any location y, the distance to the parabola from the y-axis is $\sqrt{y}$. Thus the base of the square will have length $2\sqrt{y}$. Therefore the volume is

$$\int_0^3 \left(2\sqrt{y}\right)\left(2\sqrt{y}\right)\, dy = \int_0^3 4y\, dy = 2y^2\Big|_0^3 = 18.$$

14. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are rectangles of height y^3.

SOLUTION As in previous exercise, for each y, the width of the rectangle will be $2\sqrt{y}$. Because the height is y^3, the volume of the solid is given by

$$2\int_0^3 y^{7/2}\, dy = \frac{4}{9}y^{9/2}\Big|_0^3 = 36\sqrt{3}.$$

15. Find the volume of the solid whose base is the region $|x| + |y| \le 1$ and whose vertical cross sections perpendicular to the y-axis are semicircles (with diameter along the base).

SOLUTION The region R in question is a diamond shape connecting the points $(1, 0)$, $(0, -1)$, $(-1, 0)$, and $(0, 1)$. Thus, in the lower half of the xy-plane, the radius of the circles is $y + 1$ and in the upper half, the radius is $1 - y$. Therefore, the volume is

$$\frac{\pi}{2}\int_{-1}^0 (y+1)^2\, dy + \frac{\pi}{2}\int_0^1 (1-y)^2\, dy = \frac{\pi}{2}\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{\pi}{3}.$$

16. Show that a pyramid of height h whose base is an equilateral triangle of side s has volume $\frac{\sqrt{3}}{12}hs^2$.

SOLUTION Using similar triangles, the side length of the equilateral triangle at height x above the base is

$$\frac{s(h-x)}{h};$$

the area of the cross section is therefore given by

$$\frac{\sqrt{3}}{4}\left(\frac{s(h-x)}{h}\right)^2.$$

Thus, the volume of the pyramid is

$$\frac{s^2\sqrt{3}}{4h^2}\int_0^h (h-x)^2\, dx = \left(-\frac{s^2\sqrt{3}}{12h^2}(h-x)^3\right)\Bigg|_0^h = \frac{\sqrt{3}}{12}s^2h.$$

17. The area of an ellipse is πab, where a and b are the lengths of the semimajor and semiminor axes (Figure 21). Compute the volume of a cone of height 12 whose base is an ellipse with semimajor axis $a = 6$ and semiminor axis $b = 4$.

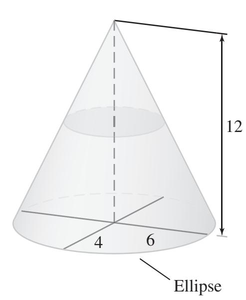


FIGURE 21

SOLUTION At each height y, the elliptical cross section has major axis $\frac{1}{2}(12 - y)$ and minor axis $\frac{1}{3}(12 - y)$. The cross-sectional area is then $\frac{\pi}{6}(12 - y)^2$, and the volume is

$$\int_0^{12} \frac{\pi}{6}(12-y)^2\, dy = -\frac{\pi}{18}(12-y)^3\Big|_0^{12} = 96\pi.$$

18. Find the volume V of a *regular* tetrahedron (Figure 22) whose face is an equilateral triangle of side s. The tetrahedron has height $h = \sqrt{2/3}s$.

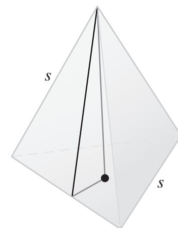


FIGURE 22

SOLUTION Our first task is to determine the relationship between the height of the tetrahedron, h, and the side length of the equilateral triangles, s. Let B be the orthocenter of the tetrahedron (the point directly below the apex), and let b denote the distance from B to each corner of the base triangle. By the Law of Cosines, we have

$$s^2 = b^2 + b^2 - 2b^2 \cos 120^\circ = 3b^2,$$

so $b^2 = \frac{1}{3}s^2$. Thus

$$h^2 = s^2 - b^2 = \frac{2}{3}s^2 \quad \text{or} \quad h = s\sqrt{\frac{2}{3}}.$$

Therefore, using similar triangles, the side length of the equilateral triangle at height z above the base is

$$s\left(\frac{h-z}{h}\right) = s - \frac{z}{\sqrt{2/3}}.$$

The volume of the tetrahedron is then given by

$$\int_0^{s\sqrt{2/3}} \frac{\sqrt{3}}{4}\left(s - \frac{z}{\sqrt{2/3}}\right)^2 dz = -\frac{\sqrt{2}}{12}\left(s - \frac{z}{\sqrt{2/3}}\right)^3\Bigg|_0^{s\sqrt{2/3}} = \frac{s^3\sqrt{2}}{12}.$$

19. A frustum of a pyramid is a pyramid with its top cut off [Figure 23(A)]. Let V be the volume of a frustum of height h whose base is a square of side a and whose top is a square of side b with $a > b \geq 0$.

(a) Show that if the frustum were continued to a full pyramid, it would have height $ha/(a-b)$ [Figure 23(B)].

(b) Show that the cross section at height x is a square of side $(1/h)(a(h-x)+bx)$.

(c) Show that $V = \frac{1}{3}h(a^2 + ab + b^2)$. A papyrus dating to the year 1850 BCE indicates that Egyptian mathematicians had discovered this formula almost 4000 years ago.

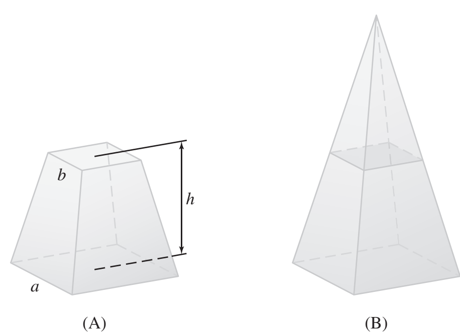


FIGURE 23

SOLUTION

(a) Let H be the height of the full pyramid. Using similar triangles, we have the proportion

$$\frac{H}{a} = \frac{H-h}{b}$$

which gives

$$H = \frac{ha}{a-b}.$$

(b) Let w denote the side length of the square cross section at height x. By similar triangles, we have

$$\frac{a}{H} = \frac{w}{H-x}.$$

Substituting the value for H from part (a) gives

$$w = \frac{a(h-x)+bx}{h}.$$

(c) The volume of the frustrum is

$$\int_0^h \left(\frac{1}{h}(a(h-x)+bx)\right)^2 dx = \frac{1}{h^2}\int_0^h \left(a^2(h-x)^2 + 2ab(h-x)x + b^2x^2\right) dx$$
$$= \frac{1}{h^2}\left(-\frac{a^2}{3}(h-x)^3 + abhx^2 - \frac{2}{3}abx^3 + \frac{1}{3}b^2x^3\right)\Bigg|_0^h = \frac{h}{3}\left(a^2+ab+b^2\right).$$

20. A plane inclined at an angle of 45° passes through a diameter of the base of a cylinder of radius r. Find the volume of the region within the cylinder and below the plane (Figure 24).

FIGURE 24

SOLUTION Place the center of the base at the origin. Then, for each x, the vertical cross section taken perpendicular to the x-axis is a rectangle of base $2\sqrt{r^2-x^2}$ and height x. The volume of the solid enclosed by the plane and the cylinder is therefore

$$\int_0^r 2x\sqrt{r^2-x^2}\,dx = \int_0^{r^2}\sqrt{u}\,du = \left(\frac{2}{3}u^{3/2}\right)\Bigg|_0^{r^2} = \frac{2}{3}r^3.$$

21. The solid S in Figure 25 is the intersection of two cylinders of radius r whose axes are perpendicular.

(a) The horizontal cross section of each cylinder at distance y from the central axis is a rectangular strip. Find the strip's width.

(b) Find the area of the horizontal cross section of S at distance y.

(c) Find the volume of S as a function of r.

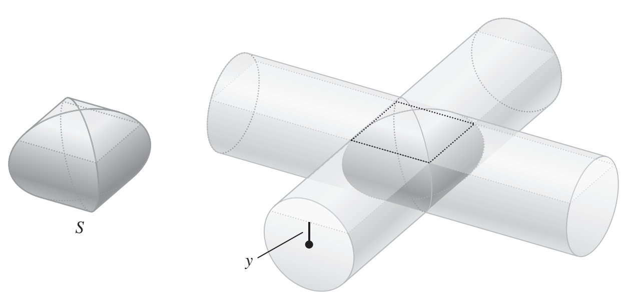


FIGURE 25 Two cylinders intersecting at right angles.

SOLUTION

(a) The horizontal cross section at distance y from the central axis (for $-r \le y \le r$) is a square of width $w = 2\sqrt{r^2-y^2}$.

(b) The area of the horizontal cross section of S at distance y from the central axis is $w^2 = 4(r^2-y^2)$.

(c) The volume of the solid S is then

$$4\int_{-r}^{r}\left(r^2-y^2\right)dy = 4\left(r^2y - \frac{1}{3}y^3\right)\Bigg|_{-r}^{r} = \frac{16}{3}r^3.$$

22. Let S be the intersection of two cylinders of radius r whose axes intersect at an angle θ. Find the volume of S as a function of r and θ.

SOLUTION Each cross section at distance y from the central axis (for $-r \le y \le r$) is a rhombus with side length $\frac{2\sqrt{r^2 - y^2}}{\sin\theta}$. The area of each rhombus is $\frac{4(r^2 - y^2)}{\sin\theta}$, and thus the volume of the solid will be

$$\frac{4}{\sin\theta}\int_{-r}^{r}\left(r^2 - y^2\right)dy = \frac{16r^3}{3\sin\theta}.$$

23. Calculate the volume of a cylinder inclined at an angle $\theta = 30°$ with height 10 and base of radius 4 (Figure 26).

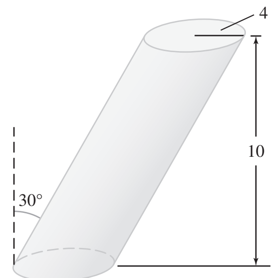


FIGURE 26 Cylinder inclined at an angle $\theta = 30°$.

SOLUTION The area of each circular cross section is $\pi(4)^2 = 16\pi$, hence the volume of the cylinder is

$$\int_0^{10} 16\pi\,dx = (16\pi x)\Big|_0^{10} = 160\pi$$

24. The areas of cross sections of Lake Nogebow at 5-meter intervals are given in the table below. Figure 27 shows a contour map of the lake. Estimate the volume V of the lake by taking the average of the right- and left-endpoint approximations to the integral of cross-sectional area.

Depth (m)	0	5	10	15	20
Area (million m^2)	2.1	1.5	1.1	0.835	0.217

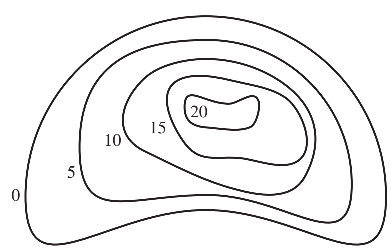


FIGURE 27 Depth contour map of Lake Nogebow.

SOLUTION The volume of the lake is

$$\int_0^{20} A(z)\,dz,$$

where $A(z)$ denotes the cross-sectional area of the lake at depth z. The right- and left-endpoint approximations to this integral, with $\Delta z = 5$, are

$$R = 5\,(1.5 + 1.1 + 0.835 + 0.217) = 18.26$$
$$L = 5\,(2.1 + 1.5 + 1.1 + 0.835) = 27.675$$

Thus

$$V \approx \frac{1}{2}(18.26 + 27.675) = 22.97 \text{ million m}^3.$$

25. Find the total mass of a 1-m rod whose linear density function is $\rho(x) = 10(x+1)^{-2}$ kg/m for $0 \le x \le 1$.

SOLUTION The total mass of the rod is

$$\int_0^1 \rho(x)\,dx = \int_0^1 \left(10(x+1)^{-2}\right)dx = \left(-10(x+1)^{-1}\right)\Big|_0^1 = 5 \text{ kg}.$$

26. Find the total mass of a 2-m rod whose linear density function is $\rho(x) = 1 + 0.5\sin(\pi x)$ kg/m for $0 \le x \le 2$.

SOLUTION The total mass of the rod is

$$\int_0^2 \rho(x)\,dx = \int_0^2 (1 + 0.5\sin\pi x)\,dx = \left(x - 0.5\frac{\cos\pi x}{\pi}\right)\Big|_0^2 = 2 \text{ kg},$$

27. A mineral deposit along a strip of length 6 cm has density $s(x) = 0.01x(6 - x)$ g/cm for $0 \le x \le 6$. Calculate the total mass of the deposit.

SOLUTION The total mass of the deposit is

$$\int_0^6 s(x)\,dx = \int_0^6 0.01x(6 - x)\,dx = \left(0.03x^2 - \frac{0.01}{3}x^3\right)\Big|_0^6 = 0.36 \text{ g}.$$

28. Charge is distributed along a glass tube of length 10 cm with linear charge density $\rho(x) = x(x^2 + 1)^{-2} \times 10^{-4}$ coulombs per centimeter for $0 \le x \le 10$. Calculate the total charge.

SOLUTION The total charge along the tube is

$$\int_0^{10} \rho(x)\,dx = 10^{-4}\int_0^{10} \frac{x}{(x^2+1)^2}\,dx = 10^{-4}\left(-\frac{1}{2}(x^2+1)^{-1}\right)\Big|_0^{10} = 5 \times 10^{-5}\left(1 - \frac{1}{101}\right) = 4.95 \times 10^{-5}$$

coulombs.

29. Calculate the population within a 10-mile radius of the city center if the radial population density is $\rho(r) = 4(1 + r^2)^{1/3}$ (in thousands per square mile).

SOLUTION The total population is

$$2\pi\int_0^{10} r\cdot\rho(r)\,dr = 2\pi\int_0^{10} 4r(1+r^2)^{1/3}\,dr = 3\pi(1+r^2)^{4/3}\Big|_0^{10}$$
$$\approx 4423.59 \text{ thousand} \approx 4.4 \text{ million}.$$

30. Odzala National Park in the Republic of the Congo has a high density of gorillas. Suppose that the radial population density is $\rho(r) = 52(1 + r^2)^{-2}$ gorillas per square kilometer, where r is the distance from a grassy clearing with a source of water. Calculate the number of gorillas within a 5-km radius of the clearing.

SOLUTION The number of gorillas within a 5-km radius of the clearing is

$$2\pi\int_0^5 r\cdot\rho(r)\,dr = \int_0^5 \frac{104\pi r}{(1+r^2)^2} = -\frac{52\pi}{1+r^2}\Big|_0^5 = 50\pi \approx 157.$$

31. Table 1 lists the population density (in people per square kilometer) as a function of distance r (in kilometers) from the center of a rural town. Estimate the total population within a 1.2-km radius of the center by taking the average of the left- and right-endpoint approximations.

TABLE 1 **Population Density**

r	$\rho(r)$	r	$\rho(r)$
0.0	125.0	0.8	56.2
0.2	102.3	1.0	46.0
0.4	83.8	1.2	37.6
0.6	68.6		

SOLUTION The total population is given by

$$2\pi\int_0^{1.2} r\cdot\rho(r)\,dr.$$

With $\Delta r = 0.2$, the left- and right-endpoint approximations to the required definite integral are

$$\begin{aligned} L_6 &= 0.2(2\pi)[0(125) + (0.2)(102.3) + (0.4)(83.8) + (0.6)(68.6) + (0.8)(56.2) + (1)(46)] \\ &= 233.86; \\ R_{10} &= 0.2(2\pi)[(0.2)(102.3) + (0.4)(83.8) + (0.6)(68.6) + (0.8)(56.2) + (1)(46) + (1.2)(37.6)] \\ &= 290.56. \end{aligned}$$

This gives an average of 262.21. Thus, there are roughly 262 people within a 1.2-km radius of the town center.

32. Find the total mass of a circular plate of radius 20 cm whose mass density is the radial function $\rho(r) = 0.03 + 0.01\cos(\pi r^2)$ g/cm^2.

SOLUTION The total mass of the plate is

$$2\pi \int_0^{20} r \cdot \rho(r)\, dr = 2\pi \int_0^{20} \left(0.03r + 0.01r\cos(\pi r^2)\right) dr = 2\pi \left(0.015r^2 + \frac{0.01}{2\pi}\sin(\pi r^2)\right)\Big|_0^{20} = 12\pi \text{ grams.}$$

33. The density of deer in a forest is the radial function $\rho(r) = 150(r^2+2)^{-2}$ deer per square kilometer, where r is the distance (in kilometers) to a small meadow. Calculate the number of deer in the region $2 \le r \le 5$ km.

SOLUTION The number of deer in the region $2 \le r \le 5$ km is

$$2\pi \int_2^5 r\,(150)\left(r^2+2\right)^{-2} dr = -150\pi \left(\frac{1}{r^2+2}\right)\Big|_2^5 = -150\pi\left(\frac{1}{27} - \frac{1}{6}\right) \approx 61 \text{ deer.}$$

34. Show that a circular plate of radius 2 cm with radial mass density $\rho(r) = \frac{4}{r}$ g/cm^2 has finite total mass, even though the density becomes infinite at the origin.

SOLUTION The total mass of the plate is

$$2\pi \int_0^2 r\left(\frac{4}{r}\right) dr = 2\pi \int_0^2 4\, dr = 16\pi \text{ g.}$$

35. Find the flow rate through a tube of radius 4 cm, assuming that the velocity of fluid particles at a distance r cm from the center is $v(r) = (16 - r^2)$ cm/s.

SOLUTION The flow rate is

$$2\pi \int_0^R r v(r)\, dr = 2\pi \int_0^4 r\left(16 - r^2\right) dr = 2\pi\left(8r^2 - \frac{1}{4}r^4\right)\Big|_0^4 = 128\pi \frac{\text{cm}^3}{\text{s}}.$$

36. The velocity of fluid particles flowing through a tube of radius 5 cm is $v(r) = (10 - 0.3r - 0.34r^2)$ cm/s, where r cm is the distance from the center. What quantity per second of fluid flows through the portion of the tube where $0 \le r \le 2$?

SOLUTION The flow rate through the portion of the tube where $0 \le r \le 2$ is

$$\begin{aligned} 2\pi \int_0^2 r v(r)\, dr &= 2\pi \int_0^2 r\left(10 - 0.3r - 0.34r^2\right) dr = 2\pi \int_0^2 \left(10r - 0.3r^2 - 0.34r^3\right) dr \\ &= 2\pi\left(5r^2 - 0.1r^3 - 0.085r^4\right)\Big|_0^2 \\ &= 112.09 \frac{\text{cm}^3}{\text{s}} \end{aligned}$$

37. A solid rod of radius 1 cm is placed in a pipe of radius 3 cm so that their axes are aligned. Water flows through the pipe and around the rod. Find the flow rate if the velocity of the water is given by the radial function $v(r) = 0.5(r-1)(3-r)$ cm/s.

SOLUTION The flow rate is

$$2\pi \int_1^3 r(0.5)(r-1)(3-r)\, dr = \pi \int_1^3 \left(-r^3 + 4r^2 - 3r\right) dr = \pi\left(-\frac{1}{4}r^4 + \frac{4}{3}r^3 - \frac{3}{2}r^2\right)\Big|_1^3 = \frac{8\pi}{3}\,\frac{\text{cm}^3}{\text{s}}.$$

38. Let $v(r)$ be the velocity of blood in an arterial capillary of radius $R = 4 \times 10^{-5}$ m. Use Poiseuille's Law (Example 6) with $k = 10^6$ (m-s)$^{-1}$ to determine the velocity at the center of the capillary and the flow rate (use correct units).

SOLUTION According to Poiseuille's Law, $v(r) = k(R^2 - r^2)$. With $R = 4 \times 10^{-5}$ m and $k = 10^6$ (m-s)$^{-1}$,

$$v(0) = 0.0016 \text{ m/s.}$$

The flow rate through the capillary is

$$2\pi \int_0^R kr(R^2 - r^2)\, dr = 2\pi k\left(\frac{R^2r^2}{2} - \frac{r^4}{4}\right)\Big|_0^R = 2\pi k\frac{R^4}{4} \approx 4.02 \times 10^{-12}\frac{\text{m}^3}{\text{s}}.$$

In Exercises 39–48, calculate the average over the given interval.

39. $f(x) = x^3$, $[0, 4]$

SOLUTION The average is

$$\frac{1}{4-0}\int_0^4 x^3\,dx = \frac{1}{4}\int_0^4 x^3\,dx = \frac{1}{16}x^4\Big|_0^4 = 16.$$

40. $f(x) = x^3$, $[-1, 1]$

SOLUTION The average is

$$\frac{1}{1-(-1)}\int_{-1}^1 x^3\,dx = \frac{1}{2}\int_{-1}^1 x^3\,dx = \frac{1}{8}x^4\Big|_{-1}^1 = 0.$$

41. $f(x) = \cos x$, $\left[0, \frac{\pi}{6}\right]$

SOLUTION The average is

$$\frac{1}{\pi/6-0}\int_0^{\pi/6} \cos x\,dx = \frac{6}{\pi}\int_0^{\pi/6} \cos x\,dx = \frac{6}{\pi}\sin x\Big|_0^{\pi/6} = \frac{3}{\pi}.$$

42. $f(x) = \sec^2 x$, $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

SOLUTION The average is

$$\frac{1}{\pi/3-\pi/6}\int_{\pi/6}^{\pi/3} \sec^2 x\,dx = \frac{6}{\pi}\int_{\pi/6}^{\pi/3} \sec^2 x\,dx = \frac{6}{\pi}\tan x\Big|_{\pi/6}^{\pi/3} = \frac{6}{\pi}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{4\sqrt{3}}{\pi}.$$

43. $f(s) = s^{-2}$, $[2, 5]$

SOLUTION The average is

$$\frac{1}{5-2}\int_2^5 s^{-2}\,ds = -\frac{1}{3}s^{-1}\Big|_2^5 = \frac{1}{10}.$$

44. $f(x) = \dfrac{\sin(\pi/x)}{x^2}$, $[1, 2]$

SOLUTION The average is

$$\frac{1}{2-1}\int_1^2 \frac{\sin(\pi/x)}{x^2}\,dx = \frac{1}{\pi}\int_{\pi/2}^{\pi} \sin u\,du = -\frac{1}{\pi}\cos u\Big|_{\pi/2}^{\pi} = \frac{1}{\pi}.$$

45. $f(x) = 2x^3 - 6x^2$, $[-1, 3]$

SOLUTION The average is

$$\frac{1}{3-(-1)}\int_{-1}^3 (2x^3 - 6x^2)\,dx = \frac{1}{4}\int_{-1}^3 (2x^3 - 6x^2)\,dx = \frac{1}{4}\left(\frac{1}{2}x^4 - 2x^3\right)\Big|_{-1}^3 = \frac{1}{4}\left(-\frac{27}{2} - \frac{5}{2}\right) = -4.$$

46. $f(x) = \dfrac{1}{x^2+1}$, $[-1, 1]$

SOLUTION The average is

$$\frac{1}{1-(-1)}\int_{-1}^1 \frac{1}{x^2+1}\,dx = \frac{1}{2}\tan^{-1} x\Big|_{-1}^1 = \frac{1}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{4}.$$

47. $f(x) = x^n$ for $n \ge 0$, $[0, 1]$

SOLUTION For $n > -1$, the average is

$$\frac{1}{1-0}\int_0^1 x^n\,dx = \int_0^1 x^n\,dx = \frac{1}{n+1}x^{n+1}\Big|_0^1 = \frac{1}{n+1}.$$

48. $f(x) = e^{-nx}, \quad [-1, 1]$

SOLUTION The average is

$$\frac{1}{1-(-1)} \int_{-1}^{1} e^{-nx}\, dx = \frac{1}{2}\left(-\frac{1}{n}e^{-nx}\right)\bigg|_{-1}^{1} = \frac{1}{2}\left(-\frac{1}{n}e^{-n} + \frac{1}{n}e^{n}\right) = \frac{1}{n}\sinh n.$$

49. The temperature (in °C) at time t (in hours) in an art museum varies according to $T(t) = 20 + 5\cos\left(\frac{\pi}{12}t\right)$. Find the average over the time periods [0, 24] and [2, 6].

SOLUTION

- The average temperature over the 24-hour period is

$$\frac{1}{24-0}\int_0^{24}\left(20 + 5\cos\left(\frac{\pi}{12}t\right)\right) dt = \frac{1}{24}\left(20t + \frac{60}{\pi}\sin\left(\frac{\pi}{12}t\right)\right)\bigg|_0^{24} = 20°\text{C}.$$

- The average temperature over the 4-hour period is

$$\frac{1}{6-2}\int_2^{6}\left(20 + 5\cos\left(\frac{\pi}{12}t\right)\right) dt = \frac{1}{4}\left(20t + \frac{60}{\pi}\sin\left(\frac{\pi}{12}t\right)\right)\bigg|_2^{6} = 22.4°\text{C}.$$

50. A ball thrown in the air vertically from ground level with initial velocity 18 m/s has height $h(t) = 18t - 9.8t^2$ at time t (in seconds). Find the average height and the average speed over the time interval extending from the ball's release to its return to ground level.

SOLUTION Let $h(t) = 18t - 9.8t^2$. The ball is at ground level when $t = 0$ s and when

$$t = \frac{18}{9.8} = \frac{9}{4.9}\text{ s.}$$

The average height of the ball is then

$$\frac{1}{\frac{9}{4.9} - 0}\int_0^{9/4.9}(18t - 9.8t^2)\, dt = \frac{4.9}{9}\left(9t^2 - \frac{9.8}{3}t^3\right)\bigg|_0^{9/4.9}$$

$$= \frac{4.9}{9}\left[9\left(\frac{9}{4.9}\right)^2 - \frac{9.8}{3}\left(\frac{9}{4.9}\right)^3\right]$$

$$= 5.51\text{ m.}$$

The average speed is given by

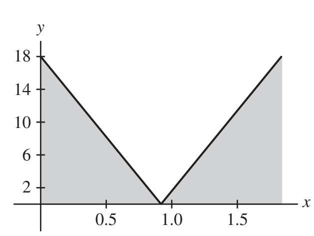
$$\frac{1}{\frac{9}{4.9} - 0}\int_0^{9/4.9}|v(t)|\, dt.$$

Now, $v(t) = h'(t) = 18 - 19.6t$. From the figure below, which shows the graph of $|v(t)|$ over the interval $[0, 9/4.9]$, we see that

$$\int_0^{9/4.9}|v(t)|\, dt = \left(\frac{9}{9.8}\right)18.$$

Thus, the average speed is

$$\frac{4.9}{9}\left(\frac{9}{9.8}\right)18 = 9\text{ m/s.}$$



51. Find the average speed over the time interval $[1, 5]$ of a particle whose position at time t is $s(t) = t^3 - 6t^2$ m/s.

SOLUTION The average speed over the time interval $[1, 5]$ is

$$\frac{1}{5-1}\int_1^5 |s'(t)|\,dt.$$

Because $s'(t) = 3t^2 - 12t = 3t(t-4)$, it follows that

$$\begin{aligned}\int_1^5 |s'(t)|\,dt &= \int_1^4 (12t - 3t^2)\,dt + \int_4^5 (3t^2 - 12t)\,dt\\ &= (6t^2 - t^3)\Big|_1^4 + (t^3 - 6t^2)\Big|_4^5\\ &= (96-64) - (6-1) + (125-150) - (64-96)\\ &= 34.\end{aligned}$$

Thus, the average speed is

$$\frac{34}{4} = \frac{17}{2}\ \text{m/s}.$$

52. An object with zero initial velocity accelerates at a constant rate of 10 m/s^2. Find its average velocity during the first 15 seconds.

SOLUTION An acceleration $a(t) = 10$ gives $v(t) = 10t + c$ for some constant c and zero initial velocity implies $c = 0$. Thus the average velocity is given by

$$\frac{1}{15-0}\int_0^{15} 10t\,dt = \frac{1}{3}t^2\Big|_0^{15} = 75\ \text{m/s}.$$

53. The acceleration of a particle is $a(t) = 60t - 4t^3$ m/s^2. Compute the average acceleration and the average speed over the time interval $[2, 6]$, assuming that the particle's initial velocity is zero.

SOLUTION The average acceleration over the time interval $[2, 6]$ is

$$\begin{aligned}\frac{1}{6-2}\int_2^6 (60t - 4t^3)\,dt &= \frac{1}{4}(30t^2 - t^4)\Big|_2^6\\ &= \frac{1}{4}\left[(1080 - 1296) - (120 - 16)\right]\\ &= -\frac{320}{4} = -80\ \text{m/s}^2.\end{aligned}$$

Given $a(t) = 60t - 4t^3$ and $v(0) = 0$, it follows that $v(t) = 30t^2 - t^4$. Now, average speed is given by

$$\frac{1}{6-2}\int_2^6 |v(t)|\,dt.$$

Based on the formula for $v(t)$,

$$\begin{aligned}\int_2^6 |v(t)|\,dt &= \int_2^{\sqrt{30}} (30t^2 - t^4)\,dt + \int_{\sqrt{30}}^6 (t^4 - 30t^2)\,dt\\ &= \left(10t^3 - \frac{1}{5}t^5\right)\Big|_2^{\sqrt{30}} + \left(\frac{1}{5}t^5 - 10t^3\right)\Big|_{\sqrt{30}}^6\\ &= 120\sqrt{30} - \frac{368}{5} - \frac{3024}{5} + 120\sqrt{30}\\ &= 240\sqrt{30} - \frac{3392}{5}.\end{aligned}$$

Finally, the average speed is

$$\frac{1}{4}\left(240\sqrt{30} - \frac{3392}{5}\right) = 60\sqrt{30} - \frac{848}{5} \approx 159.03\ \text{m/s}.$$

54. What is the average area of the circles whose radii vary from 0 to R?

SOLUTION The average area is

$$\frac{1}{R-0}\int_0^R \pi r^2\,dr = \frac{\pi}{3R}r^3\Big|_0^R = \frac{1}{3}\pi R^2.$$

55. Let M be the average value of $f(x) = x^4$ on [0, 3]. Find a value of c in [0, 3] such that $f(c) = M$.

SOLUTION We have

$$M = \frac{1}{3-0}\int_0^3 x^4\,dx = \frac{1}{3}\int_0^3 x^4\,dx = \frac{1}{15}x^5\Big|_0^3 = \frac{81}{5}.$$

Then $M = f(c) = c^4 = \frac{81}{5}$ implies $c = \frac{3}{5^{1/4}} = 2.006221$.

56. Let $f(x) = \sqrt{x}$. Find a value of c in [4, 9] such that $f(c)$ is equal to the average of f on [4, 9].

SOLUTION The average value is

$$\frac{1}{9-4}\int_4^9 \sqrt{x}\,dx = \frac{1}{5}\int_4^9 \sqrt{x}\,dx = \frac{2}{15}x^{3/2}\Big|_4^9 = \frac{38}{15}.$$

Then $f(c) = \sqrt{c} = \frac{38}{15}$ implies

$$c = \left(\frac{38}{15}\right)^2 = \frac{1444}{225} \approx 6.417778.$$

57. Let M be the average value of $f(x) = x^3$ on $[0, A]$, where $A > 0$. Which theorem guarantees that $f(c) = M$ has a solution c in $[0, A]$? Find c.

SOLUTION The Mean Value Theorem for Integrals guarantees that $f(c) = M$ has a solution c in $[0, A]$. With $f(x) = x^3$ on $[0, A]$,

$$M = \frac{1}{A-0}\int_0^A x^3\,dx = \frac{1}{A}\,\frac{1}{4}x^4\Big|_0^A = \frac{A^3}{4}.$$

Solving $f(c) = c^3 = \frac{A^3}{4}$ for c yields

$$c = \frac{A}{\sqrt[3]{4}}.$$

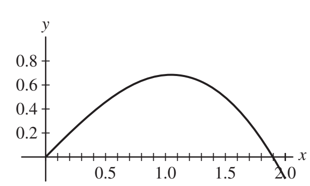
58. CAS Let $f(x) = 2\sin x - x$. Use a computer algebra system to plot $f(x)$ and estimate:

(a) The positive root α of $f(x)$.

(b) The average value M of $f(x)$ on $[0, \alpha]$.

(c) A value $c \in [0, \alpha]$ such that $f(c) = M$.

SOLUTION Let $f(x) = 2\sin x - x$. A graph of $y = f(x)$ is shown below. From this graph, the positive root of $f(x)$ appears to be roughly $x = 1.9$.



(a) Using a computer algebra system, solving the equation

$$2\sin\alpha - \alpha = 0$$

yields $\alpha = 1.895494267$.

(b) The average value of $f(x)$ on $[0, \alpha]$ is

$$M = \frac{1}{\alpha - 0}\int_0^\alpha f(x)\,dx = 0.4439980667.$$

(c) Solving

$$f(c) = 2\sin c - c = 0.4439980667$$

yields either $c = 0.4805683082$ or $c = 1.555776337$.

59. Which of $f(x) = x\sin^2 x$ and $g(x) = x^2 \sin^2 x$ has a larger average value over [0, 1]? Over [1, 2]?

SOLUTION The functions f and g differ only in the power of x multiplying $\sin^2 x$. It is also important to note that $\sin^2 x \geq 0$ for all x. Now, for each $x \in (0, 1)$, $x > x^2$ so

$$f(x) = x\sin^2 x > x^2 \sin^2 x = g(x).$$

Thus, over [0, 1], $f(x)$ will have a larger average value than $g(x)$. On the other hand, for each $x \in (1, 2)$, $x^2 > x$, so

$$g(x) = x^2 \sin^2 x > x\sin^2 x = f(x).$$

Thus, over [1, 2], $g(x)$ will have the larger average value.

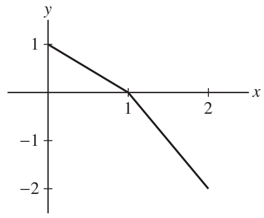
60. Find the average of $f(x) = ax + b$ over the interval $[-M, M]$, where a, b, and M are arbitrary constants.

SOLUTION The average is

$$\frac{1}{M - (-M)} \int_{-M}^{M} (ax + b)\,dx = \frac{1}{2M} \int_{-M}^{M} (ax + b)\,dx = \frac{1}{2M} \left(\frac{a}{2}x^2 + bx\right)\Bigg|_{-M}^{M} = b.$$

61. Sketch the graph of a function $f(x)$ such that $f(x) \geq 0$ on [0, 1] and $f(x) \leq 0$ on [1, 2], whose average on [0, 2] is negative.

SOLUTION Many solutions will exist. One could be



62. Give an example of a function (necessarily discontinuous) that does not satisfy the conclusion of the MVT for Integrals.

SOLUTION There are an infinite number of discontinuous functions that do not satisfy the conclusion of the Mean Value Theorem for Integrals. Consider the function on $[-1, 1]$ such that for $x < 0$, $f(x) = -1$ and for $x \geq 0$, $f(x) = 1$. Clearly the average value is 0 but $f(c) \neq 0$ for all c in $[-1, 1]$.

Further Insights and Challenges

63. An object is tossed into the air vertically from ground level with initial velocity v_0 ft/s at time $t = 0$. Find the average speed of the object over the time interval $[0, T]$, where T is the time the object returns to earth.

SOLUTION The height is given by $h(t) = v_0 t - 16t^2$. The ball is at ground level at time $t = 0$ and $T = v_0/16$. The velocity is given by $v(t) = v_0 - 32t$ and thus the speed is given by $s(t) = |v_0 - 32t|$. The average speed is

$$\frac{1}{v_0/16 - 0} \int_0^{v_0/16} |v_0 - 32t|\,dt = \frac{16}{v_0} \int_0^{v_0/32} (v_0 - 32t)\,dt + \frac{16}{v_0} \int_{v_0/32}^{v_0/16} (32t - v_0)\,dt$$

$$= \frac{16}{v_0} \left(v_0 t - 16t^2\right)\Big|_0^{v_0/32} + \frac{16}{v_0} \left(16t^2 - v_0 t\right)\Big|_{v_0/32}^{v_0/16} = v_0/2.$$

64. Review the MVT stated in Section 4.3 (Theorem 1, p. 266) and show how it can be used, together with the Fundamental Theorem of Calculus, to prove the MVT for Integrals.

SOLUTION The Mean Value Theorem essentially states that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $c \in (a, b)$. Let F be any antiderivative of f. Then

$$f(c) = F'(c) = \frac{F(b) - F(a)}{b - a} = \frac{1}{b - a}(F(b) - F(a)) = \frac{1}{b - a} \int_a^b f(x)\,dx.$$

6.3 Volumes of Revolution

Preliminary Questions

1. Which of the following is a solid of revolution?

(a) Sphere **(b)** Pyramid **(c)** Cylinder **(d)** Cube

SOLUTION The sphere and the cylinder have circular cross sections; hence, these are solids of revolution. The pyramid and cube do not have circular cross sections, so these are not solids of revolution.

2. True or false? When the region under a single graph is rotated about the x-axis, the cross sections of the solid perpendicular to the x-axis are circular disks.

SOLUTION True. The cross sections will be disks with radius equal to the value of the function.

3. True or false? When the region between two graphs is rotated about the x-axis, the cross sections to the solid perpendicular to the x-axis are circular disks.

SOLUTION False. The cross sections may be washers.

4. Which of the following integrals expresses the volume obtained by rotating the area between $y = f(x)$ and $y = g(x)$ over $[a, b]$ around the x-axis? [Assume $f(x) \geq g(x) \geq 0$.]

(a) $\pi \int_a^b \left(f(x) - g(x)\right)^2 dx$

(b) $\pi \int_a^b \left(f(x)^2 - g(x)^2\right) dx$

SOLUTION The correct answer is **(b)**. Cross sections of the solid will be washers with outer radius $f(x)$ and inner radius $g(x)$. The area of the washer is then $\pi f(x)^2 - \pi g(x)^2 = \pi(f(x)^2 - g(x)^2)$.

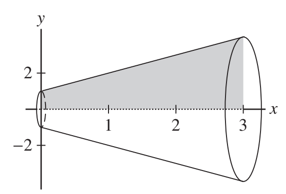
Exercises

In Exercises 1–4, (a) sketch the solid obtained by revolving the region under the graph of $f(x)$ about the x-axis over the given interval, (b) describe the cross section perpendicular to the x-axis located at x, and (c) calculate the volume of the solid.

1. $f(x) = x + 1$, $[0, 3]$

SOLUTION

(a) A sketch of the solid of revolution is shown below:



(b) Each cross section is a disk with radius $x + 1$.

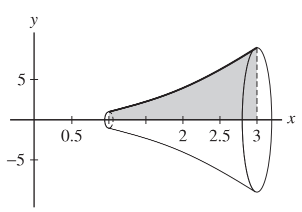
(c) The volume of the solid of revolution is

$$\pi \int_0^3 (x+1)^2\, dx = \pi \int_0^3 (x^2 + 2x + 1)\, dx = \pi \left(\frac{1}{3}x^3 + x^2 + x\right)\Big|_0^3 = 21\pi.$$

2. $f(x) = x^2$, $[1, 3]$

SOLUTION

(a) A sketch of the solid of revolution is shown below:



(b) Each cross section is a disk of radius x^2.

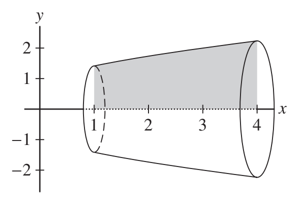
(c) The volume of the solid of revolution is

$$\pi \int_1^3 \left(x^2\right)^2 dx = \pi \left(\frac{x^5}{5}\right)\Big|_1^3 = \frac{242\pi}{5}.$$

3. $f(x) = \sqrt{x+1}$, $[1, 4]$

SOLUTION

(a) A sketch of the solid of revolution is shown below:



(b) Each cross section is a disk with radius $\sqrt{x+1}$.

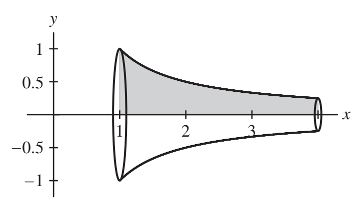
(c) The volume of the solid of revolution is

$$\pi \int_1^4 (\sqrt{x+1})^2\, dx = \pi \int_1^4 (x+1)\, dx = \pi \left(\frac{1}{2}x^2 + x\right)\Big|_1^4 = \frac{21\pi}{2}.$$

4. $f(x) = x^{-1}$, $[1, 4]$

SOLUTION

(a) A sketch of the solid of revolution is shown below:



(b) Each cross section is a disk with radius x^{-1}.

(c) The volume of the solid of revolution is

$$\pi \int_1^4 (x^{-1})^2\, dx = \pi \int_1^4 x^{-2}\, dx = \pi\, (-x)^{-1}\Big|_1^4 = \frac{3\pi}{4}.$$

In Exercises 5–12, find the volume of revolution about the x-axis for the given function and interval.

5. $f(x) = x^2 - 3x$, $[0, 3]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^3 (x^2 - 3x)^2\, dx = \pi \int_0^3 (x^4 - 6x^3 + 9x^2)\, dx = \pi \left(\frac{1}{5}x^5 - \frac{3}{2}x^4 + 3x^3\right)\Big|_0^3 = \frac{81\pi}{10}.$$

6. $f(x) = \dfrac{1}{x^2}$, $[1, 4]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^4 (x^{-2})^2\, dx = \pi \int_1^4 x^{-4}\, dx = \pi \left(-\frac{1}{3}x^{-3}\right)\Big|_1^4 = \frac{21\pi}{64}.$$

7. $f(x) = x^{5/3}$, $[1, 8]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^8 (x^{5/3})^2\, dx = \pi \int_1^8 x^{10/3}\, dx = \frac{3\pi}{13}x^{13/3}\Big|_1^8 = \frac{3\pi}{13}(2^{13} - 1) = \frac{24573\pi}{13}.$$

8. $f(x) = 4 - x^2, \quad [0, 2]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^2 (4 - x^2)^2\, dx = \pi \int_0^2 (16 - 8x^2 + x^4)\, dx = \pi \left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_0^2 = \frac{256\pi}{15}.$$

9. $f(x) = \dfrac{2}{x+1}, \quad [1, 3]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^3 \left(\frac{2}{x+1} \right)^2 dx = 4\pi \int_1^3 (x+1)^{-2}\, dx = -4\pi\, (x+1)^{-1} \Big|_1^3 = \pi.$$

10. $f(x) = \sqrt{x^4 + 1}, \quad [1, 3]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^3 (\sqrt{x^4 + 1})^2\, dx = \pi \int_1^3 (x^4 + 1)\, dx = \pi \left(\frac{1}{5}x^5 + x \right) \Big|_1^3 = \frac{252\pi}{5}.$$

11. $f(x) = e^x, \quad [0, 1]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^1 (e^x)^2\, dx = \frac{1}{2}\pi e^{2x} \Big|_0^1 = \frac{1}{2}\pi(e^2 - 1).$$

12. $f(x) = \sqrt{\cos x \sin x}, \quad \left[0, \frac{\pi}{2}\right]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^{\pi/2} (\sqrt{\cos x \sin x})^2\, dx = \pi \int_0^{\pi/2} (\cos x \sin x)\, dx = \frac{\pi}{2} \int_0^{\pi/2} \sin 2x\, dx = \frac{\pi}{4} (-\cos 2x) \Big|_0^{\pi/2} = \frac{\pi}{2}.$$

In Exercises 13 and 14, R is the shaded region in Figure 11.

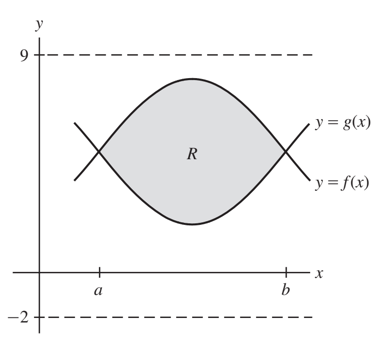


FIGURE 11

13. Which of the integrands (i)–(iv) is used to compute the volume obtained by rotating region R about $y = -2$?

(i) $(f(x)^2 + 2^2) - (g(x)^2 + 2^2)$
(ii) $(f(x) + 2)^2 - (g(x) + 2)^2$
(iii) $(f(x)^2 - 2^2) - (g(x)^2 - 2^2)$
(iv) $(f(x) - 2)^2 - (g(x) - 2)^2$

SOLUTION when the region R is rotated about $y = -2$, the outer radius is $f(x) - (-2) = f(x) + 2$ and the inner radius is $g(x) - (-2) = g(x) + 2$. Thus, the appropriate integrand is **(ii)**: $(f(x) + 2)^2 - (g(x) + 2)^2$.

14. Which of the integrands (i)–(iv) is used to compute the volume obtained by rotating R about $y = 9$?

(i) $(9 + f(x))^2 - (9 + g(x))^2$
(ii) $(9 + g(x))^2 - (9 + f(x))^2$
(iii) $(9 - f(x))^2 - (9 - g(x))^2$
(iv) $(9 - g(x))^2 - (9 - f(x))^2$

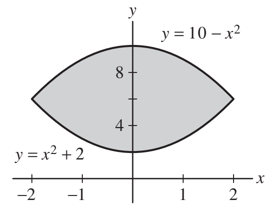
SOLUTION when the region R is rotated about $y = 9$, the outer radius is $9 - g(x)$ and the inner radius is $9 - f(x)$. Thus, the appropriate integrand is **(iv)**: $(9 - g(x))^2 - (9 - f(x))^2$.

In Exercises 15–20, (a) sketch the region enclosed by the curves, (b) describe the cross section perpendicular to the x-axis located at x, and (c) find the volume of the solid obtained by rotating the region about the x-axis.

15. $y = x^2 + 2, \quad y = 10 - x^2$

SOLUTION

(a) Setting $x^2 + 2 = 10 - x^2$ yields $2x^2 = 8$, or $x^2 = 4$. The two curves therefore intersect at $x = \pm 2$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = 10 - x^2$ and inner radius $r = x^2 + 2$.

(c) The volume of the solid of revolution is

$$\pi \int_{-2}^{2} \left((10 - x^2)^2 - (x^2 + 2)^2\right) dx = \pi \int_{-2}^{2} (96 - 24x^2)\, dx = \pi \left(96x - 8x^3\right)\Big|_{-2}^{2} = 256\pi.$$

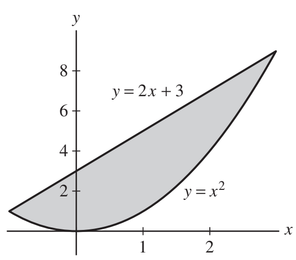
16. $y = x^2, \quad y = 2x + 3$

SOLUTION

(a) Setting $x^2 = 2x + 3$ yields

$$0 = x^2 - 2x - 3 = (x - 3)(x + 1).$$

The two curves therefore intersect at $x = -1$ and $x = 3$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = 2x + 3$ and inner radius $r = x^2$.

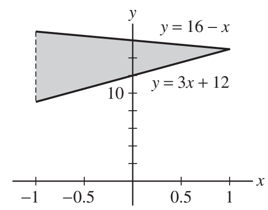
(c) The volume of the solid of revolution is

$$\pi \int_{-1}^{3} \left((2x + 3)^2 - (x^2)^2\right) dx = \pi \int_{-1}^{3} (4x^2 + 12x + 9 - x^4)\, dx = \pi \left(\frac{4}{3}x^3 + 6x^2 + 9x - \frac{1}{5}x^5\right)\Big|_{-1}^{3} = \frac{1088\pi}{15}.$$

17. $y = 16 - x, \quad y = 3x + 12, \quad x = -1$

SOLUTION

(a) Setting $16 - x = 3x + 12$, we find that the two lines intersect at $x = 1$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = 16 - x$ and inner radius $r = 3x + 12$.

(c) The volume of the solid of revolution is

$$\pi \int_{-1}^{1} \left((16-x)^2 - (3x+12)^2\right) dx = \pi \int_{-1}^{1} (112 - 104x - 8x^2)\, dx = \pi \left(112x - 52x^2 - \frac{8}{3}x^3\right)\Bigg|_{-1}^{1} = \frac{656\pi}{3}.$$

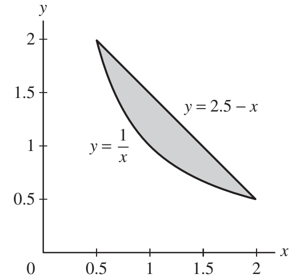
18. $y = \frac{1}{x}, \quad y = \frac{5}{2} - x$

SOLUTION

(a) Setting $\frac{1}{x} = \frac{5}{2} - x$ yields

$$0 = x^2 - \frac{5}{2}x + 1 = (x-2)\left(x - \frac{1}{2}\right).$$

The two curves therefore intersect at $x = 2$ and $x = \frac{1}{2}$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = \frac{5}{2} - x$ and inner radius $r = x^{-1}$.

(c) The volume of the solid of revolution is

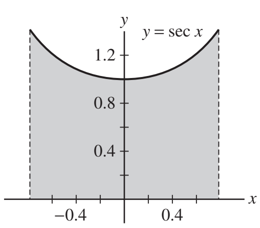
$$\pi \int_{1/2}^{2} \left(\left(\frac{5}{2} - x\right)^2 - \left(\frac{1}{x}\right)^2\right) dx = \pi \int_{1/2}^{2} \left(\frac{25}{4} - 5x + x^2 - x^{-2}\right) dx$$

$$= \pi \left(\frac{25}{4}x - \frac{5}{2}x^2 + \frac{1}{3}x^3 + x^{-1}\right)\Bigg|_{1/2}^{2} = \frac{9\pi}{8}.$$

19. $y = \sec x, \quad y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

SOLUTION

(a) The region in question is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a circular disk with radius $R = \sec x$.

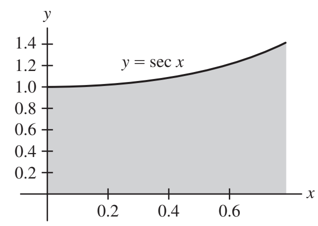
(c) The volume of the solid of revolution is

$$\pi \int_{-\pi/4}^{\pi/4} (\sec x)^2\, dx = \pi\, (\tan x)\Big|_{-\pi/4}^{\pi/4} = 2\pi.$$

20. $y = \sec x, \quad y = 0, \quad x = 0, \quad x = \frac{\pi}{4}$

SOLUTION

(a) The region in question is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a circular disk with radius $R = \sec x$.

(c) The volume of the solid of revolution is

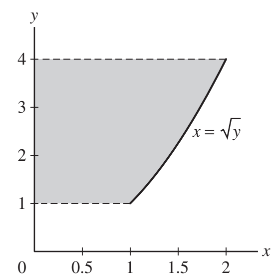
$$\pi \int_0^{\pi/4} (\sec x)^2 \, dx = \pi \, (\tan x) \Big|_0^{\pi/4} = \pi.$$

In Exercises 21–24, find the volume of the solid obtained by rotating the region enclosed by the graphs about the y-axis over the given interval.

21. $x = \sqrt{y}, \quad x = 0; \quad 1 \le y \le 4$

SOLUTION When the region in question (shown in the figure below) is rotated about the y-axis, each cross section is a disk with radius $\sqrt{y}$. The volume of the solid of revolution is

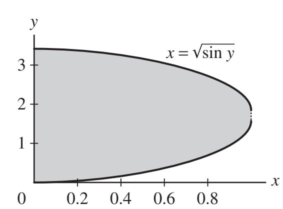
$$\pi \int_1^4 \left(\sqrt{y}\right)^2 \, dy = \frac{\pi y^2}{2}\Big|_1^4 = \frac{15\pi}{2}.$$



22. $x = \sqrt{\sin y}, \quad x = 0; \quad 0 \le y \le \pi$

SOLUTION When the region in question (shown in the figure below) is rotated about the y-axis, each cross section is a disk with radius $\sqrt{\sin y}$. The volume of the solid of revolution is

$$\pi \int_0^{\pi} \left(\sqrt{\sin y}\right)^2 \, dy = \pi \, (-\cos y) \Big|_0^{\pi} = 2\pi.$$



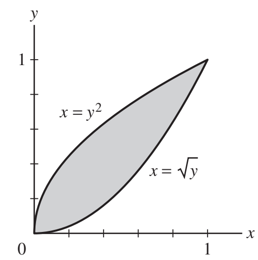
23. $x = y^2, \quad x = \sqrt{y}$

SOLUTION Setting $y^2 = \sqrt{y}$ and then squaring both sides yields

$$y^4 = y \quad \text{or} \quad y^4 - y = y(y^3 - 1) = 0,$$

so the two curves intersect at $y = 0$ and $y = 1$. When the region in question (shown in the figure below) is rotated about the y-axis, each cross section is a washer with outer radius $R = \sqrt{y}$ and inner radius $r = y^2$. The volume of the solid of revolution is

$$\pi \int_0^1 \left((\sqrt{y})^2 - (y^2)^2\right) \, dy = \pi \left(\frac{y^2}{2} - \frac{y^5}{5}\right)\Big|_0^1 = \frac{3\pi}{10}.$$



24. $x = 4 - y, \quad x = 16 - y^2$

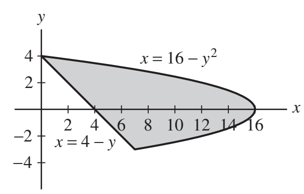
SOLUTION Setting $4 - y = 16 - y^2$ yields

$$0 = y^2 - y - 12 = (y-4)(y+3),$$

so the two curves intersect at $y = -3$ and $y = 4$. When the region enclosed by the two curves (shown in the figure below) is rotated about the y-axis, each cross section is a washer with outer radius $R = 16 - y^2$ and inner radius $r = 4 - y$. The volume of the solid of revolution is

$$\pi \int_{-3}^{4} \left((16 - y^2)^2 - (4 - y)^2 \right) dy = \pi \int_{-3}^{4} \left(y^4 - 33y^2 + 8y + 240 \right) dy$$

$$= \pi \left(\frac{1}{5}y^5 - 11y^3 + 4y^2 + 240y \right) \Big|_{-3}^{4} = \frac{4802\pi}{5}.$$



25. Rotation of the region in Figure 12 about the y-axis produces a solid with two types of different cross sections. Compute the volume as a sum of two integrals, one for $-12 \le y \le 4$ and one for $4 \le y \le 12$.

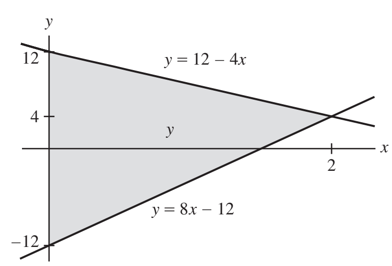


FIGURE 12

SOLUTION For $-12 \le y \le 4$, the cross section is a disk with radius $\frac{1}{8}(y + 12)$; for $4 \le y \le 12$, the cross section is a disk with radius $\frac{1}{4}(12 - y)$. Therefore, the volume of the solid of revolution is

$$V = \frac{\pi}{8} \int_{-12}^{4} (y+12)^2 \, dy + \frac{\pi}{4} \int_{4}^{12} (12 - y)^2 \, dy$$

$$= \frac{\pi}{24}(y+12)^3 \Big|_{-12}^{4} - \frac{\pi}{12}(12 - y)^3 \Big|_{4}^{12}$$

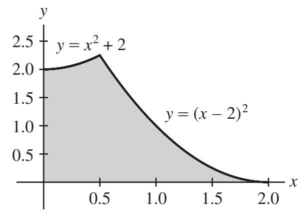
$$= \frac{512\pi}{3} + \frac{128\pi}{3} = \frac{640\pi}{3}.$$

26. Let R be the region enclosed by $y = x^2 + 2$, $y = (x-2)^2$ and the axes $x = 0$ and $y = 0$. Compute the volume V obtained by rotating R about the x-axis. *Hint:* Express V as a sum of two integrals.

SOLUTION Setting $x^2 + 2 = (x-2)^2$ yields $4x = 2$ or $x = 1/2$. When the region enclosed by the two curves and the coordinate axes (shown in the figure below) is rotated about the x-axis, there are two different cross sections. For

$0 \le x \le 1/2$, the cross section is a disk of radius x^2+2; for $1/2 \le x \le 2$, the cross section is a disk of radius $(x-2)^2$. The volume of the solid of revolution is therefore

$$V = \pi \int_0^{1/2} (x^2+2)\,dx + \pi \int_{1/2}^{2} (x-2)^2\,dx$$
$$= \pi \left(\frac{1}{3}x^3 + 2x \right) \Big|_0^{1/2} + \frac{\pi}{3}(x-2)^3 \Big|_{1/2}^{2}$$
$$= \frac{25\pi}{24} + \frac{9\pi}{8} = \frac{13\pi}{6}.$$



In Exercises 27–32, find the volume of the solid obtained by rotating region A in Figure 13 about the given axis.

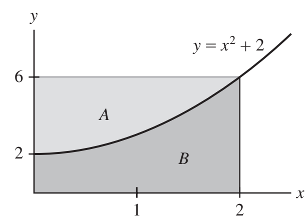


FIGURE 13

27. x-axis

SOLUTION Rotating region A about the x-axis produces a solid whose cross sections are washers with outer radius $R=6$ and inner radius $r = x^2+2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left((6)^2 - (x^2+2)^2 \right) dx = \pi \int_0^2 (32 - 4x^2 - x^4)\,dx = \pi \left(32x - \frac{4}{3}x^3 - \frac{1}{5}x^5 \right) \Big|_0^2 = \frac{704\pi}{15}.$$

28. $y = -2$

SOLUTION Rotating region A about $y=-2$ produces a solid whose cross sections are washers with outer radius $R = 6-(-2) = 8$ and inner radius $r = x^2+2-(-2) = x^2+4$. The volume of the solid of revolution is

$$\pi \int_0^2 \left((8)^2 - (x^2+4)^2 \right) dx = \pi \int_0^2 (48 - 8x^2 - x^4)\,dx = \pi \left(48x - \frac{8}{3}x^3 - \frac{1}{5}x^5 \right) \Big|_0^2 = \frac{1024\pi}{15}.$$

29. $y = 2$

SOLUTION Rotating the region A about $y=2$ produces a solid whose cross sections are washers with outer radius $R = 6-2 = 4$ and inner radius $r = x^2+2-2 = x^2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left(4^2 - (x^2)^2 \right) dx = \pi \left(16x - \frac{1}{5}x^5 \right) \Big|_0^2 = \frac{128\pi}{5}.$$

30. y-axis

SOLUTION Rotating region A about the y-axis produces a solid whose cross sections are disks with radius $R = \sqrt{y-2}$. Note that here we need to integrate along the y-axis. The volume of the solid of revolution is

$$\pi \int_2^6 (\sqrt{y-2})^2\,dy = \pi \int_2^6 (y-2)\,dy = \pi \left(\frac{1}{2}y^2 - 2y \right) \Big|_2^6 = 8\pi.$$

31. $x = -3$

SOLUTION Rotating region A about $x = -3$ produces a solid whose cross sections are washers with outer radius $R = \sqrt{y-2} - (-3) = \sqrt{y-2} + 3$ and inner radius $r = 0 - (-3) = 3$. The volume of the solid of revolution is

$$\pi \int_2^6 \left((3 + \sqrt{y-2})^2 - (3)^2 \right) dy = \pi \int_2^6 (6\sqrt{y-2} + y - 2)\, dy = \pi \left(4(y-2)^{3/2} + \frac{1}{2}y^2 - 2y \right)\Big|_2^6 = 40\pi.$$

32. $x = 2$

SOLUTION Rotating region A about $x = 2$ produces a solid whose cross sections are washers with outer radius $R = 2 - 0 = 2$ and inner radius $r = 2 - \sqrt{y-2}$. The volume of the solid of revolution is

$$\pi \int_2^6 \left(2^2 - (2 - \sqrt{y-2})^2 \right) dy = \pi \int_2^6 \left(4\sqrt{y-2} - y + 2 \right) dy = \pi \left(\frac{8}{3}(y-2)^{3/2} - \frac{1}{2}y^2 + 2y \right)\Big|_2^6 = \frac{40\pi}{3}.$$

In Exercises 33–38, find the volume of the solid obtained by rotating region B in Figure 13 about the given axis.

33. x-axis

SOLUTION Rotating region B about the x-axis produces a solid whose cross sections are disks with radius $R = x^2 + 2$. The volume of the solid of revolution is

$$\pi \int_0^2 (x^2+2)^2\, dx = \pi \int_0^2 (x^4 + 4x^2 + 4)\, dx = \pi \left(\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x \right)\Big|_0^2 = \frac{376\pi}{15}.$$

34. $y = -2$

SOLUTION Rotating region B about $y = -2$ produces a solid whose cross sections are washers with outer radius $R = x^2 + 2 - (-2) = x^2 + 4$ and inner radius $r = 0 - (-2) = 2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left((x^2+4)^2 - (2)^2 \right) dx = \pi \int_0^2 (x^4 + 8x^2 + 12)\, dx = \pi \left(\frac{1}{5}x^5 + \frac{8}{3}x^3 + 12x \right)\Big|_0^2 = \frac{776\pi}{15}.$$

35. $y = 6$

SOLUTION Rotating region B about $y = 6$ produces a solid whose cross sections are washers with outer radius $R = 6 - 0 = 6$ and inner radius $r = 6 - (x^2 + 2) = 4 - x^2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left(6^2 - (4 - x^2)^2 \right) dy = \pi \int_0^2 \left(20 + 8x^2 - x^4 \right) dy = \pi \left(20x + \frac{8}{3}x^3 - \frac{1}{5}x^5 \right)\Big|_0^2 = \frac{824\pi}{15}.$$

36. y-axis

Hint for Exercise 36: Express the volume as a sum of two integrals along the y-axis or use Exercise 30.

SOLUTION Rotating region B about the y-axis produces a solid with two different cross sections. For each $y \in [0, 2]$, the cross section is a disk with radius $R = 2$; for each $y \in [2, 6]$, the cross section is a washer with outer radius $R = 2$ and inner radius $r = \sqrt{y-2}$. The volume of the solid of revolution is

$$\pi \int_0^2 (2)^2\, dy + \pi \int_2^6 \left((2)^2 - (\sqrt{y-2})^2 \right) dy = \pi \int_0^2 4\, dy + \pi \int_2^6 (6 - y)\, dy$$
$$= \pi\,(4y)\Big|_0^2 + \pi \left(6y - \frac{1}{2}y^2 \right)\Big|_2^6 = 16\pi.$$

Alternately, we recognize that rotating both region A and region B about the y-axis produces a cylinder of radius $R = 2$ and height $h = 6$. The volume of this cylinder is $\pi(2)^2 \cdot 6 = 24\pi$. In Exercise 30, we found that the volume of the solid generated by rotating region A about the y-axis to be 8π. Therefore, the volume of the solid generated by rotating region B about the y-axis is $24\pi - 8\pi = 16\pi$.

37. $x = 2$

SOLUTION Rotating region B about $x = 2$ produces a solid with two different cross sections. For each $y \in [0, 2]$, the cross section is a disk with radius $R = 2$; for each $y \in [2, 6]$, the cross section is a disk with radius $R = 2 - \sqrt{y-2}$. The volume of the solid of revolution is

$$\pi \int_0^2 (2)^2\, dy + \pi \int_2^6 (2 - \sqrt{y-2})^2\, dy = \pi \int_0^2 4\, dy + \pi \int_2^6 (2 + y - 4\sqrt{y-2})\, dy$$
$$= \pi\,(4y)\Big|_0^2 + \pi \left(2y + \frac{1}{2}y^2 - \frac{8}{3}(y-2)^{3/2} \right)\Big|_2^6 = \frac{32\pi}{3}.$$

38. $x = -3$

SOLUTION Rotating region B about $x = -3$ produces a solid with two different cross sections. For each $y \in [0, 2]$, the cross section is a washer with outer radius $R = 2 - (-3) = 5$ and inner radius $r = 0 - (-3) = 3$; for each $y \in [2, 6]$, the cross section is a washer with outer radius $R = 2 - (-3) = 5$ and inner radius $r = \sqrt{y-2} - (-3) = \sqrt{y-2} + 3$. The volume of the solid of revolution is

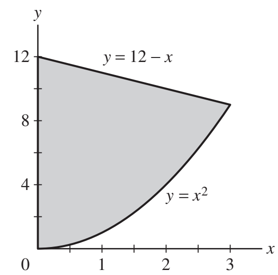
$$\pi \int_0^2 \left((5)^2 - (3)^2\right) dy + \pi \int_2^6 \left((5)^2 - (\sqrt{y-2}+3)^2\right) dy$$
$$= \pi \int_0^2 16\, dy + \pi \int_2^6 (18 - y - 6\sqrt{y-2})\, dy$$
$$= \pi\, (16y)\Big|_0^2 + \pi \left(18y - \frac{1}{2}y^2 - 4(y-2)^{3/2}\right)\Big|_2^6 = 56\pi.$$

In Exercises 39–52, find the volume of the solid obtained by rotating the region enclosed by the graphs about the given axis.

39. $y = x^2, \quad y = 12 - x, \quad x = 0, \quad$ about $y = -2$

SOLUTION Rotating the region enclosed by $y = x^2$, $y = 12 - x$ and the y-axis (shown in the figure below) about $y = -2$ produces a solid whose cross sections are washers with outer radius $R = 12 - x - (-2) = 14 - x$ and inner radius $r = x^2 - (-2) = x^2 + 2$. The volume of the solid of revolution is

$$\pi \int_0^3 \left((14-x)^2 - (x^2+2)^2\right) dx = \pi \int_0^3 (192 - 28x - 3x^2 - x^4)\, dx$$
$$= \pi \left(192x - 14x^2 - x^3 - \frac{1}{5}x^5\right)\Big|_0^3 = \frac{1872\pi}{5}.$$



40. $y = x^2, \quad y = 12 - x, \quad x = 0, \quad$ about $y = 15$

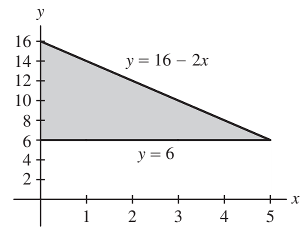
SOLUTION Rotating the region enclosed by $y = x^2$, $y = 12 - x$ and the y-axis (see the figure in the previous exercise) about $y = 15$ produces a solid whose cross sections are washers with outer radius $R = 15 - x^2$ and inner radius $r = 15 - (12 - x) = 3 + x$. The volume of the solid of revolution is

$$\pi \int_0^3 \left((15-x^2)^2 - (3+x)^2\right) dx = \pi \int_0^3 (216 - 6x - 31x^2 + x^4)\, dx$$
$$= \pi \left(216x - 3x^2 - \frac{31}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^3 = \frac{1953\pi}{5}.$$

41. $y = 16 - 2x, \quad y = 6, \quad x = 0, \quad$ about x-axis

SOLUTION Rotating the region enclosed by $y = 16 - 2x$, $y = 6$ and the y-axis (shown in the figure below) about the x-axis produces a solid whose cross sections are washers with outer radius $R = 16 - 2x$ and inner radius $r = 6$. The volume of the solid of revolution is

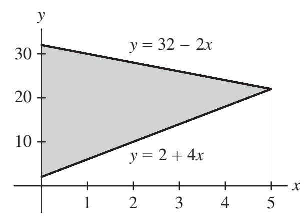
$$\pi \int_0^5 \left((16-2x)^2 - 6^2\right) dx = \pi \int_0^5 (220 - 64x + 4x^2)\, dx$$
$$= \pi \left(220x - 32x^2 + \frac{4}{3}x^3\right)\Big|_0^5 = \frac{1400\pi}{3}.$$



42. $y = 32 - 2x, \quad y = 2 + 4x, \quad x = 0,$ about y-axis

SOLUTION Rotating the region enclosed by $y = 32 - 2x$, $y = 2 + 4x$ and the y-axis (shown in the figure below) about the y-axis produces a solid with two different cross sections. For $2 \le y \le 22$, the cross section is a disk of radius $\frac{1}{4}(y-2)$; for $22 \le y \le 32$, the cross section is a disk of radius $\frac{1}{2}(32-y)$. The volume of the solid of revolution is

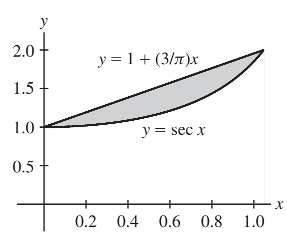
$$V = \frac{\pi}{4}\int_2^{22}(y-2)^2\,dy + \frac{\pi}{2}\int_{22}^{32}(32-y)^2\,dy$$
$$= \frac{\pi}{12}(y-2)^3\Big|_2^{22} - \frac{\pi}{6}(32-y)^3\Big|_{22}^{32}$$
$$= \frac{2000\pi}{3} + \frac{500\pi}{3} = \frac{2500\pi}{3}.$$



43. $y = \sec x, \quad y = 1 + \dfrac{3}{\pi}x,$ about x-axis

SOLUTION We first note that $y = \sec x$ and $y = 1 + (3/\pi)x$ intersect at $x = 0$ and $x = \pi/3$. Rotating the region enclosed by $y = \sec x$ and $y = 1 + (3/\pi)x$ (shown in the figure below) about the x-axis produces a cross section that is a washer with outer radius $R = 1 + (3/\pi)x$ and inner radius $r = \sec x$. The volume of the solid of revolution is

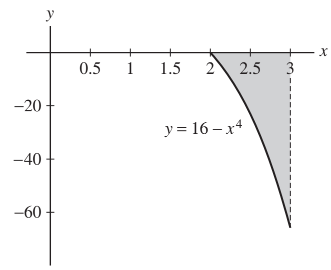
$$V = \pi\int_0^{\pi/3}\left(\left(1+\frac{3}{\pi}x\right)^2 - \sec^2 x\right)dx$$
$$= \pi\int_0^{\pi/3}\left(1 + \frac{6}{\pi}x + \frac{9}{\pi^2}x^2 - \sec^2 x\right)dx$$
$$= \pi\left(x + \frac{3}{\pi}x^2 + \frac{3}{\pi^2}x^3 - \tan x\right)\Big|_0^{\pi/3}$$
$$= \pi\left(\frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{9} - \sqrt{3}\right) = \frac{7\pi^2}{9} - \sqrt{3}\pi.$$



44. $x = 2, \quad x = 3, \quad y = 16 - x^4, \quad y = 0,$ about y-axis

SOLUTION Rotating the region enclosed by $x = 2, x = 3, y = 16 - x^4$ and the x-axis (shown in the figure below) about the y-axis produces a solid whose cross sections are washers with outer radius $R = 3$ and inner radius $r = \sqrt[4]{16-y}$. The volume of the solid of revolution is

$$\pi\int_{-65}^{0}\left(9 - \sqrt{16-y}\right)dy = \left(9y + \frac{2}{3}(16-y)^{3/2}\right)\Big|_{-65}^{0} = \frac{425\pi}{3}.$$



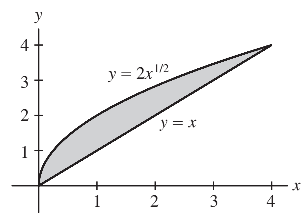
45. $y = 2\sqrt{x}$, $y = x$, about $x = -2$

SOLUTION Setting $2\sqrt{x} = x$ and squaring both sides yields

$$4x = x^2 \quad \text{or} \quad x(x-4) = 0,$$

so the two curves intersect at $x = 0$ and $x = 4$. Rotating the region enclosed by $y = 2\sqrt{x}$ and $y = x$ (see the figure below) about $x = -2$ produces a solid whose cross sections are washers with outer radius $R = y - (-2) = y + 2$ and inner radius $r = \frac{1}{4}y^2 - (-2) = \frac{1}{4}y^2 + 2$. The volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_0^4 \left((y+2)^2 - \left(\frac{1}{4}y^2 + 2\right)^2 \right) dy \\ &= \pi \int_0^4 \left(4y - \frac{1}{16}y^4 \right) dy \\ &= \pi \left(2y^2 - \frac{1}{80}y^5 \right)\Big|_0^4 \\ &= \pi \left(32 - \frac{64}{5} \right) = \frac{96\pi}{5}. \end{aligned}$$



46. $y = 2\sqrt{x}$, $y = x$, about $y = 4$

SOLUTION Setting $2\sqrt{x} = x$ and squaring both sides yields

$$4x = x^2 \quad \text{or} \quad x(x-4) = 0,$$

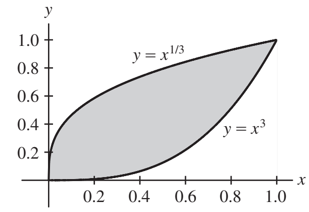
so the two curves intersect at $x = 0$ and $x = 4$. Rotating the region enclosed by $y = 2\sqrt{x}$ and $y = x$ (see the figure from the previous exercise) about $y = 4$ produces a solid whose cross sections are washers with outer radius $R = 4 - x$ and inner radius $r = 4 - 2\sqrt{x}$. The volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_0^4 \left((4-x)^2 - (4 - 2\sqrt{x})^2 \right) dy \\ &= \pi \int_0^4 \left(x^2 - 12x + 16\sqrt{x} \right) dy \\ &= \pi \left(\frac{1}{3}x^3 - 6x^2 + \frac{32}{3}x^{3/2} \right)\Big|_0^4 \\ &= \pi \left(\frac{64}{3} - 96 + \frac{256}{3} \right) = \frac{32\pi}{3}. \end{aligned}$$

47. $y = x^3$, $y = x^{1/3}$, for $x \geq 0$, about y-axis

SOLUTION Rotating the region enclosed by $y = x^3$ and $y = x^{1/3}$ (shown in the figure below) about the y-axis produces a solid whose cross sections are washers with outer radius $R = y^{1/3}$ and inner radius $r = y^3$. The volume of the solid of revolution is

$$\pi \int_0^1 \left((y^{1/3})^2 - (y^3)^2 \right) dy = \pi \int_0^1 (y^{2/3} - y^6)\, dy = \pi \left(\frac{3}{5}y^{5/3} - \frac{1}{7}y^7 \right)\Big|_0^1 = \frac{16\pi}{35}.$$



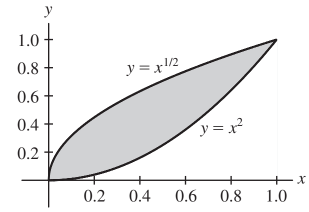
48. $y = x^2, \quad y = x^{1/2}, \quad \text{about } x = -2$

SOLUTION Rotating the region enclosed by $y = x^2$ and $y = x^{1/2}$ (shown in the figure below) about $x = -2$ produces a solid whose cross sections are washers with outer radius $R = \sqrt{y} - (-2) = \sqrt{y} + 2$ and inner radius $r = y^2 - (-2) = y^2 + 2$. The volume of the solid of revolution is

$$\pi \int_0^1 \left((\sqrt{y} + 2)^2 - (y^2 + 2)^2 \right) dy = \pi \int_0^1 \left(y + 4\sqrt{y} - y^4 - 4y^2 \right) dy$$

$$= \pi \left(\frac{1}{2}y^2 + \frac{8}{3}y^{3/2} - \frac{1}{5}y^5 - \frac{4}{3}y^3 \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{2} + \frac{8}{3} - \frac{1}{5} - \frac{4}{3} \right) = \frac{49\pi}{30}.$$

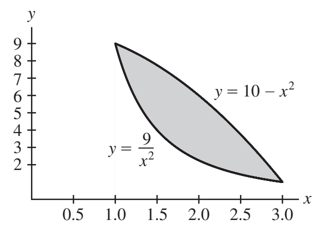


49. $y = \dfrac{9}{x^2}, \quad y = 10 - x^2, \quad x \geq 0, \quad \text{about } y = 12$

SOLUTION The region enclosed by the two curves is shown in the figure below. Rotating this region about $y = 12$ produces a solid whose cross sections are washers with outer radius $R = 12 - 9x^{-2}$ and inner radius $r = 12 - (10 - x^2) = 2 + x^2$. The volume of the solid of revolution is

$$\pi \int_1^3 \left((12 - 9x^{-2})^2 - (x^2 + 2)^2 \right) dx = \pi \int_1^3 \left(140 - 4x^2 - x^4 - 216x^{-2} + 81x^{-4} \right) dx$$

$$= \pi \left(140x - \frac{4}{3}x^3 - \frac{1}{5}x^5 + 216x^{-1} - 27x^{-3} \right) \Big|_1^3 = \frac{1184\pi}{15}.$$



50. $y = \dfrac{9}{x^2}, \quad y = 10 - x^2, \quad x \geq 0, \quad \text{about } x = -1$

SOLUTION The region enclosed by the two curves is shown in the figure from the previous exercise. Rotating this region about $x = -1$ produces a solid whose cross sections are washers with outer radius $R = \sqrt{10 - y} - (-1) = \sqrt{10 - y} + 1$ and inner radius $r = 3y^{-1/2} - (-1) = 3y^{-1/2} + 1$. The volume of the solid of revolution is

$$V = \pi \int_1^9 \left((\sqrt{10 - y} + 1)^2 - (3y^{-1/2} + 1)^2 \right) dy$$

$$= \pi \int_1^9 \left(10 - y + 2\sqrt{10 - y} - 9y^{-1} - 6y^{-1/2} \right) dy$$

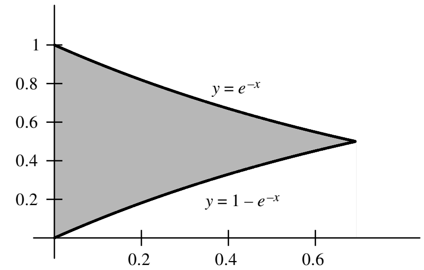
$$= \pi \left(10y - \frac{1}{2}y^2 - \frac{4}{3}(10 - y)^{3/2} - 9 \ln y - 12\sqrt{y} \right) \Big|_1^9$$

$$= \pi\left(\left(90 - \frac{81}{2} - \frac{4}{3} - 9\ln 9 - 36\right) - \left(10 - \frac{1}{2} - 36 - 12\right)\right)$$
$$= \pi\left(\frac{73}{6} - 9\ln 9 + \frac{77}{2}\right) = \left(\frac{152}{3} - 9\ln 9\right)\pi.$$

51. $y = e^{-x}$, $\quad y = 1 - e^{-x}$, $\quad x = 0$, $\quad$ about $y = 4$

SOLUTION Rotating the region enclosed by $y = 1 - e^{-x}$, $y = e^{-x}$ and the y-axis (shown in the figure below) about the line $y = 4$ produces a solid whose cross sections are washers with outer radius $R = 4 - (1 - e^{-x}) = 3 + e^{-x}$ and inner radius $r = 4 - e^{-x}$. The volume of the solid of revolution is

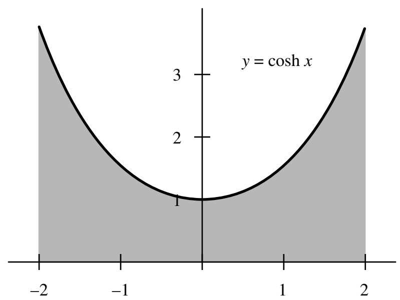
$$\pi \int_0^{\ln 2} \left((3 + e^{-x})^2 - (4 - e^{-x})^2\right) dx = \pi \int_0^{\ln 2} (14e^{-x} - 7)\, dx = \pi(-14e^{-x} - 7x)\Big|_0^{\ln 2}$$
$$= \pi(-7 - 7\ln 2 + 14) = 7\pi(1 - \ln 2).$$



52. $y = \cosh x$, $\quad x = \pm 2$, $\quad$ about x-axis

SOLUTION Rotating the region enclosed by $y = \cosh x$, $x = \pm 2$ and the x-axis (shown in the figure below) about the x-axis produces a solid whose cross sections are disks with radius $R = \cosh x$. The volume of the solid of revolution is

$$\pi \int_{-2}^{2} \cosh^2 x\, dx = \frac{1}{2}\pi \int_{-2}^{2} (1 + \cosh 2x)\, dx = \frac{1}{2}\pi\left(x + \frac{1}{2}\sinh 2x\right)\Big|_{-2}^{2}$$
$$= \frac{1}{2}\pi\left[\left(2 + \frac{1}{2}\sinh 4\right) - \left(-2 + \frac{1}{2}\sinh(-4)\right)\right] = \frac{1}{2}\pi(4 + \sinh 4).$$



53. The bowl in Figure 14(A) is 21 cm high, obtained by rotating the curve in Figure 14(B) as indicated. Estimate the volume capacity of the bowl shown by taking the average of right- and left-endpoint approximations to the integral with $N = 7$.

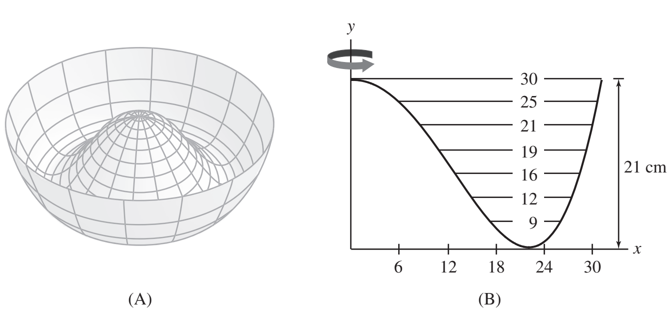


FIGURE 14

SOLUTION Using the given values for the inner radii and the values in Figure 14(B), which indicate the difference between the inner and outer radii, we find

$$R_7 = 3\pi\left((23^2-14^2)+(25^2-13^2)+(26^2-10^2)+(27^2-8^2)+(28^2-7^2)+(29^2-4^2)+(30^2-0^2)\right)$$
$$= 3\pi(4490) = 13470\pi$$

and

$$L_7 = 3\pi\left((20^2-20^2)+(23^2-14^2)+(25^2-13^2)+(26^2-10^2)+(27^2-8^2)+(28^2-7^2)+(29^2-4^2)\right)$$
$$= 3\pi(3590) = 10770\pi$$

Averaging these two values, we estimate that the volume capacity of the bowl is

$$V = 12120\pi \approx 38076.1 \text{ cm}^3.$$

54. The region between the graphs of $f(x)$ and $g(x)$ over [0, 1] is revolved about the line $y = -3$. Use the midpoint approximation with values from the following table to estimate the volume V of the resulting solid.

x	0.1	0.3	0.5	0.7	0.9
$f(x)$	8	7	6	7	8
$g(x)$	2	3.5	4	3.5	2

SOLUTION The volume of the resulting solid is

$$V = \pi\int_0^1 \left((f(x)+3)^2-(g(x)+3)^2\right)dx$$
$$\approx 0.2\pi\left((11^2-5^2)+(10^2-6.5^2)+(9^2-7^2)+(10^2-6.5^2)+(11^2-5^2)\right)$$
$$= 0.2\pi(96+57.75+32+57.75+96) = 67.9\pi.$$

55. Find the volume of the cone obtained by rotating the region under the segment joining $(0, h)$ and $(r, 0)$ about the y-axis.

SOLUTION The segment joining $(0, h)$ and $(r, 0)$ has the equation

$$y = -\frac{h}{r}x + h \quad \text{or} \quad x = \frac{r}{h}(h-y).$$

Rotating the region under this segment about the y-axis produces a cone with volume

$$\frac{\pi r^2}{h^2}\int_0^h (h-y)^2\,dx = -\frac{\pi r^2}{3h^2}(h-y)^3\Big|_0^h$$
$$= \frac{1}{3}\pi r^2 h.$$

56. The **torus** (doughnut-shaped solid) in Figure 15 is obtained by rotating the circle $(x-a)^2 + y^2 = b^2$ around the y-axis (assume that $a > b$). Show that it has volume $2\pi^2ab^2$. *Hint:* Evaluate the integral by interpreting it as the area of a circle.

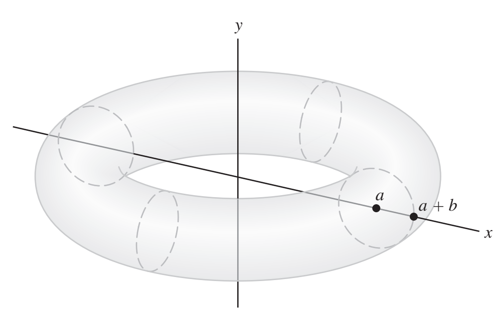


FIGURE 15 Torus obtained by rotating a circle about the y-axis.

SOLUTION Rotating the region enclosed by the circle $(x-a)^2+y^2=b^2$ about the y-axis produces a torus whose cross sections are washers with outer radius $R=a+\sqrt{b^2-y^2}$ and inner radius $r=a-\sqrt{b^2-y^2}$. The volume of the torus is then

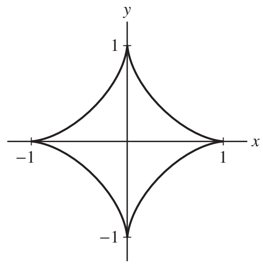
$$\pi\int_{-b}^{b}\left(\left(a+\sqrt{b^2-y^2}\right)^2-\left(a-\sqrt{b^2-y^2}\right)^2\right)dy=4a\pi\int_{-b}^{b}\sqrt{b^2-y^2}\,dy.$$

Now, the remaining definite integral is one-half the area of a circle of radius b; therefore, the volume of the torus is

$$4a\pi\cdot\frac{1}{2}\pi b^2=2\pi^2ab^2.$$

57. GU Sketch the hypocycloid $x^{2/3}+y^{2/3}=1$ and find the volume of the solid obtained by revolving it about the x-axis.

SOLUTION A sketch of the hypocycloid is shown below.



For the hypocycloid, $y=\pm\left(1-x^{2/3}\right)^{3/2}$. Rotating this region about the x-axis will produce a solid whose cross sections are disks with radius $R=\left(1-x^{2/3}\right)^{3/2}$. Thus the volume of the solid of revolution will be

$$\pi\int_{-1}^{1}\left((1-x^{2/3})^{3/2}\right)^2dx=\pi\left(\frac{-x^3}{3}+\frac{9}{7}x^{7/3}-\frac{9}{5}x^{5/3}+x\right)\Bigg|_{-1}^{1}=\frac{32\pi}{105}.$$

58. The solid generated by rotating the region between the branches of the hyperbola $y^2-x^2=1$ about the x-axis is called a **hyperboloid** (Figure 16). Find the volume of the hyperboloid for $-a\le x\le a$.

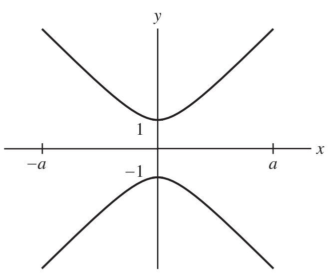


FIGURE 16 The hyperbola with equation $y^2-x^2=1$.

SOLUTION Each cross section is a disk of radius $R=\sqrt{1+x^2}$, so the volume of the hyperboloid is

$$\pi\int_{-a}^{a}\left(\sqrt{1+x^2}\right)^2dx=\pi\int_{-a}^{a}(1+x^2)\,dx=\pi\left(x+\frac{1}{3}x^3\right)\Bigg|_{-a}^{a}=\pi\left(\frac{2a^3+6a}{3}\right)$$

59. A "bead" is formed by removing a cylinder of radius r from the center of a sphere of radius R (Figure 17). Find the volume of the bead with $r=1$ and $R=2$.

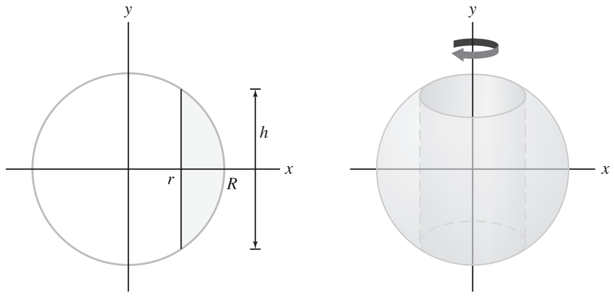


FIGURE 17 A bead is a sphere with a cylinder removed.

SOLUTION The equation of the outer circle is $x^2 + y^2 = 2^2$, and the inner cylinder intersects the sphere when $y = \pm\sqrt{3}$. Each cross section of the bead is a washer with outer radius $\sqrt{4 - y^2}$ and inner radius 1, so the volume is given by

$$\pi \int_{-\sqrt{3}}^{\sqrt{3}} \left(\left(\sqrt{4 - y^2}\right)^2 - 1^2\right) dy = \pi \int_{-\sqrt{3}}^{\sqrt{3}} \left(3 - y^2\right) dy = 4\pi\sqrt{3}.$$

Further Insights and Challenges

60. Find the volume V of the bead (Figure 17) in terms of r and R. Then show that $V = \frac{\pi}{6}h^3$, where h is the height of the bead. This formula has a surprising consequence: Since V can be expressed in terms of h alone, it follows that two beads of height 1 cm, one formed from a sphere the size of an orange and the other from a sphere the size of the earth, would have the same volume! Can you explain intuitively how this is possible?

SOLUTION The equation for the outer circle of the bead is $x^2 + y^2 = R^2$, and the inner cylinder intersects the sphere when $y = \pm\sqrt{R^2 - r^2}$. Each cross section of the bead is a washer with outer radius $\sqrt{R^2 - y^2}$ and inner radius r, so the volume is

$$\pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left(\left(\sqrt{R^2 - y^2}\right)^2 - r^2\right) dy = \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} (R^2 - r^2 - y^2)\, dy$$

$$= \pi \left((R^2 - r^2)y - \frac{1}{3}y^3\right)\Bigg|_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} = \frac{4}{3}(R^2 - r^2)^{3/2}\pi.$$

Now, $h = 2\sqrt{R^2 - r^2} = 2(R^2 - r^2)^{1/2}$, which gives $h^3 = 8(R^2 - r^2)^{3/2}$ and finally $(R^2 - r^2)^{3/2} = \frac{1}{8}h^3$. Substituting into the expression for the volume gives $V = \frac{\pi}{6}h^3$. The beads may have the same volume but clearly the wall of the earth-sized bead must be extremely thin while the orange-sized bead would be thicker.

61. The solid generated by rotating the region inside the ellipse with equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ around the x-axis is called an **ellipsoid**. Show that the ellipsoid has volume $\frac{4}{3}\pi ab^2$. What is the volume if the ellipse is rotated around the y-axis?

SOLUTION

- Rotating the ellipse about the x-axis produces an ellipsoid whose cross sections are disks with radius $R = b\sqrt{1 - (x/a)^2}$. The volume of the ellipsoid is then

$$\pi \int_{-a}^{a} \left(b\sqrt{1 - (x/a)^2}\right)^2 dx = b^2\pi \int_{-a}^{a} \left(1 - \frac{1}{a^2}x^2\right) dx = b^2\pi \left(x - \frac{1}{3a^2}x^3\right)\Bigg|_{-a}^{a} = \frac{4}{3}\pi ab^2.$$

- Rotating the ellipse about the y-axis produces an ellipsoid whose cross sections are disks with radius $R = a\sqrt{1 - (y/b)^2}$. The volume of the ellipsoid is then

$$\int_{-b}^{b} \left(a\sqrt{1 - (y/b)^2}\right)^2 dy = a^2\pi \int_{-b}^{b} \left(1 - \frac{1}{b^2}y^2\right) dy = a^2\pi \left(y - \frac{1}{3b^2}y^3\right)\Bigg|_{-b}^{b} = \frac{4}{3}\pi a^2 b.$$

62. The curve $y = f(x)$ in Figure 18, called a **tractrix**, has the following property: the tangent line at each point (x, y) on the curve has slope

$$\frac{dy}{dx} = \frac{-y}{\sqrt{1 - y^2}}$$

Let R be the shaded region under the graph of $0 \le x \le a$ in Figure 18. Compute the volume V of the solid obtained by revolving R around the x-axis in terms of the constant $c = f(a)$. *Hint:* Use the substitution $u = f(x)$ to show that

$$V = \pi \int_{c}^{1} u\sqrt{1 - u^2}\, du$$

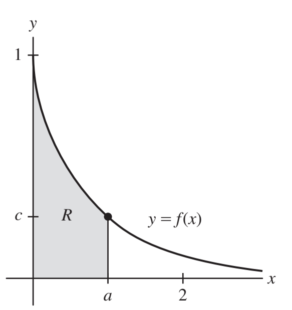


FIGURE 18 The tractrix.

SOLUTION Let $y = f(x)$ be the tractrix depicted in Figure 18. Rotating the region R about the x-axis produces a solid whose cross sections are disks with radius $f(x)$. The volume of the resulting solid is then

$$V = \pi \int_0^a [f(x)]^2 \, dx.$$

Now, let $u = f(x)$. Then

$$du = f'(x)\,dx = \frac{-f(x)}{\sqrt{1-[f(x)]^2}}\,dx = \frac{-u}{\sqrt{1-u^2}}\,dx;$$

hence,

$$dx = -\frac{\sqrt{1-u^2}}{u}\,du,$$

and

$$V = \pi \int_1^c u^2 \left(-\frac{\sqrt{1-u^2}}{u}\,du\right) = \pi \int_c^1 u\sqrt{1-u^2}\,du.$$

Carrying out the integration, we find

$$V = -\frac{\pi}{3}(1-u^2)^{3/2}\Big|_c^1 = \frac{\pi}{3}(1-c^2)^{3/2}.$$

63. Verify the formula

$$\int_{x_1}^{x_2} (x-x_1)(x-x_2)\,dx = \frac{1}{6}(x_1-x_2)^3 \qquad \boxed{3}$$

Then prove that the solid obtained by rotating the shaded region in Figure 19 about the x-axis has volume $V = \frac{\pi}{6}BH^2$, with B and H as in the figure. *Hint:* Let x_1 and x_2 be the roots of $f(x) = ax + b - (mx + c)^2$, where $x_1 < x_2$. Show that

$$V = \pi \int_{x_1}^{x_2} f(x)\,dx$$

and use Eq. (3).

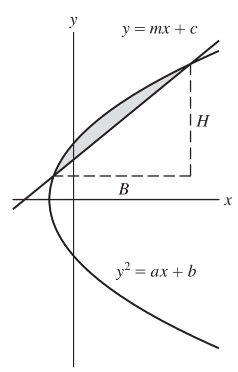


FIGURE 19 The line $y = mx + c$ intersects the parabola $y^2 = ax + b$ at two points above the x-axis.

SOLUTION First, we calculate

$$\int_{x_1}^{x_2} (x-x_1)(x-x_2)\,dx = \left(\frac{1}{3}x^3 - \frac{1}{2}(x_1+x_2)x^2 + x_1x_2x\right)\Big|_{x_1}^{x_2} = \frac{1}{6}x_1^3 - \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_1x_2^2 - \frac{1}{6}x_2^3$$
$$= \frac{1}{6}\left(x_1^3 - 3x_1^2x_2 + 3x_1x_2^2 - x_2^3\right) = \frac{1}{6}(x_1-x_2)^3.$$

Now, consider the region enclosed by the parabola $y^2 = ax + b$ and the line $y = mx + c$, and let x_1 and x_2 denote the x-coordinates of the points of intersection between the two curves with $x_1 < x_2$. Rotating the region about the y-axis produces a solid whose cross sections are washers with outer radius $R = \sqrt{ax+b}$ and inner radius $r = mx + c$. The volume of the solid of revolution is then

$$V = \pi \int_{x_1}^{x_2} \left(ax + b - (mx+c)^2\right)\,dx$$

Because x_1 and x_2 are roots of the equation $ax + b - (mx + c)^2 = 0$ and $ax + b - (mx + c)^2$ is a quadratic polynomial in x with leading coefficient $-m^2$, it follows that $ax + b - (mx + c)^2 = -m^2(x - x_1)(x - x_2)$. Therefore,

$$V = -\pi m^2 \int_{x_1}^{x_2} (x - x_1)(x - x_2)\,dx = \frac{\pi}{6} m^2 (x_2 - x_1)^3,$$

where we have used Eq. (3). From the diagram, we see that

$$B = x_2 - x_1 \qquad \text{and} \qquad H = mB,$$

so

$$V = \frac{\pi}{6} m^2 B^3 = \frac{\pi}{6} B\,(mB)^2 = \frac{\pi}{6} BH^2.$$

64. Let R be the region in the unit circle lying above the cut with the line $y = mx + b$ (Figure 20). Assume the points where the line intersects the circle lie above the x-axis. Use the method of Exercise 63 to show that the solid obtained by rotating R about the x-axis has volume $V = \frac{\pi}{6} hd^2$, with h and d as in the figure.

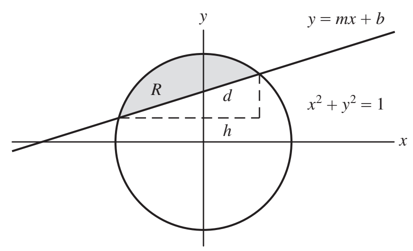


FIGURE 20

SOLUTION Let x_1 and x_2 denote the x-coordinates of the points of intersection between the circle $x^2 + y^2 = 1$ and the line $y = mx + b$ with $x_1 < x_2$. Rotating the region enclosed by the two curves about the x-axis produces a solid whose cross sections are washers with outer radius $R = \sqrt{1 - x^2}$ and inner radius $r = mx + b$. The volume of the resulting solid is then

$$V = \pi \int_{x_1}^{x_2} \left((1 - x^2) - (mx + b)^2 \right) dx$$

Because x_1 and x_2 are roots of the equation $(1 - x^2) - (mx + b)^2 = 0$ and $(1 - x^2) - (mx + b)^2$ is a quadratic polynomial in x with leading coefficient $-(1 + m^2)$, it follows that $(1 - x^2) - (mx + b)^2 = -(1 + m^2)(x - x_1)(x - x_2)$. Therefore,

$$V = -\pi(1 + m^2) \int_{x_1}^{x_2} (x - x_1)(x - x_2)\,dx = \frac{\pi}{6}(1 + m^2)(x_2 - x_1)^3.$$

From the diagram, we see that $h = x_2 - x_1$. Moreover, by the Pythagorean theorem, $d^2 = h^2 + (mh)^2 = (1 + m^2)h^2$. Thus,

$$V = \frac{\pi}{6}(1 + m^2)h^3 = \frac{\pi}{6} h \left[(1 + m^2)h^2 \right] = \frac{\pi}{6} hd^2.$$

6.4 The Method of Cylindrical Shells

Preliminary Questions

1. Consider the region $\mathcal{R}$ under the graph of the constant function $f(x) = h$ over the interval $[0, r]$. Give the height and the radius of the cylinder generated when $\mathcal{R}$ is rotated about:

(a) the x-axis **(b)** the y-axis

SOLUTION

(a) When the region is rotated about the x-axis, each shell will have radius h and height r.
(b) When the region is rotated about the y-axis, each shell will have radius r and height h.

2. Let V be the volume of a solid of revolution about the y-axis.

(a) Does the Shell Method for computing V lead to an integral with respect to x or y?
(b) Does the Disk or Washer Method for computing V lead to an integral with respect to x or y?

SOLUTION

(a) The Shell method requires slicing the solid parallel to the axis of rotation. In this case, that will mean slicing the solid in the vertical direction, so integration will be with respect to x.
(b) The Disk or Washer method requires slicing the solid perpendicular to the axis of rotation. In this case, that means slicing the solid in the horizontal direction, so integration will be with respect to y.

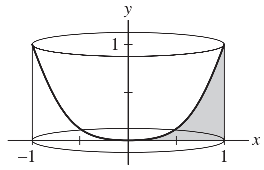
Exercises

In Exercises 1–6, sketch the solid obtained by rotating the region underneath the graph of the function over the given interval about the y-axis, and find its volume.

1. $f(x) = x^3$, $[0, 1]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height x^3, so the volume of the solid is

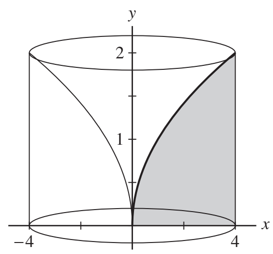
$$2\pi \int_0^1 x \cdot x^3\,dx = 2\pi \int_0^1 x^4\,dx = 2\pi \left(\frac{1}{5}x^5\right)\Big|_0^1 = \frac{2}{5}\pi.$$



2. $f(x) = \sqrt{x}$, $[0, 4]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height $\sqrt{x}$, so the volume of the solid is

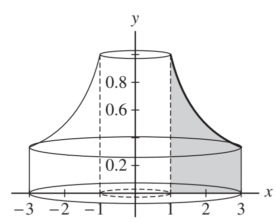
$$2\pi \int_0^4 x\sqrt{x}\,dx = 2\pi \int_0^4 x^{3/2}\,dx = 2\pi \left(\frac{2}{5}x^{5/2}\right)\Big|_0^4 = \frac{128}{5}\pi.$$



3. $f(x) = x^{-1}$, $[1, 3]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height x^{-1}, so the volume of the solid is

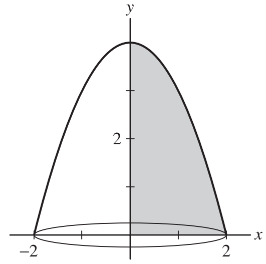
$$2\pi \int_1^3 x(x^{-1})\,dx = 2\pi \int_1^3 1\,dx = 2\pi\,(x)\Big|_1^3 = 4\pi.$$



4. $f(x) = 4 - x^2$, $[0, 2]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height $4 - x^2$, so the volume of the solid is

$$2\pi \int_0^2 x(4 - x^2)\,dx = 2\pi \int_0^2 (4x - x^3)\,dx = 2\pi \left(2x^2 - \frac{1}{4}x^4\right)\Big|_0^2 = 8\pi.$$



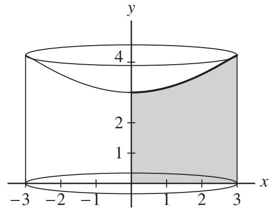
5. $f(x) = \sqrt{x^2+9}$, $[0, 3]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height $\sqrt{x^2+9}$, so the volume of the solid is

$$2\pi \int_0^3 x\sqrt{x^2+9}\,dx.$$

Let $u = x^2 + 9$. Then $du = 2x\,dx$ and

$$2\pi \int_0^3 x\sqrt{x^2+9}\,dx = \pi \int_9^{18} \sqrt{u}\,du = \pi \left(\frac{2}{3}u^{3/2}\right)\Big|_9^{18} = 18\pi(2\sqrt{2}-1).$$



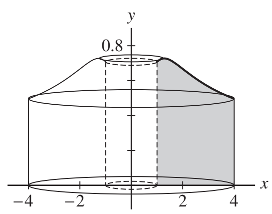
6. $f(x) = \dfrac{x}{\sqrt{1+x^3}}$, $[1, 4]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height $\dfrac{x}{\sqrt{1+x^3}}$, so the volume of the solid is

$$2\pi \int_1^4 x\left(\frac{x}{\sqrt{1+x^3}}\right)dx = 2\pi \int_1^4 \frac{x^2}{\sqrt{1+x^3}}\,dx.$$

Let $u = 1 + x^3$. Then $du = 3x^2\,dx$ and

$$2\pi \int_1^4 \frac{x^2}{\sqrt{1+x^3}}\,dx = \frac{2}{3}\pi \int_2^{65} u^{-1/2}\,du = \frac{2}{3}\pi\left(2u^{1/2}\right)\Big|_2^{65} = \frac{4\pi}{3}\left(\sqrt{65}-\sqrt{2}\right).$$

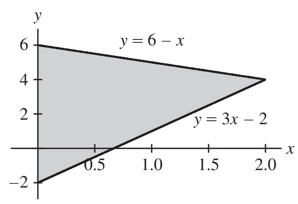


In Exercises 7–12, use the Shell Method to compute the volume obtained by rotating the region enclosed by the graphs as indicated, about the y-axis.

7. $y = 3x - 2$, $y = 6 - x$, $x = 0$

SOLUTION The region enclosed by $y = 3x - 2$, $y = 6 - x$ and $x = 0$ is shown below. When rotating this region about the y-axis, each shell has radius x and height $6 - x - (3x - 2) = 8 - 4x$. The volume of the resulting solid is

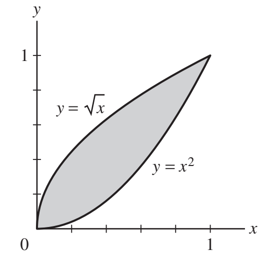
$$2\pi \int_0^2 x(8-4x)\,dx = 2\pi \int_0^2 (8x - 4x^2)\,dx = 2\pi\left(4x^2 - \frac{4}{3}x^3\right)\Big|_0^2 = \frac{32}{3}\pi.$$



8. $y = \sqrt{x}, \quad y = x^2$

SOLUTION The region enclosed by $y = \sqrt{x}$ and $y = x^2$ is shown below. When rotating this region about the y-axis, each shell has radius x and height $\sqrt{x} - x^2$. The volume of the resulting solid is

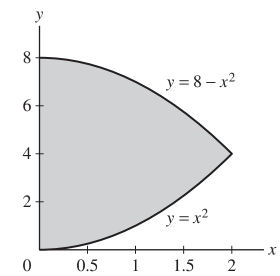
$$2\pi \int_0^1 x(\sqrt{x} - x^2)\,dx = 2\pi \int_0^1 (x^{3/2} - x^3)\,dx = 2\pi \left(\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right)\Big|_0^1 = \frac{3}{10}\pi.$$



9. $y = x^2, \quad y = 8 - x^2, \quad x = 0, \quad$ for $x \geq 0$

SOLUTION The region enclosed by $y = x^2$, $y = 8 - x^2$ and the y-axis is shown below. When rotating this region about the y-axis, each shell has radius x and height $8 - x^2 - x^2 = 8 - 2x^2$. The volume of the resulting solid is

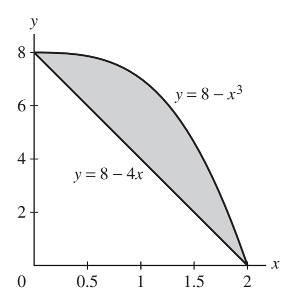
$$2\pi \int_0^2 x(8 - 2x^2)\,dx = 2\pi \int_0^2 (8x - 2x^3)\,dx = 2\pi \left(4x^2 - \frac{1}{2}x^4\right)\Big|_0^2 = 16\pi.$$



10. $y = 8 - x^3, \quad y = 8 - 4x, \quad$ for $x \geq 0$

SOLUTION The region enclosed by $y = 8 - x^3$ and $y = 8 - 4x$ is shown below. When rotating this region about the y-axis, each shell has radius x and height $(8 - x^3) - (8 - 4x) = 4x - x^3$. The volume of the resulting solid is

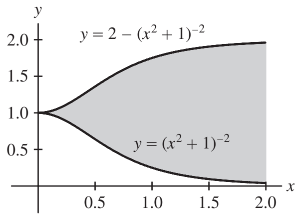
$$2\pi \int_0^2 x(4x - x^3)\,dx = 2\pi \int_0^2 (4x^2 - x^4)\,dx = 2\pi \left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{128\pi}{15}.$$



11. $y = (x^2 + 1)^{-2}, \quad y = 2 - (x^2 + 1)^{-2}, \quad x = 2$

SOLUTION The region enclosed by $y = (x^2 + 1)^{-2}$, $y = 2 - (x^2 + 1)^{-2}$ and $x = 2$ is shown below. When rotating this region about the y-axis, each shell has radius x and height $2 - (x^2 + 1)^{-2} - (x^2 + 1)^{-2} = 2 - 2(x^2 + 1)^{-2}$. The volume of the resulting solid is

$$2\pi \int_0^2 x(2 - 2(x^2 + 1)^{-2})\,dx = 2\pi \int_0^2 \left(2x - \frac{2x}{(x^2 + 1)^{-2}}\right)\,dx = 2\pi \left(x^2 + \frac{1}{x^2 + 1}\right)\Big|_0^2 = \frac{32}{5}\pi.$$

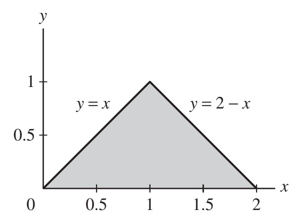


12. $y = 1 - |x - 1|, \quad y = 0$

SOLUTION The region enclosed by $y = 1 - |x - 1|$ and the x-axis is shown below. When rotating this region about the y-axis, two different shells are generated. For each $x \in [0, 1]$, the shell has radius x and height x; for each $x \in [1, 2]$, the shell has radius x and height $2 - x$. The volume of the resulting solid is

$$2\pi \int_0^1 x(x)\,dx + 2\pi \int_1^2 x(2-x)\,dx = 2\pi \int_0^1 (x^2)\,dx + 2\pi \int_1^2 (2x - x^2)\,dx$$

$$= 2\pi \left(\frac{1}{3}x^3\right)\bigg|_0^1 + 2\pi \left(x^2 - \frac{1}{3}x^3\right)\bigg|_1^2 = 2\pi.$$

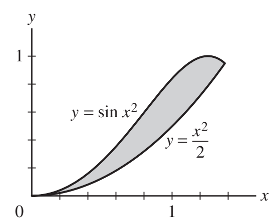


In Exercises 13 and 14, use a graphing utility to find the points of intersection of the curves numerically and then compute the volume of rotation of the enclosed region about the y-axis.

13. GU $y = \frac{1}{2}x^2, \quad y = \sin(x^2)$

SOLUTION The region enclosed by $y = \frac{1}{2}x^2$ and $y = \sin x^2$ is shown below. When rotating this region about the y-axis, each shell has radius x and height $\sin x^2 - \frac{1}{2}x^2$. Using a computer algebra system, we find that the x-coordinate of the point of intersection on the right is $x = 1.376769504$. Thus, the volume of the resulting solid of revolution is

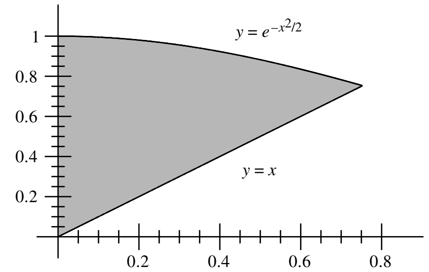
$$2\pi \int_0^{1.376769504} x\left(\sin x^2 - \frac{1}{2}x^2\right)\,dx = 1.321975576.$$



14. GU $y = e^{-x^2/2}, \quad y = x, \quad x = 0$

SOLUTION The region enclosed by $y = e^{-x^2/2}$, $y = x$ and the y-axis is shown below. When rotating this region about the y-axis, each shell has radius x and height $e^{-x^2/2} - x$. Using a computer algebra system, we find that the x-coordinate of the point of intersection on the right is $x = 0.7530891650$. Thus, volume of the resulting solid of revolution is

$$2\pi \int_0^{0.7530891650} x(e^{-x^2/2} - x)\,dx = 0.6568505551$$

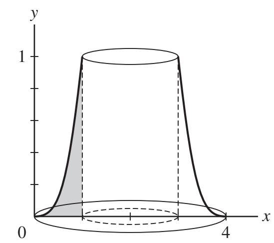


In Exercises 15–20, sketch the solid obtained by rotating the region underneath the graph of $f(x)$ over the interval about the given axis, and calculate its volume using the Shell Method.

15. $f(x) = x^3$, $[0, 1]$, about $x = 2$

SOLUTION A sketch of the solid is shown below. Each shell has radius $2 - x$ and height x^3, so the volume of the solid is

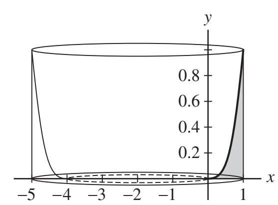
$$2\pi \int_0^1 (2 - x)\left(x^3\right) dx = 2\pi \int_0^1 (2x^3 - x^4)\, dx = 2\pi \left(\frac{x^4}{2} - \frac{x^5}{5}\right)\bigg|_0^1 = \frac{3\pi}{5}.$$



16. $f(x) = x^3$, $[0, 1]$, about $x = -2$

SOLUTION A sketch of the solid is shown below. Each shell has radius $x - (-2) = x + 2$ and height x^3, so the volume of the solid is

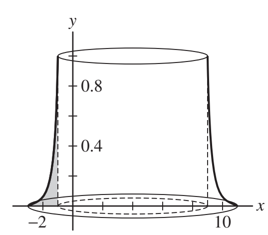
$$2\pi \int_0^1 (2 + x)\left(x^3\right) dx = 2\pi \int_0^1 (2x^3 + x^4)\, dx = 2\pi \left(\frac{x^4}{2} + \frac{x^5}{5}\right)\bigg|_0^1 = \frac{7\pi}{5}.$$



17. $f(x) = x^{-4}$, $[-3, -1]$, about $x = 4$

SOLUTION A sketch of the solid is shown below. Each shell has radius $4 - x$ and height x^{-4}, so the volume of the solid is

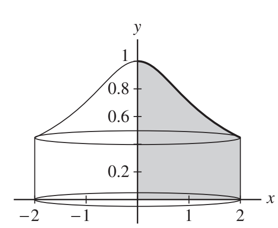
$$2\pi \int_{-3}^{-1} (4 - x)\left(x^{-4}\right) dx = 2\pi \int_{-3}^{-1} (4x^{-4} - x^{-3})\, dx = 2\pi \left(\frac{1}{2}x^{-2} - \frac{4}{3}x^{-3}\right)\bigg|_{-3}^{-1} = \frac{280\pi}{81}.$$



18. $f(x) = \dfrac{1}{\sqrt{x^2 + 1}}$, $[0, 2]$, about $x = 0$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height $\dfrac{1}{\sqrt{x^2 + 1}}$, so the volume of the solid is

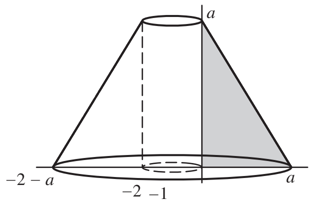
$$2\pi \int_0^2 x \left(\frac{1}{\sqrt{x^2 + 1}}\right) dx = 2\pi \left(\sqrt{x^2 + 1}\right)\bigg|_0^2 = 2\pi(\sqrt{5} - 1).$$



19. $f(x) = a - x$ with $a > 0$, $[0, a]$, about $x = -1$

SOLUTION A sketch of the solid is shown below. Each shell has radius $x - (-1) = x + 1$ and height $a - x$, so the volume of the solid is

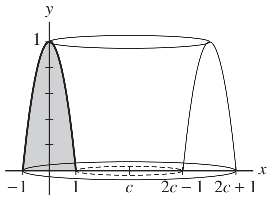
$$2\pi \int_0^a (x+1)(a-x)\,dx = 2\pi \int_0^a \left(a + (a-1)x - x^2\right) dx$$
$$= 2\pi \left(ax + \frac{a-1}{2}x^2 - \frac{1}{3}x^3 \right)\Big|_0^a$$
$$= 2\pi \left(a^2 + \frac{a^2(a-1)}{2} - \frac{a^3}{3} \right) = \frac{a^2(a+3)}{3}\pi.$$



20. $f(x) = 1 - x^2$, $[-1, 1]$, $x = c$ with $c > 1$

SOLUTION A sketch of the solid is shown below. Each shell has radius $c - x$ and height $1 - x^2$, so the volume of the solid is

$$2\pi \int_{-1}^1 (c-x)\left(1-x^2\right) dx = 2\pi \int_{-1}^1 \left(x^3 - cx^2 - x + c\right) dx = 2\pi \left(\frac{1}{4}x^4 - \frac{c}{3}x^3 - \frac{1}{2}x^2 + cx \right)\Big|_{-1}^1 = \frac{8c\pi}{3}.$$

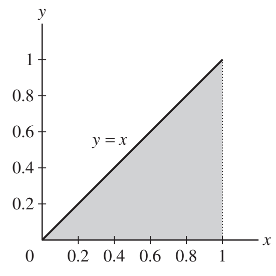


In Exercises 21–26, sketch the enclosed region and use the Shell Method to calculate the volume of rotation about the x-axis.

21. $x = y$, $y = 0$, $x = 1$

SOLUTION When the region shown below is rotated about the x-axis, each shell has radius y and height $1 - y$. The volume of the resulting solid is

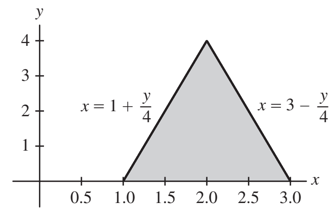
$$2\pi \int_0^1 y(1-y)\,dy = 2\pi \int_0^1 (y - y^2)\,dy = 2\pi \left(\frac{1}{2}y^2 - \frac{1}{3}y^3 \right)\Big|_0^1 = \frac{\pi}{3}.$$



22. $x = \frac{1}{4}y + 1$, $x = 3 - \frac{1}{4}y$, $y = 0$

SOLUTION When the region shown below is rotated about the x-axis, each shell has radius y and height $2 - \frac{1}{2}y$. The volume of the resulting solid is

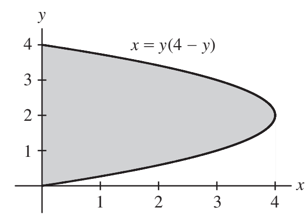
$$2\pi \int_0^4 y\left(2 - \frac{1}{2}y\right) dy = 2\pi \int_0^4 \left(2y - \frac{1}{2}y^2\right) dy = 2\pi \left(y^2 - \frac{1}{6}y^3 \right)\Big|_0^4 = \frac{32\pi}{3}.$$



23. $x = y(4 - y), \quad y = 0$

SOLUTION When the region shown below is rotated about the x-axis, each shell has radius y and height $y(4 - y)$. The volume of the resulting solid is

$$2\pi \int_0^4 y^2(4 - y)\, dy = 2\pi \int_0^4 (4y^2 - y^3)\, dy = 2\pi \left(\frac{4}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^4 = \frac{128\pi}{3}.$$



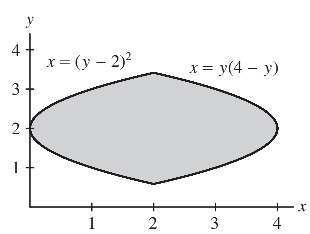
24. $x = y(4 - y), \quad x = (y - 2)^2$

SOLUTION Setting $y(4 - y) = (y - 2)^2$ yields

$$y^2 - 4y + 2 = 0 \quad \text{or} \quad y = 2 \pm \sqrt{2}.$$

When the region shown below is rotated about the x-axis, each shell has radius y and height $-2y^2 + 8y - 4$. The volume of the resulting solid is

$$2\pi \int_{2-\sqrt{2}}^{2+\sqrt{2}} y(-2y^2 + 8y - 4)\, dy = 2\pi \int_{2-\sqrt{2}}^{2+\sqrt{2}} (-2y^3 + 8y^2 - 4y)\, dy = 2\pi \left(-\frac{1}{2}y^4 + \frac{8}{3}y^3 - 2y^2\right)\Big|_{2-\sqrt{2}}^{2+\sqrt{2}} = \frac{64\pi\sqrt{2}}{3}.$$



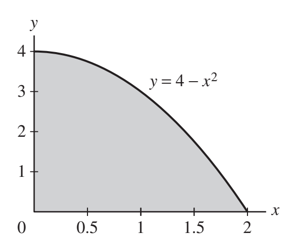
25. $y = 4 - x^2, \quad x = 0, \quad y = 0$

SOLUTION When the region shown below is rotated about the x-axis, each shell has radius y and height $\sqrt{4 - y}$. The volume of the resulting solid is

$$2\pi \int_0^4 y\sqrt{4 - y}\, dy.$$

Let $u = 4 - y$. Then $du = -dy$, $y = 4 - u$, and

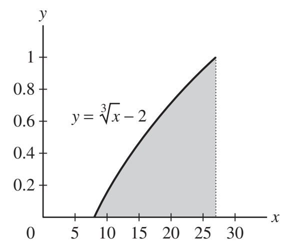
$$2\pi \int_0^4 y\sqrt{4 - y}\, dy = -2\pi \int_4^0 (4 - u)\sqrt{u}\, du = 2\pi \int_0^4 \left(4\sqrt{u} - u^{3/2}\right) du$$
$$= 2\pi \left(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_0^4 = \frac{256\pi}{15}.$$



26. $y = x^{1/3} - 2, \quad y = 0, \quad x = 27$

SOLUTION When the region shown below is rotated about the x-axis, each shell has radius y and height $27 - (y+2)^3$. The volume of the resulting solid is

$$2\pi \int_0^1 y \cdot \left(27 - (y+2)^3\right) dy = 2\pi \int_0^1 \left(19y - 12y^2 - 6y^3 - y^4\right) dy$$
$$= 2\pi \left(\frac{19}{2}y^2 - 4y^3 - \frac{3}{2}y^4 - \frac{1}{5}y^5\right)\Big|_0^1 = \frac{38\pi}{5}.$$



27. Use both the Shell and Disk Methods to calculate the volume obtained by rotating the region under the graph of $f(x) = 8 - x^3$ for $0 \le x \le 2$ about:

(a) the x-axis

(b) the y-axis

SOLUTION

(a) x-axis: Using the disk method, the cross sections are disks with radius $R = 8 - x^3$; hence the volume of the solid is

$$\pi \int_0^2 (8 - x^3)^2 \, dx = \pi \left(64x - 4x^4 + \frac{1}{7}x^7\right)\Big|_0^2 = \frac{576\pi}{7}.$$

With the shell method, each shell has radius y and height $(8-y)^{1/3}$. The volume of the solid is

$$2\pi \int_0^8 y\,(8-y)^{1/3} \, dy$$

Let $u = 8 - y$. Then $dy = -du$, $y = 8 - u$ and

$$2\pi \int_0^8 y\,(8-y)^{1/3} \, dy = 2\pi \int_0^8 (8-u)\cdot u^{1/3} \, du = 2\pi \int_0^8 (8u^{1/3} - u^{4/3}) \, du$$
$$= 2\pi \left(6u^{4/3} - \frac{3}{7}u^{7/3}\right)\Big|_0^8 = \frac{576\pi}{7}.$$

(b) y-axis: With the shell method, each shell has radius x and height $8 - x^3$. The volume of the solid is

$$2\pi \int_0^2 x(8 - x^3) \, dx = 2\pi \left(4x^2 - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{96\pi}{5}.$$

Using the disk method, the cross sections are disks with radius $R = (8-y)^{1/3}$. The volume is then given by

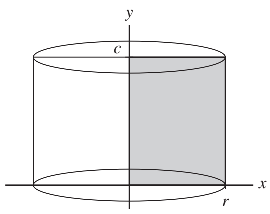
$$\pi \int_0^8 (8-y)^{2/3} \, dy = -\frac{3\pi}{5}(8-y)^{5/3}\Big|_0^8 = \frac{96\pi}{5}.$$

28. Sketch the solid of rotation about the y-axis for the region under the graph of the constant function $f(x) = c$ (where $c > 0$) for $0 \le x \le r$.

(a) Find the volume without using integration.

(b) Use the Shell Method to compute the volume.

SOLUTION



(a) The solid is simply a cylinder with height c and radius r. The volume is given by $\pi r^2 c$.
(b) Each shell has radius x and height c, so the volume is

$$2\pi \int_0^r cx\,dx = 2\pi \left(c\frac{1}{2}x^2 \right)\Big|_0^r = \pi r^2 c.$$

29. The graph in Figure 11(A) can be described by both $y = f(x)$ and $x = h(y)$, where h is the inverse of f. Let V be the volume obtained by rotating the region under the graph about the y-axis.
(a) Describe the figures generated by rotating segments $\overline{AB}$ and $\overline{CB}$ about the y-axis.
(b) Set up integrals that compute V by the Shell and Disk Methods.

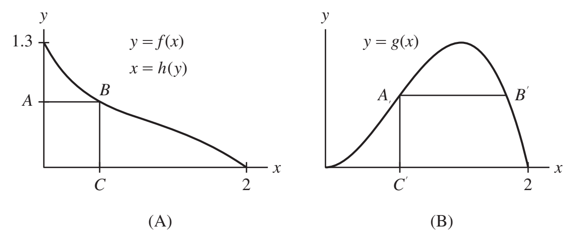


FIGURE 11

SOLUTION
(a) When rotated about the y-axis, the segment $\overline{AB}$ generates a disk with radius $R = h(y)$ and the segment $\overline{CB}$ generates a shell with radius x and height $f(x)$.
(b) Based on Figure 11(A) and the information from part (a), when using the Shell Method,

$$V = 2\pi \int_0^2 x f(x)\,dx;$$

when using the Disk Method,

$$V = \pi \int_0^{1.3} (h(y))^2\,dy.$$

30. Let W be the volume of the solid obtained by rotating the region under the graph in Figure 11(B) about the y-axis.
(a) Describe the figures generated by rotating segments $\overline{A'B'}$ and $\overline{A'C'}$ about the y-axis.
(b) Set up an integral that computes W by the Shell Method.
(c) Explain the difficulty in computing W by the Washer Method.

SOLUTION
(a) When rotated about the y-axis, the segment $\overline{A'B'}$ generates a washer and the segment $\overline{C'A'}$ generates a shell with radius x and height $g(x)$.
(b) Using Figure 11(B) and the information from part (a),

$$W = 2\pi \int_0^2 x g(x)\,dx.$$

(c) The function $g(x)$ is not one-to-one, which makes it difficult to determine the inner and outer radius of each washer.

31. Let R be the region under the graph of $y = 9 - x^2$ for $0 \le x \le 2$. Use the Shell Method to compute the volume of rotation of R about the x-axis as a sum of two integrals along the y-axis. *Hint:* The shells generated depend on whether $y \in [0, 5]$ or $y \in [5, 9]$.

SOLUTION The region R is sketched below. When rotating this region about the x-axis, we produce a solid with two different shell structures. For $0 \le y \le 5$, the shell has radius y and height 2; for $5 \le y \le 9$, the shell has radius y and height $\sqrt{9 - y}$. The volume of the solid is therefore

$$V = 2\pi \int_0^5 2y\,dy + 2\pi \int_5^9 y\sqrt{9 - y}\,dy$$

For the first integral, we calculate

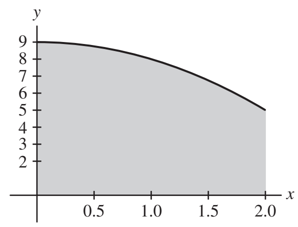
$$2\pi \int_0^5 2y\,dy = 2\pi y^2\Big|_0^5 = 50\pi.$$

For the second integral, we make the substitution $u = 9 - y$, $du = -dy$ and find

$$2\pi \int_5^9 y\sqrt{9-y}\,dy = -2\pi \int_4^0 (9-u)\sqrt{u}\,du$$
$$= 2\pi \int_0^4 (9u^{1/2} - u^{3/2})\,du$$
$$= 2\pi \left(6u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_0^4$$
$$= 2\pi \left(48 - \frac{64}{5}\right) = \frac{352\pi}{5}.$$

Thus, the total volume is

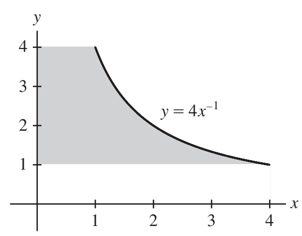
$$V = 50\pi + \frac{352\pi}{5} = \frac{602\pi}{5}.$$



32. Let R be the region under the graph of $y = 4x^{-1}$ for $1 \le y \le 4$. Use the Shell Method to compute the volume of rotation of R about the y-axis as a sum of two integrals along the x-axis.

SOLUTION The region R is sketched below. When rotating this region about the y-axis, we produce a solid with two different shell structures. For $0 \le x \le 1$, the shell has radius x and height 3; for $1 \le x \le 4$, the shell has radius x and height $4x^{-1} - 1$. The volume of the solid is therefore

$$V = 2\pi \int_0^1 3x\,dx + 2\pi \int_1^4 x(4x^{-1} - 1)\,dx$$
$$= 2\pi \int_0^1 3x\,dx + 2\pi \int_1^4 (4 - x)\,dx$$
$$= 2\pi \,\frac{3}{2}x^2\Big|_0^1 + 2\pi \left(4x - \frac{1}{2}x^2\right)\Big|_1^4$$
$$= 3\pi + 2\pi \left(8 - \frac{7}{2}\right) = 12\pi.$$



In Exercises 33–38, use the Shell Method to find the volume obtained by rotating region A in Figure 12 about the given axis.

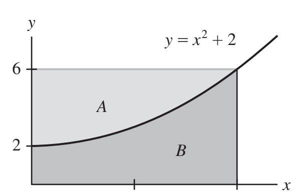


FIGURE 12

33. y-axis

SOLUTION When rotating region A about the y-axis, each shell has radius x and height $6-(x^2+2)=4-x^2$. The volume of the resulting solid is

$$2\pi\int_0^2 x(4-x^2)\,dx = 2\pi\int_0^2 (4x-x^3)\,dx = 2\pi\left(2x^2-\frac{1}{4}x^4\right)\Big|_0^2 = 8\pi.$$

34. $x=-3$

SOLUTION When rotating region A about $x=-3$, each shell has radius $x-(-3)=x+3$ and height $6-(x^2+2)=4-x^2$. The volume of the resulting solid is

$$2\pi\int_0^2 (x+3)(4-x^2)\,dx = 2\pi\int_0^2 (4x-x^3+12-3x^2)\,dx = 2\pi\left(2x^2-\frac{1}{4}x^4+12x-x^3\right)\Big|_0^2 = 40\pi.$$

35. $x=2$

SOLUTION When rotating region A about $x=2$, each shell has radius $2-x$ and height $6-(x^2+2)=4-x^2$. The volume of the resulting solid is

$$2\pi\int_0^2 (2-x)\left(4-x^2\right)\,dx = 2\pi\int_0^2 \left(8-2x^2-4x+x^3\right)\,dx = 2\pi\left(8x-\frac{2}{3}x^3-2x^2+\frac{1}{4}x^4\right)\Big|_0^2 = \frac{40\pi}{3}.$$

36. x-axis

SOLUTION When rotating region A about the x-axis, each shell has radius y and height $\sqrt{y-2}$. The volume of the resulting solid is

$$2\pi\int_2^6 y\sqrt{y-2}\,dy$$

Let $u=y-2$. Then $du=dy$, $y=u+2$ and

$$2\pi\int_2^6 y\sqrt{y-2}\,dy = 2\pi\int_0^4 (u+2)\sqrt{u}\,du = 2\pi\left(\frac{2}{5}u^{5/2}+\frac{4}{3}u^{3/2}\right)\Big|_0^4 = \frac{704\pi}{15}.$$

37. $y=-2$

SOLUTION When rotating region A about $y=-2$, each shell has radius $y-(-2)=y+2$ and height $\sqrt{y-2}$. The volume of the resulting solid is

$$2\pi\int_2^6 (y+2)\sqrt{y-2}\,dy$$

Let $u=y-2$. Then $du=dy$, $y+2=u+4$ and

$$2\pi\int_2^6 (y+2)\sqrt{y-2}\,dy = 2\pi\int_0^4 (u+4)\sqrt{u}\,du = 2\pi\left(\frac{2}{5}u^{5/2}+\frac{8}{3}u^{3/2}\right)\Big|_0^4 = \frac{1024\pi}{15}.$$

38. $y=6$

SOLUTION When rotating region A about $y=6$, each shell has radius $6-y$ and height $\sqrt{y-2}$. The volume of the resulting solid is

$$2\pi\int_2^6 (6-y)\sqrt{y-2}\,dy$$

Let $u=y-2$. Then $du=dy$, $6-y=4-u$ and

$$2\pi\int_2^6 (6-y)\sqrt{y-2}\,dy = 2\pi\int_0^4 (4-u)\sqrt{u}\,du = 2\pi\left(\frac{8}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right)\Big|_0^4 = \frac{256\pi}{15}.$$

In Exercises 39–44, use the most convenient method (Disk or Shell Method) to find the volume obtained by rotating region B in Figure 12 about the given axis.

39. y-axis

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is parallel to the axis of rotation, we will use the Shell Method. Each shell has radius x and height $x^2 + 2$. The volume of the resulting solid is

$$2\pi \int_0^2 x(x^2+2)\,dx = 2\pi \int_0^2 (x^3+2x)\,dx = 2\pi\left(\frac{1}{4}x^4 + x^2\right)\Big|_0^2 = 16\pi.$$

40. $x = -3$

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is parallel to the axis of rotation, we will use the Shell Method. Each shell has radius $x - (-3) = x + 3$ and height $x^2 + 2$. The volume of the resulting solid is

$$2\pi \int_0^2 (x+3)(x^2+2)\,dx = 2\pi \int_0^2 (x^3+3x^2+2x+6)\,dx = 2\pi\left(\frac{1}{4}x^4 + x^3 + x^2 + 6x\right)\Big|_0^2 = 56\pi.$$

41. $x = 2$

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is parallel to the axis of rotation, we will use the Shell Method. Each shell has radius $2 - x$ and height $x^2 + 2$. The volume of the resulting solid is

$$2\pi \int_0^2 (2-x)\left(x^2+2\right)\,dx = 2\pi \int_0^2 \left(2x^2 - x^3 + 4 - 2x\right)\,dx = 2\pi\left(\frac{2}{3}x^3 - \frac{1}{4}x^4 + 4x - x^2\right)\Big|_0^2 = \frac{32\pi}{3}.$$

42. x-axis

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is perpendicular to the axis of rotation, we will use the Disk Method. Each disk has outer radius $R = x^2 + 2$ and inner radius $r = 0$. The volume of the solid is then

$$\begin{aligned}\pi \int_0^2 (x^2+2)^2\,dx &= \pi \int_0^2 (x^4+4x^2+4)\,dx \\ &= \pi\left(\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x\right)\Big|_0^2 \\ &= \pi\left(\frac{32}{5} + \frac{32}{3} + 8\right) = \frac{376\pi}{15}.\end{aligned}$$

43. $y = -2$

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is perpendicular to the axis of rotation, we will use the Disk Method. Each disk has outer radius $R = x^2 + 2 - (-2) = x^2 + 4$ and inner radius $r = 0 - (-2) = 2$. The volume of the solid is then

$$\begin{aligned}\pi \int_0^2 \left((x^2+4)^2 - 2^2\right)\,dx &= \pi \int_0^2 (x^4+8x^2+12)\,dx \\ &= \pi\left(\frac{1}{5}x^5 + \frac{8}{3}x^3 + 12x\right)\Big|_0^2 \\ &= \pi\left(\frac{32}{5} + \frac{64}{3} + 24\right) = \frac{776\pi}{15}.\end{aligned}$$

44. $y = 8$

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is perpendicular to the axis of rotation, we will use the Disk Method. Each disk has outer radius $R = 8 - 0 = 8$ and inner radius $r = 8 - (x^2 + 2) = 6 - x^2$. The volume of the solid is then

$$\pi \int_0^2 \left(8^2 - (6 - x^2)^2\right) dx = \pi \int_0^2 (28 + 12x^2 - x^4)\, dx$$
$$= \pi \left(28x + 4x^3 - \frac{1}{5}x^5\right)\Big|_0^2$$
$$= \pi \left(56 + 32 - \frac{32}{5}\right) = \frac{408\pi}{5}.$$

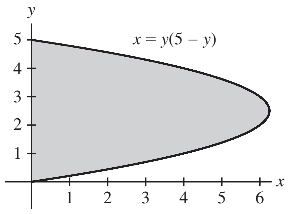
In Exercises 45–50, use the most convenient method (Disk or Shell Method) to find the given volume of rotation.

45. Region between $x = y(5 - y)$ and $x = 0$, rotated about the y-axis

SOLUTION Examine the figure below, which shows the region bounded by $x = y(5 - y)$ and $x = 0$. If the indicated region is sliced vertically, then the top of the slice lies along one branch of the parabola $x = y(5 - y)$ and the bottom lies along the other branch. On the other hand, if the region is sliced horizontally, then the right endpoint of the slice always lies along the parabola and left endpoint always lies along the y-axis. Clearly, it will be easier to slice the region horizontally.

Now, suppose the region is rotated about the y-axis. Because a horizontal slice is perpendicular to the y-axis, we will calculate the volume of the resulting solid using the disk method. Each cross section is a disk of radius $R = y(5 - y)$, so the volume is

$$\pi \int_0^5 y^2(5 - y)^2\, dy = \pi \int_0^5 (25y^2 - 10y^3 + y^4)\, dy = \pi \left(\frac{25}{3}y^3 - \frac{5}{2}y^4 + \frac{1}{5}y^5\right)\Big|_0^5 = \frac{625\pi}{6}.$$



46. Region between $x = y(5 - y)$ and $x = 0$, rotated about the x-axis

SOLUTION Examine the figure from the previous exercise, which shows the region bounded by $x = y(5 - y)$ and $x = 0$. If the indicated region is sliced vertically, then the top of the slice lies along one branch of the parabola $x = y(5 - y)$ and the bottom lies along the other branch. On the other hand, if the region is sliced horizontally, then the right endpoint of the slice always lies along the parabola and left endpoint always lies along the y-axis. Clearly, it will be easier to slice the region horizontally.

Now, suppose the region is rotated about the x-axis. Because a horizontal slice is parallel to the x-axis, we will calculate the volume of the resulting solid using the shell method. Each shell has a radius of y and a height of $y(5 - y)$, so the volume is

$$2\pi \int_0^5 y^2(5 - y)\, dy = 2\pi \int_0^5 (5y^2 - y^3)\, dy = 2\pi \left(\frac{5}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^5 = \frac{625\pi}{6}.$$

47. Region in Figure 13, rotated about the x-axis

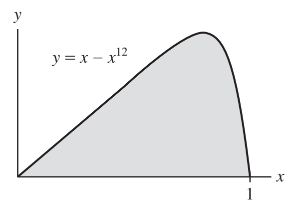


FIGURE 13

SOLUTION Examine Figure 13. If the indicated region is sliced vertically, then the top of the slice lies along the curve $y = x - x^{12}$ and the bottom lies along the curve $y = 0$ (the x-axis). On the other hand, if the region is sliced horizontally, the equation $y = x - x^{12}$ must be solved for x in order to determine the endpoint locations. Clearly, it will be easier to slice the region vertically.

Now, suppose the region in Figure 13 is rotated about the x-axis. Because a vertical slice is perpendicular to the x-axis, we will calculate the volume of the resulting solid using the disk method. Each cross section is a disk of radius $R = x - x^{12}$, so the volume is

$$\pi \int_0^1 \left(x - x^{12}\right)^2 dx = \pi \left(\frac{1}{3}x^3 - \frac{1}{7}x^{14} + \frac{1}{25}x^{25}\right)\Big|_0^1 = \frac{121\pi}{525}.$$

48. Region in Figure 13, rotated about the y-axis

SOLUTION Examine Figure 13. If the indicated region is sliced vertically, then the top of the slice lies along the curve $y = x - x^{12}$ and the bottom lies along the curve $y = 0$ (the x-axis). On the other hand, if the region is sliced horizontally, the equation $y = x - x^{12}$ must be solved for x in order to determine the endpoint locations. Clearly, it will be easier to slice the region vertically.

Now suppose the region is rotated about the y-axis. Because a vertical slice is parallel to the y-axis, we will calculate the volume of the resulting solid using the shell method. Each shell has radius x and height $x - x^{12}$, so the volume is

$$2\pi \int_0^1 x(x - x^{12})\,dx = 2\pi \left(\frac{1}{3}x^3 - \frac{1}{14}x^{14}\right)\Big|_0^1 = \frac{11\pi}{21}.$$

49. Region in Figure 14, rotated about $x = 4$

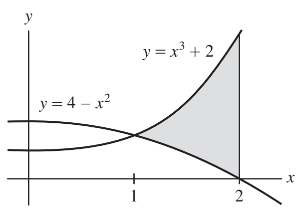


FIGURE 14

SOLUTION Examine Figure 14. If the indicated region is sliced vertically, then the top of the slice lies along the curve $y = x^3 + 2$ and the bottom lies along the curve $y = 4 - x^2$. On the other hand, the left end of a horizontal slice switches from $y = 4 - x^2$ to $y = x^3 + 2$ at $y = 3$. Here, vertical slices will be more convenient.

Now, suppose the region in Figure 14 is rotated about $x = 4$. Because a vertical slice is parallel to $x = 4$, we will calculate the volume of the resulting solid using the shell method. Each shell has radius $4 - x$ and height $x^3 + 2 - (4 - x^2) = x^3 + x^2 - 2$, so the volume is

$$2\pi \int_1^2 (4 - x)(x^3 + x^2 - 2)\,dx = 2\pi \left(-\frac{1}{5}x^5 + \frac{3}{4}x^4 + \frac{4}{3}x^3 + x^2 - 8x\right)\Big|_1^2 = \frac{563\pi}{30}.$$

50. Region in Figure 14, rotated about $y = -2$

SOLUTION Examine Figure 14. If the indicated region is sliced vertically, then the top of the slice lies along the curve $y = x^3 + 2$ and the bottom lies along the curve $y = 4 - x^2$. On the other hand, the left end of a horizontal slice switches from $y = 4 - x^2$ to $y = x^3 + 2$ at $y = 3$. Here, vertical slices will be more convenient.

Now suppose the region is rotated about $y = -2$. Because a vertical slice is perpendicular to $y = -2$, we will calculate the volume of the resulting solid using the disk method. Each cross section is a washer with outer radius $R = x^3 + 2 - (-2) = x^3 + 4$ and inner radius $r = 4 - x^2 - (-2) = 6 - x^2$, so the volume is

$$\pi \int_1^2 \left((x^3 + 4)^2 - (6 - x^2)^2\right)\,dx = \pi \left(\frac{1}{7}x^7 - \frac{1}{5}x^5 + 2x^4 + 4x^3 - 20x\right)\Big|_1^2 = \frac{1748\pi}{35}.$$

In Exercises 51–54, use the Shell Method to find the given volume of rotation.

51. A sphere of radius r

SOLUTION A sphere of radius r can be generated by rotating the region under the semicircle $y = \sqrt{r^2 - x^2}$ about the x-axis. Each shell has radius y and height

$$\sqrt{r^2 - y^2} - \left(-\sqrt{r^2 - y^2}\right) = 2\sqrt{r^2 - y^2}.$$

Thus, the volume of the sphere is

$$2\pi \int_0^r 2y\sqrt{r^2 - y^2}\,dy.$$

Let $u = r^2 - y^2$. Then $du = -2y\,dy$ and

$$2\pi \int_0^r 2y\sqrt{r^2 - y^2}\,dy = 2\pi \int_0^{r^2} \sqrt{u}\,du = 2\pi \left(\frac{2}{3}u^{3/2}\right)\Big|_0^{r^2} = \frac{4}{3}\pi r^3.$$

52. The "bead" formed by removing a cylinder of radius r from the center of a sphere of radius R (compare with Exercise 59 in Section 6.3)

SOLUTION Each shell has radius x and height $2\sqrt{R^2 - x^2}$. The volume of the bead is then

$$2\pi \int_r^R 2x\sqrt{R^2 - x^2}\,dx.$$

Let $u = R^2 - x^2$. Then $du = -2x\,dx$ and

$$2\pi \int_r^R 2x\sqrt{R^2 - x^2}\,dx = 2\pi \int_0^{R^2-r^2} \sqrt{u}\,du = 2\pi \left(\frac{2}{3}u^{3/2}\right)\Bigg|_0^{R^2-r^2} = \frac{4}{3}\pi(R^2 - r^2)^{3/2}.$$

53. The torus obtained by rotating the circle $(x-a)^2 + y^2 = b^2$ about the y-axis, where $a > b$ (compare with Exercise 53 in Section 5.3). *Hint:* Evaluate the integral by interpreting part of it as the area of a circle.

SOLUTION When rotating the region enclosed by the circle $(x-a)^2 + y^2 = b^2$ about the y-axis each shell has radius x and height

$$\sqrt{b^2 - (x-a)^2} - \left(-\sqrt{b^2 - (x-a)^2}\right) = 2\sqrt{b^2 - (x-a)^2}.$$

The volume of the resulting torus is then

$$2\pi \int_{a-b}^{a+b} 2x\sqrt{b^2 - (x-a)^2}\,dx.$$

Let $u = x - a$. Then $du = dx$, $x = u + a$ and

$$2\pi \int_{a-b}^{a+b} 2x\sqrt{b^2 - (x-a)^2}\,dx = 2\pi \int_{-b}^{b} 2(u+a)\sqrt{b^2 - u^2}\,du$$
$$= 4\pi \int_{-b}^{b} u\sqrt{b^2 - u^2}\,du + 4a\pi \int_{-b}^{b} \sqrt{b^2 - u^2}\,du.$$

Now,

$$\int_{-b}^{b} u\sqrt{b^2 - u^2}\,du = 0$$

because the integrand is an odd function and the integration interval is symmetric with respect to zero. Moreover, the other integral is one-half the area of a circle of radius b; thus,

$$\int_{-b}^{b} \sqrt{b^2 - u^2}\,du = \frac{1}{2}\pi b^2.$$

Finally, the volume of the torus is

$$4\pi(0) + 4a\pi\left(\frac{1}{2}\pi b^2\right) = 2\pi^2 ab^2.$$

54. The "paraboloid" obtained by rotating the region between $y = x^2$ and $y = c$ ($c > 0$) about the y-axis

SOLUTION When we rotate the region in the first quadrant bounded by $y = x^2$ and $y = c$ about the y-axis, each shell has a radius of x and a height of $c - x^2$. The volume of the paraboloid is then

$$2\pi \int_0^{\sqrt{c}} x(c - x^2)\,dx = 2\pi \int_0^{\sqrt{c}} (cx - x^3)\,dx = 2\pi \left(\frac{1}{2}cx^2 - \frac{1}{4}x^4\right)\Bigg|_0^{\sqrt{c}} = \frac{1}{2}\pi c^2.$$

Further Insights and Challenges

55. The surface area of a sphere of radius r is $4\pi r^2$. Use this to derive the formula for the volume V of a sphere of radius R in a new way.

(a) Show that the volume of a thin spherical shell of inner radius r and thickness Δr is approximately $4\pi r^2 \Delta r$.

(b) Approximate V by decomposing the sphere of radius R into N thin spherical shells of thickness $\Delta r = R/N$.

(c) Show that the approximation is a Riemann sum that converges to an integral. Evaluate the integral.

SOLUTION

(a) The volume of a thin spherical shell of inner radius r and thickness Δx is given by the product of the surface area of the shell, $4\pi r^2$ and the thickness. Thus, we have $4\pi r^2 \Delta x$.

(b) The volume of the sphere is approximated by

$$R_N = 4\pi \left(\frac{R}{N}\right) \sum_{k=1}^{N} (x_k)^2$$

where $x_k = k\frac{R}{N}$.

(c) $V = 4\pi \lim_{N\to\infty} \left(\frac{R}{N}\right) \sum_{k=1}^{N} (x_k)^2 = 4\pi \int_0^R x^2\,dx = 4\pi \left(\frac{1}{3}x^3\right)\Big|_0^R = \frac{4}{3}\pi R^3.$

56. Show that the solid (an **ellipsoid**) obtained by rotating the region R in Figure 15 about the y-axis has volume $\frac{4}{3}\pi a^2 b$.

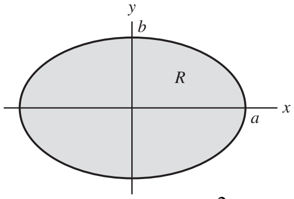


FIGURE 15 The ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

SOLUTION Let's slice the portion of the ellipse in the first and fourth quadrants horizontally and rotate the slices about the y-axis. The resulting ellipsoid has cross sections that are disks with radius

$$R = \sqrt{a^2 - \frac{a^2 y^2}{b^2}}.$$

Thus, the volume of the ellipsoid is

$$\pi \int_{-b}^{b} \left(a^2 - \frac{a^2y^2}{b^2}\right) dy = \pi \left(a^2 y - \frac{a^2 y^3}{3b^2}\right)\Bigg|_{-b}^{b} = \pi \left[\left(a^2 b - \frac{a^2 b}{3}\right) - \left(-a^2 b + \frac{a^2 b}{3}\right)\right] = \frac{4}{3}\pi a^2 b.$$

57. The bell-shaped curve $y = f(x)$ in Figure 16 satisfies $dy/dx = -xy$. Use the Shell Method and the substitution $u = f(x)$ to show that the solid obtained by rotating the region R about the y-axis has volume $V = 2\pi(1 - c)$, where $c = f(a)$. Observe that as $c \to 0$, the region R becomes infinite but the volume V approaches 2π.

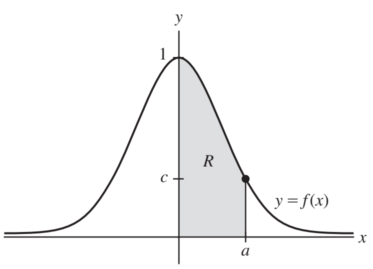


FIGURE 16 The bell-shaped curve.

SOLUTION Let $y = f(x)$ be the exponential function depicted in Figure 16. When rotating the region R about the y-axis, each shell in the resulting solid has radius x and height $f(x)$. The volume of the solid is then

$$V = 2\pi \int_0^a x f(x)\,dx.$$

Now, let $u = f(x)$. Then $du = f'(x)\,dx = -xf(x)\,dx$; hence, $xf(x)dx = -du$, and

$$V = 2\pi \int_1^c (-du) = 2\pi \int_c^1 du = 2\pi(1 - c).$$

6.5 Work and Energy

Preliminary Questions

1. Why is integration needed to compute the work performed in stretching a spring?

SOLUTION Recall that the force needed to extend or compress a spring depends on the amount by which the spring has already been extended or compressed from its equilibrium position. In other words, the force needed to move a spring is variable. Whenever the force is variable, work needs to be computed with an integral.

2. Why is integration needed to compute the work performed in pumping water out of a tank but not to compute the work performed in lifting up the tank?

SOLUTION To lift a tank through a vertical distance d, the force needed to move the tank remains constant; hence, no integral is needed to calculate the work done in lifting the tank. On the other hand, pumping water from a tank requires that different layers of the water be lifted through different distances, and, depending on the shape of the tank, may require different forces. Thus, pumping water from a tank requires that an integral be evaluated.

3. Which of the following represents the work required to stretch a spring (with spring constant k) a distance x beyond its equilibrium position: kx, $-kx$, $\frac{1}{2}mk^2$, $\frac{1}{2}kx^2$, or $\frac{1}{2}mx^2$?

SOLUTION The work required to stretch a spring with spring constant k a distance x beyond its equilibrium position is

$$\int_0^x ky\,dy = \frac{1}{2}ky^2\Big|_0^x = \frac{1}{2}kx^2.$$

Exercises

1. How much work is done raising a 4-kg mass to a height of 16 m above ground?

SOLUTION The force needed to lift a 4-kg object is a constant

$$(4\text{ kg})(9.8\text{ m/s}^2) = 39.2\text{ N}.$$

The work done in lifting the object to a height of 16 m is then

$$(39.2\text{ N})(16\text{ m}) = 627.2\text{ J}.$$

2. How much work is done raising a 4-lb mass to a height of 16 ft above ground?

SOLUTION The force needed to lift a 4-lb object is a constant 4 lb. The work done in lifting the object to a height of 16 ft is then

$$(4\text{ lb})(16\text{ ft}) = 64\text{ ft-lb}.$$

In Exercises 3–6, compute the work (in joules) required to stretch or compress a spring as indicated, assuming a spring constant of $k = 800$ N/m.

3. Stretching from equilibrium to 12 cm past equilibrium

SOLUTION The work required to stretch the spring 12 cm past equilibrium is

$$\int_0^{0.12} 800x\,dx = 400x^2\Big|_0^{0.12} = 5.76\text{ J}.$$

4. Compressing from equilibrium to 4 cm past equilibrium

SOLUTION The work required to compress the spring 4 cm past equilibrium is

$$\int_0^{-0.04} 800x\,dx = 400x^2\Big|_0^{-0.04} = 0.64\text{ J}.$$

5. Stretching from 5 cm to 15 cm past equilibrium

SOLUTION The work required to stretch the spring from 5 cm to 15 cm past equilibrium is

$$\int_{0.05}^{0.15} 800x\,dx = 400x^2\Big|_{0.05}^{0.15} = 8\text{ J}.$$

6. Compressing 4 cm more when it is already compressed 5 cm

SOLUTION The work required to compress the spring from 5 cm to 9 cm past equilibrium is

$$\int_{-0.05}^{-0.09} 800x\,dx = 400x^2\Big|_{-0.05}^{-0.09} = 2.24\text{ J}.$$

7. If 5 J of work are needed to stretch a spring 10 cm beyond equilibrium, how much work is required to stretch it 15 cm beyond equilibrium?

SOLUTION First, we determine the value of the spring constant as follows:

$$\int_0^{0.1} kx\,dx = \frac{1}{2}kx^2\Big|_0^{0.1} = 0.005k = 5\text{ J}.$$

Thus, $k = 1000$ N/m. Next, we calculate the work required to stretch the spring 15 cm beyond equilibrium:

$$\int_0^{0.15} 1000x\,dx = 500x^2\Big|_0^{0.15} = 11.25 \text{ J}.$$

8. To create images of samples at the molecular level, atomic force microscopes use silicon micro-cantilevers that obey Hooke's Law $F(x) = -kx$, where x is the distance through which the tip is deflected (Figure 6). Suppose that 10^{-17} J of work are required to deflect the tip a distance 10^{-8} m. Find the deflection if a force of 10^{-9} N is applied to the tip.

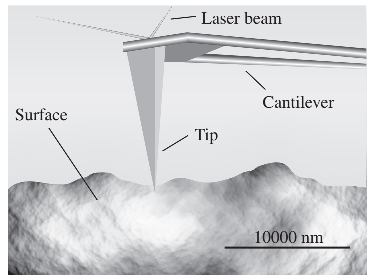


FIGURE 6

SOLUTION First, we determine the value of the constant k. Knowing it takes 10^{-17} J of work to deflect the tip a distance 10^{-8} m, it follows that

$$\frac{1}{2}k(10^{-8})^2 = 10^{-17} \quad \text{or} \quad k = \frac{1}{5} \text{ N/m}.$$

Now, the deflection produced by a force of 10^{-9} N can be determined as

$$x = \frac{F}{k} = \frac{10^{-9}}{1/5} = 5 \times 10^{-9} \text{ m}.$$

9. A spring obeys a force law $F(x) = -kx^{1.1}$ with $k = 100$ N/m. Find the work required to stretch a spring 0.3 m past equilibrium.

SOLUTION The work required to stretch this spring 0.3 m past equilibrium is

$$\int_0^{0.3} 100x^{1.1}\,dx = \frac{100}{1.1}x^{2.1}\Big|_0^{0.3} \approx 7.25 \text{ J}.$$

10. Show that the work required to stretch a spring from position a to position b is $\frac{1}{2}k(b^2 - a^2)$, where k is the spring constant. How do you interpret the negative work obtained when $|b| < |a|$?

SOLUTION The work required to stretch a spring from position a to position b is

$$\int_a^b kx\,dx = \frac{1}{2}kx^2\Big|_a^b = \frac{1}{2}k(b^2 - a^2).$$

When $|b| < |a|$, the "negative work" is the work done by the spring to return to its equilibrium position.

In Exercises 11–14, use the method of Examples 2 and 3 to calculate the work against gravity required to build the structure out of a lightweight material of density 600 kg/m^3.

11. Box of height 3 m and square base of side 2 m

SOLUTION The volume of one layer is $4\Delta y$ m^3 and so the weight of one layer is $23520\Delta y$ N. Thus, the work done against gravity to build the tower is

$$W = \int_0^3 23520y\,dy = 11760y^2\Big|_0^3 = 105840 \text{ J}.$$

12. Cylindrical column of height 4 m and radius 0.8 m

SOLUTION The area of the base is 0.64π m^2, so the volume of each small layer is $0.64\pi\,\Delta y$ m^3. The weight of one layer is then $3763.2\pi\,\Delta y$ N. Finally, the total work done against gravity to build the tower is

$$\int_0^4 3763.2\pi y\,dy = 30105.6\pi \text{ J} \approx 94579.5 \text{ J}.$$

13. Right circular cone of height 4 m and base of radius 1.2 m

SOLUTION By similar triangles, the layer of the cone at a height y above the base has radius $r = 0.3(4 - y)$ meters. Thus, the volume of the small layer at this height is $0.09\pi(4 - y)^2 \Delta y$ m^3, and the weight is $529.2\pi(4 - y)^2 \Delta y$ N. Finally, the total work done against gravity to build the tower is

$$\int_0^4 529.2\pi(4 - y)^2 y\, dy = 11289.6\pi \text{ J} \approx 35467.3 \text{ J}.$$

14. Hemisphere of radius 0.8 m

SOLUTION The area of one layer is $\pi(0.64 - y^2)$ m^2, so the volume of each small layer is $\pi(0.64 - y^2)\Delta y$ m^3. The weight of one layer is then $5880\pi(0.64 - y^2)\Delta y$ N. Finally, the total work done against gravity to build the tower is

$$\int_0^{0.8} 5880\pi(0.64 - y^2) y\, dy = 602.112\pi \text{ J} \approx 1891.6 \text{ J}.$$

15. Built around 2600 BCE, the Great Pyramid of Giza in Egypt (Figure 7) is 146 m high and has a square base of side 230 m. Find the work (against gravity) required to build the pyramid if the density of the stone is estimated at 2000 kg/m^3.

FIGURE 7 The Great Pyramid in Giza, Egypt.

SOLUTION From similar triangles, the area of one layer is

$$\left(230 - \frac{230}{146} y\right)^2 \text{ m}^2,$$

so the volume of each small layer is

$$\left(230 - \frac{230}{146} y\right)^2 \Delta y \text{ m}^3.$$

The weight of one layer is then

$$19600 \left(230 - \frac{230}{146} y\right)^2 \Delta y \text{ N}.$$

Finally, the total work needed to build the pyramid was

$$\int_0^{146} 19600 \left(230 - \frac{230}{146} y\right)^2 y\, dy \approx 1.84 \times 10^{12} \text{ J}.$$

16. Calculate the work (against gravity) required to build a box of height 3 m and square base of side 2 m out of material of variable density, assuming that the density at height y is $f(y) = 1000 - 100y$ kg/m^3.

SOLUTION The volume of one layer is $4\Delta y$ m^3 and so the weight of one layer is $(4000 - 400y)\Delta y$ N. Thus, the work done against gravity to build the tower is

$$W = \int_0^3 (4000 - 400y) y\, dy = \left(2000y^2 - \frac{400}{3} y^3\right)\Big|_0^3 = 14400 \text{ J}.$$

In Exercises 17–22, calculate the work (in joules) required to pump all of the water out of a full tank. Distances are in meters, and the density of water is 1000 kg/m^3.

17. Rectangular tank in Figure 8; water exits from a small hole at the top.

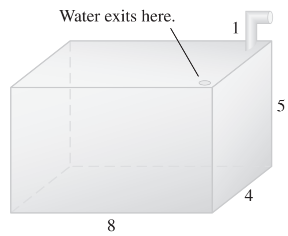


FIGURE 8

SOLUTION Place the origin on the top of the box, and let the positive y-axis point downward. The volume of one layer of water is $32\Delta y$ m^3, so the force needed to lift each layer is

$$(9.8)(1000)32\Delta y = 313600\Delta y \text{ N}.$$

Each layer must be lifted y meters, so the total work needed to empty the tank is

$$\int_0^5 313600y\,dy = 156800y^2\Big|_0^5 = 3.92 \times 10^6 \text{ J}.$$

18. Rectangular tank in Figure 8; water exits through the spout.

SOLUTION Place the origin on the top of the box, and let the positive y-axis point downward. The volume of one layer of water is $32\Delta y$ m^3, so the force needed to lift each layer is

$$(9.8)(1000)32\Delta y = 313600\Delta y \text{ N}.$$

Each layer must be lifted $y+1$ meters, so the total work needed to empty the tank is

$$\int_0^5 313600(y+1)\,dy = 156800(y+1)^2\Big|_0^5 = 5.488 \times 10^6 \text{ J}.$$

19. Hemisphere in Figure 9; water exits through the spout.

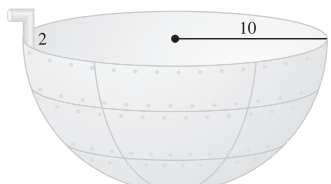


FIGURE 9

SOLUTION Place the origin at the center of the hemisphere, and let the positive y-axis point downward. The radius of a layer of water at depth y is $\sqrt{100-y^2}$ m, so the volume of the layer is $\pi(100-y^2)\Delta y$ m^3, and the force needed to lift the layer is $9800\pi(100-y^2)\Delta y$ N. The layer must be lifted $y+2$ meters, so the total work needed to empty the tank is

$$\int_0^{10} 9800\pi(100-y^2)(y+2)\,dy = \frac{112700000\pi}{3} \text{ J} \approx 1.18 \times 10^8 \text{ J}.$$

20. Conical tank in Figure 10; water exits through the spout.

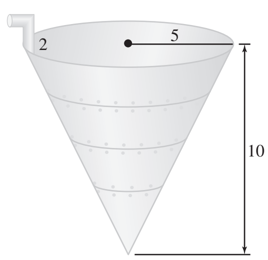


FIGURE 10

SOLUTION Place the origin at the vertex of the inverted cone, and let the positive y-axis point upward. Consider a layer of water at a height of y meters. From similar triangles, the area of the layer is

$$\pi\left(\frac{y}{2}\right)^2 \text{ m}^2,$$

so the volume is

$$\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ m}^3.$$

Thus the weight of one layer is

$$9800\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ N}.$$

The layer must be lifted $12 - y$ meters, so the total work needed to empty the tank is

$$\int_0^{10} 9800\pi\left(\frac{y}{2}\right)^2 (12 - y)\, dy = \pi(3.675 \times 10^6) \text{ J} \approx 1.155 \times 10^7 \text{ J}.$$

21. Horizontal cylinder in Figure 11; water exits from a small hole at the top. *Hint:* Evaluate the integral by interpreting part of it as the area of a circle.

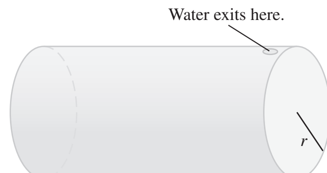


FIGURE 11

SOLUTION Place the origin along the axis of the cylinder. At location y, the layer of water is a rectangular slab of length ℓ, width $2\sqrt{r^2 - y^2}$ and thickness Δy. Thus, the volume of the layer is $2\ell\sqrt{r^2 - y^2}\Delta y$, and the force needed to lift the layer is $19{,}600\ell\sqrt{r^2 - y^2}\Delta y$. The layer must be lifted a distance $r - y$, so the total work needed to empty the tank is given by

$$\int_{-r}^{r} 19{,}600\ell\sqrt{r^2 - y^2}(r - y)\, dy = 19{,}600\ell r \int_{-r}^{r} \sqrt{r^2 - y^2}\, dy - 19{,}600\ell \int_{-r}^{r} y\sqrt{r^2 - y^2}\, dy.$$

Now,

$$\int_{-r}^{r} y\sqrt{r^2 - y^2}\, du = 0$$

because the integrand is an odd function and the integration interval is symmetric with respect to zero. Moreover, the other integral is one-half the area of a circle of radius r; thus,

$$\int_{-r}^{r} \sqrt{r^2 - y^2}\, dy = \frac{1}{2}\pi r^2.$$

Finally, the total work needed to empty the tank is

$$19{,}600\ell r\left(\frac{1}{2}\pi r^2\right) - 19{,}600\ell(0) = 9800\ell\pi r^3 \text{ J}.$$

22. Trough in Figure 12; water exits by pouring over the sides.

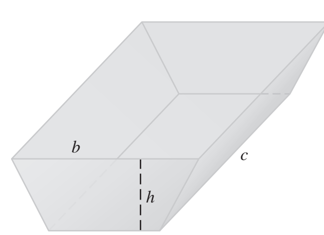


FIGURE 12

SOLUTION Place the origin along the bottom edge of the trough, and let the positive y-axis point upward. From similar triangles, the width of a layer of water at a height of y meters is

$$w = a + \frac{y\,(b-a)}{h}\ \mathrm{m}^2,$$

so the volume of each layer is

$$c\left(a + \frac{y\,(b-a)}{h}\right)\Delta y\ \mathrm{m}^3.$$

Thus, the force needed to lift the layer is

$$9800c\left(a + \frac{y\,(b-a)}{h}\right)\Delta y\ \mathrm{N}.$$

Each layer must be lifted $h - y$ meters, so the total work needed to empty the tank is

$$\int_0^h 9800(h-y)c\left(a + \frac{y\,(b-a)}{h}\right)dy = 9800\,c\left(\frac{ah^2}{3} + \frac{bh^2}{6}\right)\ \mathrm{J}.$$

23. Find the work W required to empty the tank in Figure 8 through the hole at the top if the tank is half full of water.

SOLUTION Place the origin on the top of the box, and let the positive y-axis point downward. Note that with this coordinate system, the bottom half of the box corresponds to y values from 2.5 to 5. The volume of one layer of water is $32\Delta y\ \mathrm{m}^3$, so the force needed to lift each layer is

$$(9.8)(1000)32\Delta y = 313{,}600\Delta y\ \mathrm{N}.$$

Each layer must be lifted y meters, so the total work needed to empty the tank is

$$\int_{2.5}^{5} 313{,}600y\,dy = 156{,}800y^2\Big|_{2.5}^{5} = 2.94\times 10^6\ \mathrm{J}.$$

24. Assume the tank in Figure 8 is full of water and let W be the work required to pump out half of the water through the hole at the top. Do you expect W to equal the work computed in Exercise 23? Explain and then compute W.

SOLUTION Recall that the origin was placed at the top of the box with the positive y-axis pointing downward. Pumping out half the water from a full tank would involve y values ranging from $y = 0$ to $y = 2.5$, whereas pumping out a half-full tank would involve y values ranging from $y = 2.5$ to $y = 5$. Because pumping out half the water from a full tank requires moving the layers of water a shorter distance than pumping out a half-full tank, we do not expect that W would be equal to the work computed in Exercise 23.

To compute W, we proceed as in Exercise 17 and Exercise 23, to find

$$W = \int_0^{2.5} 313{,}600y\,dy = 980{,}000\ \mathrm{J}.$$

It is reassuring to note that

$$\text{Work(Exercise 23)} + \text{Work(Exercise 24)} = \text{Work(Exercise 17)}.$$

25. Assume the tank in Figure 10 is full. Find the work required to pump out half of the water. *Hint:* First, determine the level H at which the water remaining in the tank is equal to one-half the total capacity of the tank.

SOLUTION Our first step is to determine the level H at which the water remaining in the tank is equal to one-half the total capacity of the tank. From Figure 10 and similar triangles, we see that the radius of the cone at level H is $H/2$ so the volume of water is

$$V = \frac{1}{3}\pi r^2 H = \frac{1}{3}\pi\left(\frac{H}{2}\right)^2 H = \frac{1}{12}\pi H^3.$$

The total capacity of the tank is $250\pi/3\ \mathrm{m}^3$, so the water level when the water remaining in the tank is equal to one-half the total capacity of the tank satisfies

$$\frac{1}{12}\pi H^3 = \frac{125}{3}\pi \quad \text{or} \quad H = \frac{10}{2^{1/3}}\ \mathrm{m}.$$

Place the origin at the vertex of the inverted cone, and let the positive y-axis point upward. Now, consider a layer of water at a height of y meters. From similar triangles, the area of the layer is

$$\pi\left(\frac{y}{2}\right)^2\ \mathrm{m}^2,$$

so the volume is

$$\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ m}^3.$$

Thus the weight of one layer is

$$9800\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ N}.$$

The layer must be lifted $12 - y$ meters, so the total work needed to empty the half-full tank is

$$\int_{10/2^{1/3}}^{10} 9800\pi\left(\frac{y}{2}\right)^2 (12-y)\,dy \approx 3.79 \times 10^6 \text{ J}.$$

26. Assume that the tank in Figure 10 is full.

(a) Calculate the work $F(y)$ required to pump out water until the water level has reached level y.

(b) CAS Plot $F(y)$.

(c) What is the significance of $F'(y)$ as a rate of change?

(d) CAS If your goal is to pump out all of the water, at which water level y_0 will half of the work be done?

SOLUTION

(a) Place the origin at the vertex of the inverted cone, and let the positive y-axis point upward. Consider a layer of water at a height of y meters. From similar triangles, the area of the layer is

$$\pi\left(\frac{y}{2}\right)^2 \text{ m}^2,$$

so the volume is

$$\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ m}^3.$$

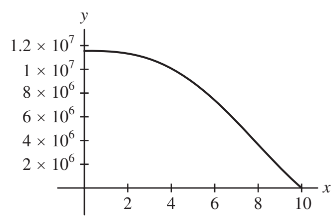
Thus the weight of one layer is

$$9800\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ N}.$$

The layer must be lifted $12 - y$ meters, so the total work needed to pump out water until the water level has reached level y is

$$\int_{y}^{10} 9800\pi\left(\frac{y}{2}\right)^2 (12-y)\,dy = 3{,}675{,}000\pi - 9800\pi y^3 + \frac{1225\pi}{2} y^4 \text{ J}.$$

(b) A plot of $F(y)$ is shown below.



(c) First, note that $F'(y) < 0$; as y increases, less water is being pumped from the tank, so $F(y)$ decreases. Therefore, when the water level in the tank has reached level y, we can interpret $-F'(y)$ as the amount of work per meter needed to remove the next layer of water from the tank. In other words, $-F'(y)$ is a "marginal work" function.

(d) The amount of work needed to empty the tank is $3{,}675{,}000\pi$ J. Half of this work will be done when the water level reaches height y_0 satisfying

$$3{,}675{,}000\pi - 9800\pi y_0^3 + \frac{1225\pi}{2} y_0^4 = 1{,}837{,}500\pi.$$

Using a computer algebra system, we find $y_0 = 6.91$ m.

27. Calculate the work required to lift a 10-m chain over the side of a building (Figure 13) Assume that the chain has a density of 8 kg/m. *Hint:* Break up the chain into N segments, estimate the work performed on a segment, and compute the limit as $N \to \infty$ as an integral.

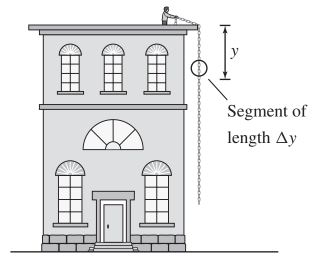


FIGURE 13 The small segment of the chain of length Δy located y meters from the top is lifted through a vertical distance y.

SOLUTION In this example, each part of the chain is lifted a different distance. Therefore, we divide the chain into N small segments of length $\Delta y = 10/N$. Suppose that the ith segment is located a distance y_i from the top of the building. This segment weighs $8(9.8)\Delta y$ kilograms and it must be lifted approximately y_i meters (not exactly y_i meters, because each point along the segment is a slightly different distance from the top). The work W_i done on this segment is approximately $W_i \approx 78.4 y_i \Delta y$ N. The total work W is the sum of the W_i and we have

$$W = \sum_{j=1}^{N} W_i \approx \sum_{j=1}^{N} 78.4 y_j \, \Delta y.$$

Passing to the limit as $N \to \infty$, we obtain

$$W = \int_0^{10} 78.4\, y\, dy = 39.2 y^2 \Big|_0^{10} = 3920 \text{ J}.$$

28. How much work is done lifting a 3-m chain over the side of a building if the chain has mass density 4 kg/m?

SOLUTION Consider a segment of the chain of length Δy located a distance y_j meters from the top of the building. The work needed to lift this segment of the chain to the top of the building is approximately

$$W_j \approx (4\Delta y)(9.8) y_j \text{ J}.$$

Summing over all segments of the chain and passing to the limit as $\Delta y \to 0$, it follows that the total work is

$$\int_0^3 4 \cdot 9.8 y\, dy = 19.6 y^2 \Big|_0^3 = 176.4 \text{ J}.$$

29. A 6-m chain has mass 18 kg. Find the work required to lift the chain over the side of a building.

SOLUTION First, note that the chain has a mass density of 3 kg/m. Now, consider a segment of the chain of length Δy located a distance y_j feet from the top of the building. The work needed to lift this segment of the chain to the top of the building is approximately

$$W_j \approx (3\Delta y) 9.8 y_j \text{ ft-lb}.$$

Summing over all segments of the chain and passing to the limit as $\Delta y \to 0$, it follows that the total work is

$$\int_0^6 29.4 y\, dy = 14.7 y^2 \Big|_0^6 = 529.2 \text{ J}.$$

30. A 10-m chain with mass density 4 kg/m is initially coiled on the ground. How much work is performed in lifting the chain so that it is fully extended (and one end touches the ground)?

SOLUTION Consider a segment of the chain of length Δy that must be lifted y_j feet off the ground. The work needed to lift this segment of the chain is approximately

$$W_j \approx (4\Delta y) 9.8 y_j \text{ J}.$$

Summing over all segments of the chain and passing to the limit as $\Delta y \to 0$, it follows that the total work is

$$\int_0^{10} 39.2 y\, dy = 19.6 y^2 \Big|_0^{10} = 1960 \text{ J}.$$

31. How much work is done lifting a 12-m chain that has mass density 3 kg/m (initially coiled on the ground) so that its top end is 10 m above the ground?

SOLUTION Consider a segment of the chain of length Δy that must be lifted y_j feet off the ground. The work needed to lift this segment of the chain is approximately

$$W_j \approx (3\Delta y)9.8y_j \text{ J}.$$

Summing over all segments of the chain and passing to the limit as $\Delta y \to 0$, it follows that the total work is

$$\int_0^{10} 29.4y\,dy = 14.7y^2\Big|_0^{10} = 1470 \text{ J}.$$

32. A 500-kg wrecking ball hangs from a 12-m cable of density 15 kg/m attached to a crane. Calculate the work done if the crane lifts the ball from ground level to 12 m in the air by drawing in the cable.

SOLUTION We will treat the cable and the wrecking ball separately. Consider a segment of the cable of length Δy that must be lifted y_j feet. The work needed to lift the cable segment is approximately

$$W_j \approx (15\Delta y)9.8y_j \text{ J}.$$

Summing over all of the segments of the cable and passing to the limit as $\Delta y \to 0$, it follows that lifting the cable requires

$$\int_0^{12} 147y\,dy = 73.5y^2\Big|_0^{12} = 10{,}584 \text{ J}.$$

Lifting the 500 kg wrecking ball 12 meters requires an additional 58,800 J. Thus, the total work is 69,384 J.

33. Calculate the work required to lift a 3-m chain over the side of a building if the chain has variable density of $\rho(x) = x^2 - 3x + 10$ kg/m for $0 \le x \le 3$.

SOLUTION Consider a segment of the chain of length Δx that must be lifted x_j feet. The work needed to lift this segment is approximately

$$W_j \approx \left(\rho(x_j)\Delta x\right)9.8x_j \text{ J}.$$

Summing over all segments of the chain and passing to the limit as $\Delta x \to 0$, it follows that the total work is

$$\begin{aligned}\int_0^3 9.8\rho(x)x\,dx &= 9.8\int_0^3 \left(x^3 - 3x^2 + 10x\right)dx \\ &= 9.8\left(\frac{1}{4}x^4 - x^3 + 5x^2\right)\Big|_0^3 = 374.85 \text{ J}.\end{aligned}$$

34. A 3-m chain with linear mass density $\rho(x) = 2x(4-x)$ kg/m lies on the ground. Calculate the work required to lift the chain so that its bottom is 2 m above ground.

SOLUTION Consider a segment of the chain of length Δx that must be lifted x_j feet. The work needed to lift this segment is approximately

$$W_j \approx \left(\rho(x_j)\Delta x\right)9.8x_j \text{ J}.$$

Summing over all segments of the chain and passing to the limit as $\Delta x \to 0$, it follows that the total work needed to fully extend the chain is

$$\begin{aligned}\int_0^3 9.8\rho(x)x\,dx &= 9.8\int_0^3 \left(8x^2 - 2x^3\right)dx \\ &= 9.8\left(\frac{8}{3}x^3 - \frac{1}{2}x^4\right)\Big|_0^3 = 308.7 \text{ J}.\end{aligned}$$

Lifting the entire chain, which weighs

$$\int_0^3 9.8\rho(x)\,dx = 9.8\int_0^3 \left(8x - 2x^2\right)dx = 9.8\left(4x^2 - \frac{2}{3}x^3\right)\Big|_0^3 = 176.4 \text{ N}$$

another two meters requires an additional 352.8 J of work. The total work is therefore 661.5 J.

Exercises 35–37: The gravitational force between two objects of mass m and M, separated by a distance r, has magnitude GMm/r^2, where $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-1}.

35. Show that if two objects of mass M and m are separated by a distance r_1, then the work required to increase the separation to a distance r_2 is equal to $W = GMm(r_1^{-1} - r_2^{-1})$.

SOLUTION The work required to increase the separation from a distance r_1 to a distance r_2 is

$$\int_{r_1}^{r_2} \frac{GMm}{r^2}\,dr = -\frac{GMm}{r}\bigg|_{r_1}^{r_2} = GMm(r_1^{-1} - r_2^{-1}).$$

36. Use the result of Exercise 35 to calculate the work required to place a 2000-kg satellite in an orbit 1200 km above the surface of the earth. Assume that the earth is a sphere of radius $R_e = 6.37 \times 10^6$ m and mass $M_e = 5.98 \times 10^{24}$ kg. Treat the satellite as a point mass.

SOLUTION The satellite will move from a distance $r_1 = R_e$ to a distance $r_2 = R_e + 1{,}200{,}000$. Thus, from Exercise 35,

$$W = (6.67 \times 10^{-11})(5.98 \times 10^{24})(2000)\left(\frac{1}{6.37 \times 10^6} - \frac{1}{6.37 \times 10^6 + 1{,}200{,}000}\right) \approx 1.99 \times 10^{10} \text{ J}.$$

37. Use the result of Exercise 35 to compute the work required to move a 1500-kg satellite from an orbit 1000 to an orbit 1500 km above the surface of the earth.

SOLUTION The satellite will move from a distance $r_1 = R_e + 1{,}000{,}000$ to a distance $r_2 = R_e + 1{,}500{,}000$. Thus, from Exercise 35,

$$W = (6.67 \times 10^{-11})(5.98 \times 10^{24})(1500) \times \left(\frac{1}{6.37 \times 10^6 + 1{,}000{,}000} - \frac{1}{6.37 \times 10^6 + 1{,}500{,}000}\right)$$
$$\approx 5.16 \times 10^9 \text{ J}.$$

38. The pressure P and volume V of the gas in a cylinder of length 0.8 meters and radius 0.2 meters, with a movable piston, are related by $PV^{1.4} = k$, where k is a constant (Figure 14). When the piston is fully extended, the gas pressure is 2000 kilopascals (one kilopascal is 10^3 newtons per square meter).

(a) Calculate k.

(b) The force on the piston is PA, where A is the piston's area. Calculate the force as a function of the length x of the column of gas.

(c) Calculate the work required to compress the gas column from 0.8 m to 0.5 m.

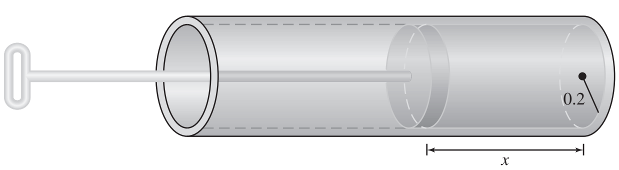


FIGURE 14 Gas in a cylinder with a piston.

SOLUTION

(a) We have $P = 2 \times 10^6$ and $V = 0.032\pi$. Thus

$$k = 2 \times 10^6 (0.032\pi)^{1.4} = 80{,}213.9.$$

(b) The area of the piston is $A = 0.04\pi$ and the volume of the cylinder as a function of x is $V = 0.04\pi x$, which gives $P = k/V^{1.4} = k/(0.04\pi x)^{1.4}$. Thus

$$F = PA = \frac{k}{(0.04\pi x)^{1.4}} 0.04\pi = k(0.04\pi)^{-0.4} x^{-1.4}.$$

(c) Since the force is pushing against the piston, in order to calculate work, we must calculate the integral of the opposite force, i.e., we have

$$W = -k(0.04\pi)^{-0.4} \int_{0.8}^{0.5} x^{-1.4}\,dx = -k(0.04\pi)^{-0.4} \frac{1}{-0.4} x^{-0.4}\bigg|_{0.8}^{0.5} = 103{,}966.7 \text{ J}.$$

Further Insights and Challenges

39. Work-Energy Theorem An object of mass m moves from x_1 to x_2 during the time interval $[t_1, t_2]$ due to a force $F(x)$ acting in the direction of motion. Let $x(t)$, $v(t)$, and $a(t)$ be the position, velocity, and acceleration at time t. The object's kinetic energy is $\text{KE} = \frac{1}{2}mv^2$.

(a) Use the change-of-variables formula to show that the work performed is equal to

$$W = \int_{x_1}^{x_2} F(x)\,dx = \int_{t_1}^{t_2} F(x(t))v(t)\,dt$$

(b) Use Newton's Second Law, $F(x(t)) = ma(t)$, to show that

$$\frac{d}{dt}\left(\frac{1}{2}mv(t)^2\right) = F(x(t))v(t)$$

(c) Use the FTC to prove the Work-Energy Theorem: The change in kinetic energy during the time interval $[t_1, t_2]$ is equal to the work performed.

SOLUTION

(a) Let $x_1 = x(t_1)$ and $x_2 = x(t_2)$, then $x = x(t)$ gives $dx = v(t)\,dt$. By substitution we have

$$W = \int_{x_1}^{x_2} F(x)\,dx = \int_{t_1}^{t_2} F(x(t))v(t)\,dt.$$

(b) Knowing $F(x(t)) = m \cdot a(t)$, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}m \cdot v(t)^2\right) &= m \cdot v(t)\,v'(t) && \text{(Chain Rule)}\\ &= m \cdot v(t)\,a(t)\\ &= v(t) \cdot F(x(t)) && \text{(Newton's 2nd law)}\end{aligned}$$

(c) From the FTC,

$$\frac{1}{2}m \cdot v(t)^2 = \int F(x(t))\,v(t)\,dt.$$

Since $KE = \frac{1}{2}\,m\,v^2$,

$$\Delta KE = KE(t_2) - KE(t_1) = \frac{1}{2}m\,v(t_2)^2 - \frac{1}{2}m\,v(t_1)^2 = \int_{t_1}^{t_2} F(x(t))\,v(t)\,dt.$$

(d)

$$\begin{aligned}W &= \int_{x_1}^{x_2} F(x)\,dx = \int_{t_1}^{t_2} F(x(t))\,v(t)\,dt && \text{(Part (a))}\\ &= KE(t_2) - KE(t_1)\\ &= \Delta KE && \text{(as required)}\end{aligned}$$

40. A model train of mass 0.5 kg is placed at one end of a straight 3-m electric track. Assume that a force $F(x) = (3x - x^2)$ N acts on the train at distance x along the track. Use the Work-Energy Theorem (Exercise 39) to determine the velocity of the train when it reaches the end of the track.

SOLUTION We have

$$W = \int_0^3 F(x)\,dx = \int_0^3 (3x - x^2)\,dx = \left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^3 = 4.5\text{ J}.$$

Then the change in KE must be equal to W, which gives

$$4.5 = \frac{1}{2}m(v(t_2)^2 - v(t_1)^2)$$

Note that $v(t_1) = 0$ as the train was placed on the track with no initial velocity and $m = 0.5$. Thus

$$v(t_2) = \sqrt{18} = 4.242641\text{ m/sec}.$$

41. With what initial velocity v_0 must we fire a rocket so it attains a maximum height r above the earth? *Hint:* Use the results of Exercises 35 and 39. As the rocket reaches its maximum height, its KE decreases from $\frac{1}{2}mv_0^2$ to zero.

SOLUTION The work required to move the rocket a distance r from the surface of the earth is

$$W(r) = GM_e m\left(\frac{1}{R_e} - \frac{1}{r + R_e}\right).$$

As the rocket climbs to a height r, its kinetic energy is reduced by the amount $W(r)$. The rocket reaches its maximum height when its kinetic energy is reduced to zero, that is, when

$$\frac{1}{2}mv_0^2 = GM_e m\left(\frac{1}{R_e} - \frac{1}{r + R_e}\right).$$

Therefore, its initial velocity must be

$$v_0 = \sqrt{2GM_e\left(\frac{1}{R_e} - \frac{1}{r + R_e}\right)}.$$

42. With what initial velocity must we fire a rocket so it attains a maximum height of $r = 20$ km above the surface of the earth?

SOLUTION Using the result of the previous exercise with $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}, $M_e = 5.98 \times 10^{24}$ kg, $R_e = 6.37 \times 10^6$ m and $r = 20{,}000$ m,

$$v_0 = \sqrt{2GM_e\left(\frac{1}{R_e} - \frac{1}{r + R_e}\right)} = 626 \text{ m/sec.}$$

43. Calculate **escape velocity,** the minimum initial velocity of an object to ensure that it will continue traveling into space and never fall back to earth (assuming that no force is applied after takeoff). *Hint:* Take the limit as $r \to \infty$ in Exercise 41.

SOLUTION The result of Exercise 41 leads to an interesting conclusion. The initial velocity v_0 required to reach a height r does not increase beyond all bounds as r tends to infinity; rather, it approaches a finite limit, called the escape velocity:

$$v_{\text{esc}} = \lim_{r \to \infty}\sqrt{2GM_e\left(\frac{1}{R_e} - \frac{1}{r + R_e}\right)} = \sqrt{\frac{2GM_e}{R_e}}$$

In other words, v_{esc} is large enough to insure that the rocket reaches a height r for every value of r! Therefore, a rocket fired with initial velocity v_{esc} never returns to earth. It continues traveling indefinitely into outer space.

Now, let's see how large escape velocity actually is:

$$v_{\text{esc}} = \left(\frac{2 \cdot 6.67 \times 10^{-11} \cdot 5.989 \times 10^{24}}{6.37 \times 10^6}\right)^{1/2} \approx 11{,}190 \text{ m/sec.}$$

Since one meter per second is equal to 2.236 miles per hour, escape velocity is approximately $11{,}190(2.236) = 25{,}020$ miles per hour.

CHAPTER REVIEW EXERCISES

1. Compute the area of the region in Figure 1(A) enclosed by $y = 2 - x^2$ and $y = -2$.

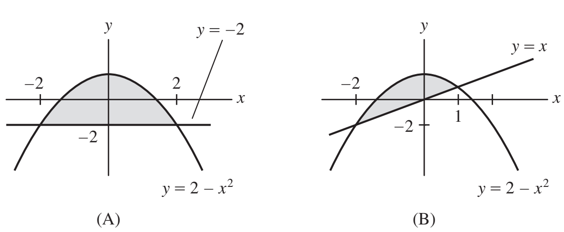


FIGURE 1

SOLUTION The graphs of $y = 2 - x^2$ and $y = -2$ intersect where $2 - x^2 = -2$, or $x = \pm 2$. Therefore, the enclosed area lies over the interval $[-2, 2]$. The region enclosed by the graphs lies below $y = 2 - x^2$ and above $y = -2$, so the area is

$$\int_{-2}^{2}\left((2 - x^2) - (-2)\right)dx = \int_{-2}^{2}(4 - x^2)\,dx = \left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2} = \frac{32}{3}.$$

2. Compute the area of the region in Figure 1(B) enclosed by $y = 2 - x^2$ and $y = x$.

SOLUTION The graphs of $y = 2 - x^2$ and $y = x$ intersect where $2 - x^2 = x$, which simplifies to

$$0 = x^2 + x - 2 = (x + 2)(x - 1).$$

Thus, the graphs intersect at $x = -2$ and $x = 1$. As the graph of $y = x$ lies below the graph of $y = 2 - x^2$ over the interval $[-2, 1]$, the area between the graphs is

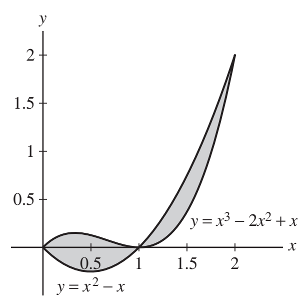
$$\int_{-2}^{1} \left((2 - x^2) - x\right) dx = \left(2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right)\bigg|_{-2}^{1} = \frac{9}{2}.$$

In Exercises 3–12, find the area of the region enclosed by the graphs of the functions.

3. $y = x^3 - 2x^2 + x, \quad y = x^2 - x$

SOLUTION The region bounded by the graphs of $y = x^3 - 2x^2 + x$ and $y = x^2 - x$ over the interval $[0, 2]$ is shown below. For $x \in [0, 1]$, the graph of $y = x^3 - 2x^2 + x$ lies above the graph of $y = x^2 - x$, whereas, for $x \in [1, 2]$, the graph of $y = x^2 - x$ lies above the graph of $y = x^3 - 2x^2 + x$. The area of the region is therefore given by

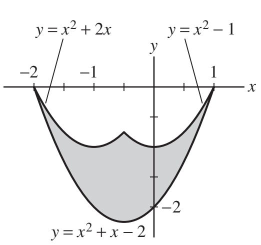
$$\begin{aligned}\int_0^1 \left((x^3 - 2x^2 + x) - (x^2 - x)\right) dx &+ \int_1^2 \left((x^2 - x) - (x^3 - 2x^2 + x)\right) dx \\ &= \left(\frac{1}{4}x^4 - x^3 + x^2\right)\bigg|_0^1 + \left(x^3 - x^2 - \frac{1}{4}x^4\right)\bigg|_1^2 \\ &= \frac{1}{4} - 1 + 1 + (8 - 4 - 4) - \left(1 - 1 - \frac{1}{4}\right) = \frac{1}{2}.\end{aligned}$$



4. $y = x^2 + 2x, \quad y = x^2 - 1, \quad h(x) = x^2 + x - 2$

SOLUTION The region bounded by the graphs of $y = x^2 + 2x$, $y = x^2 - 1$ and $y = x^2 + x - 2$ is shown below. For each $x \in [-2, -\frac{1}{2}]$, the graph of $y = x^2 + 2x$ lies above the graph of $y = x^2 + x - 2$, whereas, for each $x \in [-\frac{1}{2}, 1]$, the graph of $y = x^2 - 1$ lies above the graph of $y = x^2 + x - 2$. The area of the region is therefore given by

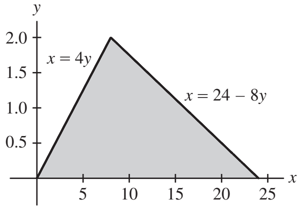
$$\begin{aligned}\int_{-2}^{-1/2} \left((x^2 + 2x) - (x^2 + x - 2)\right) dx &+ \int_{-1/2}^{1} \left((x^2 - 1) - (x^2 + x - 2)\right) dx \\ &= \left(\frac{1}{2}x^2 + 2x\right)\bigg|_{-2}^{-1/2} + \left(-\frac{1}{2}x^2 + x\right)\bigg|_{-1/2}^{1} \\ &= \left(\frac{1}{8} - 1\right) - (2 - 4) + \left(-\frac{1}{2} + 1\right) - \left(-\frac{1}{8} - \frac{1}{2}\right) = \frac{9}{4}.\end{aligned}$$



5. $x = 4y, \quad x = 24 - 8y, \quad y = 0$

SOLUTION The region bounded by the graphs $x = 4y$, $x = 24 - 8y$ and $y = 0$ is shown below. For each $0 \le y \le 2$, the graph of $x = 24 - 8y$ lies to the right of $x = 4y$. The area of the region is therefore

$$A = \int_0^2 (24 - 8y - 4y)\,dy = \int_0^2 (24 - 12y)\,dy$$
$$= (24y - 6y^2)\Big|_0^2 = 24.$$



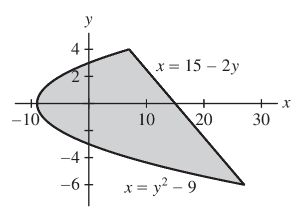
6. $x = y^2 - 9, \quad x = 15 - 2y$

SOLUTION Setting $y^2 - 9 = 15 - 2y$ yields

$$y^2 + 2y - 24 = (y + 6)(y - 4) = 0,$$

so the two curves intersect at $y = -6$ and $y = 4$. The region bounded by the graphs $x = y^2 - 9$ and $x = 15 - 2y$ is shown below. For each $-6 \le y \le 4$, the graph of $x = 15 - 2y$ lies to the right of $x = y^2 - 9$. The area of the region is therefore

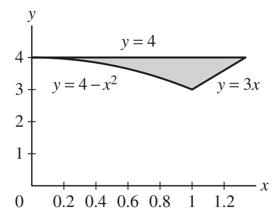
$$A = \int_{-6}^4 \left(15 - 2y - (y^2 - 9)\right)\,dy = \int_{-6}^4 (24 - 2y - y^2)\,dy$$
$$= \left(24y - y^2 - \frac{1}{3}y^3\right)\Bigg|_{-6}^4$$
$$= \left(\frac{176}{3} - (-108)\right) = \frac{500}{3}.$$



7. $y = 4 - x^2, \quad y = 3x, \quad y = 4$

SOLUTION The region bounded by the graphs of $y = 4 - x^2$, $y = 3x$ and $y = 4$ is shown below. For $x \in [0, 1]$, the graph of $y = 4$ lies above the graph of $y = 4 - x^2$, whereas, for $x \in [1, \frac{4}{3}]$, the graph of $y = 4$ lies above the graph of $y = 3x$. The area of the region is therefore given by

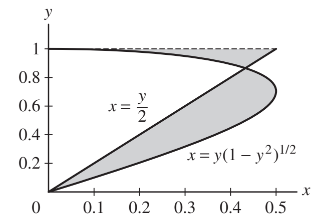
$$\int_0^1 (4 - (4 - x^2))\,dx + \int_1^{4/3} (4 - 3x)\,dx = \frac{1}{3}x^3\Big|_0^1 + \left(4x - \frac{3}{2}x^2\right)\Bigg|_1^{4/3} = \frac{1}{3} + \left(\frac{16}{3} - \frac{8}{3}\right) - \left(4 - \frac{3}{2}\right) = \frac{1}{2}.$$



8. GU $x = \frac{1}{2}y, \quad x = y\sqrt{1-y^2}, \quad 0 \le y \le 1$

SOLUTION The region bounded by the graphs of $x = y/2$ and $x = y\sqrt{1-y^2}$ over the interval [0, 1] is shown below. For $y \in [0, \frac{\sqrt{3}}{2}]$, the graph of $x = y\sqrt{1-y^2}$ lies to the right of the graph of $x = y/2$, whereas, for $y \in [\frac{\sqrt{3}}{2}, 1]$, the graph of $x = y/2$ lies to the right of the graph of $x = y\sqrt{1-y^2}$. The area of the region is therefore given by

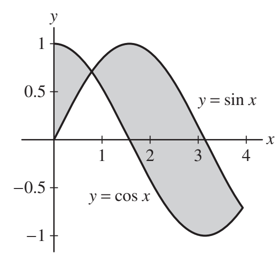
$$\int_0^{\sqrt{3}/2} \left(y\sqrt{1-y^2} - \frac{y}{2} \right) dy + \int_{\sqrt{3}/2}^1 \left(\frac{y}{2} - y\sqrt{1-y^2} \right) dy$$
$$= \left(-\frac{1}{3}(1-y^2)^{3/2} - \frac{y^2}{4} \right)\Big|_0^{\sqrt{3}/2} + \left(\frac{y^2}{4} + \frac{1}{3}(1-y^2)^{3/2} \right)\Big|_{\sqrt{3}/2}^1$$
$$= -\frac{1}{24} - \frac{3}{16} + \frac{1}{3} + \frac{1}{4} - \frac{3}{16} - \frac{1}{24} = \frac{1}{8}.$$



9. $y = \sin x, \quad y = \cos x, \quad 0 \le x \le \frac{5\pi}{4}$

SOLUTION The region bounded by the graphs of $y = \sin x$ and $y = \cos x$ over the interval $[0, \frac{5\pi}{4}]$ is shown below. For $x \in [0, \frac{\pi}{4}]$, the graph of $y = \cos x$ lies above the graph of $y = \sin x$, whereas, for $x \in [\frac{\pi}{4}, \frac{5\pi}{4}]$, the graph of $y = \sin x$ lies above the graph of $y = \cos x$. The area of the region is therefore given by

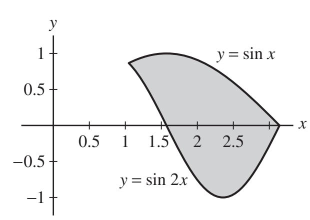
$$\int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$$
$$= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4}$$
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0+1) + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) = 3\sqrt{2} - 1.$$



10. $f(x) = \sin x, \quad g(x) = \sin 2x, \quad \frac{\pi}{3} \le x \le \pi$

SOLUTION The region bounded by the graphs of $y = \sin x$ and $y = \sin 2x$ over the interval $[\frac{\pi}{3}, \pi]$ is shown below. As the graph of $y = \sin x$ lies above the graph of $y = \sin 2x$, the area of the region is given by

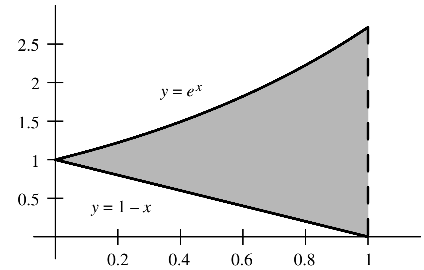
$$\int_{\pi/3}^{\pi} (\sin x - \sin 2x)\, dx = \left(-\cos x + \frac{1}{2}\cos 2x \right)\Big|_{\pi/3}^{\pi} = \left(1 + \frac{1}{2} \right) - \left(-\frac{1}{2} - \frac{1}{4} \right) = \frac{9}{4}.$$



11. $y = e^x, \quad y = 1 - x, \quad x = 1$

SOLUTION The region bounded by the graphs of $y = e^x$, $y = 1 - x$ and $x = 1$ is shown below. As the graph of $y = e^x$ lies above the graph of $y = 1 - x$, the area of the region is given by

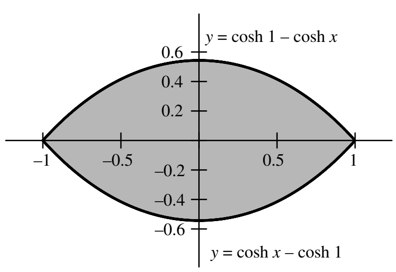
$$\int_0^1 \left(e^x - (1 - x)\right) dx = \left(e^x - x + \frac{1}{2}x^2\right)\Big|_0^1 = \left(e - 1 + \frac{1}{2}\right) - 1 = e - \frac{3}{2}.$$



12. $y = \cosh 1 - \cosh x, \quad y = \cosh x - \cosh 1$

SOLUTION The region bounded by the graphs of $y = \cosh 1 - \cosh x$, $y = \cosh x - \cosh 1$ is shown below. As the graph of $y = \cosh 1 - \cosh x$ lies above the graph of $y = \cosh x - \cosh 1$, the area of the region is given by

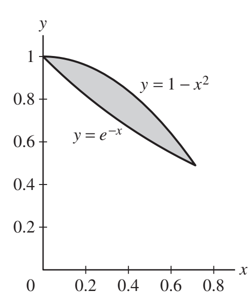
$$\begin{aligned}\int_{-1}^1 ((\cosh 1 - \cosh x) - (\cosh x - \cosh 1))\, dx &= (2x \cosh 1 - 2 \sinh x)\Big|_{-1}^1 \\ &= (2\cosh 1 - 2\sinh 1) - (-2\cosh 1 + 2\sinh 1) \\ &= 4\cosh 1 - 4\sinh 1 = 4e^{-1}.\end{aligned}$$



13. GU Use a graphing utility to locate the points of intersection of $y = e^{-x}$ and $y = 1 - x^2$ and find the area between the two curves (approximately).

SOLUTION The region bounded by the graphs of $y = e^{-x}$ and $y = 1 - x^2$ is shown below. One point of intersection clearly occurs at $x = 0$. Using a computer algebra system, we find that the other point of intersection occurs at $x = 0.7145563847$. As the graph of $y = 1 - x^2$ lies above the graph of $y = e^{-x}$, the area of the region is given by

$$\int_0^{0.7145563847} \left(1 - x^2 - e^{-x}\right) dx = 0.08235024596$$



14. Figure 2 shows a solid whose horizontal cross section at height y is a circle of radius $(1+y)^{-2}$ for $0 \le y \le H$. Find the volume of the solid.

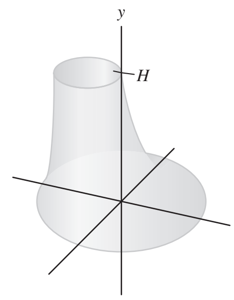


FIGURE 2

SOLUTION The area of each horizontal cross section is $A(y) = \pi(1+y)^{-4}$. Therefore, the volume of the solid is

$$\int_0^H \pi(1+y)^{-4}\,dy = \pi \left.\frac{(1+y)^{-3}}{-3}\right|_0^H = \pi\left(\frac{(1+H)^{-3}}{-3} + \frac{1}{3}\right) = \frac{\pi}{3}\left(1 - \frac{1}{(1+H)^3}\right).$$

15. The base of a solid is the unit circle $x^2 + y^2 = 1$, and its cross sections perpendicular to the x-axis are rectangles of height 4. Find its volume.

SOLUTION Because the cross sections are rectangles of constant height 4, the figure is a cylinder of radius 1 and height 4. The volume is therefore $\pi r^2 h = 4\pi$.

16. The base of a solid is the triangle bounded by the axes and the line $2x + 3y = 12$, and its cross sections perpendicular to the y-axis have area $A(y) = (y+2)$. Find its volume.

SOLUTION The volume of this solid is

$$V = \int_0^4 A(y)\,dy = \int_0^4 (y+2)\,dy = \left.\left(\frac{1}{2}y^2 + 2y\right)\right|_0^4 = 16.$$

17. Find the total mass of a rod of length 1.2 m with linear density $\rho(x) = (1 + 2x + \frac{2}{9}x^3)$ kg/m.

SOLUTION The total weight of the rod is

$$\int_0^{1.2} \rho(x)\,dx = \left.\left(x + x^2 + \frac{1}{18}x^4\right)\right|_0^{1.2} = 2.7552 \text{ kg}.$$

18. Find the flow rate (in the correct units) through a pipe of diameter 6 cm if the velocity of fluid particles at a distance r from the center of the pipe is $v(r) = (3 - r)$ cm/s.

SOLUTION The flow rate through the pipe is

$$2\pi \int_0^3 r v(r)\,dr = 2\pi \int_0^3 (3r - r^2)\,dr = 2\pi \left.\left(\frac{3}{2}r^2 - \frac{1}{3}r^3\right)\right|_0^3 = 2\pi\left(\frac{27}{2} - 9\right) = 9\pi\,\frac{\text{cm}^3}{\text{s}}.$$

In Exercises 19–24, find the average value of the function over the interval.

19. $f(x) = x^3 - 2x + 2, \quad [-1, 2]$

SOLUTION The average value is

$$\frac{1}{2-(-1)} \int_{-1}^2 \left(x^3 - 2x + 2\right) dx = \frac{1}{3}\left.\left(\frac{1}{4}x^4 - x^2 + 2x\right)\right|_{-1}^2 = \frac{1}{3}\left[(4 - 4 + 4) - \left(\frac{1}{4} - 1 - 2\right)\right] = \frac{9}{4}.$$

20. $f(x) = |x|, \quad [-4, 4]$

SOLUTION The average value is

$$\frac{1}{4-(-4)} \int_{-4}^4 |x|\,dx = \frac{1}{8}\left(\int_{-4}^0 (-x)\,dx + \int_0^4 x\,dx\right) = \frac{1}{8}\left(-\frac{1}{2}x^2\Big|_{-4}^0 + \frac{1}{2}x^2\Big|_0^4\right) = \frac{1}{8}\left[(0+8) + (8-0)\right] = 2.$$

21. $f(x) = x\cosh(x^2)$, $[0, 1]$

SOLUTION The average value is

$$\frac{1}{1-0}\int_0^1 x\cosh(x^2)\,dx.$$

To evaluate the integral, let $u = x^2$. Then $du = 2x\,dx$ and

$$\frac{1}{1-0}\int_0^1 x\cosh(x^2)\,dx = \frac{1}{2}\int_0^1 \cosh u\,du = \frac{1}{2}\sinh u\Big|_0^1 = \frac{1}{2}\sinh 1.$$

22. $f(x) = \dfrac{e^x}{1+e^{2x}}$, $\left[0, \frac{1}{2}\right]$

SOLUTION The average value is

$$\frac{1}{\frac{1}{2}-0}\int_0^{1/2} \frac{e^x}{1+e^{2x}}\,dx.$$

To evaluate the integral, let $u = e^x$. Then $du = e^x\,dx$ and

$$\frac{1}{\frac{1}{2}-0}\int_0^{1/2} \frac{e^x}{1+e^{2x}}\,dx = 2\int_1^{\sqrt{e}} \frac{du}{1+u^2} = 2\tan^{-1} u\Big|_1^{\sqrt{e}} = 2\left(\tan^{-1}\sqrt{e} - \frac{\pi}{4}\right).$$

23. $f(x) = \sqrt{9-x^2}$, $[0, 3]$ *Hint:* Use geometry to evaluate the integral.

SOLUTION The region below the graph of $y = \sqrt{9-x^2}$ but above the x-axis over the interval $[0, 3]$ is one-quarter of a circle of radius 3; consequently,

$$\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4}.$$

The average value is then

$$\frac{1}{3-0}\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{3}\left(\frac{9\pi}{4}\right) = \frac{3\pi}{4}.$$

24. $f(x) = x[x]$, $[0, 3]$, where $[x]$ is the greatest integer function.

SOLUTION The average value is

$$\frac{1}{3-0}\int_0^3 x[x]\,dx = \frac{1}{3}\left(\int_0^1 x\cdot 0\,dx + \int_1^2 x\cdot 1\,dx + \int_2^3 x\cdot 2\,dx\right)$$
$$= \frac{1}{3}\left(\frac{1}{2}x^2\Big|_1^2 + x^2\Big|_2^3\right) = \frac{1}{3}\left(2 - \frac{1}{2} + 9 - 4\right) = \frac{13}{6}.$$

25. Find $\int_2^5 g(t)\,dt$ if the average value of $g(t)$ on $[2, 5]$ is 9.

SOLUTION The average value of the function $g(t)$ on $[2, 5]$ is given by

$$\frac{1}{5-2}\int_2^5 g(t)\,dt = \frac{1}{3}\int_2^5 g(t)\,dt.$$

Therefore,

$$\int_2^5 g(t)\,dt = 3(\text{average value}) = 3(9) = 27.$$

26. The average value of $R(x)$ over $[0, x]$ is equal to x for all x. Use the FTC to determine $R(x)$.

SOLUTION The average value of the function $R(x)$ over $[0, x]$ is

$$\frac{1}{x-0}\int_0^x R(t)\,dt = \frac{1}{x}\int_0^x R(t)\,dt.$$

Given that the average value is equal to x, it follows that

$$\int_0^x R(t)\,dt = x^2.$$

Differentiating both sides of this equation and using the Fundamental Theorem of Calculus on the left-hand side yields

$$R(x) = 2x.$$

27. Use the Washer Method to find the volume obtained by rotating the region in Figure 3 about the x-axis.

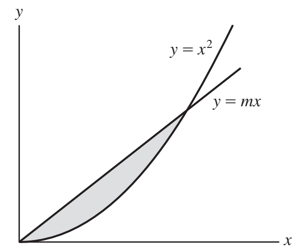


FIGURE 3

SOLUTION Setting $x^2 = mx$ yields $x(x-m) = 0$, so the two curves intersect at $(0, 0)$ and (m, m^2). To use the washer method, we must slice the solid perpendicular to the axis of rotation; as we are revolving about the y-axis, this implies a horizontal slice and integration in y. For each $y \in [0, m^2]$, the cross section is a washer with outer radius $R = \sqrt{y}$ and inner radius $r = \frac{y}{m}$. The volume of the solid is therefore given by

$$\pi\int_0^{m^2}\left((\sqrt{y})^2 - \left(\frac{y}{m}\right)^2\right)dy = \pi\left(\frac{1}{2}y^2 - \frac{y^3}{3m^2}\right)\Bigg|_0^{m^2} = \pi\left(\frac{m^4}{2} - \frac{m^4}{3}\right) = \frac{\pi}{6}m^4.$$

28. Use the Shell Method to find the volume obtained by rotating the region in Figure 3 about the x-axis.

SOLUTION Setting $x^2 = mx$ yields $x(x-m) = 0$, so the two curves intersect at $(0, 0)$ and (m, m^2). To use the shell method, we must slice the solid parallel to the axis of rotation; as we are revolving about the x-axis, this implies a horizontal slice and integration in y. For each $y \in [0, m^2]$, the shell has radius y and height $\sqrt{y} - \frac{y}{m}$. The volume of the solid is therefore given by

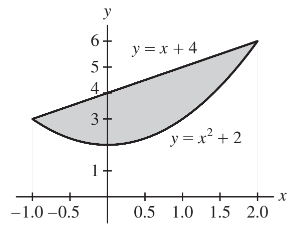
$$2\pi\int_0^{m^2} y\left(\sqrt{y} - \frac{y}{m}\right)dy = 2\pi\left(\frac{2}{5}y^{5/2} - \frac{y^3}{3m}\right)\Bigg|_0^{m^2} = 2\pi\left(\frac{2m^5}{5} - \frac{m^5}{3}\right) = \frac{2\pi}{15}m^5.$$

In Exercises 29–40, use any method to find the volume of the solid obtained by rotating the region enclosed by the curves about the given axis.

29. $y = x^2 + 2, \quad y = x + 4, \quad x\text{-axis}$

SOLUTION Let's choose to slice the region bounded by the graphs of $y = x^2 + 2$ and $y = x + 4$ (see the figure below) vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [-1, 2]$, the washer has outer radius $x + 4$ and inner radius $x^2 + 2$. The volume of the solid is therefore given by

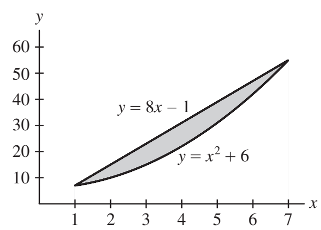
$$\begin{aligned}\pi\int_{-1}^{2}((x+4)^2 - (x^2+2)^2)\,dx &= \pi\int_{-1}^{2}(-x^4 - 3x^2 + 8x + 12)\,dx\\ &= \pi\left(-\frac{1}{5}x^5 - x^3 + 4x^2 + 12x\right)\Bigg|_{-1}^{2}\\ &= \pi\left(\frac{128}{5} + \frac{34}{5}\right) = \frac{162\pi}{5}.\end{aligned}$$



30. $y = x^2 + 6, \quad y = 8x - 1, \quad y\text{-axis}$

SOLUTION Let's choose to slice the region bounded by the graphs of $y = x^2 + 6$ and $y = 8x - 1$ (see the figure below) vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [1, 7]$, the shell has radius x and height $8x - 1 - (x^2 + 6) = -x^2 + 8x - 7$. The volume of the solid is therefore given by

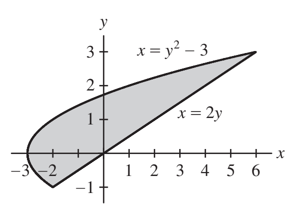
$$\begin{aligned} 2\pi \int_1^7 x(-x^2 + 8x - 7)\,dx &= 2\pi \int_1^7 (-x^3 + 8x^2 - 7x)\,dx \\ &= 2\pi \left(-\frac{1}{4}x^4 + \frac{8}{3}x^3 - \frac{7}{2}x^2 \right)\Bigg|_1^7 \\ &= 2\pi \left(\frac{1715}{12} + \frac{13}{12} \right) = 288\pi. \end{aligned}$$



31. $x = y^2 - 3, \quad x = 2y, \quad \text{axis } y = 4$

SOLUTION Let's choose to slice the region bounded by the graphs of $x = y^2 - 3$ and $x = 2y$ (see the figure below) horizontally. Because a horizontal slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $y \in [-1, 3]$, the shell has radius $4 - y$ and height $2y - (y^2 - 3) = 3 + 2y - y^2$. The volume of the solid is therefore given by

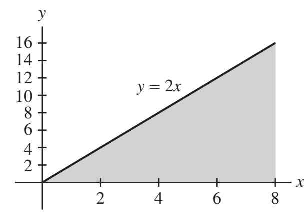
$$\begin{aligned} 2\pi \int_{-1}^3 (4 - y)(3 + 2y - y^2)\,dy &= 2\pi \int_{-1}^3 (12 + 5y - 6y^2 + y^3)\,dy \\ &= 2\pi \left(12y + \frac{5}{2}y^2 - 2y^3 + \frac{1}{4}y^4 \right)\Bigg|_{-1}^3 \\ &= 2\pi \left(\frac{99}{4} + \frac{29}{4} \right) = 64\pi. \end{aligned}$$



32. $y = 2x, \quad y = 0, \quad x = 8, \quad \text{axis } x = -3$

SOLUTION Let's choose to slice the region bounded by the graphs of $y = 2x$, $y = 0$ and $x = 8$ (see the figure below) vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [0, 8]$, the shell has radius $x - (-3) = x + 3$ and height $2x$. The volume of the solid is therefore given by

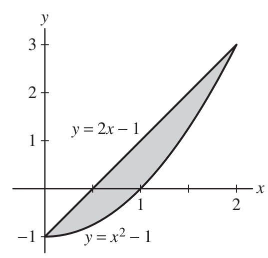
$$2\pi \int_0^8 (x + 3)(2x)\,dx = 4\pi \left(\frac{1}{3}x^3 + \frac{3}{2}x^2 \right)\Bigg|_0^8 = 4\pi \left(\frac{512}{3} + 96 \right) = \frac{3200\pi}{3}.$$



33. $y = x^2 - 1, \quad y = 2x - 1, \quad \text{axis } x = -2$

SOLUTION The region bounded by the graphs of $y = x^2 - 1$ and $y = 2x - 1$ is shown below. Let's choose to slice the region vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [0, 2]$, the shell has radius $x - (-2) = x + 2$ and height $(2x - 1) - (x^2 - 1) = 2x - x^2$. The volume of the solid is therefore given by

$$2\pi \int_0^2 (x+2)(2x - x^2)\,dx = 2\pi \left(2x^2 - \frac{1}{4}x^4\right)\Big|_0^2 = 2\pi(8-4) = 8\pi.$$



34. $y = x^2 - 1, \quad y = 2x - 1, \quad \text{axis } y = 4$

SOLUTION Let's choose to slice the region bounded by the graphs of $y = x^2 - 1$ and $y = 2x - 1$ (see the figure in the previous exercise) vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [0, 2]$, the cross section is a washer with outer radius $R = 4 - (x^2 - 1) = 5 - x^2$ and inner radius $r = 4 - (2x - 1) = 5 - 2x$. The volume of the solid is therefore given by

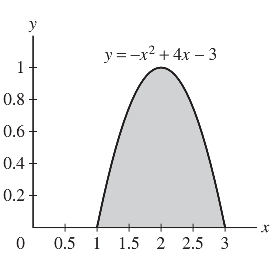
$$\pi \int_0^2 \left((5 - x^2)^2 - (5 - 2x)^2\right) dx = \pi \left(10x^2 - \frac{14}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^2 = \pi\left(40 - \frac{112}{3} + \frac{32}{5}\right) = \frac{136\pi}{15}.$$

35. $y = -x^2 + 4x - 3, \quad y = 0, \quad \text{axis } y = -1$

SOLUTION The region bounded by the graph of $y = -x^2 + 4x - 3$ and the x-axis is shown below. Let's choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [1, 3]$, the cross section is a washer with outer radius $R = -x^2 + 4x - 3 - (-1) = -x^2 + 4x - 2$ and inner radius $r = 0 - (-1) = 1$. The volume of the solid is therefore given by

$$\pi \int_1^3 \left((-x^2 + 4x - 2)^2 - 1\right) dx = \pi \left(\frac{1}{5}x^5 - 2x^4 + \frac{20}{3}x^3 - 8x^2 + 3x\right)\Big|_1^3$$

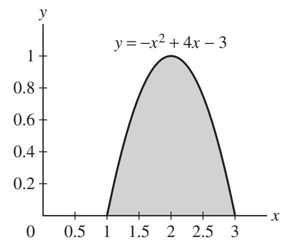
$$= \pi\left[\left(\frac{243}{5} - 162 + 180 - 72 + 9\right) - \left(\frac{1}{5} - 2 + \frac{20}{3} - 8 + 3\right)\right] = \frac{56\pi}{15}.$$



36. $y = -x^2 + 4x - 3, \quad y = 0, \quad \text{axis } x = 4$

SOLUTION The region bounded by the graph of $y = -x^2 + 4x - 3$ and the x-axis is shown in the previous exercise. Let's choose to slice the region vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [1, 3]$, the shell has radius $4 - x$ and height $-x^2 + 4x - 3$. The volume of the solid is therefore given by

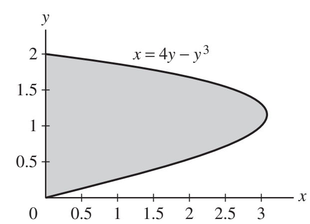
$$2\pi \int_1^3 (4 - x)(-x^2 + 4x - 3)\, dx = 2\pi \int_1^3 (x^3 - 8x^2 + 19x - 12)\, dx$$
$$= 2\pi \left(\frac{1}{4}x^4 - \frac{8}{3}x^3 + \frac{19}{2}x^2 - 12x \right)\Bigg|_1^3$$
$$= 2\pi \left(-\frac{9}{4} + \frac{59}{12} \right) = \frac{16\pi}{3}.$$



37. $x = 4y - y^3, \quad x = 0, \quad y \geq 0, \quad x\text{-axis}$

SOLUTION The region bounded by the graphs of $x = 4y - y^3$ and $x = 0$ for $y \geq 0$ is shown below. Let's choose to slice this region horizontally. Because a horizontal slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $y \in [0, 2]$, the shell has radius y and height $4y - y^3$. The volume of the solid is therefore given by

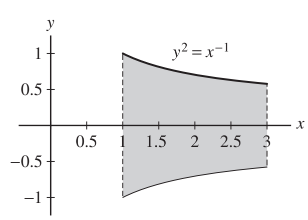
$$2\pi \int_0^2 y(4y - y^3)\, dy = 2\pi \int_0^2 (4y^2 - y^4)\, dy$$
$$= 2\pi \left(\frac{4}{3}y^3 - \frac{1}{5}y^5 \right)\Bigg|_0^2$$
$$= 2\pi \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{128\pi}{15}.$$



38. $y^2 = x^{-1}, \quad x = 1, \quad x = 3, \quad \text{axis } y = -3$

SOLUTION The region bounded by the graphs of $y^2 = x^{-1}$, $x = 1$ and $x = 3$ is shown below. Let's choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [1, 3]$, the cross section is a washer with outer radius $R = \frac{1}{\sqrt{x}} - (-3) = 3 + \frac{1}{\sqrt{x}}$ and inner radius $r = -\frac{1}{\sqrt{x}} - (-3) = 3 - \frac{1}{\sqrt{x}}$. The volume of the solid is therefore given by

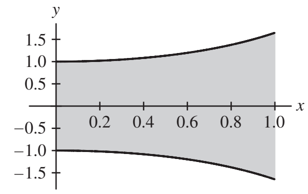
$$\pi \int_1^3 \left(\left(3 + \frac{1}{\sqrt{x}}\right)^2 - \left(3 - \frac{1}{\sqrt{x}}\right)^2 \right) dx = 12\pi \int_1^3 x^{-1/2}\, dx = 24\pi \sqrt{x}\Big|_1^3 = 24\pi(\sqrt{3} - 1).$$



39. $y = e^{-x^2/2}$, $y = -e^{-x^2/2}$, $x = 0$, $x = 1$, y-axis

SOLUTION Let's choose to slice the region bounded by the graphs of $y = e^{-x^2/2}$ and $y = -e^{-x^2/2}$ (see the figure below) vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [0, 1]$, the shell has radius x and height $e^{-x^2/2} - (-e^{-x^2/2}) = 2e^{-x^2/2}$. The volume of the solid is therefore given by

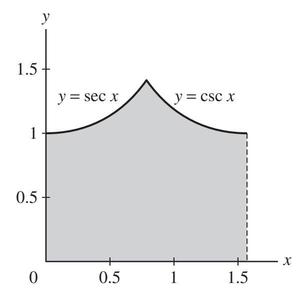
$$2\pi \int_0^1 2xe^{-x^2/2}\,dx = -4\pi e^{-x^2/2}\Big|_0^1$$
$$= -4\pi(e^{-1/2} - 1) = 4\pi(1 - e^{-1/2}).$$



40. $y = \sec x$, $y = \csc x$, $y = 0$, $x = 0$, $x = \dfrac{\pi}{2}$, x-axis

SOLUTION

(a) The region in question is shown in the figure below.



(b) When the region is rotated about the x-axis, cross sections for $x \in [0, \pi/4]$ are circular disks with radius $R = \sec x$, whereas cross sections for $x \in [\pi/4, \pi/2]$ are circular disks with radius $R = \csc x$.

(c) The volume of the solid of revolution is

$$\pi \int_0^{\pi/4} \sec^2 x\,dx + \pi \int_{\pi/4}^{\pi/2} \csc^2 x\,dx = \pi\,(\tan x)\Big|_0^{\pi/4} + \pi\,(-\cot x)\Big|_{\pi/4}^{\pi/2} = \pi\,(1) + \pi\,(1) = 2\pi.$$

In Exercises 41–44, find the volume obtained by rotating the region about the given axis. The regions refer to the graph of the hyperbola $y^2 - x^2 = 1$ in Figure 4.

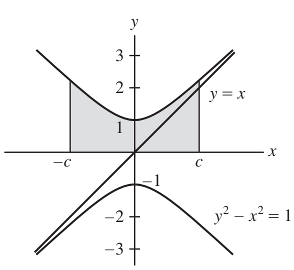


FIGURE 4

41. The shaded region between the upper branch of the hyperbola and the x-axis for $-c \le x \le c$, about the x-axis.

SOLUTION Let's choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [-c, c]$, cross sections are circular disks with radius $R = \sqrt{1 + x^2}$. The volume of the solid is therefore given by

$$\pi \int_{-c}^{c} (1 + x^2)\,dx = \pi \left(x + \frac{1}{3}x^3\right)\Big|_{-c}^{c} = \pi\left[\left(c + \frac{c^3}{3}\right) - \left(-c - \frac{c^3}{3}\right)\right] = 2\pi\left(c + \frac{c^3}{3}\right).$$

42. The region between the upper branch of the hyperbola and the x-axis for $0 \le x \le c$, about the y-axis.

SOLUTION Let's choose to slice the region vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [0, c]$, the shell has radius x and height $\sqrt{1+x^2}$. The volume of the solid is therefore given by

$$2\pi \int_0^c x\sqrt{1+x^2}\,dx = \frac{2\pi}{3}(1+x^2)^{3/2}\Big|_0^c = \frac{2\pi}{3}\left((1+c^2)^{3/2} - 1\right).$$

43. The region between the upper branch of the hyperbola and the line $y = x$ for $0 \le x \le c$, about the x-axis.

SOLUTION Let's choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [0, c]$, cross sections are washers with outer radius $R = \sqrt{1+x^2}$ and inner radius $r = x$. The volume of the solid is therefore given by

$$\pi \int_0^c \left((1+x^2) - x^2\right)\,dx = \pi x\Big|_0^c = c\pi.$$

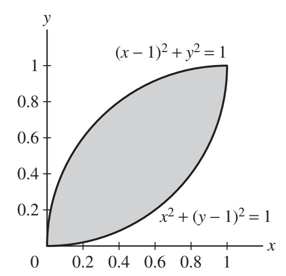
44. The region between the upper branch of the hyperbola and $y = 2$, about the y-axis.

SOLUTION The upper branch of the hyperbola and the horizontal line $y = 2$ intersect when $x = \pm\sqrt{3}$. Using the shell method, each shell has radius x and height $2 - \sqrt{1+x^2}$. The volume of the solid is therefore given by

$$2\pi \int_0^{\sqrt{3}} x\left(2 - \sqrt{1+x^2}\right)\,dx = 2\pi\left(x^2 - \frac{1}{3}(1+x^2)^{3/2}\right)\Big|_0^{\sqrt{3}} = 2\pi\left(3 - \frac{8}{3} + \frac{1}{3}\right) = \frac{4\pi}{3}.$$

45. Let R be the intersection of the circles of radius 1 centered at $(1, 0)$ and $(0, 1)$. Express as an integral (but do not evaluate): **(a)** the area of R and **(b)** the volume of revolution of R about the x-axis.

SOLUTION The region R is shown below.



(a) A vertical slice of R has its top along the upper left arc of the circle $(x-1)^2 + y^2 = 1$ and its bottom along the lower right arc of the circle $x^2 + (y-1)^2 = 1$. The area of R is therefore given by

$$\int_0^1 \left(\sqrt{1-(x-1)^2} - (1 - \sqrt{1-x^2})\right)\,dx.$$

(b) If we revolve R about the x-axis and use the washer method, each cross section is a washer with outer radius $\sqrt{1-(x-1)^2}$ and inner radius $1 - \sqrt{1-x^2}$. The volume of the solid is therefore given by

$$\pi \int_0^1 \left[(1-(x-1)^2) - (1-\sqrt{1-x^2})^2\right]\,dx.$$

46. Let $a > 0$. Show that the volume obtained when the region between $y = a\sqrt{x - ax^2}$ and the x-axis is rotated about the x-axis is independent of the constant a.

SOLUTION Setting $a\sqrt{x-ax^2} = 0$ yields $x = 0$ and $x = 1/a$. Using the washer method, cross sections are circular disks with radius $R = a\sqrt{x-ax^2}$. The volume of the solid is therefore given by

$$\pi \int_0^{1/a} a^2(x - ax^2)\,dx = \pi\left(\frac{1}{2}a^2x^2 - \frac{1}{3}a^3x^3\right)\Big|_0^{1/a} = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6},$$

which is independent of the constant a.

47. If 12 J of work are needed to stretch a spring 20 cm beyond equilibrium, how much work is required to compress it 6 cm beyond equilibrium?

SOLUTION First, we determine the value of the spring constant k as follows:

$$\frac{1}{2}k(0.2)^2 = 12 \quad \text{so} \quad k = 600 \text{ N/m}.$$

Now, the work needed to compress the spring 6 cm beyond equilibrium is

$$W = \int_0^{0.06} 600x\,dx = 300x^2\Big|_0^{0.06} = 1.08 \text{ J}.$$

48. A spring whose equilibrium length is 15 cm exerts a force of 50 N when it is stretched to 20 cm. Find the work required to stretch the spring from 22 to 24 cm.

SOLUTION A force of 50 N is exerted when the spring is stretched 5 cm = 0.05 m from its equilibrium length; therefore, the value of the spring constant is $k = 1000$ N/m. The work required to stretch the spring from a length of 22 cm to a length of 24 cm is then

$$\int_{0.07}^{0.09} 1000x\,dx = 500x^2\Big|_{0.07}^{0.09} = 500(0.09^2 - 0.07^2) = 1.6 \text{ J}.$$

49. If 18 ft-lb of work are needed to stretch a spring 1.5 ft beyond equilibrium, how far will the spring stretch if a 12-lb weight is attached to its end?

SOLUTION First, we determine the value of the spring constant as follows:

$$\frac{1}{2}k(1.5)^2 = 18 \quad \text{so} \quad k = 16 \text{ lb/ft}.$$

Now, if a 12-lb weight is attached to the end of the spring, balancing the forces acting on the weight, we have $12 = 16d$, which implies $d = 0.75$ ft. A 12-lb weight will therefore stretch the spring 9 inches.

50. Let W be the work (against the sun's gravitational force) required to transport an 80-kg person from Earth to Mars when the two planets are aligned with the sun at their minimal distance of 55.7×10^6 km. Use Newton's Universal Law of Gravity (see Exercises 35–37 in Section 6.5) to express W as an integral and evaluate it. The sun has mass $M_s = 1.99 \times 10^{30}$ kg, and the distance from the sun to the earth is 149.6×10^6 km.

SOLUTION According to Newton's Universal Law of Gravity, the gravitational force between the person and the sun is

$$\frac{GM_s m}{r^2},$$

where $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-1} is a constant, $M_s = 1.99 \times 10^{30}$ kg is the mass of the sun, $m = 80$ kg is the mass of the person, and r is the distance between the sun and the person. The work against the sun's gravitational force required to transport the person from Earth to Mars when the two planets are aligned with the sun is therefore given by

$$W = \int_{r_{se}}^{r_{se}+r_{em}} \frac{GM_s m}{r^2}\,dr = GM_s m\left(\frac{1}{r_{se}} - \frac{1}{r_{se}+r_{em}}\right),$$

where $r_{se} = 149.6 \times 10^6$ km is the distance from the sun to Earth and $r_{em} = 55.7 \times 10^6$ km is the distance from Earth to Mars. Converting the distances to meters and substituting the known values into the formula for W yields

$$W = (6.67 \times 10^{-11})(1.99 \times 10^{30})(80)\left(\frac{1}{149.6 \times 10^9} - \frac{1}{205.3 \times 10^9}\right) \approx 1.93 \times 10^{10} \text{ J}.$$

In Exercises 51 and 52, water is pumped into a spherical tank of radius 2 m *from a source located* 1 m *below a hole at the bottom (Figure 5). The density of water is* 1000 kg/m^3.

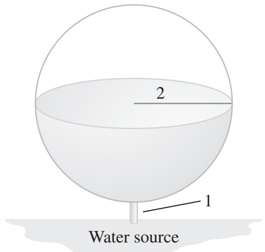


FIGURE 5

51. Calculate the work required to fill the tank.

SOLUTION Place the origin at the base of the sphere with the positive y-axis pointing upward. The equation for the great circle of the sphere is then $x^2 + (y-2)^2 = 4$. At location y, the horizontal cross section is a circle of radius $\sqrt{4-(y-2)^2} = \sqrt{4y-y^2}$; the volume of the layer is then $\pi(4y-y^2)\Delta y$ m^3, and the force needed to lift the layer is $1000(9.8)\pi(4y-y^2)\Delta y$ N. The layer of water must be lifted $y+1$ meters, so the work required to fill the tank is given by

$$9800\pi \int_0^4 (y+1)(4y-y^2)\,dy = 9800\pi \int_0^4 (3y^2+4y-y^3)\,dy$$

$$= 9800\pi \left(y^3 + 2y^2 - \frac{1}{4}y^4 \right)\Big|_0^4$$

$$= 313{,}600\pi \approx 985{,}203.5 \text{ J}.$$

52. Calculate the work $F(h)$ required to fill the tank to level h meters in the sphere.

SOLUTION Place the origin at the base of the sphere with the positive y-axis pointing upward. The equation for the great circle of the sphere is then $x^2 + (y-2)^2 = 4$. At location y, the horizontal cross section is a circle of radius $\sqrt{4-(y-2)^2} = \sqrt{4y-y^2}$; the volume of the layer is then $\pi(4y-y^2)\Delta y$ m^3, and the force needed to lift the layer is $1000(9.8)\pi(4y-y^2)\Delta y$ N. The layer of water must be lifted $y+1$ meters, so the work required to fill the tank is given by

$$9800\pi \int_0^h (y+1)(4y-y^2)\,dy = 9800\pi \int_0^h (3y^2+4y-y^3)\,dy$$

$$= 9800\pi \left(y^3 + 2y^2 - \frac{1}{4}y^4 \right)\Big|_0^h$$

$$= 9800\pi \left(h^3 + 2h^2 - \frac{1}{4}h^4 \right) \text{ J}.$$

53. A tank of mass 20 kg containing 100 kg of water (density 1000 kg/m^3) is raised vertically at a constant speed of 100 m/min for one minute, during which time it leaks water at a rate of 40 kg/min. Calculate the total work performed in raising the container.

SOLUTION Let t denote the elapsed time in minutes and let y denote the height of the container. Given that the speed of ascent is 100 m/min, $y = 100t$; moreover, the mass of water in the container is

$$100 - 40t = 100 - 0.4y\text{kg}.$$

The force needed to lift the container and its contents is then

$$9.8\,(20 + (100 - 0.4y)) = 1176 - 3.92y \text{ N},$$

and the work required to lift the container and its contents is

$$\int_0^{100} (1176 - 3.92y)\,dy = (1176y - 1.96y^2)\Big|_0^{100} = 98{,}000\text{J}.$$