4 APPLICATIONS OF THE DERIVATIVE

4.1 Linear Approximation and Applications

Preliminary Questions

1. True or False? The Linear Approximation says that the vertical change in the graph is approximately equal to the vertical change in the tangent line.

SOLUTION This statement is true. The linear approximation does say that the vertical change in the graph is approximately equal to the vertical change in the tangent line.

2. Estimate $g(1.2) - g(1)$ if $g'(1) = 4$.

SOLUTION Using the Linear Approximation,

$$g(1.2) - g(1) \approx g'(1)(1.2 - 1) = 4(0.2) = 0.8.$$

3. Estimate $f(2.1)$ if $f(2) = 1$ and $f'(2) = 3$.

SOLUTION Using the Linearization,

$$f(2.1) \approx f(2) + f'(2)(2.1 - 2) = 1 + 3(0.1) = 1.3$$

4. Complete the sentence: The Linear Approximation shows that up to a small error, the change in output Δf is directly proportional to ….

SOLUTION The Linear Approximation tells us that up to a small error, the change in output Δf is directly proportional to the change in input Δx when Δx is small.

Exercises

In Exercises 1–6, use Eq. (1) to estimate $\Delta f = f(3.02) - f(3)$.

1. $f(x) = x^2$

SOLUTION Let $f(x) = x^2$. Then $f'(x) = 2x$ and $\Delta f \approx f'(3)\Delta x = 6(0.02) = 0.12$.

2. $f(x) = x^4$

SOLUTION Let $f(x) = x^4$. Then $f'(x) = 4x^3$ and $\Delta f \approx f'(3)\Delta x = 4(27)(0.02) = 2.16$.

3. $f(x) = x^{-1}$

SOLUTION Let $f(x) = x^{-1}$. Then $f'(x) = -x^{-2}$ and $\Delta f \approx f'(3)\Delta x = -\frac{1}{9}(0.02) = -0.00222$.

4. $f(x) = \dfrac{1}{x+1}$

SOLUTION Let $f(x) = (x+1)^{-1}$. Then $f'(x) = -(x+1)^{-2}$ and $\Delta f \approx f'(3)\Delta x = -\frac{1}{16}(0.02) = -0.00125$.

5. $f(x) = \sqrt{x+6}$

SOLUTION Let $f(x) = \sqrt{x+6}$. Then $f'(x) = \frac{1}{2}(x+6)^{-1/2}$ and

$$\Delta f \approx f'(3)\Delta x = \frac{1}{2}9^{-1/2}(0.02) = 0.003333.$$

6. $f(x) = \tan\dfrac{\pi x}{3}$

SOLUTION Let $f(x) = \tan\frac{\pi x}{3}$. Then $f'(x) = \frac{\pi}{3}\sec^2\frac{\pi x}{3}$ and

$$\Delta f \approx f'(3)\Delta x = \frac{\pi}{3}(0.02) = 0.020944.$$

7. The cube root of 27 is 3. How much larger is the cube root of 27.2? Estimate using the Linear Approximation.

SOLUTION Let $f(x) = x^{1/3}$, $a = 27$, and $\Delta x = 0.2$. Then $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(a) = f'(27) = \frac{1}{27}$. The Linear Approximation is

$$\Delta f \approx f'(a)\Delta x = \frac{1}{27}(0.2) = 0.0074074$$

8. Estimate $\ln(e^3 + 0.1) - \ln(e^3)$ using differentials.

SOLUTION Let $f(x) = \ln x$, $a = e^3$, and $\Delta x = 0.1$. Then $f'(x) = x^{-1}$ and $f'(a) = e^{-3}$. Thus,

$$\ln(e^3 + 0.1) - \ln(e^3) = \Delta f \approx f'(a)\Delta x = e^{-3}(0.1) = 0.00498.$$

In Exercises 9–12, use Eq. (1) to estimate Δf. Use a calculator to compute both the error and the percentage error.

9. $f(x) = \sqrt{1+x}$, $\quad a = 3$, $\quad \Delta x = 0.2$

SOLUTION Let $f(x) = (1+x)^{1/2}$, $a = 3$, and $\Delta x = 0.2$. Then $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, $f'(a) = f'(3) = \frac{1}{4}$ and $\Delta f \approx f'(a)\Delta x = \frac{1}{4}(0.2) = 0.05$. The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(3.2) - f(3) = \sqrt{4.2} - 2 \approx 0.049390.$$

The error in the Linear Approximation is therefore $|0.049390 - 0.05| = 0.000610$; in percentage terms, the error is

$$\frac{0.000610}{0.049390} \times 100\% \approx 1.24\%.$$

10. $f(x) = 2x^2 - x$, $\quad a = 5$, $\quad \Delta x = -0.4$

SOLUTION Let $f(x) = 2x^2 - x$, $a = 5$ and $\Delta x = -0.4$. Then $f'(x) = 4x - 1$, $f'(a) = 19$ and $\Delta f \approx f'(a)\Delta x = 19(-0.4) = -7.6$. The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(4.6) - f(5) = 37.72 - 45 = -7.28.$$

The error in the Linear Approximation is therefore $|-7.28 - (-7.6)| = 0.32$; in percentage terms, the error is

$$\frac{0.32}{7.28} \times 100\% \approx 4.40\%.$$

11. $f(x) = \dfrac{1}{1+x^2}$, $\quad a = 3$, $\quad \Delta x = 0.5$

SOLUTION Let $f(x) = \frac{1}{1+x^2}$, $a = 3$, and $\Delta x = .5$. Then $f'(x) = -\frac{2x}{(1+x^2)^2}$, $f'(a) = f'(3) = -0.06$ and $\Delta f \approx f'(a)\Delta x = -0.06(0.5) = -0.03$. The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(3.5) - f(3) \approx -0.0245283.$$

The error in the Linear Approximation is therefore $|-0.0245283 - (-0.03)| = 0.0054717$; in percentage terms, the error is

$$\left|\frac{0.0054717}{-0.0245283}\right| \times 100\% \approx 22.31\%$$

12. $f(x) = \ln(x^2 + 1)$, $\quad a = 1$, $\quad \Delta x = 0.1$

SOLUTION Let $f(x) = \ln(x^2 + 1)$, $a = 1$, and $\Delta x = 0.1$. Then $f'(x) = \frac{2x}{x^2+1}$, $f'(a) = f'(1) = 1$, and $\Delta f \approx f'(a)\Delta x = 1(0.1) = 0.1$. The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(1.1) - f(1) = 0.099845.$$

The error in the Linear Approximation is therefore $|0.099845 - 0.1| = 0.000155$; in percentage terms, the error is

$$\frac{0.000155}{0.099845} \times 100\% \approx 0.16\%.$$

In Exercises 13–16, estimate Δy using differentials [Eq. (3)].

13. $y = \cos x$, $\quad a = \frac{\pi}{6}$, $\quad dx = 0.014$

SOLUTION Let $f(x) = \cos x$. Then $f'(x) = -\sin x$ and

$$\Delta y \approx dy = f'(a)dx = -\sin\left(\frac{\pi}{6}\right)(0.014) = -0.007.$$

14. $y = \tan^2 x, \quad a = \frac{\pi}{4}, \quad dx = -0.02$

SOLUTION Let $f(x) = \tan^2 x$. Then $f'(x) = 2\tan x \sec^2 x$ and

$$\Delta y \approx dy = f'(a)dx = 2\tan\frac{\pi}{4}\sec^2\frac{\pi}{4}(-0.02) = -0.08.$$

15. $y = \dfrac{10 - x^2}{2 + x^2}, \quad a = 1, \quad dx = 0.01$

SOLUTION Let $f(x) = \dfrac{10 - x^2}{2 + x^2}$. Then

$$f'(x) = \frac{(2 + x^2)(-2x) - (10 - x^2)(2x)}{(2 + x^2)^2} = -\frac{24x}{(2 + x^2)^2}$$

and

$$\Delta y \approx dy = f'(a)dx = -\frac{24}{9}(0.01) = -0.026667.$$

16. $y = x^{1/3}e^{x-1}, \quad a = 1, \quad dx = 0.1$

SOLUTION Let $y = x^{1/3}e^{x-1}$, $a = 1$, and $dx = 0.1$. Then $y'(x) = \frac{1}{3}x^{-2/3}e^{x-1}(3x + 1)$, $y'(a) = y'(1) = \frac{4}{3}$, and $\Delta y \approx dy = y'(a)\,dx = \frac{4}{3}(0.1) = 0.133333$.

In Exercises 17–24, estimate using the Linear Approximation and find the error using a calculator.

17. $\sqrt{26} - \sqrt{25}$

SOLUTION Let $f(x) = \sqrt{x}$, $a = 25$, and $\Delta x = 1$. Then $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(a) = f'(25) = \frac{1}{10}$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{10}(1) = 0.1$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(26) - f(25) \approx 0.0990195$.
- The error in this estimate is $|0.0990195 - 0.1| = 0.000980486$.

18. $16.5^{1/4} - 16^{1/4}$

SOLUTION Let $f(x) = x^{1/4}$, $a = 16$, and $\Delta x = .5$. Then $f'(x) = \frac{1}{4}x^{-3/4}$ and $f'(a) = f'(16) = \frac{1}{32}$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{32}(0.5) = 0.015625$.
- The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(16.5) - f(16) \approx 2.015445 - 2 = 0.015445$$

- The error in this estimate is $|0.015625 - 0.015445| \approx 0.00018$.

19. $\dfrac{1}{\sqrt{101}} - \dfrac{1}{10}$

SOLUTION Let $f(x) = \frac{1}{\sqrt{x}}$, $a = 100$, and $\Delta x = 1$. Then $f'(x) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-3/2}$ and $f'(a) = -\frac{1}{2}(\frac{1}{1000}) = -0.0005$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = -0.0005(1) = -0.0005$.
- The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = \frac{1}{\sqrt{101}} - \frac{1}{10} = -0.000496281.$$

- The error in this estimate is $|-0.0005 - (-0.000496281)| = 3.71902 \times 10^{-6}$.

20. $\dfrac{1}{\sqrt{98}} - \dfrac{1}{10}$

SOLUTION Let $f(x) = \frac{1}{\sqrt{x}}$, $a = 100$, and $\Delta x = -2$. Then $f'(x) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-3/2}$ and $f'(a) = -\frac{1}{2}(\frac{1}{1000}) = -0.0005$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = -0.0005(-2) = 0.001$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(98) - f(100) = 0.00101525$.
- The error in this estimate is $|0.001 - 0.00101525| \approx 0.00001525$.

21. $9^{1/3} - 2$

SOLUTION Let $f(x) = x^{1/3}$, $a = 8$, and $\Delta x = 1$. Then $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(a) = f'(8) = \frac{1}{12}$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{12}(1) = 0.083333$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(9) - f(8) = 0.080084$.
- The error in this estimate is $|0.080084 - 0.083333| \approx 3.25 \times 10^{-3}$.

22. $\tan^{-1}(1.05) - \frac{\pi}{4}$

SOLUTION Let $f(x) = \tan^{-1} x$, $a = 1$, and $\Delta x = 0.05$. Then $f'(x) = (1 + x^2)^{-1}$ and $f'(a) = f'(1) = \frac{1}{2}$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{2}(0.05) = 0.025$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(1.05) - f(1) = 0.024385$.
- The error in this estimate is $|0.024385 - 0.025| \approx 6.15 \times 10^{-4}$.

23. $e^{-0.1} - 1$

SOLUTION Let $f(x) = e^x$, $a = 0$, and $\Delta x = -0.1$. Then $f'(x) = e^x$ and $f'(a) = f'(0) = 1$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = 1(-0.1) = -0.1$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(-0.1) - f(0) = -0.095163$.
- The error in this estimate is $| - 0.095163 - (-0.1)| \approx 4.84 \times 10^{-3}$.

24. $\ln(0.97)$

SOLUTION Let $f(x) = \ln x$, $a = 1$, and $\Delta x = -0.03$. Then $f'(x) = \frac{1}{x}$ and $f'(a) = f'(1) = 1$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = (1)(-0.03) = -0.03$, so $\ln(0.97) \approx \ln 1 - 0.03 = -0.03$.
- The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(0.97) - f(1) \approx -0.030459 - 0 = -0.030459.$$

- The error is $|\Delta f - f'(a)\Delta x| \approx 0.000459$.

25. Estimate $f(4.03)$ for $f(x)$ as in Figure 8.

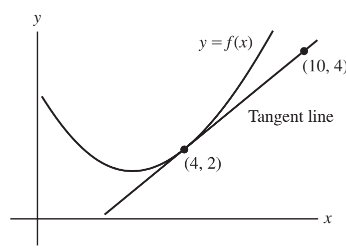


FIGURE 8

SOLUTION Using the Linear Approximation, $f(4.03) \approx f(4) + f'(4)(0.03)$. From the figure, we find that $f(4) = 2$ and

$$f'(4) = \frac{4-2}{10-4} = \frac{1}{3}.$$

Thus,

$$f(4.03) \approx 2 + \frac{1}{3}(0.03) = 2.01.$$

26. At a certain moment, an object in linear motion has velocity 100 m/s. Estimate the distance traveled over the next quarter-second, and explain how this is an application of the Linear Approximation.

SOLUTION Because the velocity is 100 m/s, we estimate the object will travel

$$\left(100\ \frac{\text{m}}{\text{s}}\right)\left(\frac{1}{4}\ \text{s}\right) = 25\ \text{m}$$

in the next quarter-second. Recall that velocity is the derivative of position, so we have just estimated the change in position, Δs, using the product $s'\Delta t$, which is just the Linear Approximation.

27. Which is larger: $\sqrt{2.1} - \sqrt{2}$ or $\sqrt{9.1} - \sqrt{9}$? Explain using the Linear Approximation.

SOLUTION Let $f(x) = \sqrt{x}$, and $\Delta x = 0.1$. Then $f'(x) = \frac{1}{2}x^{-1/2}$ and the Linear Approximation at $x = a$ gives

$$\Delta f = \sqrt{a + 0.1} - \sqrt{a} \approx f'(a)(0.1) = \frac{1}{2}a^{-1/2}(0.1) = \frac{0.05}{\sqrt{a}}$$

We see that Δf decreases as a increases. In particular

$$\sqrt{2.1} - \sqrt{2} \approx \frac{0.05}{\sqrt{2}} \qquad \text{is larger than} \qquad \sqrt{9.1} - \sqrt{9} \approx \frac{0.05}{3}$$

28. Estimate $\sin 61^\circ - \sin 60^\circ$ using the Linear Approximation. *Hint:* Express $\Delta\theta$ in radians.

SOLUTION Let $f(x) = \sin x$, $a = \frac{\pi}{3}$, and $\Delta x = \frac{\pi}{180}$. Then $f'(x) = \cos x$ and $f'(a) = f'(\frac{\pi}{3}) = \frac{1}{2}$. Finally, the Linear Approximation is

$$\Delta f \approx f'(a)\Delta x = \frac{1}{2}\left(\frac{\pi}{180}\right) = \frac{\pi}{360} \approx 0.008727$$

29. Box office revenue at a multiplex cinema in Paris is $R(p) = 3600p - 10p^3$ euros per showing when the ticket price is p euros. Calculate $R(p)$ for $p = 9$ and use the Linear Approximation to estimate ΔR if p is raised or lowered by 0.5 euros.

SOLUTION Let $R(p) = 3600p - 10p^3$. Then $R(9) = 3600(9) - 10(9)^3 = 25110$ euros. Moreover, $R'(p) = 3600 - 30p^2$, so by the Linear Approximation,

$$\Delta R \approx R'(9)\Delta p = 1170\Delta p.$$

If p is raised by 0.5 euros, then $\Delta R \approx 585$ euros; on the other hand, if p is lowered by 0.5 euros, then $\Delta R \approx -585$ euros.

30. The *stopping distance* for an automobile is $F(s) = 1.1s + 0.054s^2$ ft, where s is the speed in mph. Use the Linear Approximation to estimate the change in stopping distance per additional mph when $s = 35$ and when $s = 55$.

SOLUTION Let $F(s) = 1.1s + 0.054s^2$.

- The Linear Approximation at $s = 35$ mph is

  $$\Delta F \approx F'(35)\Delta s = (1.1 + 0.108 \times 35)\Delta s = 4.88\Delta s \text{ ft}$$

  The change in stopping distance per additional mph for $s = 35$ mph is approximately 4.88 ft.
- The Linear Approximation at $s = 55$ mph is

  $$\Delta F \approx F'(55)\Delta s = (1.1 + 0.108 \times 55)\Delta s = 7.04\Delta s \text{ ft}$$

  The change in stopping distance per additional mph for $s = 55$ mph is approximately 7.04 ft.

31. A thin silver wire has length $L = 18$ cm when the temperature is $T = 30^\circ$C. Estimate ΔL when T decreases to 25°C if the coefficient of thermal expansion is $k = 1.9 \times 10^{-5}\,^\circ\text{C}^{-1}$ (see Example 3).

SOLUTION We have

$$\frac{dL}{dT} = kL = (1.9 \times 10^{-5})(18) = 3.42 \times 10^{-4} \text{ cm}/^\circ\text{C}$$

The change in temperature is $\Delta T = -5^\circ$ C, so by the Linear Approximation, the change in length is approximately

$$\Delta L \approx 3.42 \times 10^{-4}\Delta T = (3.42 \times 10^{-4})(-5) = -0.00171 \text{ cm}$$

At $T = 25^\circ$ C, the length of the wire is approximately 17.99829 cm.

32. At a certain moment, the temperature in a snake cage satisfies $dT/dt = 0.008^\circ$C/s. Estimate the rise in temperature over the next 10 seconds.

SOLUTION Using the Linear Approximation, the rise in temperature over the next 10 seconds will be

$$\Delta T \approx \frac{dT}{dt}\Delta t = 0.008(10) = 0.08^\circ\text{C}.$$

33. The atmospheric pressure at altitude h (kilometers) for $11 \le h \le 25$ is approximately

$$P(h) = 128e^{-0.157h} \text{ kilopascals.}$$

(a) Estimate ΔP at $h = 20$ when $\Delta h = 0.5$.

(b) Compute the actual change, and compute the percentage error in the Linear Approximation.

SOLUTION

(a) Let $P(h) = 128e^{-0.157h}$. Then $P'(h) = -20.096e^{-0.157h}$. Using the Linear Approximation,

$$\Delta P \approx P'(h)\Delta h = P'(20)(0.5) = -0.434906 \text{ kilopascals.}$$

(b) The actual change in pressure is

$$P(20.5) - P(20) = -0.418274 \text{ kilopascals.}$$

The percentage error in the Linear Approximation is

$$\left|\frac{-0.434906 - (-0.418274)}{-0.418274}\right| \times 100\% \approx 3.98\%.$$

34. The resistance R of a copper wire at temperature $T = 20°\text{C}$ is $R = 15\ \Omega$. Estimate the resistance at $T = 22°\text{C}$, assuming that $dR/dT\big|_{T=20} = 0.06\ \Omega/°\text{C}$.

SOLUTION $\Delta T = 2°\text{C}$. The Linear Approximation gives us:

$$R(22) - R(20) \approx dR/dT\bigg|_{T=20} \Delta T = 0.06\ \Omega/°\text{C}(2°\text{C}) = 0.12\ \Omega.$$

Therefore, $R(22) \approx 15\ \Omega + 0.12\ \Omega = 15.12\ \Omega$.

35. Newton's Law of Gravitation shows that if a person weighs w pounds on the surface of the earth, then his or her weight at distance x from the center of the earth is

$$W(x) = \frac{wR^2}{x^2} \qquad (\text{for } x \geq R)$$

where $R = 3960$ miles is the radius of the earth (Figure 9).

(a) Show that the weight lost at altitude h miles above the earth's surface is approximately $\Delta W \approx -(0.0005w)h$. *Hint:* Use the Linear Approximation with $dx = h$.

(b) Estimate the weight lost by a 200-lb football player flying in a jet at an altitude of 7 miles.

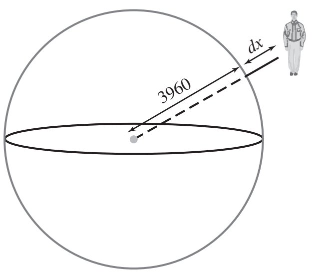


FIGURE 9 The distance to the center of the earth is $3960 + h$ miles.

SOLUTION

(a) Using the Linear Approximation

$$\Delta W \approx W'(R)\Delta x = -\frac{2wR^2}{R^3}h = -\frac{2wh}{R} \approx -0.0005wh.$$

(b) Substitute $w = 200$ and $h = 7$ into the result from part (a) to obtain

$$\Delta W \approx -0.0005(200)(7) = -0.7 \text{ pounds.}$$

36. Using Exercise 35(a), estimate the altitude at which a 130-lb pilot would weigh 129.5 lb.

SOLUTION From Exercise 35(a), the weight loss ΔW at altitude h (in miles) for a person weighing w at the surface of the earth is approximately

$$\Delta W \approx -0.0005wh$$

If $w = 130$ pounds, then $\Delta W \approx -0.065h$. Accordingly, the pilot loses approximately 0.065 pounds per mile of altitude gained. The pilot will weigh 129.5 pounds at the altitude h such that $-0.065h = -0.5$, or $h = 0.5/0.065 \approx 7.7$ miles.

37. A stone tossed vertically into the air with initial velocity v cm/s reaches a maximum height of $h = v^2/1960$ cm.

(a) Estimate Δh if $v = 700$ cm/s and $\Delta v = 1$ cm/s.

(b) Estimate Δh if $v = 1{,}000$ cm/s and $\Delta v = 1$ cm/s.

(c) In general, does a 1 cm/s increase in v lead to a greater change in h at low or high initial velocities? Explain.

SOLUTION A stone tossed vertically with initial velocity v cm/s attains a maximum height of $h(v) = v^2/1960$ cm. Thus, $h'(v) = v/980$.

(a) If $v = 700$ and $\Delta v = 1$, then $\Delta h \approx h'(v)\Delta v = \frac{1}{980}(700)(1) \approx 0.71$ cm.

(b) If $v = 1000$ and $\Delta v = 1$, then $\Delta h \approx h'(v)\Delta v = \frac{1}{980}(1000)(1) = 1.02$ cm.

(c) A one centimeter per second increase in initial velocity v increases the maximum height by approximately $v/980$ cm. Accordingly, there is a bigger effect at higher velocities.

38. The side s of a square carpet is measured at 6 m. Estimate the maximum error in the area A of the carpet if s is accurate to within 2 centimeters.

SOLUTION Let s be the length in meters of the side of the square carpet. Then $A(s) = s^2$ is the area of the carpet. With $a = 6$ and $\Delta s = 0.02$ (note that 1 cm equals 0.01 m), an estimate of the size of the error in the area is given by the Linear Approximation:

$$\Delta A \approx A'(6)\Delta s = 12\,(0.02) = 0.24 \text{ m}^2$$

In Exercises 39 and 40, use the following fact derived from Newton's Laws: An object released at an angle θ with initial velocity v ft/s travels a horizontal distance

$$s = \frac{1}{32}v^2 \sin 2\theta \text{ ft} \quad \text{(Figure 10)}$$

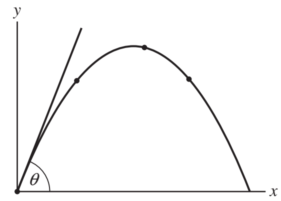


FIGURE 10 Trajectory of an object released at an angle θ.

39. A player located 18.1 ft from the basket launches a successful jump shot from a height of 10 ft (level with the rim of the basket), at an angle $\theta = 34°$ and initial velocity $v = 25$ ft/s.)

(a) Show that $\Delta s \approx 0.255\Delta\theta$ ft for a small change of $\Delta\theta$.

(b) Is it likely that the shot would have been successful if the angle had been off by 2°?

SOLUTION Using Newton's laws and the given initial velocity of $v = 25$ ft/s, the shot travels $s = \frac{1}{32}v^2 \sin 2t = \frac{625}{32}\sin 2t$ ft, where t is in radians.

(a) If $\theta = 34°$ (i.e., $t = \frac{17}{90}\pi$), then

$$\Delta s \approx s'(t)\Delta t = \frac{625}{16}\cos\left(\frac{17}{45}\pi\right)\Delta t = \frac{625}{16}\cos\left(\frac{17}{45}\pi\right)\Delta\theta \cdot \frac{\pi}{180} \approx 0.255\Delta\theta.$$

(b) If $\Delta\theta = 2°$, this gives $\Delta s \approx 0.51$ ft, in which case the shot would not have been successful, having been off half a foot.

40. Estimate Δs if $\theta = 34°$, $v = 25$ ft/s, and $\Delta v = 2$.

SOLUTION Using Newton's laws and the fixed angle of $\theta = 34° = \frac{17}{90}\pi$, the shot travels

$$s = \frac{1}{32}v^2 \sin\frac{17}{45}\pi.$$

With $v = 25$ ft/s and $\Delta v = 2$ ft/s, we find

$$\Delta s \approx s'(v)\Delta v = \frac{1}{16}(25)\sin\frac{17\pi}{45} \cdot 2 = 2.897 \text{ ft}.$$

41. The radius of a spherical ball is measured at $r = 25$ cm. Estimate the maximum error in the volume and surface area if r is accurate to within 0.5 cm.

SOLUTION The volume and surface area of the sphere are given by $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, respectively. If $r = 25$ and $\Delta r = \pm 0.5$, then

$$\Delta V \approx V'(25)\Delta r = 4\pi(25)^2(0.5) \approx 3927 \text{ cm}^3,$$

and

$$\Delta S \approx S'(25)\Delta r = 8\pi(25)(0.5) \approx 314.2 \text{ cm}^2.$$

42. The dosage D of diphenhydramine for a dog of body mass w kg is $D = 4.7w^{2/3}$ mg. Estimate the maximum allowable error in w for a cocker spaniel of mass $w = 10$ kg if the percentage error in D must be less than 3%.

SOLUTION We have $D = kw^{2/3}$ where $k = 4.7$. The Linear Approximation yields

$$\Delta D \approx \frac{2}{3}kw^{-1/3}\Delta w,$$

so

$$\frac{\Delta D}{D} \approx \frac{\frac{2}{3}kw^{-1/3}\Delta w}{kw^{2/3}} = \frac{2}{3}\cdot\frac{\Delta w}{w}$$

If the percentage error in D must be less than 3%, we estimate the maximum allowable error in w to be

$$\Delta w \approx \frac{3w}{2}\cdot\frac{\Delta D}{D} = \frac{3(10)}{2}(.03) = 0.45 \text{ kg}$$

43. The volume (in liters) and pressure P (in atmospheres) of a certain gas satisfy $PV = 24$. A measurement yields $V = 4$ with a possible error of ± 0.3 L. Compute P and estimate the maximum error in this computation.

SOLUTION Given $PV = 24$ and $V = 4$, it follows that $P = 6$ atmospheres. Solving $PV = 24$ for P yields $P = 24V^{-1}$. Thus, $P' = -24V^{-2}$ and

$$\Delta P \approx P'(4)\Delta V = -24(4)^{-2}(\pm 0.3) = \pm 0.45 \text{ atmospheres.}$$

44. In the notation of Exercise 43, assume that a measurement yields $V = 4$. Estimate the maximum allowable error in V if P must have an error of less than 0.2 atm.

SOLUTION From Exercise 43, with $V = 4$, we have

$$\Delta P \approx -\frac{3}{2}\Delta V \quad \text{or} \quad \Delta V = -\frac{2}{3}\Delta P.$$

If we require $|\Delta P| \le 0.2$, then we must have

$$|\Delta V| \le \frac{2}{3}(0.2) = 0.133333 \text{ L.}$$

In Exercises 45–54, find the linearization at $x = a$.

45. $f(x) = x^4, \quad a = 1$

SOLUTION Let $f(x) = x^4$. Then $f'(x) = 4x^3$. The linearization at $a = 1$ is

$$L(x) = f'(a)(x - a) + f(a) = 4(x - 1) + 1 = 4x - 3.$$

46. $f(x) = \dfrac{1}{x}, \quad a = 2$

SOLUTION Let $f(x) = \frac{1}{x} = x^{-1}$. Then $f'(x) = -x^{-2}$. The linearization at $a = 2$ is

$$L(x) = f'(a)(x - a) + f(a) = -\frac{1}{4}(x - 2) + \frac{1}{2} = -\frac{1}{4}x + 1.$$

47. $f(\theta) = \sin^2\theta, \quad a = \frac{\pi}{4}$

SOLUTION Let $f(\theta) = \sin^2\theta$. Then $f'(\theta) = 2\sin\theta\cos\theta = \sin 2\theta$. The linearization at $a = \frac{\pi}{4}$ is

$$L(\theta) = f'(a)(\theta - a) + f(a) = 1\left(\theta - \frac{\pi}{4}\right) + \frac{1}{2} = \theta - \frac{\pi}{4} + \frac{1}{2}.$$

48. $g(x) = \dfrac{x^2}{x - 3}, \quad a = 4$

SOLUTION Let $g(x) = \frac{x^2}{x-3}$. Then

$$g'(x) = \frac{(x - 3)(2x) - x^2}{(x - 3)^2} = \frac{x^2 - 6x}{(x - 3)^2}.$$

The linearization at $a = 4$ is

$$L(x) = g'(a)(x - a) + g(a) = -8(x - 4) + 16 = -8x + 48.$$

49. $y = (1+x)^{-1/2}$, $a = 0$

SOLUTION Let $f(x) = (1+x)^{-1/2}$. Then $f'(x) = -\frac{1}{2}(1+x)^{-3/2}$. The linearization at $a = 0$ is

$$L(x) = f'(a)(x-a) + f(a) = -\frac{1}{2}x + 1.$$

50. $y = (1+x)^{-1/2}$, $a = 3$

SOLUTION Let $f(x) = (1+x)^{-1/2}$. Then $f'(x) = -\frac{1}{2}(1+x)^{-3/2}$, $f(a) = 4^{-1/2} = \frac{1}{2}$, and $f'(a) = -\frac{1}{2}(4^{-3/2}) = -\frac{1}{16}$, so the linearization at $a = 3$ is

$$L(x) = f'(a)(x-a) + f(a) = -\frac{1}{16}(x-3) + \frac{1}{2} = -\frac{1}{16}x + \frac{11}{16}.$$

51. $y = (1+x^2)^{-1/2}$, $a = 0$

SOLUTION Let $f(x) = (1+x^2)^{-1/2}$. Then $f'(x) = -x(1+x^2)^{-3/2}$, $f(a) = 1$ and $f'(a) = 0$, so the linearization at a is

$$L(x) = f'(a)(x-a) + f(a) = 1.$$

52. $y = \tan^{-1} x$, $a = 1$

SOLUTION Let $f(x) = \tan^{-1} x$. Then

$$f'(x) = \frac{1}{1+x^2}, \quad f(a) = \frac{\pi}{4}, \text{ and } f'(a) = \frac{1}{2},$$

so the linearization of $f(x)$ at a is

$$L(x) = f'(a)(x-a) + f(a) = \frac{1}{2}(x-1) + \frac{\pi}{4}.$$

53. $y = e^{\sqrt{x}}$, $a = 1$

SOLUTION Let $f(x) = e^{\sqrt{x}}$. Then

$$f'(x) = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}, \quad f(a) = e, \text{ and } f'(a) = \frac{1}{2}e,$$

so the linearization of $f(x)$ at a is

$$L(x) = f'(a)(x-a) + f(a) = \frac{1}{2}e(x-1) + e = \frac{1}{2}e(x+1).$$

54. $y = e^x \ln x$, $a = 1$

SOLUTION Let $f(x) = e^x \ln x$. Then

$$f'(x) = \frac{e^x}{x} + e^x \ln x, \quad f(a) = 0, \text{ and } f'(a) = e,$$

so the linearization of $f(x)$ at a is

$$L(x) = f'(a)(x-a) + f(a) = e(x-1).$$

55. What is $f(2)$ if the linearization of $f(x)$ at $a = 2$ is $L(x) = 2x + 4$?

SOLUTION $f(2) = L(2) = 2(2) + 4 = 8.$

56. Compute the linearization of $f(x) = 3x - 4$ at $a = 0$ and $a = 2$. Prove more generally that a linear function coincides with its linearization at $x = a$ for all a.

SOLUTION Let $f(x) = 3x - 4$. Then $f'(x) = 3$. With $a = 0$, $f(a) = -4$ and $f'(a) = 3$, so the linearization of $f(x)$ at $a = 0$ is

$$L(x) = -4 + 3(x-0) = 3x - 4 = f(x).$$

With $a = 2$, $f(a) = 2$ and $f'(a) = 3$, so the linearization of $f(x)$ at $a = 2$ is

$$L(x) = 2 + 3(x-2) = 2 + 3x - 6 = 3x - 4 = f(x).$$

More generally, let $g(x) = bx + c$ be any linear function. The linearization $L(x)$ of $g(x)$ at $x = a$ is

$$L(x) = g'(a)(x-a) + g(a) = b(x-a) + ba + c = bx + c = g(x);$$

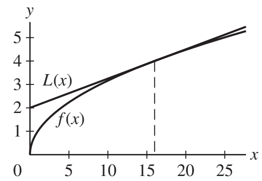
i.e., $L(x) = g(x)$.

57. Estimate $\sqrt{16.2}$ using the linearization $L(x)$ of $f(x) = \sqrt{x}$ at $a = 16$. Plot $f(x)$ and $L(x)$ on the same set of axes and determine whether the estimate is too large or too small.

SOLUTION Let $f(x) = x^{1/2}$, $a = 16$, and $\Delta x = 0.2$. Then $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(a) = f'(16) = \frac{1}{8}$. The linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = \frac{1}{8}(x-16) + 4 = \frac{1}{8}x + 2.$$

Thus, we have $\sqrt{16.2} \approx L(16.2) = 4.025$. Graphs of $f(x)$ and $L(x)$ are shown below. Because the graph of $L(x)$ lies above the graph of $f(x)$, we expect that the estimate from the Linear Approximation is too large.



58. GU Estimate $1/\sqrt{15}$ using a suitable linearization of $f(x) = 1/\sqrt{x}$. Plot $f(x)$ and $L(x)$ on the same set of axes and determine whether the estimate is too large or too small. Use a calculator to compute the percentage error.

SOLUTION The nearest perfect square to 15 is 16. Let $f(x) = \frac{1}{\sqrt{x}}$ and $a = 16$. Then $f'(x) = -\frac{1}{2}x^{-3/2}$ and $f'(a) = f'(16) = -\frac{1}{128}$. The linearization is

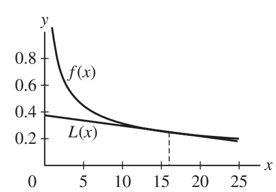
$$L(x) = f'(a)(x-a) + f(a) = -\frac{1}{128}(x-16) + \frac{1}{4}.$$

Then

$$\frac{1}{\sqrt{15}} \approx L(15) = -\frac{1}{128}(-1) + \frac{1}{4} = \frac{33}{128} = 0.257813.$$

Graphs of $f(x)$ and $L(x)$ are shown below. Because the graph of $L(x)$ lies below the graph of $f(x)$, we expect that the estimate from the Linear Approximation is too small. The percentage error in the estimate is

$$\left|\frac{\frac{1}{\sqrt{15}} - 0.257813}{\frac{1}{\sqrt{15}}}\right| \times 100\% \approx 0.15\%$$



In Exercises 59–67, approximate using linearization and use a calculator to compute the percentage error.

59. $\dfrac{1}{\sqrt{17}}$

SOLUTION Let $f(x) = x^{-1/2}$, $a = 16$, and $\Delta x = 1$. Then $f'(x) = -\frac{1}{2}x^{-3/2}$, $f'(a) = f'(16) = -\frac{1}{128}$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = -\frac{1}{128}(x-16) + \frac{1}{4} = -\frac{1}{128}x + \frac{3}{8}.$$

Thus, we have $\frac{1}{\sqrt{17}} \approx L(17) \approx 0.24219$. The percentage error in this estimate is

$$\left|\frac{\frac{1}{\sqrt{17}} - 0.24219}{\frac{1}{\sqrt{17}}}\right| \times 100\% \approx 0.14\%$$

60. $\dfrac{1}{101}$

SOLUTION Let $f(x) = x^{-1}$, $a = 100$ and $\Delta x = 1$. Then $f'(x) = -x^{-2}$, $f'(a) = f'(100) = -0.0001$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x - a) + f(a) = -0.0001(x - 100) + 0.01 = -0.0001x + 0.02.$$

Thus, we have

$$\frac{1}{101} \approx L(101) = -0.0001(101) + 0.02 = 0.0099.$$

The percentage error in this estimate is

$$\left|\frac{\frac{1}{101} - 0.0099}{\frac{1}{101}}\right| \times 100\% \approx 0.01\%$$

61. $\dfrac{1}{(10.03)^2}$

SOLUTION Let $f(x) = x^{-2}$, $a = 10$ and $\Delta x = 0.03$. Then $f'(x) = -2x^{-3}$, $f'(a) = f'(10) = -0.002$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x - a) + f(a) = -0.002(x - 10) + 0.01 = -0.002x + 0.03.$$

Thus, we have

$$\frac{1}{(10.03)^2} \approx L(10.03) = -0.002(10.03) + 0.03 = 0.00994.$$

The percentage error in this estimate is

$$\left|\frac{\frac{1}{(10.03)^2} - 0.00994}{\frac{1}{(10.03)^2}}\right| \times 100\% \approx 0.0027\%$$

62. $(17)^{1/4}$

SOLUTION Let $f(x) = x^{1/4}$, $a = 16$, and $\Delta x = 1$. Then $f'(x) = \frac{1}{4}x^{-3/4}$, $f'(a) = f'(16) = \frac{1}{32}$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x - a) + f(a) = \frac{1}{32}(x - 16) + 2 = \frac{1}{32}x + \frac{3}{2}.$$

Thus, we have $(17)^{1/4} \approx L(17) = 2.03125$. The percentage error in this estimate is

$$\left|\frac{(17)^{1/4} - 2.03125}{(17)^{1/4}}\right| \times 100\% \approx 0.035\%$$

63. $(64.1)^{1/3}$

SOLUTION Let $f(x) = x^{1/3}$, $a = 64$, and $\Delta x = 0.1$. Then $f'(x) = \frac{1}{3}x^{-2/3}$, $f'(a) = f'(64) = \frac{1}{48}$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x - a) + f(a) = \frac{1}{48}(x - 64) + 4 = \frac{1}{48}x + \frac{8}{3}.$$

Thus, we have $(64.1)^{1/3} \approx L(64.1) \approx 4.002083$. The percentage error in this estimate is

$$\left|\frac{(64.1)^{1/3} - 4.002083}{(64.1)^{1/3}}\right| \times 100\% \approx 0.000019\%$$

64. $(1.2)^{5/3}$

SOLUTION Let $f(x) = (1 + x)^{5/3}$ and $a = 0$. Then $f'(x) = \frac{5}{3}(1 + x)^{2/3}$, $f'(a) = f'(0) = \frac{5}{3}$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x - a) + f(a) = \frac{5}{3}x + 1.$$

Thus, we have $(1.2)^{5/3} \approx L(0.2) = \frac{5}{3}(0.2) + 1 = 1.3333$. The percentage error in this estimate is

$$\left|\frac{(1.2)^{5/3} - 1.3333}{(1.2)^{5/3}}\right| \times 100\% \approx 1.61\%$$

65. $\cos^{-1}(0.52)$

SOLUTION Let $f(x) = \cos^{-1} x$ and $a = 0.5$. Then

$$f'(x) = -\frac{1}{\sqrt{1-x^2}}, \quad f'(a) = f'(0) = -\frac{2\sqrt{3}}{3},$$

and the linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = -\frac{2\sqrt{3}}{3}(x - 0.5) + \frac{\pi}{3}.$$

Thus, we have $\cos^{-1}(0.52) \approx L(0.02) = 1.024104$. The percentage error in this estimate is

$$\left|\frac{\cos^{-1}(0.52) - 1.024104}{\cos^{-1}(0.52)}\right| \times 100\% \approx 0.015\%.$$

66. $\ln 1.07$

SOLUTION Let $f(x) = \ln(1+x)$ and $a = 0$. Then $f'(x) = \frac{1}{1+x}$, $f'(a) = f'(0) = 1$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = x.$$

Thus, we have $\ln 1.07 \approx L(0.07) = 0.07$. The percentage error in this estimate is

$$\left|\frac{\ln 1.07 - 0.07}{\ln 1.07}\right| \times 100\% \approx 3.46\%.$$

67. $e^{-0.012}$

SOLUTION Let $f(x) = e^x$ and $a = 0$. Then $f'(x) = e^x$, $f'(a) = f'(0) = 1$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = 1(x-0) + 1 = x + 1.$$

Thus, we have $e^{-0.012} \approx L(-0.012) = 1 - 0.012 = 0.988$. The percentage error in this estimate is

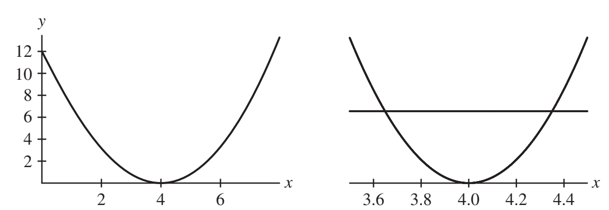
$$\left|\frac{e^{-0.012} - 0.988}{e^{-0.012}}\right| \times 100\% \approx 0.0073\%.$$

68. GU Compute the linearization $L(x)$ of $f(x) = x^2 - x^{3/2}$ at $a = 4$. Then plot $f(x) - L(x)$ and find an interval I around $a = 4$ such that $|f(x) - L(x)| \le 0.1$ for $x \in I$.

SOLUTION Let $f(x) = x^2 - x^{3/2}$ and $a = 4$. Then $f'(x) = 2x - \frac{3}{2}x^{1/2}$, $f'(4) = 5$ and

$$L(x) = f(a) + f'(a)(x-a) = 8 + 5(x-4) = 5x - 12.$$

The graph of $y = f(x) - L(x)$ is shown below at the left, and portions of the graphs of $y = f(x) - L(x)$ and $y = 0.1$ are shown below at the right. From the graph on the right, we see that $|f(x) - L(x)| < 0.1$ roughly for $3.6 < x < 4.4$.



69. Show that the Linear Approximation to $f(x) = \sqrt{x}$ at $x = 9$ yields the estimate $\sqrt{9+h} - 3 \approx \frac{1}{6}h$. Set $K = 0.01$ and show that $|f''(x)| \le K$ for $x \ge 9$. Then verify numerically that the error E satisfies Eq. (5) for $h = 10^{-n}$, for $1 \le n \le 4$.

SOLUTION Let $f(x) = \sqrt{x}$. Then $f(9) = 3$, $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(9) = \frac{1}{6}$. Therefore, by the Linear Approximation,

$$f(9+h) - f(9) = \sqrt{9+h} - 3 \approx \frac{1}{6}h.$$

Moreover, $f''(x) = -\frac{1}{4}x^{-3/2}$, so $|f''(x)| = \frac{1}{4}x^{-3/2}$. Because this is a decreasing function, it follows that for $x \geq 9$,

$$K = \max|f''(x)| \leq |f''(9)| = \frac{1}{108} < 0.01.$$

From the following table, we see that for $h = 10^{-n}$, $1 \leq n \leq 4$, $E \leq \frac{1}{2}Kh^2$.

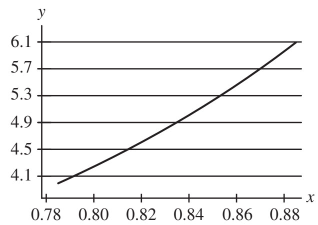
h	$E = \lvert\sqrt{9+h} - 3 - \frac{1}{6}h\rvert$	$\frac{1}{2}Kh^2$
10^{-1}	4.604×10^{-5}	5.00×10^{-5}
10^{-2}	4.627×10^{-7}	5.00×10^{-7}
10^{-3}	4.629×10^{-9}	5.00×10^{-9}
10^{-4}	4.627×10^{-11}	5.00×10^{-11}

70. GU The Linear Approximation to $f(x) = \tan x$ at $x = \frac{\pi}{4}$ yields the estimate $\tan\left(\frac{\pi}{4} + h\right) - 1 \approx 2h$. Set $K = 6.2$ and show, using a plot, that $|f''(x)| \leq K$ for $x \in [\frac{\pi}{4}, \frac{\pi}{4} + 0.1]$. Then verify numerically that the error E satisfies Eq. (5) for $h = 10^{-n}$, for $1 \leq n \leq 4$.

SOLUTION Let $f(x) = \tan x$. Then $f(\frac{\pi}{4}) = 1$, $f'(x) = \sec^2 x$ and $f'(\frac{\pi}{4}) = 2$. Therefore, by the Linear Approximation,

$$f\left(\frac{\pi}{4} + h\right) - f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4} + h\right) - 1 \approx 2h.$$

Moreover, $f''(x) = 2\sec^2 x \tan x$. The graph of the second derivative over the interval $[\pi/4, \pi/4 + 0.1]$ is shown below. From this graph we see that $K = \max|f''(x)| \approx 6.1 < 6.2$.



Finally, from the following table, we see that for $h = 10^{-n}$, $1 \leq n \leq 4$, $E \leq \frac{1}{2}Kh^2$.

h	$E = \lvert\tan(\frac{\pi}{4} + h) - 1 - 2h\rvert$	$\frac{1}{2}Kh^2$
10^{-1}	2.305×10^{-2}	3.10×10^{-2}
10^{-2}	2.027×10^{-4}	3.10×10^{-4}
10^{-3}	2.003×10^{-6}	3.10×10^{-6}
10^{-4}	2.000×10^{-8}	3.10×10^{-8}

Further Insights and Challenges

71. Compute dy/dx at the point $P = (2, 1)$ on the curve $y^3 + 3xy = 7$ and show that the linearization at P is $L(x) = -\frac{1}{3}x + \frac{5}{3}$. Use $L(x)$ to estimate the y-coordinate of the point on the curve where $x = 2.1$.

SOLUTION Differentiating both sides of the equation $y^3 + 3xy = 7$ with respect to x yields

$$3y^2\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0,$$

so

$$\frac{dy}{dx} = -\frac{y}{y^2 + x}.$$

Thus,

$$\left.\frac{dy}{dx}\right|_{(2,1)} = -\frac{1}{1^2 + 2} = -\frac{1}{3},$$

and the linearization at $P = (2, 1)$ is

$$L(x) = 1 - \frac{1}{3}(x - 2) = -\frac{1}{3}x + \frac{5}{3}.$$

Finally, when $x = 2.1$, we estimate that the y-coordinate of the point on the curve is

$$y \approx L(2.1) = -\frac{1}{3}(2.1) + \frac{5}{3} = 0.967.$$

72. Apply the method of Exercise 71 to $P = (0.5, 1)$ on $y^5 + y - 2x = 1$ to estimate the y-coordinate of the point on the curve where $x = 0.55$.

SOLUTION Differentiating both sides of the equation $y^5 + y - 2x = 1$ with respect to x yields

$$5y^4 \frac{dy}{dx} + \frac{dy}{dx} - 2 = 0,$$

so

$$\frac{dy}{dx} = \frac{2}{5y^4 + 1}.$$

Thus,

$$\left.\frac{dy}{dx}\right|_{(0.5,1)} = \frac{2}{5(1)^2 + 1} = \frac{1}{3},$$

and the linearization at $P = (0.5, 1)$ is

$$L(x) = 1 + \frac{1}{3}\left(x - \frac{1}{2}\right) = \frac{1}{3}x + \frac{5}{6}.$$

Finally, when $x = 0.55$, we estimate that the y-coordinate of the point on the curve is

$$y \approx L(0.55) = \frac{1}{3}(0.55) + \frac{5}{6} = 1.017.$$

73. Apply the method of Exercise 71 to $P = (-1, 2)$ on $y^4 + 7xy = 2$ to estimate the solution of $y^4 - 7.7y = 2$ near $y = 2$.

SOLUTION Differentiating both sides of the equation $y^4 + 7xy = 2$ with respect to x yields

$$4y^3 \frac{dy}{dx} + 7x\frac{dy}{dx} + 7y = 0,$$

so

$$\frac{dy}{dx} = -\frac{7y}{4y^3 + 7x}.$$

Thus,

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = -\frac{7(2)}{4(2)^3 + 7(-1)} = -\frac{14}{25},$$

and the linearization at $P = (-1, 2)$ is

$$L(x) = 2 - \frac{14}{25}(x + 1) = -\frac{14}{25}x + \frac{36}{25}.$$

Finally, the equation $y^4 - 7.7y = 2$ corresponds to $x = -1.1$, so we estimate the solution of this equation near $y = 2$ is

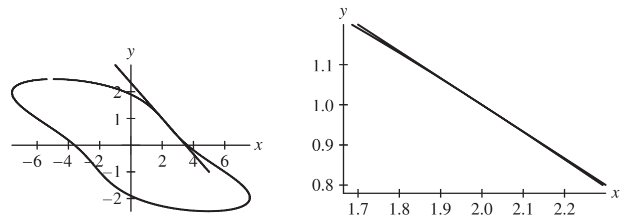
$$y \approx L(-1.1) = -\frac{14}{25}(-1.1) + \frac{36}{25} = 2.056.$$

74. Show that for any real number k, $(1 + \Delta x)^k \approx 1 + k\Delta x$ for small Δx. Estimate $(1.02)^{0.7}$ and $(1.02)^{-0.3}$.

SOLUTION Let $f(x) = (1 + x)^k$. Then for small Δx, we have

$$f(\Delta x) \approx L(\Delta x) = f'(0)(\Delta x - 0) + f(0) = k(1 + 0)^{k-1}(\Delta x - 0) + 1 = 1 + k\Delta x$$

- Let $k = 0.7$ and $\Delta x = 0.02$. Then $L(0.02) = 1 + (0.7)(0.02) = 1.014$.
- Let $k = -0.3$ and $\Delta x = 0.02$. Then $L(0.02) = 1 + (-0.3)(0.02) = 0.994$.



75. Let $\Delta f = f(5+h) - f(5)$, where $f(x) = x^2$. Verify directly that $E = |\Delta f - f'(5)h|$ satisfies (5) with $K = 2$.

SOLUTION Let $f(x) = x^2$. Then

$$\Delta f = f(5+h) - f(5) = (5+h)^2 - 5^2 = h^2 + 10h$$

and

$$E = |\Delta f - f'(5)h| = |h^2 + 10h - 10h| = h^2 = \frac{1}{2}(2)h^2 = \frac{1}{2}Kh^2.$$

76. Let $\Delta f = f(1+h) - f(1)$ where $f(x) = x^{-1}$. Show directly that $E = |\Delta f - f'(1)h|$ is equal to $h^2/(1+h)$. Then prove that $E \le 2h^2$ if $-\frac{1}{2} \le h \le \frac{1}{2}$. *Hint:* In this case, $\frac{1}{2} \le 1 + h \le \frac{3}{2}$.

SOLUTION Let $f(x) = x^{-1}$. Then

$$\Delta f = f(1+h) - f(1) = \frac{1}{1+h} - 1 = -\frac{h}{1+h}$$

and

$$E = |\Delta f - f'(1)h| = \left| -\frac{h}{1+h} + h \right| = \frac{h^2}{1+h}.$$

If $-\frac{1}{2} \le h \le \frac{1}{2}$, then $\frac{1}{2} \le 1 + h \le \frac{3}{2}$ and $\frac{2}{3} \le \frac{1}{1+h} \le 2$. Thus, $E \le 2h^2$ for $-\frac{1}{2} \le h \le \frac{1}{2}$.

4.2 Extreme Values

Preliminary Questions

1. What is the definition of a critical point?

SOLUTION A critical point is a value of the independent variable x in the domain of a function f at which either $f'(x) = 0$ or $f'(x)$ does not exist.

In Questions 2 and 3, choose the correct conclusion.

2. If $f(x)$ is not continuous on [0, 1], then

(a) $f(x)$ has no extreme values on [0, 1].

(b) $f(x)$ might not have any extreme values on [0, 1].

SOLUTION The correct response is **(b)**: $f(x)$ might not have any extreme values on [0, 1]. Although [0, 1] is closed, because f is not continuous, the function is not guaranteed to have any extreme values on [0, 1].

3. If $f(x)$ is continuous but has no critical points in [0, 1], then

(a) $f(x)$ has no min or max on [0, 1].

(b) Either $f(0)$ or $f(1)$ is the minimum value on [0, 1].

SOLUTION The correct response is **(b)**: either $f(0)$ or $f(1)$ is the minimum value on [0, 1]. Remember that extreme values occur either at critical points or endpoints. If a continuous function on a closed interval has no critical points, the extreme values must occur at the endpoints.

4. Fermat's Theorem *does not* claim that if $f'(c) = 0$, then $f(c)$ is a local extreme value (this is false). What *does* Fermat's Theorem assert?

SOLUTION Fermat's Theorem claims: If $f(c)$ is a local extreme value, then either $f'(c) = 0$ or $f'(c)$ does not exist.

Exercises

1. The following questions refer to Figure 15.

(a) How many critical points does $f(x)$ have on [0, 8]?

(b) What is the maximum value of $f(x)$ on [0, 8]?

(c) What are the local maximum values of $f(x)$?

(d) Find a closed interval on which both the minimum and maximum values of $f(x)$ occur at critical points.

(e) Find an interval on which the minimum value occurs at an endpoint.

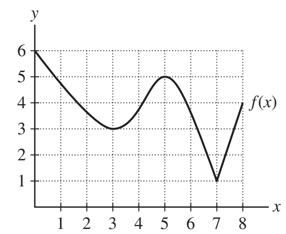


FIGURE 15

SOLUTION

(a) $f(x)$ has three critical points on the interval [0, 8]: at $x = 3$, $x = 5$ and $x = 7$. Two of these, $x = 3$ and $x = 5$, are where the derivative is zero and one, $x = 7$, is where the derivative does not exist.

(b) The maximum value of $f(x)$ on [0, 8] is 6; the function takes this value at $x = 0$.

(c) $f(x)$ achieves a local maximum of 5 at $x = 5$.

(d) Answers may vary. One example is the interval [4, 8]. Another is [2, 6].

(e) Answers may vary. The easiest way to ensure this is to choose an interval on which the graph takes no local minimum. One example is [0, 2].

2. State whether $f(x) = x^{-1}$ (Figure 16) has a minimum or maximum value on the following intervals:

(a) $(0, 2)$ **(b)** $(1, 2)$ **(c)** $[1, 2]$

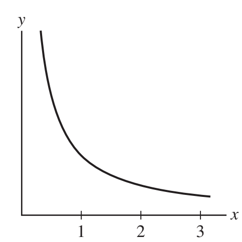


FIGURE 16 Graph of $f(x) = x^{-1}$.

SOLUTION $f(x)$ has no local minima or maxima. Hence, $f(x)$ only takes minimum and maximum values on an interval if it takes them at the endpoints.

(a) $f(x)$ takes no minimum or maximum value on this interval, since the interval does not contain its endpoints.

(b) $f(x)$ takes no minimum or maximum value on this interval, since the interval does not contain its endpoints.

(c) The function is decreasing on the whole interval [1, 2]. Hence, $f(x)$ takes on its maximum value of 1 at $x = 1$ and $f(x)$ takes on its minimum value of $\frac{1}{2}$ at $x = 2$.

In Exercises 3–20, find all critical points of the function.

3. $f(x) = x^2 - 2x + 4$

SOLUTION Let $f(x) = x^2 - 2x + 4$. Then $f'(x) = 2x - 2 = 0$ implies that $x = 1$ is the lone critical point of f.

4. $f(x) = 7x - 2$

SOLUTION Let $f(x) = 7x - 2$. Then $f'(x) = 7$, which is never zero, so $f(x)$ has no critical points.

5. $f(x) = x^3 - \frac{9}{2}x^2 - 54x + 2$

SOLUTION Let $f(x) = x^3 - \frac{9}{2}x^2 - 54x + 2$. Then $f'(x) = 3x^2 - 9x - 54 = 3(x + 3)(x - 6) = 0$ implies that $x = -3$ and $x = 6$ are the critical points of f.

6. $f(t) = 8t^3 - t^2$

SOLUTION Let $f(t) = 8t^3 - t^2$. Then $f'(t) = 24t^2 - 2t = 2t(12t - 1) = 0$ implies that $t = 0$ and $t = \frac{1}{12}$ are the critical points of f.

7. $f(x) = x^{-1} - x^{-2}$

SOLUTION Let $f(x) = x^{-1} - x^{-2}$. Then

$$f'(x) = -x^{-2} + 2x^{-3} = \frac{2-x}{x^3} = 0$$

implies that $x = 2$ is the only critical point of f. Though $f'(x)$ does not exist at $x = 0$, this is not a critical point of f because $x = 0$ is not in the domain of f.

8. $g(z) = \dfrac{1}{z-1} - \dfrac{1}{z}$

SOLUTION Let

$$g(z) = \frac{1}{z-1} - \frac{1}{z} = \frac{z-(z-1)}{z(z-1)} = \frac{1}{z^2-z}.$$

Then

$$g'(z) = -\frac{1}{(z^2-z)^2}(2z-1) = -\frac{2z-1}{(z^2-z)^2} = 0$$

implies that $z = 1/2$ is the only critical point of g. Though $g'(z)$ does not exist at either $z = 0$ or $z = 1$, neither is a critical point of g because neither is in the domain of g.

9. $f(x) = \dfrac{x}{x^2+1}$

SOLUTION Let $f(x) = \dfrac{x}{x^2+1}$. Then $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} = 0$ implies that $x = \pm 1$ are the critical points of f.

10. $f(x) = \dfrac{x^2}{x^2-4x+8}$

SOLUTION Let $f(x) = \dfrac{x^2}{x^2-4x+8}$. Then

$$f'(x) = \frac{(x^2-4x+8)(2x) - x^2(2x-4)}{(x^2-4x+8)^2} = \frac{4x(4-x)}{(x^2-4x+8)^2} = 0$$

implies that $x = 0$ and $x = 4$ are the critical points of f.

11. $f(t) = t - 4\sqrt{t+1}$

SOLUTION Let $f(t) = t - 4\sqrt{t+1}$. Then

$$f'(t) = 1 - \frac{2}{\sqrt{t+1}} = 0$$

implies that $t = 3$ is a critical point of f. Because $f'(t)$ does not exist at $t = -1$, this is another critical point of f.

12. $f(t) = 4t - \sqrt{t^2+1}$

SOLUTION Let $f(t) = 4t - \sqrt{t^2+1}$. Then

$$f'(t) = 4 - \frac{t}{(t^2+1)^{1/2}} = \frac{4(t^2+1)^{1/2} - t}{(t^2+1)^{1/2}} = 0$$

implies that there are no critical points of f since neither the numerator nor denominator equals 0 for any value of t.

13. $f(x) = x^2\sqrt{1-x^2}$

SOLUTION Let $f(x) = x^2\sqrt{1-x^2}$. Then

$$f'(x) = -\frac{x^3}{\sqrt{1-x^2}} + 2x\sqrt{1-x^2} = \frac{2x-3x^3}{\sqrt{1-x^2}}.$$

This derivative is 0 when $x = 0$ and when $x = \pm\sqrt{2/3}$; the derivative does not exist when $x = \pm 1$. All five of these values are critical points of f

14. $f(x) = x + |2x + 1|$

SOLUTION Removing the absolute values, we see that

$$f(x) = \begin{cases} -x - 1, & x < -\frac{1}{2} \\ 3x + 1, & x \geq -\frac{1}{2} \end{cases}$$

Thus,

$$f'(x) = \begin{cases} -1, & x < -\frac{1}{2} \\ 3, & x \geq -\frac{1}{2} \end{cases}$$

and we see that $f'(0)$ is never equal to 0. However, $f'(-1/2)$ does not exist, so $x = -1/2$ is the only critical point of f.

15. $g(\theta) = \sin^2 \theta$

SOLUTION Let $g(\theta) = \sin^2 \theta$. Then $g'(\theta) = 2 \sin \theta \cos \theta = \sin 2\theta = 0$ implies that

$$\theta = \frac{n\pi}{2}$$

is a critical value of g for all integer values of n.

16. $R(\theta) = \cos \theta + \sin^2 \theta$

SOLUTION Let $R(\theta) = \cos \theta + \sin^2 \theta$. Then

$$R'(\theta) = -\sin \theta + 2 \sin \theta \cos \theta = \sin \theta (2 \cos \theta - 1) = 0$$

implies that $\theta = n\pi$,

$$\theta = \frac{\pi}{3} + 2n\pi \quad \text{and} \quad \theta = \frac{5\pi}{3} + 2n\pi$$

are critical points of R for all integer values of n.

17. $f(x) = x \ln x$

SOLUTION Let $f(x) = x \ln x$. Then $f'(x) = 1 + \ln x = 0$ implies that $x = e^{-1} = \frac{1}{e}$ is the only critical point of f.

18. $f(x) = xe^{2x}$

SOLUTION Let $f(x) = xe^{2x}$. Then $f'(x) = (2x + 1)e^{2x} = 0$ implies that $x = -\frac{1}{2}$ is the only critical point of f.

19. $f(x) = \sin^{-1} x - 2x$

SOLUTION Let $f(x) = \sin^{-1} x - 2x$. Then

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - 2 = 0$$

implies that $x = \pm\frac{\sqrt{3}}{2}$ are the critical points of f.

20. $f(x) = \sec^{-1} x - \ln x$

SOLUTION Let $f(x) = \sec^{-1} x - \ln x$. Then

$$f'(x) = \frac{1}{x\sqrt{x^2 - 1}} - \frac{1}{x}.$$

This derivative is equal to zero when $\sqrt{x^2 - 1} = 1$, or when $x = \pm\sqrt{2}$. Moreover, the derivative does not exist at $x = 0$ and at $x = \pm 1$. Among these numbers, $x = 1$ and $x = \sqrt{2}$ are the only critical points of f. $x = -\sqrt{2}$, $x = -1$, and $x = 0$ are not critical points of f because none are in the domain of f.

21. Let $f(x) = x^2 - 4x + 1$.

(a) Find the critical point c of $f(x)$ and compute $f(c)$.

(b) Compute the value of $f(x)$ at the endpoints of the interval $[0, 4]$.

(c) Determine the min and max of $f(x)$ on $[0, 4]$.

(d) Find the extreme values of $f(x)$ on $[0, 1]$.

SOLUTION Let $f(x) = x^2 - 4x + 1$.

(a) Then $f'(c) = 2c - 4 = 0$ implies that $c = 2$ is the sole critical point of f. We have $f(2) = -3$.

(b) $f(0) = f(4) = 1$.

(c) Using the results from (a) and (b), we find the maximum value of f on $[0, 4]$ is 1 and the minimum value is -3.

(d) We have $f(1) = -2$. Hence the maximum value of f on $[0, 1]$ is 1 and the minimum value is -2.

22. Find the extreme values of $f(x) = 2x^3 - 9x^2 + 12x$ on [0, 3] and [0, 2].

SOLUTION Let $f(x) = 2x^3 - 9x^2 + 12x$. First, we find the critical points. Setting $f'(x) = 6x^2 - 18x + 12 = 0$ yields $x^2 - 3x + 2 = 0$ so that $x = 2$ or $x = 1$. Next, we compare: first, for [0, 3]:

x-value	Value of f	
1 (critical point)	$f(1) = 5$	
2 (critical point)	$f(2) = 4$	
0 (endpoint)	$f(0) = 0$	min
3 (endpoint)	$f(3) = 9$	max

Then, for [0, 2]:

x-value	Value of f	
1 (critical point)	$f(1) = 5$	max
2 (endpoint)	$f(2) = 4$	
0 (endpoint)	$f(0) = 0$	min

23. Find the critical points of $f(x) = \sin x + \cos x$ and determine the extreme values on $\left[0, \frac{\pi}{2}\right]$.

SOLUTION

- Let $f(x) = \sin x + \cos x$. Then on the interval $\left[0, \frac{\pi}{2}\right]$, we have $f'(x) = \cos x - \sin x = 0$ at $x = \frac{\pi}{4}$, the only critical point of f in this interval.
- Since $f(\frac{\pi}{4}) = \sqrt{2}$ and $f(0) = f(\frac{\pi}{2}) = 1$, the maximum value of f on $\left[0, \frac{\pi}{2}\right]$ is $\sqrt{2}$, while the minimum value is 1.

24. Compute the critical points of $h(t) = (t^2 - 1)^{1/3}$. Check that your answer is consistent with Figure 17. Then find the extreme values of $h(t)$ on [0, 1] and [0, 2].

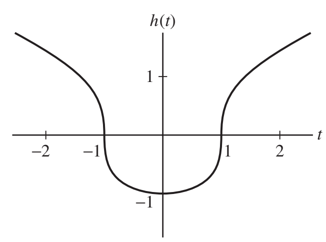


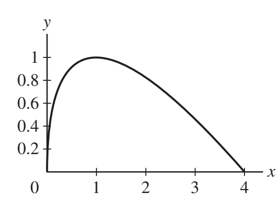
FIGURE 17 Graph of $h(t) = (t^2 - 1)^{1/3}$.

SOLUTION

- Let $h(t) = (t^2 - 1)^{1/3}$. Then $h'(t) = \dfrac{2t}{3(t^2 - 1)^{2/3}} = 0$ implies critical points at $t = 0$ and $t = \pm 1$. These results are consistent with Figure 17 which shows a horizontal tangent at $t = 0$ and vertical tangents at $t = \pm 1$.
- Since $h(0) = -1$ and $h(1) = 0$, the maximum value on [0, 1] is $h(1) = 0$ and the minimum is $h(0) = -1$. Similarly, the minimum on [0, 2] is $h(0) = -1$ and the maximum is $h(2) = 3^{1/3} \approx 1.44225$.

25. GU Plot $f(x) = 2\sqrt{x} - x$ on [0, 4] and determine the maximum value graphically. Then verify your answer using calculus.

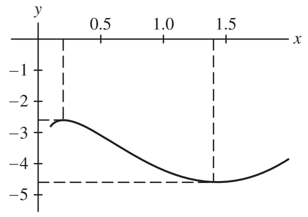
SOLUTION The graph of $y = 2\sqrt{x} - x$ over the interval [0, 4] is shown below. From the graph, we see that at $x = 1$, the function achieves its maximum value of 1.



To verify the information obtained from the plot, let $f(x) = 2\sqrt{x} - x$. Then $f'(x) = x^{-1/2} - 1$. Solving $f'(x) = 0$ yields the critical points $x = 0$ and $x = 1$. Because $f(0) = f(4) = 0$ and $f(1) = 1$, we see that the maximum value of f on $[0, 4]$ is 1.

26. GU Plot $f(x) = \ln x - 5\sin x$ on $[0.1, 2]$ and approximate both the critical points and the extreme values.

SOLUTION The graph of $f(x) = \ln x - 5\sin x$ is shown below. From the graph, we see that critical points occur at approximately $x = 0.2$ and $x = 1.4$. The maximum value of approximately -2.6 occurs at $x \approx 0.2$; the minimum value of approximately -4.6 occurs at $x \approx 1.4$.



27. CAS Approximate the critical points of $g(x) = x\cos^{-1} x$ and estimate the maximum value of $g(x)$.

SOLUTION $g'(x) = \frac{-x}{\sqrt{1-x^2}} + \cos^{-1} x$, so $g'(x) = 0$ when $x \approx 0.652185$. Evaluating g at the endpoints of its domain, $x = \pm 1$, and at the critical point $x \approx 0.652185$, we find $g(-1) = -\pi$, $g(0.652185) \approx 0.561096$, and $g(1) = 0$. Hence, the maximum value of $g(x)$ is approximately 0.561096.

28. CAS Approximate the critical points of $g(x) = 5e^x - \tan x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

SOLUTION Let $g(x) = 5e^x - \tan x$. Then $g'(x) = 5e^x - \sec^2 x$. The derivative is defined for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and is equal to 0 for $x \approx 1.339895$ and $x \approx -0.82780$. Hence, the critical points of g are $x \approx 1.339895$ and $x \approx -0.82780$.

In Exercises 29–58, find the min and max of the function on the given interval by comparing values at the critical points and endpoints.

29. $y = 2x^2 + 4x + 5$, $[-2, 2]$

SOLUTION Let $f(x) = 2x^2 + 4x + 5$. Then $f'(x) = 4x + 4 = 0$ implies that $x = -1$ is the only critical point of f. The minimum of f on the interval $[-2, 2]$ is $f(-1) = 3$, whereas its maximum is $f(2) = 21$. (*Note:* $f(-2) = 5$.)

30. $y = 2x^2 + 4x + 5$, $[0, 2]$

SOLUTION Let $f(x) = 2x^2 + 4x + 5$. Then $f'(x) = 4x + 4 = 0$ implies that $x = -1$ is the only critical point of f. The minimum of f on the interval $[0, 2]$ is $f(0) = 5$, whereas its maximum is $f(2) = 21$. (*Note:* The critical point $x = -1$ is not on the interval $[0, 2]$.)

31. $y = 6t - t^2$, $[0, 5]$

SOLUTION Let $f(t) = 6t - t^2$. Then $f'(t) = 6 - 2t = 0$ implies that $t = 3$ is the only critical point of f. The minimum of f on the interval $[0, 5]$ is $f(0) = 0$, whereas the maximum is $f(3) = 9$. (*Note:* $f(5) = 5$.)

32. $y = 6t - t^2$, $[4, 6]$

SOLUTION Let $f(t) = 6t - t^2$. Then $f'(t) = 6 - 2t = 0$ implies that $t = 3$ is the only critical point of f. The minimum of f on the interval $[4, 6]$ is $f(6) = 0$, whereas the maximum is $f(4) = 8$. (*Note:* The critical point $t = 3$ is not on the interval $[4, 6]$.)

33. $y = x^3 - 6x^2 + 8$, $[1, 6]$

SOLUTION Let $f(x) = x^3 - 6x^2 + 8$. Then $f'(x) = 3x^2 - 12x = 3x(x - 4) = 0$ implies that $x = 0$ and $x = 4$ are the critical points of f. The minimum of f on the interval $[1, 6]$ is $f(4) = -24$, whereas the maximum is $f(6) = 8$. (*Note:* $f(1) = 3$ and the critical point $x = 0$ is not in the interval $[1, 6]$.)

34. $y = x^3 + x^2 - x$, $[-2, 2]$

SOLUTION Let $f(x) = x^3 + x^2 - x$. Then $f'(x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1) = 0$ implies that $x = 1/3$ and $x = -1$ are critical points of f. The minimum of f on the interval $[-2, 2]$ is $f(-2) = -2$, whereas the maximum is $f(2) = 10$. (*Note:* $f(-1) = 1$ and $f(1/3) = -5/27$.)

35. $y = 2t^3 + 3t^2$, $[1, 2]$

SOLUTION Let $f(t) = 2t^3 + 3t^2$. Then $f'(t) = 6t^2 + 6t = 6t(t + 1) = 0$ implies that $t = 0$ and $t = -1$ are the critical points of f. The minimum of f on the interval $[1, 2]$ is $f(1) = 5$, whereas the maximum is $f(2) = 28$. (*Note:* Neither critical points are in the interval $[1, 2]$.)

36. $y = x^3 - 12x^2 + 21x$, $[0, 2]$

SOLUTION Let $f(x) = x^3 - 12x^2 + 21x$. Then $f'(x) = 3x^2 - 24x + 21 = 3(x - 7)(x - 1) = 0$ implies that $x = 1$ and $x = 7$ are the critical points of f. The minimum of f on the interval $[0, 2]$ is $f(0) = 0$, whereas its maximum is $f(1) = 10$. (*Note:* $f(2) = 2$ and the critical point $x = 7$ is not in the interval $[0, 2]$.)

37. $y = z^5 - 80z$, $[-3, 3]$

SOLUTION Let $f(z) = z^5 - 80z$. Then $f'(z) = 5z^4 - 80 = 5(z^4 - 16) = 5(z^2 + 4)(z + 2)(z - 2) = 0$ implies that $z = \pm 2$ are the critical points of f. The minimum value of f on the interval $[-3, 3]$ is $f(2) = -128$, whereas the maximum is $f(-2) = 128$. (*Note:* $f(-3) = 3$ and $f(3) = -3$.)

38. $y = 2x^5 + 5x^2$, $[-2, 2]$

SOLUTION Let $f(x) = 2x^5 + 5x^2$. Then $f'(x) = 10x^4 + 10x = 10x(x^3 + 1) = 0$ implies that $x = 0$ and $x = -1$ are critical points of f. The minimum value of f on the interval $[-2, 2]$ is $f(-2) = -44$, whereas the maximum is $f(2) = 84$. (*Note:* $f(-1) = 3$ and $f(0) = 0$.)

39. $y = \dfrac{x^2 + 1}{x - 4}$, $[5, 6]$

SOLUTION Let $f(x) = \dfrac{x^2 + 1}{x - 4}$. Then

$$f'(x) = \frac{(x - 4) \cdot 2x - (x^2 + 1) \cdot 1}{(x - 4)^2} = \frac{x^2 - 8x - 1}{(x - 4)^2} = 0$$

implies $x = 4 \pm \sqrt{17}$ are critical points of f. $x = 4$ is not a critical point because $x = 4$ is not in the domain of f. On the interval $[5, 6]$, the minimum of f is $f(6) = \frac{37}{2} = 18.5$, whereas the maximum of f is $f(5) = 26$. (*Note:* The critical points $x = 4 \pm \sqrt{17}$ are not in the interval $[5, 6]$.)

40. $y = \dfrac{1 - x}{x^2 + 3x}$, $[1, 4]$

SOLUTION Let $f(x) = \dfrac{1 - x}{x^2 + 3x}$. Then

$$f'(x) = \frac{-(x^2 + 3x) - (1 - x)(2x + 3)}{(x^2 + 3x)^2} = \frac{(x - 3)(x + 1)}{(x^2 + 3x)^2} = 0$$

implies that $x = 3$ and $x = -1$ are critical points. Neither $x = 0$ nor $x = -3$ is a critical point because neither is in the domain of f. On the interval $[1, 4]$, the maximum value is $f(1) = 0$ and the minimum value is $f(3) = -\frac{1}{9}$. (*Note:* The critical point $x = -1$ is not in the interval $[1, 4]$.)

41. $y = x - \dfrac{4x}{x + 1}$, $[0, 3]$

SOLUTION Let $f(x) = x - \dfrac{4x}{x + 1}$. Then

$$f'(x) = 1 - \frac{4}{(x + 1)^2} = \frac{(x - 1)(x + 3)}{(x + 1)^2} = 0$$

implies that $x = 1$ and $x = -3$ are critical points of f. $x = -1$ is not a critical point because $x = -1$ is not in the domain of f. The minimum of f on the interval $[0, 3]$ is $f(1) = -1$, whereas the maximum is $f(0) = f(3) = 0$. (*Note:* The critical point $x = -3$ is not in the interval $[0, 3]$.)

42. $y = 2\sqrt{x^2 + 1} - x$, $[0, 2]$

SOLUTION Let $f(x) = 2\sqrt{x^2 + 1} - x$. Then

$$f'(x) = \frac{2x}{\sqrt{x^2 + 1}} - 1 = 0$$

implies that $x = \pm\sqrt{\frac{1}{3}}$ are critical points of f. On the interval $[0, 2]$, the minimum is $f\left(\sqrt{\frac{1}{3}}\right) = \sqrt{3}$ and the maximum is $f(2) = 2\sqrt{5} - 2$. (*Note:* The critical point $x = -\sqrt{\frac{1}{3}}$ is not in the interval $[0, 2]$.)

43. $y = (2+x)\sqrt{2+(2-x)^2}$, $[0, 2]$

SOLUTION Let $f(x) = (2+x)\sqrt{2+(2-x)^2}$. Then

$$f'(x) = \sqrt{2+(2-x)^2} - (2+x)(2+(2-x)^2)^{-1/2}(2-x) = \frac{2(x-1)^2}{\sqrt{2+(2-x)^2}} = 0$$

implies that $x = 1$ is the critical point of f. On the interval $[0, 2]$, the minimum is $f(0) = 2\sqrt{6} \approx 4.9$ and the maximum is $f(2) = 4\sqrt{2} \approx 5.66$. (*Note:* $f(1) = 3\sqrt{3} \approx 5.2$.)

44. $y = \sqrt{1+x^2} - 2x$, $[0, 1]$

SOLUTION Let $f(x) = \sqrt{1+x^2} - 2x$. Then

$$f'(x) = \frac{x}{\sqrt{1+x^2}} - 2 = 0$$

implies that f has no critical points. The minimum value of f on the interval $[0, 1]$ is $f(1) = \sqrt{2} - 2$, whereas the maximum is $f(0) = 1$.

45. $y = \sqrt{x+x^2} - 2\sqrt{x}$, $[0, 4]$

SOLUTION Let $f(x) = \sqrt{x+x^2} - 2\sqrt{x}$. Then

$$f'(x) = \frac{1}{2}(x+x^2)^{-1/2}(1+2x) - x^{-1/2} = \frac{1+2x-2\sqrt{1+x}}{2\sqrt{x}\sqrt{1+x}} = 0$$

implies that $x = 0$ and $x = \frac{\sqrt{3}}{2}$ are the critical points of f. Neither $x = -1$ nor $x = -\frac{\sqrt{3}}{2}$ is a critical point because neither is in the domain of f. On the interval $[0, 4]$, the minimum of f is $f\left(\frac{\sqrt{3}}{2}\right) \approx -0.589980$ and the maximum is $f(4) \approx 0.472136$. (*Note:* $f(0) = 0$.)

46. $y = (t - t^2)^{1/3}$, $[-1, 2]$

SOLUTION Let $s(t) = (t-t^2)^{1/3}$. Then $s'(t) = \frac{1}{3}(t-t^2)^{-2/3}(1-2t) = 0$ at $t = \frac{1}{2}$, a critical point of s. Other critical points of s are $t = 0$ and $t = 1$, where the derivative of s does not exist. Therefore, on the interval $[-1, 2]$, the minimum of s is $s(-1) = s(2) = -2^{1/3} \approx -1.26$ and the maximum is $s(\frac{1}{2}) = (\frac{1}{4})^{1/3} \approx 0.63$. (*Note:* $s(0) = s(1) = 0$.)

47. $y = \sin x \cos x$, $\left[0, \frac{\pi}{2}\right]$

SOLUTION Let $f(x) = \sin x \cos x = \frac{1}{2}\sin 2x$. On the interval $\left[0, \frac{\pi}{2}\right]$, $f'(x) = \cos 2x = 0$ when $x = \frac{\pi}{4}$. The minimum of f on this interval is $f(0) = f(\frac{\pi}{2}) = 0$, whereas the maximum is $f(\frac{\pi}{4}) = \frac{1}{2}$.

48. $y = x + \sin x$, $[0, 2\pi]$

SOLUTION Let $f(x) = x + \sin x$. Then $f'(x) = 1 + \cos x = 0$ implies that $x = \pi$ is the only critical point on $[0, 2\pi]$. The minimum value of f on the interval $[0, 2\pi]$ is $f(0) = 0$, whereas the maximum is $f(2\pi) = 2\pi$. (*Note:* $f(\pi) = \pi - 1$.)

49. $y = \sqrt{2}\,\theta - \sec\theta$, $\left[0, \frac{\pi}{3}\right]$

SOLUTION Let $f(\theta) = \sqrt{2}\theta - \sec\theta$. On the interval $[0, \frac{\pi}{3}]$, $f'(\theta) = \sqrt{2} - \sec\theta\tan\theta = 0$ at $\theta = \frac{\pi}{4}$. The minimum value of f on this interval is $f(0) = -1$, whereas the maximum value over this interval is $f(\frac{\pi}{4}) = \sqrt{2}(\frac{\pi}{4} - 1) \approx -0.303493$. (*Note:* $f(\frac{\pi}{3}) = \sqrt{2}\frac{\pi}{3} - 2 \approx -.519039$.)

50. $y = \cos\theta + \sin\theta$, $[0, 2\pi]$

SOLUTION Let $f(\theta) = \cos\theta + \sin\theta$. On the interval $[0, 2\pi]$, $f'(\theta) = -\sin\theta + \cos\theta = 0$ where $\sin\theta = \cos\theta$, which is at the two points $\theta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$. The minimum value on the interval is $f(\frac{5\pi}{4}) = -\sqrt{2}$, whereas the maximum value on the interval is $f(\frac{\pi}{4}) = \sqrt{2}$. (*Note:* $f(0) = f(2\pi) = 1$.)

51. $y = \theta - 2\sin\theta$, $[0, 2\pi]$

SOLUTION Let $g(\theta) = \theta - 2\sin\theta$. On the interval $[0, 2\pi]$, $g'(\theta) = 1 - 2\cos\theta = 0$ at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5}{3}\pi$. The minimum of g on this interval is $g(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3} \approx -.685$ and the maximum is $g(\frac{5}{3}\pi) = \frac{5}{3}\pi + \sqrt{3} \approx 6.968$. (*Note:* $g(0) = 0$ and $g(2\pi) = 2\pi \approx 6.283$.)

52. $y = 4\sin^3\theta - 3\cos^2\theta$, $[0, 2\pi]$

SOLUTION Let $f(\theta) = 4\sin^3\theta - 3\cos^2\theta$. Then

$$\begin{aligned} f'(\theta) &= 12\sin^2\theta\cos\theta + 6\cos\theta\sin\theta \\ &= 6\cos\theta\sin\theta(2\sin\theta + 1) = 0 \end{aligned}$$

yields $\theta = 0, \pi/2, \pi, 7\pi/6, 3\pi/2, 11\pi/6, 2\pi$ as critical points of f. The minimum value of f on the interval $[0, 2\pi]$ is $f(3\pi/2) = -4$, whereas the maximum is $f(\pi/2) = 4$. (*Note:* $f(0) = f(\pi) = f(2\pi) = -3$ and $f(7\pi/6) = f(11\pi/6) = -11/4$.)

53. $y = \tan x - 2x, \quad [0, 1]$

SOLUTION Let $f(x) = \tan x - 2x$. Then on the interval $[0, 1]$, $f'(x) = \sec^2 x - 2 = 0$ at $x = \frac{\pi}{4}$. The minimum of f is $f(\frac{\pi}{4}) = 1 - \frac{\pi}{2} \approx -0.570796$ and the maximum is $f(0) = 0$. (*Note:* $f(1) = \tan 1 - 2 \approx -0.442592$.)

54. $y = xe^{-x}, \quad [0, 2]$

SOLUTION Let $f(x) = xe^{-x}$. Then, on the interval $[0, 2]$, $f'(x) = -xe^{-x} + e^{-x} = (1 - x)e^{-x} = 0$ at $x = 1$. The minimum of f on this interval is $f(0) = 0$ and the maximum is $f(1) = e^{-1} \approx 0.367879$. (*Note:* $f(2) = 2e^{-2} \approx 0.270671$.)

55. $y = \dfrac{\ln x}{x}, \quad [1, 3]$

SOLUTION Let $f(x) = \frac{\ln x}{x}$. Then, on the interval $[1, 3]$,

$$f'(x) = \frac{1 - \ln x}{x^2} = 0$$

at $x = e$. The minimum of f on this interval is $f(1) = 0$ and the maximum is $f(e) = e^{-1} \approx 0.367879$. (*Note:* $f(3) = \frac{1}{3}\ln 3 \approx 0.366204$.)

56. $y = 3e^x - e^{2x}, \quad \left[-\frac{1}{2}, 1\right]$

SOLUTION Let $f(x) = 3e^x - e^{2x}$. Then, on the interval $\left[-\frac{1}{2}, 1\right]$, $f'(x) = 3e^x - 2e^{2x} = e^x(3 - 2e^x) = 0$ at $x = \ln(3/2)$. The minimum of f on this interval is $f(1) = 3e - e^2 \approx 0.765789$ and the maximum is $f(\ln(3/2)) = 2.25$. (*Note:* $f(-1/2) = 3e^{-1/2} - e^{-1} \approx 1.451713$.)

57. $y = 5\tan^{-1} x - x, \quad [1, 5]$

SOLUTION Let $f(x) = 5\tan^{-1} x - x$. Then, on the interval $[1, 5]$,

$$f'(x) = 5\frac{1}{1 + x^2} - 1 = 0$$

at $x = 2$. The minimum of f on this interval is $f(5) = 5\tan^{-1} 5 - 5 \approx 1.867004$ and the maximum is $f(2) = 5\tan^{-1} 2 - 2 \approx 3.535744$. (*Note:* $f(1) = \frac{5\pi}{4} - 1 \approx 2.926991$.)

58. $y = x^3 - 24\ln x, \quad \left[\frac{1}{2}, 3\right]$

SOLUTION Let $f(x) = x^3 - 24\ln x$. Then, on the interval $\left[\frac{1}{2}, 3\right]$,

$$f'(x) = 3x^2 - \frac{24}{x} = 0$$

at $x = 2$. The minimum of f on this interval is $f(2) = 8 - 24\ln 2 \approx -8.635532$ and the maximum is $f(1/2) = \frac{1}{8} + 24\ln 2 \approx 16.760532$. (*Note:* $f(3) = 27 - 24\ln 2 \approx 0.633305$.)

59. Let $f(\theta) = 2\sin 2\theta + \sin 4\theta$.

(a) Show that θ is a critical point if $\cos 4\theta = -\cos 2\theta$.

(b) Show, using a unit circle, that $\cos\theta_1 = -\cos\theta_2$ if and only if $\theta_1 = \pi \pm \theta_2 + 2\pi k$ for an integer k.

(c) Show that $\cos 4\theta = -\cos 2\theta$ if and only if $\theta = \frac{\pi}{2} + \pi k$ or $\theta = \frac{\pi}{6} + \left(\frac{\pi}{3}\right)k$.

(d) Find the six critical points of $f(\theta)$ on $[0, 2\pi]$ and find the extreme values of $f(\theta)$ on this interval.

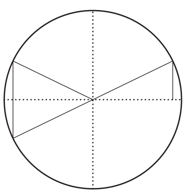
(e) GU Check your results against a graph of $f(\theta)$.

SOLUTION $f(\theta) = 2\sin 2\theta + \sin 4\theta$ is differentiable at all θ, so the way to find the critical points is to find all points such that $f'(\theta) = 0$.

(a) $f'(\theta) = 4\cos 2\theta + 4\cos 4\theta$. If $f'(\theta) = 0$, then $4\cos 4\theta = -4\cos 2\theta$, so $\cos 4\theta = -\cos 2\theta$.

(b) Given the point $(\cos\theta, \sin\theta)$ at angle θ on the unit circle, there are two points with x coordinate $-\cos\theta$. The graphic shows these two points, which are:

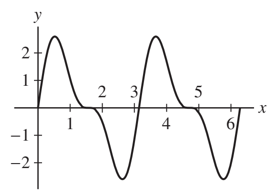
- The point $(\cos(\theta + \pi), \sin(\theta + \pi))$ on the opposite end of the unit circle.
- The point $(\cos(\pi - \theta), \sin(\theta - \pi))$ obtained by reflecting through the y axis.



If we include all angles representing these points on the circle, we find that $\cos\theta_1 = -\cos\theta_2$ if and only if $\theta_1 = (\pi + \theta_2) + 2\pi k$ or $\theta_1 = (\pi - \theta_2) + 2\pi k$ for integers k.
(c) Using (b), we recognize that $\cos 4\theta = -\cos 2\theta$ if $4\theta = 2\theta + \pi + 2\pi k$ or $4\theta = \pi - 2\theta + 2\pi k$. Solving for θ, we obtain $\theta = \frac{\pi}{2} + k\pi$ or $\theta = \frac{\pi}{6} + \frac{\pi}{3}k$.
(d) To find all θ, $0 \le \theta < 2\pi$ indicated by (c), we use the following table:

k	0	1	2	3	4	5
$\frac{\pi}{2} + k\pi$	$\frac{\pi}{2}$	$\frac{3\pi}{2}$				
$\frac{\pi}{6} + \frac{\pi}{3}k$	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$

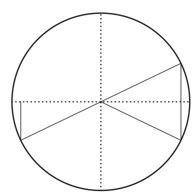
The critical points in the range $[0, 2\pi]$ are $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$, and $\frac{11\pi}{6}$. On this interval, the maximum value is $f(\frac{\pi}{6}) = f(\frac{7\pi}{6}) = \frac{3\sqrt{3}}{2}$ and the minimum value is $f(\frac{5\pi}{6}) = f(\frac{11\pi}{6}) = -\frac{3\sqrt{3}}{2}$.
(e) The graph of $f(\theta) = 2\sin 2\theta + \sin 4\theta$ is shown here:



We can see that there are six flat points on the graph between 0 and 2π, as predicted. There are 4 local extrema, and two points at $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$ where the graph has neither a local maximum nor a local minimum.

60. GU Find the critical points of $f(x) = 2\cos 3x + 3\cos 2x$ in $[0, 2\pi]$. Check your answer against a graph of $f(x)$.

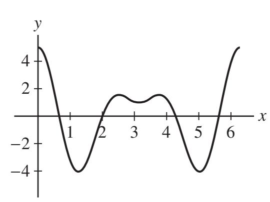
SOLUTION $f(x)$ is differentiable for all x, so we are looking for points where $f'(x) = 0$ only. Setting $f'(x) = -6\sin 3x - 6\sin 2x$, we get $\sin 3x = -\sin 2x$. Looking at a unit circle, we find the relationship between angles y and x such that $\sin y = -\sin x$. This technique is also used in Exercise 59.



From the diagram, we see that $\sin y = -\sin x$ if y is either (i.) the point antipodal to x ($y = \pi + x + 2\pi k$) or (ii.) the point obtained by reflecting x through the horizontal axis ($y = -x + 2\pi k$).

Since $\sin 3x = -\sin 2x$, we get either $3x = \pi + 2x + 2\pi k$ or $3x = -2x + 2\pi k$. Solving each of these equations for x yields $x = \pi + 2\pi k$ and $x = \frac{2\pi}{5}k$, respectively. The values of x between 0 and 2π are $0, \frac{2\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{8\pi}{5}$, and 2π.

The graph is shown below. As predicted, it has horizontal tangent lines at $\frac{2\pi}{5}k$ and at $x = \frac{\pi}{2}$. Each of these points is a local extremum.



In Exercises 61–64, find the critical points and the extreme values on $[0, 4]$. *In Exercises 63 and 64, refer to Figure 18.*

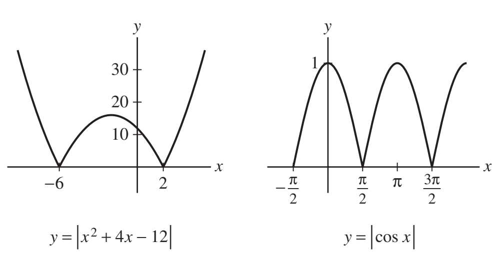
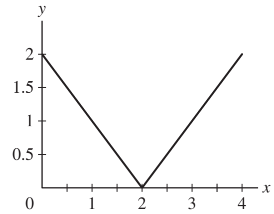


FIGURE 18

61. $y = |x - 2|$

SOLUTION Let $f(x) = |x-2|$. For $x < 2$, we have $f'(x) = -1$. For $x > 2$, we have $f'(x) = 1$. Now as $x \to 2-$, we have $\dfrac{f(x) - f(2)}{x-2} = \dfrac{(2-x) - 0}{x-2} \to -1$; whereas as $x \to 2+$, we have $\dfrac{f(x) - f(2)}{x-2} = \dfrac{(x-2) - 0}{x-2} \to 1$. Therefore, $f'(2) = \lim_{x\to 2} \dfrac{f(x) - f(2)}{x-2}$ does not exist and the lone critical point of f is $x = 2$. Alternately, we examine the graph of $f(x) = |x-2|$ shown below.

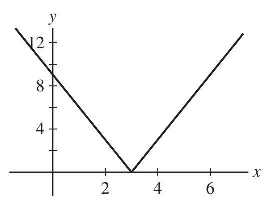
To find the extremum, we check the values of $f(x)$ at the critical point and the endpoints. $f(0) = 2$, $f(4) = 2$, and $f(2) = 0$. $f(x)$ takes its minimum value of 0 at $x = 2$, and its maximum of 2 at $x = 0$ and at $x = 4$.



62. $y = |3x - 9|$

SOLUTION Let $f(x) = |3x - 9| = 3|x-3|$. For $x < 3$, we have $f'(x) = -3$. For $x > 3$, we have $f'(x) = 3$. Now as $x \to 3-$, we have $\dfrac{f(x) - f(3)}{x-3} = \dfrac{3(3-x) - 0}{x-3} \to -3$; whereas as $x \to 3+$, we have $\dfrac{f(x) - f(3)}{x-3} = \dfrac{3(x-3) - 0}{x-3} \to 3$. Therefore, $f'(3) = \lim_{x\to 3} \dfrac{f(x) - f(3)}{x-3}$ does not exist and the lone critical point of f is $x = 3$. Alternately, we examine the graph of $f(x) = |3x - 9|$ shown below.

To find the extrema of $f(x)$ on $[0, 4]$, we test the values of $f(x)$ at the critical point and the endpoints. $f(0) = 9$, $f(3) = 0$ and $f(4) = 3$, so $f(x)$ takes its minimum value of 0 at $x = 3$, and its maximum value of 9 at $x = 0$.



63. $y = |x^2 + 4x - 12|$

SOLUTION Let $f(x) = |x^2 + 4x - 12| = |(x+6)(x-2)|$. From the graph of f in Figure 18, we see that $f'(x)$ does not exist at $x = -6$ and at $x = 2$, so these are critical points of f. There is also a critical point between $x = -6$ and $x = 2$ at which $f'(x) = 0$. For $-6 < x < 2$, $f(x) = -x^2 - 4x + 12$, so $f'(x) = -2x - 4 = 0$ when $x = -2$. On the interval $[0, 4]$ the minimum value of f is $f(2) = 0$ and the maximum value is $f(4) = 20$. (*Note:* $f(0) = 12$ and the critical points $x = -6$ and $x = -2$ are not in the interval.)

64. $y = |\cos x|$

SOLUTION Let $f(x) = |\cos x|$. There are two types of critical points: points of the form πn where the derivative is zero and points of the form $n\pi + \pi/2$ where the derivative does not exist. Only two of these, $x = \frac{\pi}{2}$ and $x = \pi$ are in the interval $[0, 4]$. Now, $f(0) = f(\pi) = 1$, $f(4) = |\cos 4| \approx 0.6536$, and $f(\frac{\pi}{2}) = 0$, so $f(x)$ takes its maximum value of 1 at $x = 0$ and $x = \pi$ and its minimum of 0 at $x = \frac{\pi}{2}$.

In Exercises 65–68, verify Rolle's Theorem for the given interval.

65. $f(x) = x + x^{-1}, \quad \left[\frac{1}{2}, 2\right]$

SOLUTION Because f is continuous on $[\frac{1}{2}, 2]$, differentiable on $(\frac{1}{2}, 2)$ and

$$f\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{\frac{1}{2}} = \frac{5}{2} = 2 + \frac{1}{2} = f(2),$$

we may conclude from Rolle's Theorem that there exists a $c \in (\frac{1}{2}, 2)$ at which $f'(c) = 0$. Here, $f'(x) = 1 - x^{-2} = \frac{x^2-1}{x^2}$, so we may take $c = 1$.

66. $f(x) = \sin x, \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

SOLUTION Because f is continuous on $[\frac{\pi}{4}, \frac{3\pi}{4}]$, differentiable on $(\frac{\pi}{4}, \frac{3\pi}{4})$ and

$$f\left(\frac{\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2},$$

we may conclude from Rolle's Theorem that there exists a $c \in (\frac{\pi}{4}, \frac{3\pi}{4})$ at which $f'(c) = 0$. Here, $f'(x) = \cos x$, so we may take $c = \frac{\pi}{2}$.

67. $f(x) = \dfrac{x^2}{8x - 15}, \quad [3, 5]$

SOLUTION Because f is continuous on $[3, 5]$, differentiable on $(3, 5)$ and $f(3) = f(5) = 1$, we may conclude from Rolle's Theorem that there exists a $c \in (3, 5)$ at which $f'(c) = 0$. Here,

$$f'(x) = \frac{(8x - 15)(2x) - 8x^2}{(8x - 15)^2} = \frac{2x(4x - 15)}{(8x - 15)^2},$$

so we may take $c = \frac{15}{4}$.

68. $f(x) = \sin^2 x - \cos^2 x, \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

SOLUTION Because f is continuous on $[\frac{\pi}{4}, \frac{3\pi}{4}]$, differentiable on $(\frac{\pi}{4}, \frac{3\pi}{4})$ and

$$f\left(\frac{\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = 0,$$

we may conclude from Rolle's Theorem that there exists a $c \in (\frac{\pi}{4}, \frac{3\pi}{4})$ at which $f'(c) = 0$. Here,

$$f'(x) = 2\sin x\cos x - 2\cos x(-\sin x) = 2\sin 2x,$$

so we may take $c = \frac{\pi}{2}$.

69. Prove that $f(x) = x^5 + 2x^3 + 4x - 12$ has precisely one real root.

SOLUTION Let's first establish the $f(x) = x^5 + 2x^3 + 4x - 12$ has at least one root. Because f is a polynomial, it is continuous for all x. Moreover, $f(0) = -12 < 0$ and $f(2) = 44 > 0$. Therefore, by the Intermediate Value Theorem, there exists a $c \in (0, 2)$ such that $f(c) = 0$.

Next, we prove that this is the only root. We will use proof by contradiction. Suppose $f(x) = x^5 + 2x^3 + 4x - 12$ has two real roots, $x = a$ and $x = b$. Then $f(a) = f(b) = 0$ and Rolle's Theorem guarantees that there exists a $c \in (a, b)$ at which $f'(c) = 0$. However, $f'(x) = 5x^4 + 6x^2 + 4 \geq 4$ for all x, so there is no $c \in (a, b)$ at which $f'(c) = 0$. Based on this contradiction, we conclude that $f(x) = x^5 + 2x^3 + 4x - 12$ cannot have more than one real root. Finally, f must have precisely one real root.

70. Prove that $f(x) = x^3 + 3x^2 + 6x$ has precisely one real root.

SOLUTION First, note that $f(0) = 0$, so f has at least one real root. We will proceed by contradiction to establish that $x = 0$ is the only real root. Suppose there exists another real root, say $x = a$. Because the polynomial f is continuous and differentiable for all real x, it follows by Rolle's Theorem that there exists a real number c between 0 and a such that $f'(c) = 0$. However, $f'(x) = 3x^2 + 6x + 6 = 3(x + 1)^2 + 3 \geq 3$ for all x. Thus, there is no c between 0 and a at which $f'(c) = 0$. Based on this contradiction, we conclude that $f(x) = x^3 + 3x^2 + 6x$ cannot have more than one real root. Finally, f must have precisely one real root.

71. Prove that $f(x) = x^4 + 5x^3 + 4x$ has no root c satisfying $c > 0$. *Hint:* Note that $x = 0$ is a root and apply Rolle's Theorem.

SOLUTION We will proceed by contradiction. Note that $f(0) = 0$ and suppose that there exists a $c > 0$ such that $f(c) = 0$. Then $f(0) = f(c) = 0$ and Rolle's Theorem guarantees that there exists a $d \in (0, c)$ such that $f'(d) = 0$. However, $f'(x) = 4x^3 + 15x^2 + 4 > 4$ for all $x > 0$, so there is no $d \in (0, c)$ such that $f'(d) = 0$. Based on this contradiction, we conclude that $f(x) = x^4 + 5x^3 + 4x$ has no root c satisfying $c > 0$.

72. Prove that $c = 4$ is the largest root of $f(x) = x^4 - 8x^2 - 128$.

SOLUTION First, note that $f(4) = 4^4 - 8(4)^2 - 128 = 256 - 128 - 128 = 0$, so $c = 4$ is a root of f. We will proceed by contradiction to establish that $c = 4$ is the largest real root. Suppose there exists real root, say $x = a$, where $a > 4$. Because the polynomial f is continuous and differentiable for all real x, it follows by Rolle's Theorem that there exists a real number $c \in (4, a)$ such that $f'(c) = 0$. However, $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) > 0$ for all $x > 4$. Thus, there is no $c \in (4, a)$ at which $f'(c) = 0$. Based on this contradiction, we conclude that $f(x) = x^4 - 8x^2 - 128$ has no real root larger than 4.

73. The position of a mass oscillating at the end of a spring is $s(t) = A\sin\omega t$, where A is the amplitude and ω is the angular frequency. Show that the speed $|v(t)|$ is at a maximum when the acceleration $a(t)$ is zero and that $|a(t)|$ is at a maximum when $v(t)$ is zero.

SOLUTION Let $s(t) = A\sin\omega t$. Then

$$v(t) = \frac{ds}{dt} = A\omega\cos\omega t$$

and

$$a(t) = \frac{dv}{dt} = -A\omega^2\sin\omega t.$$

Thus, the speed

$$|v(t)| = |A\omega\cos\omega t|$$

is a maximum when $|\cos\omega t| = 1$, which is precisely when $\sin\omega t = 0$; that is, the speed $|v(t)|$ is at a maximum when the acceleration $a(t)$ is zero. Similarly,

$$|a(t)| = |A\omega^2\sin\omega t|$$

is a maximum when $|\sin\omega t| = 1$, which is precisely when $\cos\omega t = 0$; that is, $|a(t)|$ is at a maximum when $v(t)$ is zero.

74. The concentration $C(t)$ (in mg/cm^3) of a drug in a patient's bloodstream after t hours is

$$C(t) = \frac{0.016t}{t^2+4t+4}$$

Find the maximum concentration in the time interval [0, 8] and the time at which it occurs.

SOLUTION

$$C'(t) = \frac{0.016(t^2+4t+4) - (0.016t(2t+4))}{(t^2+4t+4)^2} = 0.016\frac{-t^2+4}{(t^2+4t+4)^2} = 0.016\frac{2-t}{(t+2)^3}.$$

$C'(t)$ exists for all $t \geq 0$, so we are looking for points where $C'(t) = 0$. $C'(t) = 0$ when $t = 2$. Using a calculator, we find that $C(2) = 0.002\frac{\text{mg}}{\text{cm}^3}$. On the other hand, $C(0) = 0$ and $C(10) \approx 0.001$. Hence, the maximum concentration occurs at $t = 2$ hours and is equal to $.002\frac{\text{mg}}{\text{cm}^3}$.

75. CAS **Antibiotic Levels** A study shows that the concentration $C(t)$ (in micrograms per milliliter) of antibiotic in a patient's blood serum after t hours is $C(t) = 120(e^{-0.2t} - e^{-bt})$, where $b \geq 1$ is a constant that depends on the particular combination of antibiotic agents used. Solve numerically for the value of b (to two decimal places) for which maximum concentration occurs at $t = 1$ h. You may assume that the maximum occurs at a critical point as suggested by Figure 19.

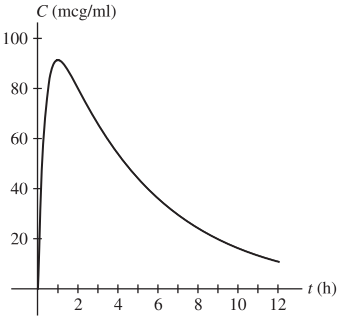


FIGURE 19 Graph of $C(t) = 120(e^{-0.2t} - e^{-bt})$ with b chosen so that the maximum occurs at $t = 1$ h.

SOLUTION Answer is $b = 2.86$. The max of $C(t)$ occurs at $t = \ln(5b)/(b-0.2)$ so we solve $\ln(5b)/(b-0.1) = 1$ numerically.

Let $C(t) = 120(e^{-0.2t} - e^{-bt})$. Then $C'(t) = 120(-0.2e^{-0.2t} + be^{-bt}) = 0$ when

$$t = \frac{\ln 5b}{b-0.2}.$$

Substituting $t = 1$ and solving for b numerically yields $b \approx 2.86$.

76. CAS In the notation of Exercise 75, find the value of b (to two decimal places) for which the maximum value of $C(t)$ is equal to 100 mcg/ml.

SOLUTION From the previous exercise, we know that $C(t)$ achieves its maximum when

$$t = \frac{\ln 5b}{b - 0.2}.$$

Substituting this expression into the formula for $C(t)$, setting the resulting expression equal to 100 and solving for b yields $b \approx 4.75$.

77. In 1919, physicist Alfred Betz argued that the maximum efficiency of a wind turbine is around 59%. If wind enters a turbine with speed v_1 and exits with speed v_2, then the power extracted is the difference in kinetic energy per unit time:

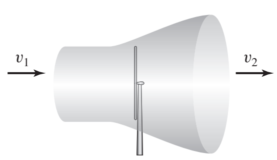
$$P = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 \quad \text{watts}$$

where m is the mass of wind flowing through the rotor per unit time (Figure 20). Betz assumed that $m = \rho A(v_1 + v_2)/2$, where ρ is the density of air and A is the area swept out by the rotor. Wind flowing undisturbed through the same area A would have mass per unit time $\rho A v_1$ and power $P_0 = \frac{1}{2}\rho A v_1^3$. The fraction of power extracted by the turbine is $F = P/P_0$.

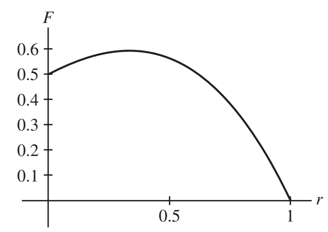
(a) Show that F depends only on the ratio $r = v_2/v_1$ and is equal to $F(r) = \frac{1}{2}(1 - r^2)(1 + r)$, where $0 \le r \le 1$.

(b) Show that the maximum value of $F(r)$, called the **Betz Limit**, is $16/27 \approx 0.59$.

(c) Explain why Betz's formula for $F(r)$ is not meaningful for r close to zero. *Hint:* How much wind would pass through the turbine if v_2 were zero? Is this realistic?



(A) Wind flowing through a turbine.



(B) F is the fraction of energy extracted by the turbine as a function of $r = v_2/v_1$.

FIGURE 20

SOLUTION

(a) We note that

$$F = \frac{P}{P_0} = \frac{\frac{1}{2}\frac{\rho A(v_1+v_2)}{2}(v_1^2 - v_2^2)}{\frac{1}{2}\rho A v_1^3}$$
$$= \frac{1}{2}\frac{v_1^2 - v_2^2}{v_1^2} \cdot \frac{v_1 + v_2}{v_1}$$
$$= \frac{1}{2}\left(1 - \frac{v_2^2}{v_1^2}\right)\left(1 + \frac{v_2}{v_1}\right)$$
$$= \frac{1}{2}(1 - r^2)(1 + r).$$

(b) Based on part (a),

$$F'(r) = \frac{1}{2}(1 - r^2) - r(1 + r) = -\frac{3}{2}r^2 - r + \frac{1}{2}.$$

The roots of this quadratic are $r = -1$ and $r = \frac{1}{3}$. Now, $F(0) = \frac{1}{2}$, $F(1) = 0$ and

$$F\left(\frac{1}{3}\right) = \frac{1}{2} \cdot \frac{8}{9} \cdot \frac{4}{3} = \frac{16}{27} \approx 0.59.$$

Thus, the Betz Limit is $16/27 \approx 0.59$.

(c) If v_2 were 0, then no air would be passing through the turbine, which is not realistic.

78. GU The **Bohr radius** a_0 of the hydrogen atom is the value of r that minimizes the energy

$$E(r) = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}$$

where $\hbar$, m, e, and ϵ_0 are physical constants. Show that $a_0 = 4\pi\epsilon_0\hbar^2/(me^2)$. Assume that the minimum occurs at a critical point, as suggested by Figure 21.

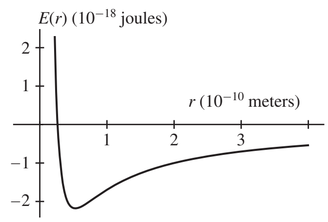


FIGURE 21

SOLUTION Let

$$E(r) = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}.$$

Then

$$\frac{dE}{dr} = -\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\epsilon_0 r^2} = 0$$

implies

$$r = \frac{4\pi\epsilon_0\hbar^2}{me^2}.$$

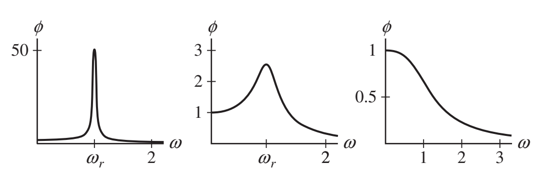
Thus,

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}.$$

79. The response of a circuit or other oscillatory system to an input of frequency ω ("omega") is described by the function

$$\phi(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4D^2\omega^2}}$$

Both ω_0 (the natural frequency of the system) and D (the damping factor) are positive constants. The graph of ϕ is called a **resonance curve**, and the positive frequency $\omega_r > 0$, where ϕ takes its maximum value, if it exists, is called the **resonant frequency**. Show that $\omega_r = \sqrt{\omega_0^2 - 2D^2}$ if $0 < D < \omega_0/\sqrt{2}$ and that no resonant frequency exists otherwise (Figure 22).



(A) $D = 0.01$ (B) $D = 0.2$ (C) $D = 0.75$ (no resonance)

FIGURE 22 Resonance curves with $\omega_0 = 1$.

SOLUTION Let $\phi(\omega) = ((\omega_0^2 - \omega^2)^2 + 4D^2\omega^2)^{-1/2}$. Then

$$\phi'(\omega) = \frac{2\omega((\omega_0^2 - \omega^2) - 2D^2)}{((\omega_0^2 - \omega^2)^2 + 4D^2\omega^2)^{3/2}}$$

and the non-negative critical points are $\omega = 0$ and $\omega = \sqrt{\omega_0^2 - 2D^2}$. The latter critical point is positive if and only if $\omega_0^2 - 2D^2 > 0$, and since we are given $D > 0$, this is equivalent to $0 < D < \omega_0/\sqrt{2}$.

Define $\omega_r = \sqrt{\omega_0^2 - 2D^2}$. Now, $\phi(0) = 1/\omega_0^2$ and $\phi(\omega) \to 0$ as $\omega \to \infty$. Finally,

$$\phi(\omega_r) = \frac{1}{2D\sqrt{\omega_0^2 - D^2}},$$

which, for $0 < D < \omega_0/\sqrt{2}$, is larger than $1/\omega_0^2$. Hence, the point $\omega = \sqrt{\omega_0^2 - 2D^2}$, if defined, is a local maximum.

80. Bees build honeycomb structures out of cells with a hexagonal base and three rhombus-shaped faces on top, as in Figure 23. We can show that the surface area of this cell is

$$A(\theta) = 6hs + \frac{3}{2}s^2(\sqrt{3}\ \csc\theta - \cot\theta)$$

with h, s, and θ as indicated in the figure. Remarkably, bees "know" which angle θ minimizes the surface area (and therefore requires the least amount of wax).

(a) Show that $\theta \approx 54.7°$ (assume h and s are constant). *Hint:* Find the critical point of $A(\theta)$ for $0 < \theta < \pi/2$.

(b) GU Confirm, by graphing $f(\theta) = \sqrt{3}\csc\theta - \cot\theta$, that the critical point indeed minimizes the surface area.

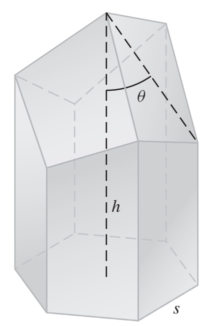
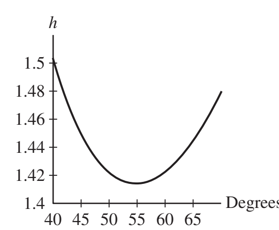


FIGURE 23 A cell in a honeycomb constructed by bees.

SOLUTION

(a) Because h and s are constant relative to θ, we have $A'(\theta) = \frac{3}{2}s^2(-\sqrt{3}\csc\theta\cot\theta + \csc^2\theta) = 0$. From this, we get $\sqrt{3}\csc\theta\cot\theta = \csc^2\theta$, or $\cos\theta = \frac{1}{\sqrt{3}}$, whence $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.955317$ radians = 54.736°.

(b) The plot of $\sqrt{3}\csc\theta - \cot\theta$, where θ is given in degrees, is given below. We can see that the minimum occurs just below 55°.



81. Find the maximum of $y = x^a - x^b$ on [0, 1] where $0 < a < b$. In particular, find the maximum of $y = x^5 - x^{10}$ on [0, 1].

SOLUTION

- Let $f(x) = x^a - x^b$. Then $f'(x) = ax^{a-1} - bx^{b-1}$. Since $a < b$, $f'(x) = x^{a-1}(a - bx^{b-a}) = 0$ implies critical points $x = 0$ and $x = (\frac{a}{b})^{1/(b-a)}$, which is in the interval [0, 1] as $a < b$ implies $\frac{a}{b} < 1$ and consequently $x = (\frac{a}{b})^{1/(b-a)} < 1$. Also, $f(0) = f(1) = 0$ and $a < b$ implies $x^a > x^b$ on the interval [0, 1], which gives $f(x) > 0$ and thus the maximum value of f on [0, 1] is

$$f\left(\left(\frac{a}{b}\right)^{1/(b-a)}\right) = \left(\frac{a}{b}\right)^{a/(b-a)} - \left(\frac{a}{b}\right)^{b/(b-a)}.$$

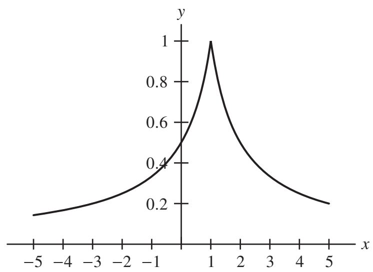
- Let $f(x) = x^5 - x^{10}$. Then by part (a), the maximum value of f on [0, 1] is

$$f\left(\left(\frac{1}{2}\right)^{1/5}\right) = \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

In Exercises 82–84, plot the function using a graphing utility and find its critical points and extreme values on $[-5, 5]$.

82. GU $y = \dfrac{1}{1 + |x - 1|}$

SOLUTION Let $f(x) = \frac{1}{1+|x-1|}$. The plot follows:



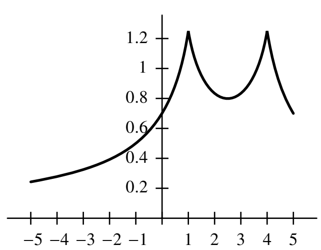
We can see on the plot that the only critical point of $f(x)$ lies at $x = 1$. With $f(-5) = \frac{1}{7}$, $f(1) = 1$ and $f(5) = \frac{1}{5}$, it follows that the maximum value of $f(x)$ on $[-5, 5]$ is $f(1) = 1$ and the minimum value is $f(-5) = \frac{1}{7}$.

83. GU $y = \dfrac{1}{1 + |x - 1|} + \dfrac{1}{1 + |x - 4|}$

SOLUTION Let

$$f(x) = \frac{1}{1 + |x - 1|} + \frac{1}{1 + |x - 4|}.$$

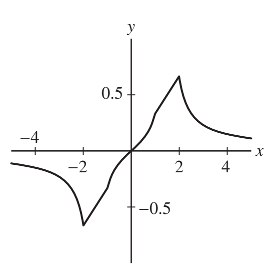
The plot follows:



We can see on the plot that the critical points of $f(x)$ lie at the cusps at $x = 1$ and $x = 4$ and at the location of the horizontal tangent line at $x = \frac{5}{2}$. With $f(-5) = \frac{17}{70}$, $f(1) = f(4) = \frac{5}{4}$, $f(\frac{5}{2}) = \frac{4}{5}$ and $f(5) = \frac{7}{10}$, it follows that the maximum value of $f(x)$ on $[-5, 5]$ is $f(1) = f(4) = \frac{5}{4}$ and the minimum value is $f(-5) = \frac{17}{70}$.

84. GU $y = \dfrac{x}{|x^2 - 1| + |x^2 - 4|}$

SOLUTION Let $f(x) = \frac{x}{|x^2-1|+|x^2-4|}$. The cusps of the graph of a function containing $|g(x)|$ are likely to lie where $g(x) = 0$, so we choose a plot range that includes $x = \pm 2$ and $x = \pm 1$:



As we can see from the graph, the function has cusps at $x = \pm 2$ and sharp corners at $x = \pm 1$. The cusps at $(2, \frac{2}{3})$ and $(-2, -\frac{2}{3})$ are the locations of the maximum and minimum values of $f(x)$, respectively.

85. **(a)** Use implicit differentiation to find the critical points on the curve $27x^2 = (x^2 + y^2)^3$.
(b) GU Plot the curve and the horizontal tangent lines on the same set of axes.

SOLUTION
(a) Differentiating both sides of the equation $27x^2 = (x^2 + y^2)^3$ with respect to x yields

$$54x = 3(x^2 + y^2)^2 \left(2x + 2y\frac{dy}{dx}\right).$$

Solving for dy/dx we obtain

$$\frac{dy}{dx} = \frac{27x - 3x(x^2 + y^2)^2}{3y(x^2 + y^2)^2} = \frac{x(9 - (x^2 + y^2)^2)}{y(x^2 + y^2)^2}.$$

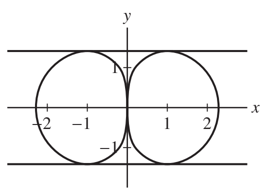
Thus, the derivative is zero when $x^2 + y^2 = 3$. Substituting into the equation for the curve, this yields $x^2 = 1$, or $x = \pm 1$. There are therefore four points at which the derivative is zero:

$$(-1, -\sqrt{2}), (-1, \sqrt{2}), (1, -\sqrt{2}), (1, \sqrt{2}).$$

There are also critical points where the derivative does not exist. This occurs when $y = 0$ and gives the following points with vertical tangents:

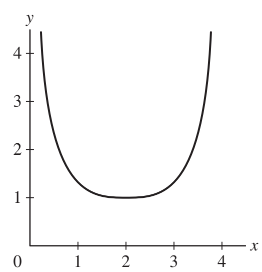
$$(0, 0), (\pm\sqrt[4]{27}, 0).$$

(b) The curve $27x^2 = (x^2 + y^2)^3$ and its horizontal tangents are plotted below.



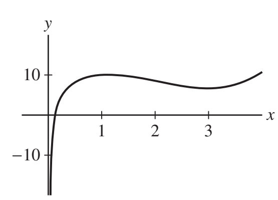
86. Sketch the graph of a continuous function on $(0, 4)$ with a minimum value but no maximum value.

SOLUTION Here is the graph of a function f on $(0, 4)$ with a minimum value [at $x = 2$] but no maximum value [since $f(x) \to \infty$ as $x \to 0+$ and as $x \to 4-$].



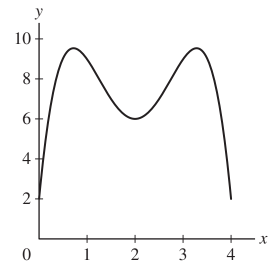
87. Sketch the graph of a continuous function on $(0, 4)$ having a local minimum but no absolute minimum.

SOLUTION Here is the graph of a function f on $(0, 4)$ with a local minimum value [between $x = 2$ and $x = 4$] but no absolute minimum [since $f(x) \to -\infty$ as $x \to 0+$].



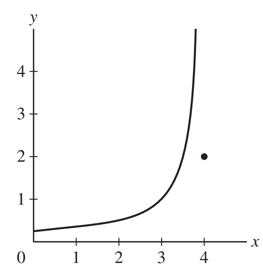
88. Sketch the graph of a function on $[0, 4]$ having
(a) Two local maxima and one local minimum.
(b) An absolute minimum that occurs at an endpoint, and an absolute maximum that occurs at a critical point.

SOLUTION Here is the graph of a function on $[0, 4]$ that (a) has two local maxima and one local minimum and (b) has an absolute minimum that occurs at an endpoint (at $x = 0$ or $x = 4$) and has an absolute maximum that occurs at a critical point.



89. Sketch the graph of a function $f(x)$ on $[0, 4]$ with a discontinuity such that $f(x)$ has an absolute minimum but no absolute maximum.

SOLUTION Here is the graph of a function f on $[0, 4]$ that (a) has a discontinuity [at $x = 4$] and (b) has an absolute minimum [at $x = 0$] but no absolute maximum [since $f(x) \to \infty$ as $x \to 4-$].



90. A rainbow is produced by light rays that enter a raindrop (assumed spherical) and exit after being reflected internally as in Figure 24. The angle between the incoming and reflected rays is $\theta = 4r - 2i$, where the angle of incidence i and refraction r are related by Snell's Law $\sin i = n \sin r$ with $n \approx 1.33$ (the index of refraction for air and water).

(a) Use Snell's Law to show that $\dfrac{dr}{di} = \dfrac{\cos i}{n \cos r}$.

(b) Show that the maximum value $\theta_{\max}$ of θ occurs when i satisfies $\cos i = \sqrt{\dfrac{n^2 - 1}{3}}$. *Hint:* Show that $\dfrac{d\theta}{di} = 0$ if $\cos i = \dfrac{n}{2} \cos r$. Then use Snell's Law to eliminate r.

(c) Show that $\theta_{\max} \approx 59.58°$.

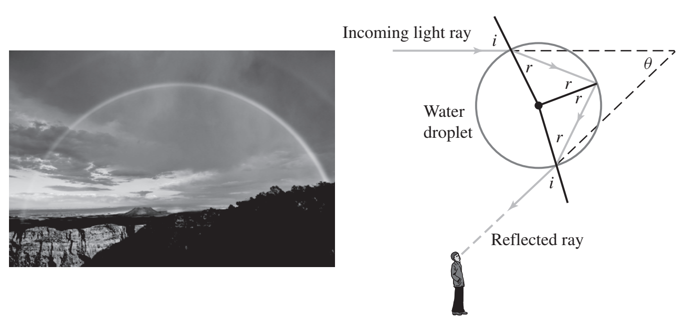


FIGURE 24

SOLUTION

(a) Differentiating Snell's Law with respect to i yields

$$\cos i = n \cos r \frac{dr}{di} \quad \text{or} \quad \frac{dr}{di} = \frac{\cos i}{n \cos r}.$$

(b) Differentiating the formula for θ with respect to i yields

$$\frac{d\theta}{di} = 4\frac{dr}{di} - 2 = 4\frac{\cos i}{n \cos r} - 2$$

by part (a). Thus,

$$\frac{d\theta}{di} = 0 \quad \text{when} \quad \cos i = \frac{n}{2} \cos r.$$

Squaring both sides of this last equation gives

$$\cos^2 i = \frac{n^2}{4}\cos^2 r,$$

while squaring both sides of Snell's Law gives

$$\sin^2 i = n^2 \sin^2 r \quad \text{or} \quad 1 - \cos^2 i = n^2(1 - \cos^2 r).$$

Solving this equation for $\cos^2 r$ gives

$$\cos^2 r = 1 - \frac{1 - \cos^2 i}{n^2};$$

Combining these last two equations and solving for $\cos i$ yields

$$\cos i = \sqrt{\frac{n^2 - 1}{3}}.$$

(c) With $n = 1.33$,

$$\cos i = \sqrt{\frac{(1.33)^2 - 1}{3}} = 0.5063$$

and

$$\cos r = \frac{2}{1.33}\cos i = 0.7613.$$

Thus, $r = 40.42°$, $i = 59.58°$ and

$$\theta_{\max} = 4r - 2i = 42.53°.$$

Further Insights and Challenges

91. Show that the extreme values of $f(x) = a\sin x + b\cos x$ are $\pm\sqrt{a^2 + b^2}$.

SOLUTION If $f(x) = a\sin x + b\cos x$, then $f'(x) = a\cos x - b\sin x$, so that $f'(x) = 0$ implies $a\cos x - b\sin x = 0$. This implies $\tan x = \frac{a}{b}$. Then,

$$\sin x = \frac{\pm a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \cos x = \frac{\pm b}{\sqrt{a^2 + b^2}}.$$

Therefore

$$f(x) = a\sin x + b\cos x = a\frac{\pm a}{\sqrt{a^2 + b^2}} + b\frac{\pm b}{\sqrt{a^2 + b^2}} = \pm\frac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \pm\sqrt{a^2 + b^2}.$$

92. Show, by considering its minimum, that $f(x) = x^2 - 2x + 3$ takes on only positive values. More generally, find the conditions on r and s under which the quadratic function $f(x) = x^2 + rx + s$ takes on only positive values. Give examples of r and s for which f takes on both positive and negative values.

SOLUTION

- Observe that $f(x) = x^2 - 2x + 3 = (x - 1)^2 + 2 > 0$ for all x. Let $f(x) = x^2 + rx + s$. Completing the square, we note that $f(x) = (x + \frac{1}{2}r)^2 + s - \frac{1}{4}r^2 > 0$ for all x provided that $s > \frac{1}{4}r^2$.
- Let $f(x) = x^2 - 4x + 3 = (x - 1)(x - 3)$. Then f takes on both positive and negative values. Here, $r = -4$ and $s = 3$.

93. Show that if the quadratic polynomial $f(x) = x^2 + rx + s$ takes on both positive and negative values, then its minimum value occurs at the midpoint between the two roots.

SOLUTION Let $f(x) = x^2 + rx + s$ and suppose that $f(x)$ takes on both positive and negative values. This will guarantee that f has two real roots. By the quadratic formula, the roots of f are

$$x = \frac{-r \pm \sqrt{r^2 - 4s}}{2}.$$

Observe that the midpoint between these roots is

$$\frac{1}{2}\left(\frac{-r+\sqrt{r^2-4s}}{2}+\frac{-r-\sqrt{r^2-4s}}{2}\right)=-\frac{r}{2}.$$

Next, $f'(x)=2x+r=0$ when $x=-\frac{r}{2}$ and, because the graph of $f(x)$ is an upward opening parabola, it follows that $f(-\frac{r}{2})$ is a minimum. Thus, f takes on its minimum value at the midpoint between the two roots.

94. Generalize Exercise 93: Show that if the horizontal line $y=c$ intersects the graph of $f(x)=x^2+rx+s$ at two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, then $f(x)$ takes its minimum value at the midpoint $M=\dfrac{x_1+x_2}{2}$ (Figure 25).

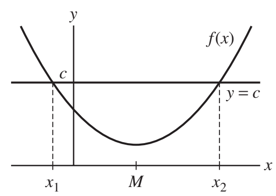


FIGURE 25

SOLUTION Suppose that a horizontal line $y=c$ intersects the graph of a quadratic function $f(x)=x^2+rx+s$ in two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Then of course $f(x_1)=f(x_2)=c$. Let $g(x)=f(x)-c$. Then $g(x_1)=g(x_2)=0$. By Exercise 93, g takes on its minimum value at $x=\frac{1}{2}(x_1+x_2)$. Hence so does $f(x)=g(x)+c$.

95. A cubic polynomial may have a local min and max, or it may have neither (Figure 26). Find conditions on the coefficients a and b of

$$f(x)=\frac{1}{3}x^3+\frac{1}{2}ax^2+bx+c$$

that ensure that f has neither a local min nor a local max. *Hint:* Apply Exercise 92 to $f'(x)$.

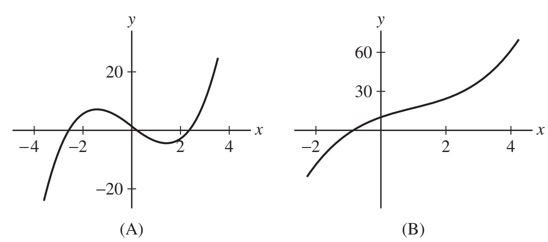


FIGURE 26 Cubic polynomials

SOLUTION Let $f(x)=\frac{1}{3}x^3+\frac{1}{2}ax^2+bx+c$. Using Exercise 92, we have $g(x)=f'(x)=x^2+ax+b>0$ for all x provided $b>\frac{1}{4}a^2$, in which case f has no critical points and hence no local extrema. (Actually $b\geq\frac{1}{4}a^2$ will suffice, since in this case [as we'll see in a later section] f has an inflection point but no local extrema.)

96. Find the min and max of

$$f(x)=x^p(1-x)^q \quad \text{on } [0,1],$$

where $p, q>0$.

SOLUTION Let $f(x)=x^p(1-x)^q$, $0\leq x\leq 1$, where p and q are positive numbers. Then

$$\begin{aligned}f'(x)&=x^p q(1-x)^{q-1}(-1)+(1-x)^q px^{p-1}\\&=x^{p-1}(1-x)^{q-1}(p(1-x)-qx)=0 \quad \text{at} \quad x=0,1,\frac{p}{p+q}\end{aligned}$$

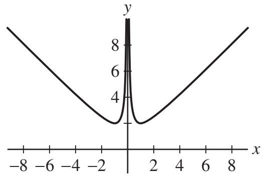
The minimum value of f on $[0,1]$ is $f(0)=f(1)=0$, whereas its maximum value is

$$f\left(\frac{p}{p+q}\right)=\frac{p^pq^q}{(p+q)^{p+q}}.$$

97. Prove that if f is continuous and $f(a)$ and $f(b)$ are local minima where $a<b$, then there exists a value c between a and b such that $f(c)$ is a local maximum. (*Hint:* Apply Theorem 1 to the interval $[a,b]$.) Show that continuity is a necessary hypothesis by sketching the graph of a function (necessarily discontinuous) with two local minima but no local maximum.

SOLUTION

- Let $f(x)$ be a continuous function with $f(a)$ and $f(b)$ local minima on the interval $[a, b]$. By Theorem 1, $f(x)$ must take on both a minimum and a maximum on $[a, b]$. Since local minima occur at $f(a)$ and $f(b)$, the maximum must occur at some other point in the interval, call it c, where $f(c)$ is a local maximum.
- The function graphed here is discontinuous at $x = 0$.



4.3 The Mean Value Theorem and Monotonicity

Preliminary Questions

1. For which value of m is the following statement correct? If $f(2) = 3$ and $f(4) = 9$, and $f(x)$ is differentiable, then f has a tangent line of slope m.

SOLUTION The Mean Value Theorem guarantees that the function has a tangent line with slope equal to

$$\frac{f(4) - f(2)}{4 - 2} = \frac{9 - 3}{4 - 2} = 3.$$

Hence, $m = 3$ makes the statement correct.

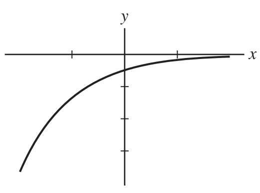
2. Assume f is differentiable. Which of the following statements does *not* follow from the MVT?

(a) If f has a secant line of slope 0, then f has a tangent line of slope 0.
(b) If $f(5) < f(9)$, then $f'(c) > 0$ for some $c \in (5, 9)$.
(c) If f has a tangent line of slope 0, then f has a secant line of slope 0.
(d) If $f'(x) > 0$ for all x, then every secant line has positive slope.

SOLUTION Conclusion **(c)** does not follow from the Mean Value Theorem. As a counterexample, consider the function $f(x) = x^3$. Note that $f'(0) = 0$, but no secant line has zero slope.

3. Can a function that takes on only negative values have a positive derivative? If so, sketch an example.

SOLUTION Yes. The figure below displays a function that takes on only negative values but has a positive derivative.



4. For $f(x)$ with derivative as in Figure 12:

(a) Is $f(c)$ a local minimum or maximum?
(b) Is $f(x)$ a decreasing function?

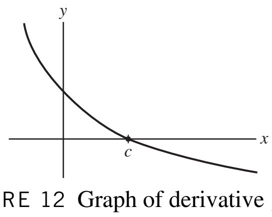


FIGURE 12 Graph of derivative $f'(x)$.

SOLUTION

(a) To the left of $x = c$, the derivative is positive, so f is increasing; to the right of $x = c$, the derivative is negative, so f is decreasing. Consequently, $f(c)$ must be a local maximum.
(b) No. The derivative is a decreasing function, but as noted in part (a), $f(x)$ is increasing for $x < c$ and decreasing for $x > c$.

Exercises

In Exercises 1–8, find a point c satisfying the conclusion of the MVT for the given function and interval.

1. $y = x^{-1}$, $[2, 8]$

SOLUTION Let $f(x) = x^{-1}$, $a = 2$, $b = 8$. Then $f'(x) = -x^{-2}$, and by the MVT, there exists a $c \in (2, 8)$ such that

$$-\frac{1}{c^2} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{8} - \frac{1}{2}}{8 - 2} = -\frac{1}{16}.$$

Thus $c^2 = 16$ and $c = \pm 4$. Choose $c = 4 \in (2, 8)$.

2. $y = \sqrt{x}$, $[9, 25]$

SOLUTION Let $f(x) = x^{1/2}$, $a = 9$, $b = 25$. Then $f'(x) = \frac{1}{2}x^{-1/2}$, and by the MVT, there exists a $c \in (9, 25)$ such that

$$\frac{1}{2}c^{-1/2} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{5 - 3}{25 - 9} = \frac{1}{8}.$$

Thus $\frac{1}{\sqrt{c}} = \frac{1}{4}$ and $c = 16 \in (9, 25)$.

3. $y = \cos x - \sin x$, $[0, 2\pi]$

SOLUTION Let $f(x) = \cos x - \sin x$, $a = 0$, $b = 2\pi$. Then $f'(x) = -\sin x - \cos x$, and by the MVT, there exists a $c \in (0, 2\pi)$ such that

$$-\sin c - \cos c = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{1 - 1}{2\pi - 0} = 0.$$

Thus $-\sin c = \cos c$. Choose either $c = \frac{3\pi}{4}$ or $c = \frac{7\pi}{4} \in (0, 2\pi)$.

4. $y = \dfrac{x}{x + 2}$, $[1, 4]$

SOLUTION Let $f(x) = x/(x + 2)$, $a = 1$, $b = 4$. Then $f'(x) = \frac{2}{(x+2)^2}$, and by the MVT, there exists a $c \in (1, 4)$ such that

$$\frac{2}{(c + 2)^2} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\frac{2}{3} - \frac{1}{3}}{4 - 1} = \frac{1}{9}.$$

Thus $(c + 2)^2 = 18$ and $c = -2 \pm 3\sqrt{2}$. Choose $c = 3\sqrt{2} - 2 \approx 2.24 \in (1, 4)$.

5. $y = x^3$, $[-4, 5]$

SOLUTION Let $f(x) = x^3$, $a = -4$, $b = 5$. Then $f'(x) = 3x^2$, and by the MVT, there exists a $c \in (-4, 5)$ such that

$$3c^2 = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{189}{9} = 21.$$

Solving for c yields $c^2 = 7$, so $c = \pm\sqrt{7}$. Both of these values are in the interval $[-4, 5]$, so either value can be chosen.

6. $y = x \ln x$, $[1, 2]$

SOLUTION Let $f(x) = x \ln x$, $a = 1$, $b = 2$. Then $f'(x) = 1 + \ln x$, and by the MVT, there exists a $c \in (1, 2)$ such that

$$1 + \ln c = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{2 \ln 2}{1} = 2 \ln 2.$$

Solving for c yields $c = e^{2ln2-1} = \frac{4}{e} \approx 1.4715 \in (1, 2)$.

7. $y = e^{-2x}$, $[0, 3]$

SOLUTION Let $f(x) = e^{-2x}$, $a = 0$, $b = 3$. Then $f'(x) = -2e^{-2x}$, and by the MVT, there exists a $c \in (0, 3)$ such that

$$-2e^{-2c} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{e^{-6} - 1}{3 - 0} = \frac{e^{-6} - 1}{3}.$$

Solving for c yields

$$c = -\frac{1}{2} \ln\left(\frac{1 - e^{-6}}{6}\right) \approx 0.8971 \in (0, 3).$$

8. $y = e^x - x, \quad [-1, 1]$

SOLUTION Let $f(x) = e^x - x$, $a = -1$, $b = 1$. Then $f'(x) = e^x - 1$, and by the MVT, there exists a $c \in (-1, 1)$ such that

$$e^c - 1 = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{(e-1) - (e^{-1} + 1)}{1 - (-1)} = \frac{1}{2}(e - e^{-1}) - 1.$$

Solving for c yields

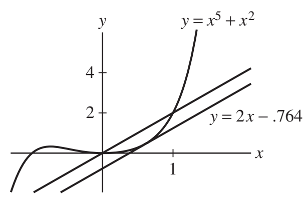
$$c = \ln\left(\frac{e - e^{-1}}{2}\right) \approx 0.1614 \in (-1, 1).$$

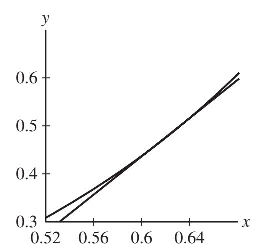
9. GU Let $f(x) = x^5 + x^2$. The secant line between $x = 0$ and $x = 1$ has slope 2 (check this), so by the MVT, $f'(c) = 2$ for some $c \in (0, 1)$. Plot $f(x)$ and the secant line on the same axes. Then plot $y = 2x + b$ for different values of b until the line becomes tangent to the graph of f. Zoom in on the point of tangency to estimate x-coordinate c of the point of tangency.

SOLUTION Let $f(x) = x^5 + x^2$. The slope of the secant line between $x = 0$ and $x = 1$ is

$$\frac{f(1) - f(0)}{1 - 0} = \frac{2 - 0}{1} = 2.$$

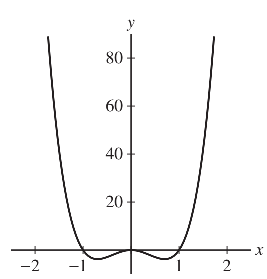
A plot of $f(x)$, the secant line between $x = 0$ and $x = 1$, and the line $y = 2x - 0.764$ is shown below at the left. The line $y = 2x - 0.764$ appears to be tangent to the graph of $y = f(x)$. Zooming in on the point of tangency (see below at the right), it appears that the x-coordinate of the point of tangency is approximately 0.62.

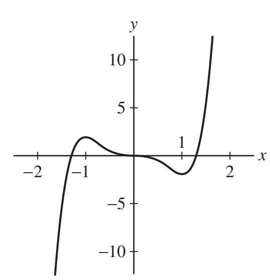




10. GU Plot the derivative of $f(x) = 3x^5 - 5x^3$. Describe its sign changes and use this to determine the local extreme values of $f(x)$. Then graph $f(x)$ to confirm your conclusions.

SOLUTION Let $f(x) = 3x^5 - 5x^3$. Then $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$. The graph of $f'(x)$ is shown below at the left. Because $f'(x)$ changes from positive to negative at $x = -1$, $f(x)$ changes from increasing to decreasing and therefore has a local maximum at $x = -1$. At $x = 1$, $f'(x)$ changes from negative to positive, so $f(x)$ changes from decreasing to increasing and therefore has a local minimum. Though $f'(x) = 0$ at $x = 0$, $f'(x)$ does not change sign at $x = 0$, so $f(x)$ has neither a local maximum nor a local minimum at $x = 0$. The graph of $f(x)$, shown below at the right, confirms each of these conclusions.





11. Determine the intervals on which $f'(x)$ is positive and negative, assuming that Figure 13 is the graph of $f(x)$.

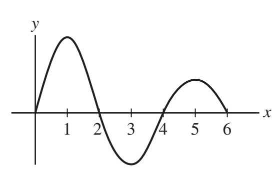


FIGURE 13

SOLUTION The derivative of f is positive on the intervals $(0, 1)$ and $(3, 5)$ where f is increasing; it is negative on the intervals $(1, 3)$ and $(5, 6)$ where f is decreasing.

12. Determine the intervals on which $f(x)$ is increasing or decreasing, assuming that Figure 13 is the graph of $f'(x)$.

SOLUTION $f(x)$ is increasing on every interval (a, b) over which $f'(x) > 0$, and is decreasing on every interval over which $f'(x) < 0$. If the graph of $f'(x)$ is given in Figure 13, then $f(x)$ is increasing on the intervals $(0, 2)$ and $(4, 6)$, and is decreasing on the interval $(2, 4)$.

13. State whether $f(2)$ and $f(4)$ are local minima or local maxima, assuming that Figure 13 is the graph of $f'(x)$.

SOLUTION

- $f'(x)$ makes a transition from positive to negative at $x = 2$, so $f(2)$ is a local maximum.
- $f'(x)$ makes a transition from negative to positive at $x = 4$, so $f(4)$ is a local minimum.

14. Figure 14 shows the graph of the derivative $f'(x)$ of a function $f(x)$. Find the critical points of $f(x)$ and determine whether they are local minima, local maxima, or neither.

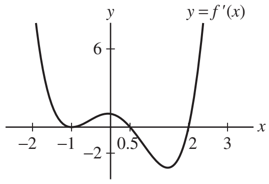


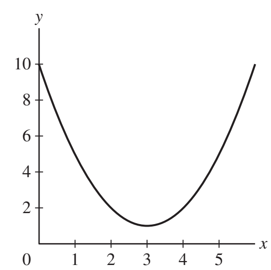
FIGURE 14

SOLUTION Since $f'(x) = 0$ when $x = -1$, $x = \frac{1}{2}$ and $x = 2$, these are the critical points of f. At $x = -1$, there is no sign transition in f', so $f(-1)$ is neither a local maximum nor a local minimum. At $x = \frac{1}{2}$, f' transitions from $+$ to $-$, so $f(\frac{1}{2})$ is a local maximum. Finally, at $x = 2$, f' transitions from $-$ to $+$, so $f(2)$ is a local minimum.

In Exercises 15–18, sketch the graph of a function $f(x)$ whose derivative $f'(x)$ has the given description.

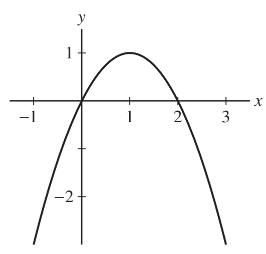
15. $f'(x) > 0$ for $x > 3$ and $f'(x) < 0$ for $x < 3$

SOLUTION Here is the graph of a function f for which $f'(x) > 0$ for $x > 3$ and $f'(x) < 0$ for $x < 3$.



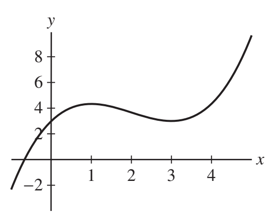
16. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$

SOLUTION Here is the graph of a function f for which $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$.



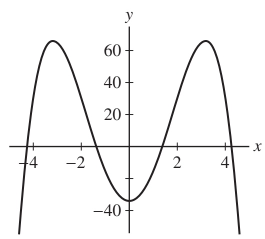
17. $f'(x)$ is negative on $(1, 3)$ and positive everywhere else.

SOLUTION Here is the graph of a function f for which $f'(x)$ is negative on $(1, 3)$ and positive elsewhere.



18. $f'(x)$ makes the sign transitions $+, -, +, -$.

SOLUTION Here is the graph of a function f for which f' makes the sign transitions $+, -, +, -$.



In Exercises 19–22, find all critical points of f and use the First Derivative Test to determine whether they are local minima or maxima.

19. $f(x) = 4 + 6x - x^2$

SOLUTION Let $f(x) = 4 + 6x - x^2$. Then $f'(x) = 6 - 2x = 0$ implies that $x = 3$ is the only critical point of f. As x increases through 3, $f'(x)$ makes the sign transition $+, -$. Therefore, $f(3) = 13$ is a local maximum.

20. $f(x) = x^3 - 12x - 4$

SOLUTION Let $f(x) = x^3 - 12x - 4$. Then, $f'(x) = 3x^2 - 12 = 3(x-2)(x+2) = 0$ implies that $x = \pm 2$ are critical points of f. As x increases through -2, $f'(x)$ makes the sign transition $+, -$; therefore, $f(-2)$ is a local maximum. On the other hand, as x increases through 2, $f'(x)$ makes the sign transition $-, +$; therefore, $f(2)$ is a local minimum.

21. $f(x) = \dfrac{x^2}{x+1}$

SOLUTION Let $f(x) = \dfrac{x^2}{x+1}$. Then

$$f'(x) = \frac{x(x+2)}{(x+1)^2} = 0$$

implies that $x = 0$ and $x = -2$ are critical points. Note that $x = -1$ is not a critical point because it is not in the domain of f. As x increases through -2, $f'(x)$ makes the sign transition $+, -$ so $f(-2) = -4$ is a local maximum. As x increases through 0, $f'(x)$ makes the sign transition $-, +$ so $f(0) = 0$ is a local minimum.

22. $f(x) = x^3 + x^{-3}$

SOLUTION Let $f(x) = x^3 + x^{-3}$. Then

$$f'(x) = 3x^2 - 3x^{-4} = \frac{3}{x^4}(x^6 - 1) = \frac{3}{x^4}(x-1)(x+1)(x^2 - x + 1)(x^2 + x + 1) = 0$$

implies that $x = \pm 1$ are critical points of f. Though $f'(x)$ does not exist at $x = 0$, $x = 0$ is not a critical point of f because it is not in the domain of f. As x increases through -1, $f'(x)$ makes the sign transition $+, -$; therefore, $f(-1)$ is a local maximum. On the other hand, as x increases through 1, $f'(x)$ makes the sign transition $-, +$; therefore, $f(1)$ is a local minimum.

In Exercises 23–52, find the critical points and the intervals on which the function is increasing or decreasing. Use the First Derivative Test to determine whether the critical point is a local min or max (or neither).

SOLUTION *Here is a table legend for Exercises 23–44.*

SYMBOL	*MEANING*
$-$	*The entity is negative on the given interval.*
0	*The entity is zero at the specified point.*
$+$	*The entity is positive on the given interval.*
U	*The entity is undefined at the specified point.*
↗	*f is increasing on the given interval.*
↘	*f is decreasing on the given interval.*
M	*f has a local maximum at the specified point.*
m	*f has a local minimum at the specified point.*
¬	*There is no local extremum here.*

23. $y = -x^2 + 7x - 17$

SOLUTION Let $f(x) = -x^2 + 7x - 17$. Then $f'(x) = 7 - 2x = 0$ yields the critical point $c = \frac{7}{2}$.

x	$\left(-\infty, \frac{7}{2}\right)$	$7/2$	$\left(\frac{7}{2}, \infty\right)$
f'	$+$	0	$-$
f	↗	M	↘

24. $y = 5x^2 + 6x - 4$

SOLUTION Let $f(x) = 5x^2 + 6x - 4$. Then $f'(x) = 10x + 6 = 0$ yields the critical point $c = -\frac{3}{5}$.

x	$\left(-\infty, -\frac{3}{5}\right)$	$-3/5$	$\left(-\frac{3}{5}, \infty\right)$
f'	$-$	0	$+$
f	↘	m	↗

25. $y = x^3 - 12x^2$

SOLUTION Let $f(x) = x^3 - 12x^2$. Then $f'(x) = 3x^2 - 24x = 3x(x - 8) = 0$ yields critical points $c = 0, 8$.

x	$(-\infty, 0)$	0	$(0, 8)$	8	$(8, \infty)$
f'	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗

26. $y = x(x - 2)^3$

SOLUTION Let $f(x) = x\,(x - 2)^3$. Then

$$f'(x) = x \cdot 3\,(x-2)^2 + (x-2)^3 \cdot 1 = (4x - 2)\,(x-2)^2 = 0$$

yields critical points $c = 2, \frac{1}{2}$.

x	$(-\infty, 1/2)$	$1/2$	$(1/2, 2)$	2	$(2, \infty)$
f'	$-$	0	$+$	0	$+$
f	↘	m	↗	¬	↗

27. $y = 3x^4 + 8x^3 - 6x^2 - 24x$

SOLUTION Let $f(x) = 3x^4 + 8x^3 - 6x^2 - 24x$. Then

$$\begin{aligned} f'(x) &= 12x^3 + 24x^2 - 12x - 24 \\ &= 12x^2(x+2) - 12(x+2) = 12(x+2)(x^2-1) \\ &= 12\,(x-1)\,(x+1)\,(x+2) = 0 \end{aligned}$$

yields critical points $c = -2, -1, 1$.

x	$(-\infty, -2)$	-2	$(-2, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
f'	$-$	0	$+$	0	$-$	0	$+$
f	↘	m	↗	M	↘	m	↗

28. $y = x^2 + (10 - x)^2$

SOLUTION Let $f(x) = x^2 + (10 - x)^2$. Then $f'(x) = 2x + 2\,(10 - x)\,(-1) = 4x - 20 = 0$ yields the critical point $c = 5$.

x	$(-\infty, 5)$	5	$(5, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

29. $y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + 4$

SOLUTION Let $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + 4$. Then $f'(x) = x^2 + 3x + 2 = (x+1)(x+2) = 0$ yields critical points $c = -2, -1$.

x	$(-\infty, -2)$	-2	$(-2, -1)$	-1	$(-1, \infty)$
f'	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗

30. $y = x^4 + x^3$

SOLUTION Let $f(x) = x^4 + x^3$. Then $f'(x) = 4x^3 + 3x^2 = x^2(4x+3)$ yields critical points $c = 0, -\frac{3}{4}$.

x	$\left(-\infty, -\frac{3}{4}\right)$	$-\frac{3}{4}$	$\left(-\frac{3}{4}, 0\right)$	0	$(0, \infty)$
f'	$-$	0	$+$	0	$+$
f	↘	m	↗	¬	↗

31. $y = x^5 + x^3 + 1$

SOLUTION Let $f(x) = x^5 + x^3 + 1$. Then $f'(x) = 5x^4 + 3x^2 = x^2(5x^2 + 3)$ yields a single critical point: $c = 0$.

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	$+$	0	$+$
f	↗	¬	↗

32. $y = x^5 + x^3 + x$

SOLUTION Let $f(x) = x^5 + x^3 + x$. Then $f'(x) = 5x^4 + 3x^2 + 1 \geq 1$ for all x. Thus, f has no critical points and is always increasing.

33. $y = x^4 - 4x^{3/2} \quad (x > 0)$

SOLUTION Let $f(x) = x^4 - 4x^{3/2}$ for $x > 0$. Then $f'(x) = 4x^3 - 6x^{1/2} = 2x^{1/2}(2x^{5/2} - 3) = 0$, which gives us the critical point $c = (\frac{3}{2})^{2/5}$. (*Note:* $c = 0$ is not in the interval under consideration.)

x	$\left(0, (\frac{3}{2})^{2/5}\right)$	$\frac{3}{2}^{2/5}$	$\left((\frac{3}{2})^{2/5}, \infty\right)$
f'	$-$	0	$+$
f	↘	m	↗

34. $y = x^{5/2} - x^2 \quad (x > 0)$

SOLUTION Let $f(x) = x^{5/2} - x^2$. Then $f'(x) = \frac{5}{2}x^{3/2} - 2x = x(\frac{5}{2}x^{1/2} - 2) = 0$, so the critical point is $c = \frac{16}{25}$. (*Note:* $c = 0$ is not in the interval under consideration.)

x	$(0, \frac{16}{25})$	$\frac{16}{25}$	$(\frac{16}{25}, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

35. $y = x + x^{-1} \quad (x > 0)$

SOLUTION Let $f(x) = x + x^{-1}$ for $x > 0$. Then $f'(x) = 1 - x^{-2} = 0$ yields the critical point $c = 1$. (*Note:* $c = -1$ is not in the interval under consideration.)

x	$(0, 1)$	1	$(1, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

36. $y = x^{-2} - 4x^{-1} \quad (x > 0)$

SOLUTION Let $f(x) = x^{-2} - 4x^{-1}$. Then $f'(x) = -2x^{-3} + 4x^{-2} = 0$ yields $-2 + 4x = 0$. Thus, $2x = 1$, and $x = \frac{1}{2}$.

x	$(0, \frac{1}{2})$	$\frac{1}{2}$	$(\frac{1}{2}, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

37. $y = \dfrac{1}{x^2 + 1}$

SOLUTION Let $f(x) = \left(x^2 + 1\right)^{-1}$. Then $f'(x) = -2x\left(x^2 + 1\right)^{-2} = 0$ yields critical point $c = 0$.

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	$+$	0	$-$
f	↗	M	↘

38. $y = \dfrac{2x + 1}{x^2 + 1}$

SOLUTION Let $f(x) = \dfrac{2x + 1}{x^2 + 1}$. Then

$$f'(x) = \frac{\left(x^2 + 1\right)(2) - (2x + 1)(2x)}{\left(x^2 + 1\right)^2} = \frac{-2\left(x^2 + x - 1\right)}{\left(x^2 + 1\right)^2} = 0$$

yields critical points $c = \dfrac{-1 \pm \sqrt{5}}{2}$.

x	$\left(-\infty, \frac{-1-\sqrt{5}}{2}\right)$	$\frac{-1-\sqrt{5}}{2}$	$\left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$	$\frac{-1+\sqrt{5}}{2}$	$\left(\frac{-1+\sqrt{5}}{2}, \infty\right)$
f'	$-$	0	$+$	0	$-$
f	↘	m	↗	M	↘

39. $y = \dfrac{x^3}{x^2 + 1}$

SOLUTION Let $f(x) = \dfrac{x^3}{x^2 + 1}$. Then

$$f'(x) = \frac{(x^2 + 1)(3x^2) - x^3(2x)}{(x^2 + 1)^2} = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2} = 0$$

yields the single critical point $c = 0$.

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	$+$	0	$+$
f	↗	¬	↗

40. $y = \dfrac{x^3}{x^2 - 3}$

SOLUTION Let $f(x) = \dfrac{x^3}{x^2 - 3}$. Then

$$f'(x) = \frac{(x^2 - 3)(3x^2) - x^3(2x)}{(x^2 - 3)^2} = \frac{x^2(x^2 - 9)}{(x^2 - 3)^2} = 0$$

yields the critical points $c = 0$ and $c = \pm 3$. $c = \pm\sqrt{3}$ are not critical points because they are not in the domain of f.

x	$(-\infty, -3)$	-3	$(-3, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, 0)$	0	$(0, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, 3)$	3	$(3, \infty)$
f'	$+$	0	$-$	∞	$-$	0	$-$	∞	$-$	0	$+$
f	↗	M	↘	¬	↘	¬	↘	¬	↘	m	↗

41. $y = \theta + \sin\theta + \cos\theta$

SOLUTION Let $f(\theta) = \theta + \sin\theta + \cos\theta$. Then $f'(\theta) = 1 + \cos\theta - \sin\theta = 0$ yields the critical points $c = \frac{\pi}{2}$ and $c = \pi$.

θ	$\left(0, \frac{\pi}{2}\right)$	$\frac{\pi}{2}$	$\left(\frac{\pi}{2}, \pi\right)$	π	$(\pi, 2\pi)$
f'	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗

42. $y = \sin\theta + \sqrt{3}\cos\theta$

SOLUTION Let $f(\theta) = \sin\theta + \sqrt{3}\cos\theta$. Then $f'(\theta) = \cos\theta - \sqrt{3}\sin\theta = 0$ yields the critical points $c = \frac{\pi}{6}$ and $c = \frac{7\pi}{6}$.

θ	$\left(0, \frac{\pi}{6}\right)$	$\frac{\pi}{6}$	$\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$	$\frac{7\pi}{6}$	$\left(\frac{7\pi}{6}, 2\pi\right)$
f'	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗

43. $y = \sin^2\theta + \sin\theta$

SOLUTION Let $f(\theta) = \sin^2\theta + \sin\theta$. Then $f'(\theta) = 2\sin\theta\cos\theta + \cos\theta = \cos\theta(2\sin\theta + 1) = 0$ yields the critical points $c = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$, and $\frac{11\pi}{6}$.

θ	$\left(0, \frac{\pi}{2}\right)$	$\frac{\pi}{2}$	$\left(\frac{\pi}{2}, \frac{7\pi}{6}\right)$	$\frac{7\pi}{6}$	$\left(\frac{7\pi}{6}, \frac{3\pi}{2}\right)$	$\frac{3\pi}{2}$	$\left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$	$\frac{11\pi}{6}$	$\left(\frac{11\pi}{6}, 2\pi\right)$
f'	$+$	0	$-$	0	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗	M	↘	m	↗

44. $y = \theta - 2\cos\theta, \quad [0, 2\pi]$

SOLUTION Let $f(\theta) = \theta - 2\cos\theta$. Then $f'(\theta) = 1 + 2\sin\theta = 0$, which yields $c = \frac{7\pi}{6}, \frac{11\pi}{6}$ on the interval $[0, 2\pi]$.

θ	$\left(0, \frac{7\pi}{6}\right)$	$\frac{7\pi}{6}$	$\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$	$\frac{11\pi}{6}$	$\left(\frac{11\pi}{6}, 2\pi\right)$
f'	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗

45. $y = x + e^{-x}$

SOLUTION Let $f(x) = x + e^{-x}$. Then $f'(x) = 1 - e^{-x}$, which yields $c = 0$ as the only critical point.

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

46. $y = \dfrac{e^x}{x} \quad (x > 0)$

SOLUTION Let $f(x) = \frac{e^x}{x}$. Then

$$f'(x) = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2},$$

which yields $c = 1$ as the only critical point.

x	$(0, 1)$	1	$(1, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

47. $y = e^{-x}\cos x, \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

SOLUTION Let $f(x) = e^{-x}\cos x$. Then

$$f'(x) = -e^{-x}\sin x - e^{-x}\cos x = -e^{-x}(\sin x + \cos x),$$

which yields $c = -\frac{\pi}{4}$ as the only critical point on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

x	$\left[-\frac{\pi}{2}, -\frac{\pi}{4}\right)$	$-\frac{\pi}{4}$	$\left(-\frac{\pi}{4}, \frac{\pi}{2}\right]$
f'	$+$	0	$-$
f	$\nearrow$	M	$\searrow$

48. $y = x^2 e^x$

SOLUTION Let $f(x) = x^2e^x$. Then $f'(x) = x^2e^x + 2xe^x = xe^x(x+2)$, which yields $c = -2$ and $c = 0$ as critical points.

x	$(-\infty, -2)$	-2	$(-2, 0)$	0	$(0, \infty)$
f'	$+$	0	$-$	0	$+$
f	$\nearrow$	M	$\searrow$	m	$\nearrow$

49. $y = \tan^{-1} x - \frac{1}{2}x$

SOLUTION Let $f(x) = \tan^{-1} x - \frac{1}{2}x$. Then

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{2},$$

which yields $c = \pm 1$ as critical points.

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
f'	$-$	0	$+$	0	$-$
f	$\searrow$	m	$\nearrow$	M	$\searrow$

50. $y = (x^2 - 2x)e^x$

SOLUTION Let $f(x) = (x^2 - 2x)e^x$. Then

$$f'(x) = (x^2 - 2x)e^x + (2x - 2)e^x = (x^2 - 2)e^x,$$

which yields $c = \pm\sqrt{2}$ as critical points.

x	$(-\infty, \sqrt{2})$	$-\sqrt{2}$	$(-\sqrt{2}, \sqrt{2})$	$\sqrt{2}$	$(\sqrt{2}, \infty)$
f'	$+$	0	$-$	0	$+$
f	$\nearrow$	M	$\searrow$	m	$\nearrow$

51. $y = x - \ln x \quad (x > 0)$

SOLUTION Let $f(x) = x - \ln x$. Then $f'(x) = 1 - x^{-1}$, which yields $c = 1$ as the only critical point.

x	$(0, 1)$	1	$(1, \infty)$
f'	$-$	0	$+$
f	$\searrow$	m	$\nearrow$

52. $y = \dfrac{\ln x}{x} \quad (x > 0)$

SOLUTION Let $f(x) = \frac{\ln x}{x}$. Then

$$f'(x) = \frac{1 - \ln x}{x^2},$$

which yields $c = e$ as the only critical point.

x	$(0, e)$	e	(e, ∞)
f'	$+$	0	$-$
f	↗	M	↘

53. Find the minimum value of $f(x) = x^x$ for $x > 0$.

SOLUTION Let $f(x) = x^x$. By logarithmic differentiation, we know that $f'(x) = x^x(1 + \ln x)$. Thus, $x = \frac{1}{e}$ is the only critical point. Because $f'(x) < 0$ for $0 < x < \frac{1}{e}$ and $f'(x) > 0$ for $x > \frac{1}{e}$,

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} \approx 0.692201$$

is the minimum value.

54. Show that $f(x) = x^2 + bx + c$ is decreasing on $\left(-\infty, -\frac{b}{2}\right)$ and increasing on $\left(-\frac{b}{2}, \infty\right)$.

SOLUTION Let $f(x) = x^2 + bx + c$. Then $f'(x) = 2x + b = 0$ yields the critical point $c = -\frac{b}{2}$.

- For $x < -\frac{b}{2}$, we have $f'(x) < 0$, so f is decreasing on $\left(-\infty, -\frac{b}{2}\right)$.
- For $x > -\frac{b}{2}$, we have $f'(x) > 0$, so f is increasing on $\left(-\frac{b}{2}, \infty\right)$.

55. Show that $f(x) = x^3 - 2x^2 + 2x$ is an increasing function. *Hint:* Find the minimum value of $f'(x)$.

SOLUTION Let $f(x) = x^3 - 2x^2 + 2x$. For all x, we have

$$f'(x) = 3x^2 - 4x + 2 = 3\left(x - \frac{2}{3}\right)^2 + \frac{2}{3} \geq \frac{2}{3} > 0.$$

Since $f'(x) > 0$ for all x, the function f is everywhere increasing.

56. Find conditions on a and b that ensure that $f(x) = x^3 + ax + b$ is increasing on $(-\infty, \infty)$.

SOLUTION Let $f(x) = x^3 + ax + b$.

- If $a > 0$, then $f'(x) = 3x^2 + a > 0$ and f is increasing for all x.
- If $a = 0$, then

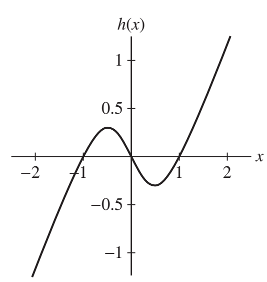
$$f(x_2) - f(x_1) = (3x_2^3 + b) - (3x_1^3 + b) = 3(x_2 - x_1)(x_2^2 + x_2x_1 + x_1^2) > 0$$

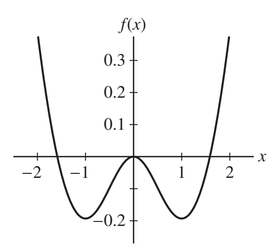
  whenever $x_2 > x_1$. Thus, f is increasing for all x.
- If $a < 0$, then $f'(x) = 3x^2 + a < 0$ and f is decreasing for $|x| < \sqrt{-\frac{a}{3}}$.

In summary, $f(x) = x^3 + ax + b$ is increasing on $(-\infty, \infty)$ whenever $a \geq 0$.

57. GU Let $h(x) = \dfrac{x(x^2 - 1)}{x^2 + 1}$ and suppose that $f'(x) = h(x)$. Plot $h(x)$ and use the plot to describe the local extrema and the increasing/decreasing behavior of $f(x)$. Sketch a plausible graph for $f(x)$ itself.

SOLUTION The graph of $h(x)$ is shown below at the left. Because $h(x)$ is negative for $x < -1$ and for $0 < x < 1$, it follows that $f(x)$ is decreasing for $x < -1$ and for $0 < x < 1$. Similarly, $f(x)$ is increasing for $-1 < x < 0$ and for $x > 1$ because $h(x)$ is positive on these intervals. Moreover, $f(x)$ has local minima at $x = -1$ and $x = 1$ and a local maximum at $x = 0$. A plausible graph for $f(x)$ is shown below at the right.





58. Sam made two statements that Deborah found dubious.

(a) "The average velocity for my trip was 70 mph; at no point in time did my speedometer read 70 mph."

(b) "A policeman clocked me going 70 mph, but my speedometer never read 65 mph."

In each case, which theorem did Deborah apply to prove Sam's statement false: the Intermediate Value Theorem or the Mean Value Theorem? Explain.

SOLUTION
(a) Deborah is applying the Mean Value Theorem here. Let $s(t)$ be Sam's distance, in miles, from his starting point, let a be the start time for Sam's trip, and let b be the end time of the same trip. Sam is claiming that at no point was

$$s'(t) = \frac{s(b) - s(a)}{b - a}.$$

This violates the MVT.
(b) Deborah is applying the Intermediate Value Theorem here. Let $v(t)$ be Sam's velocity in miles per hour. Sam started out at rest, and reached a velocity of 70 mph. By the IVT, he should have reached a velocity of 65 mph at some point.

59. Determine where $f(x) = (1000 - x)^2 + x^2$ is decreasing. Use this to decide which is larger: $800^2 + 200^2$ or $600^2 + 400^2$.

SOLUTION If $f(x) = (1000 - x)^2 + x^2$, then $f'(x) = -2(1000 - x) + 2x = 4x - 2000$. $f'(x) < 0$ as long as $x < 500$. Therefore, $800^2 + 200^2 = f(200) > f(400) = 600^2 + 400^2$.

60. Show that $f(x) = 1 - |x|$ satisfies the conclusion of the MVT on $[a, b]$ if both a and b are positive or negative, but not if $a < 0$ and $b > 0$.

SOLUTION Let $f(x) = 1 - |x|$.

- If a and b (where $a < b$) are both positive (or both negative), then f is continuous on $[a, b]$ and differentiable on (a, b). Accordingly, the hypotheses of the MVT are met and the theorem does apply. Indeed, in these cases, any point $c \in (a, b)$ satisfies the conclusion of the MVT (since f' is constant on $[a, b]$ in these instances).
- For $a = -2$ and $b = 1$, we have $\frac{f(b) - f(a)}{b - a} = \frac{0 - (-1)}{1 - (-2)} = \frac{1}{3}$. Yet there is no point $c \in (-2, 1)$ such that $f'(c) = \frac{1}{3}$. Indeed, $f'(x) = 1$ for $x < 0$, $f'(x) = -1$ for $x > 0$, and $f'(0)$ is undefined. The MVT does not apply in this case, since f is not differentiable on the open interval $(-2, 1)$.

61. Which values of c satisfy the conclusion of the MVT on the interval $[a, b]$ if $f(x)$ is a linear function?

SOLUTION Let $f(x) = px + q$, where p and q are constants. Then the slope of *every* secant line and tangent line of f is p. Accordingly, considering the interval $[a, b]$, *every* point $c \in (a, b)$ satisfies $f'(c) = p = \frac{f(b) - f(a)}{b - a}$, the conclusion of the MVT.

62. Show that if $f(x)$ is any quadratic polynomial, then the midpoint $c = \frac{a + b}{2}$ satisfies the conclusion of the MVT on $[a, b]$ for any a and b.

SOLUTION Let $f(x) = px^2 + qx + r$ with $p \neq 0$ and consider the interval $[a, b]$. Then $f'(x) = 2px + q$, and by the MVT we have

$$2pc + q = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\left(pb^2 + qb + r\right) - \left(pa^2 + qa + r\right)}{b - a}$$
$$= \frac{(b - a)(p(b + a) + q)}{b - a} = p(b + a) + q$$

Thus $2pc + q = p(a + b) + q$, and $c = \frac{a + b}{2}$.

63. Suppose that $f(0) = 2$ and $f'(x) \leq 3$ for $x > 0$. Apply the MVT to the interval $[0, 4]$ to prove that $f(4) \leq 14$. Prove more generally that $f(x) \leq 2 + 3x$ for all $x > 0$.

SOLUTION The MVT, applied to the interval $[0, 4]$, guarantees that there exists a $c \in (0, 4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} \qquad \text{or} \qquad f(4) - f(0) = 4f'(c).$$

Because $c > 0$, $f'(c) \leq 3$, so $f(4) - f(0) \leq 12$. Finally, $f(4) \leq f(0) + 12 = 14$.

More generally, let $x > 0$. The MVT, applied to the interval $[0, x]$, guarantees there exists a $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \qquad \text{or} \qquad f(x) - f(0) = f'(c)x.$$

Because $c > 0$, $f'(c) \leq 3$, so $f(x) - f(0) \leq 3x$. Finally, $f(x) \leq f(0) + 3x = 3x + 2$.

64. Show that if $f(2) = -2$ and $f'(x) \geq 5$ for $x > 2$, then $f(4) \geq 8$.

SOLUTION The MVT, applied to the interval $[2, 4]$, guarantees there exists a $c \in (2, 4)$ such that

$$f'(c) = \frac{f(4) - f(2)}{4 - 2} \qquad \text{or} \qquad f(4) - f(2) = 2f'(c).$$

Because $f'(x) \geq 5$, it follows that $f(4) - f(2) \geq 10$, or $f(4) \geq f(2) + 10 = 8$.

65. Show that if $f(2) = 5$ and $f'(x) \geq 10$ for $x > 2$, then $f(x) \geq 10x - 15$ for all $x > 2$.

SOLUTION Let $x > 2$. The MVT, applied to the interval $[2, x]$, guarantees there exists a $c \in (2, x)$ such that

$$f'(c) = \frac{f(x) - f(2)}{x - 2} \qquad \text{or} \qquad f(x) - f(2) = (x - 2)f'(c).$$

Because $f'(x) \geq 10$, it follows that $f(x) - f(2) \geq 10(x - 2)$, or $f(x) \geq f(2) + 10(x - 2) = 10x - 15$.

Further Insights and Challenges

66. Show that a cubic function $f(x) = x^3 + ax^2 + bx + c$ is increasing on $(-\infty, \infty)$ if $b > a^2/3$.

SOLUTION Let $f(x) = x^3 + ax^2 + bx + c$. Then $f'(x) = 3x^2 + 2ax + b = 3\left(x + \frac{a}{3}\right)^2 - \frac{a^2}{3} + b > 0$ for all x if $b - \frac{a^2}{3} > 0$. Therefore, if $b > a^2/3$, then $f(x)$ is increasing on $(-\infty, \infty)$.

67. Prove that if $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for $x \geq 0$, then $f(x) \leq g(x)$ for all $x \geq 0$. *Hint:* Show that $f(x) - g(x)$ is nonincreasing.

SOLUTION Let $h(x) = f(x) - g(x)$. By the sum rule, $h'(x) = f'(x) - g'(x)$. Since $f'(x) \leq g'(x)$ for all $x \geq 0$, $h'(x) \leq 0$ for all $x \geq 0$. This implies that h is nonincreasing. Since $h(0) = f(0) - g(0) = 0$, $h(x) \leq 0$ for all $x \geq 0$ (as h is nonincreasing, it cannot climb above zero). Hence $f(x) - g(x) \leq 0$ for all $x \geq 0$, and so $f(x) \leq g(x)$ for $x \geq 0$.

68. Use Exercise 67 to prove that $x \leq \tan x$ for $0 \leq x < \frac{\pi}{2}$.

SOLUTION Let $f(x) = x$ and $g(x) = \tan x$. Then $f(0) = g(0) = 0$ and $f'(x) = 1 \leq \sec^2 x = g'(x)$ for $0 \leq x < \frac{\pi}{2}$. Apply the result of Exercise 67 to conclude that $x \leq \tan x$ for $0 \leq x < \frac{\pi}{2}$.

69. Use Exercise 67 and the inequality $\sin x \leq x$ for $x \geq 0$ (established in Theorem 3 of Section 2.6) to prove the following assertions for all $x \geq 0$ (each assertion follows from the previous one).

(a) $\cos x \geq 1 - \frac{1}{2}x^2$

(b) $\sin x \geq x - \frac{1}{6}x^3$

(c) $\cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

(d) Can you guess the next inequality in the series?

SOLUTION

(a) We prove this using Exercise 67: Let $g(x) = \cos x$ and $f(x) = 1 - \frac{1}{2}x^2$. Then $f(0) = g(0) = 1$ and $g'(x) = -\sin x \geq -x = f'(x)$ for $x \geq 0$ by Exercise 68. Now apply Exercise 67 to conclude that $\cos x \geq 1 - \frac{1}{2}x^2$ for $x \geq 0$.

(b) Let $g(x) = \sin x$ and $f(x) = x - \frac{1}{6}x^3$. Then $f(0) = g(0) = 0$ and $g'(x) = \cos x \geq 1 - \frac{1}{2}x^2 = f'(x)$ for $x \geq 0$ by part (a). Now apply Exercise 67 to conclude that $\sin x \geq x - \frac{1}{6}x^3$ for $x \geq 0$.

(c) Let $g(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ and $f(x) = \cos x$. Then $f(0) = g(0) = 1$ and $g'(x) = -x + \frac{1}{6}x^3 \geq -\sin x = f'(x)$ for $x \geq 0$ by part (b). Now apply Exercise 67 to conclude that $\cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ for $x \geq 0$.

(d) The next inequality in the series is $\sin x \leq x - \frac{1}{6}x^3 + \frac{1}{120}x^5$, valid for $x \geq 0$. To construct (d) from (c), we note that the derivative of $\sin x$ is $\cos x$, and look for a polynomial (which we currently must do by educated guess) whose derivative is $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$. We know the derivative of x is 1, and that a term whose derivative is $-\frac{1}{2}x^2$ should be of the form Cx^3. $\frac{d}{dx}Cx^3 = 3Cx^2 = -\frac{1}{2}x^2$, so $C = -\frac{1}{6}$. A term whose derivative is $\frac{1}{24}x^4$ should be of the form Dx^5. From this, $\frac{d}{dx}Dx^5 = 5Dx^4 = \frac{1}{24}x^4$, so that $5D = \frac{1}{24}$, or $D = \frac{1}{120}$.

70. Let $f(x) = e^{-x}$. Use the method of Exercise 69 to prove the following inequalities for $x \geq 0$.

(a) $e^{-x} \geq 1 - x$

(b) $e^{-x} \leq 1 - x + \frac{1}{2}x^2$

(c) $e^{-x} \geq 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$

Can you guess the next inequality in the series?

SOLUTION

(a) Let $f(x) = 1 - x$ and $g(x) = e^{-x}$. Then $f(0) = g(0) = 1$ and, for $x \geq 0$,

$$f'(x) = -1 \leq -e^{-x} = g'(x).$$

Thus, by Exercise 67 we conclude that $e^{-x} \geq 1 - x$ for $x \geq 0$.

(b) Let $f(x) = e^{-x}$ and $g(x) = 1 - x + \frac{1}{2}x^2$. Then $f(0) = g(0) = 1$ and, for $x \geq 0$,

$$f'(x) = -e^{-x} \leq x - 1 = g'(x)$$

by the result from part (a). Thus, by Exercise 67 we conclude that $e^{-x} \leq 1 - x + \dfrac{1}{2}x^2$ for $x \geq 0$.

(c) Let $f(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$ and $g(x) = e^{-x}$. Then $f(0) = g(0) = 1$ and, for $x \geq 0$,

$$f'(x) = -1 + x - \frac{1}{2}x^2 \leq -e^{-x} = g'(x)$$

by the result from part (b). Thus, by Exercise 67 we conclude that $e^{-x} \geq 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$ for $x \geq 0$.

The next inequality in the series is $e^{-x} \leq 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4$ for $x \geq 0$

71. Assume that f'' exists and $f''(x) = 0$ for all x. Prove that $f(x) = mx + b$, where $m = f'(0)$ and $b = f(0)$.

SOLUTION

- Let $f''(x) = 0$ for all x. Then $f'(x) =$ constant for all x. Since $f'(0) = m$, we conclude that $f'(x) = m$ for all x.
- Let $g(x) = f(x) - mx$. Then $g'(x) = f'(x) - m = m - m = 0$ which implies that $g(x) =$ constant for all x and consequently $f(x) - mx =$ constant for all x. Rearranging the statement, $f(x) = mx +$ constant. Since $f(0) = b$, we conclude that $f(x) = mx + b$ for all x.

72. Define $f(x) = x^3 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$.

(a) Show that $f'(x)$ is continuous at $x = 0$ and that $x = 0$ is a critical point of f.

(b) GU Examine the graphs of $f(x)$ and $f'(x)$. Can the First Derivative Test be applied?

(c) Show that $f(0)$ is neither a local min nor a local max.

SOLUTION

(a) Let $f(x) = x^3 \sin(\frac{1}{x})$. Then

$$f'(x) = 3x^2 \sin\left(\frac{1}{x}\right) + x^3 \cos\left(\frac{1}{x}\right)(-x^{-2}) = x\left(3x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right).$$

This formula is not defined at $x = 0$, but its limit is. Since $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ for all x,

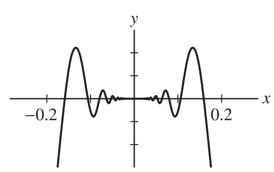
$$|f'(x)| = |x|\left|3x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right| \leq |x|\left(\left|3x \sin\left(\frac{1}{x}\right)\right| + \left|\cos\left(\frac{1}{x}\right)\right|\right) \leq |x|(3|x| + 1)$$

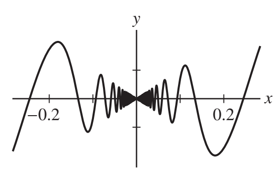
so, by the Squeeze Theorem, $\lim_{x\to 0} |f'(x)| = 0$. But does $f'(0) = 0$? We check using the limit definition of the derivative:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Thus $f'(x)$ is continuous at $x = 0$, and $x = 0$ is a critical point of f.

(b) The figure below at the left shows $f(x)$, and the figure below at the right shows $f'(x)$. Note how the two functions oscillate near $x = 0$, which implies that the First Derivative Test cannot be applied.





(c) As x approaches 0 from either direction, $f(x)$ alternates between positive and negative arbitrarily close to $x = 0$. This means that $f(0)$ cannot be a local minimum (since $f(x)$ gets lower than $f(0)$ arbitrarily close to 0), nor can $f(0)$ be a local maximum (since $f(x)$ takes values higher than $f(0)$ arbitrarily close to $x = 0$). Therefore $f(0)$ is neither a local minimum nor a local maximum of f.

73. Suppose that $f(x)$ satisfies the following equation (an example of a **differential equation**):

$$f''(x) = -f(x) \qquad \boxed{1}$$

(a) Show that $f(x)^2 + f'(x)^2 = f(0)^2 + f'(0)^2$ for all x. *Hint:* Show that the function on the left has zero derivative.

(b) Verify that $\sin x$ and $\cos x$ satisfy Eq. (1), and deduce that $\sin^2 x + \cos^2 x = 1$.

SOLUTION

(a) Let $g(x) = f(x)^2 + f'(x)^2$. Then

$$g'(x) = 2f(x)f'(x) + 2f'(x)f''(x) = 2f(x)f'(x) + 2f'(x)(-f(x)) = 0,$$

where we have used the fact that $f''(x) = -f(x)$. Because $g'(0) = 0$ for all x, $g(x) = f(x)^2 + f'(x)^2$ must be a constant function. In other words, $f(x)^2 + f'(x)^2 = C$ for some constant C. To determine the value of C, we can substitute any number for x. In particular, for this problem, we want to substitute $x = 0$ and find $C = f(0)^2 + f'(0)^2$. Hence,

$$f(x)^2 + f'(x)^2 = f(0)^2 + f'(0)^2.$$

(b) Let $f(x) = \sin x$. Then $f'(x) = \cos x$ and $f''(x) = -\sin x$, so $f''(x) = -f(x)$. Next, let $f(x) = \cos x$. Then $f'(x) = -\sin x$, $f''(x) = -\cos x$, and we again have $f''(x) = -f(x)$. Finally, if we take $f(x) = \sin x$, the result from part (a) guarantees that

$$\sin^2 x + \cos^2 x = \sin^2 0 + \cos^2 0 = 0 + 1 = 1.$$

74. Suppose that functions f and g satisfy Eq. (1) and have the same initial values—that is, $f(0) = g(0)$ and $f'(0) = g'(0)$. Prove that $f(x) = g(x)$ for all x. *Hint:* Apply Exercise 73(a) to $f - g$.

SOLUTION Let $h(x) = f(x) - g(x)$. Then

$$h''(x) = f''(x) - g''(x) = -f(x) - (-g(x)) = -(f(x) - g(x)) = -h(x).$$

Furthermore, $h(0) = f(0) - g(0) = 0$ and $h'(0) = f'(0) - g'(0) = 0$. Thus, by part (a) of Exercise 73, $h(x)^2 + h'(x)^2 = 0$. This can only happen if $h(x) = 0$ for all x, or, equivalently, $f(x) = g(x)$ for all x.

75. Use Exercise 74 to prove: $f(x) = \sin x$ is the unique solution of Eq. (1) such that $f(0) = 0$ and $f'(0) = 1$; and $g(x) = \cos x$ is the unique solution such that $g(0) = 1$ and $g'(0) = 0$. This result can be used to develop all the properties of the trigonometric functions "analytically"—that is, without reference to triangles.

SOLUTION In part (b) of Exercise 73, it was shown that $f(x) = \sin x$ satisfies Eq. (1), and we can directly calculate that $f(0) = \sin 0 = 0$ and $f'(0) = \cos 0 = 1$. Suppose there is another function, call it $F(x)$, that satisfies Eq. (1) with the same initial conditions: $F(0) = 0$ and $F'(0) = 1$. By Exercise 74, it follows that $F(x) = \sin x$ for all x. Hence, $f(x) = \sin x$ is the unique solution of Eq. (1) satisfying $f(0) = 0$ and $f'(0) = 1$. The proof that $g(x) = \cos x$ is the unique solution of Eq. (1) satisfying $g(0) = 1$ and $g'(0) = 0$ is carried out in a similar manner.

4.4 The Shape of a Graph

Preliminary Questions

1. If f is concave up, then f' is (choose one):

(a) increasing **(b)** decreasing

SOLUTION The correct response is **(a)**: increasing. If the function is concave up, then f'' is positive. Since f'' is the derivative of f', it follows that the derivative of f' is positive and f' must therefore be increasing.

2. What conclusion can you draw if $f'(c) = 0$ and $f''(c) < 0$?

SOLUTION If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum.

3. True or False? If $f(c)$ is a local min, then $f''(c)$ must be positive.

SOLUTION False. $f''(c)$ could be zero.

4. True or False? If $f''(x)$ changes from $+$ to $-$ at $x = c$, then f has a point of inflection at $x = c$.

SOLUTION False. f will have a point of inflection at $x = c$ only if $x = c$ is in the domain of f.

Exercises

1. Match the graphs in Figure 13 with the description:

(a) $f''(x) < 0$ for all x.
(b) $f''(x)$ goes from $+$ to $-$.
(c) $f''(x) > 0$ for all x.
(d) $f''(x)$ goes from $-$ to $+$.

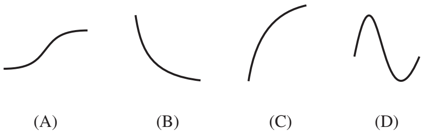


FIGURE 13

SOLUTION

(a) In C, we have $f''(x) < 0$ for all x.
(b) In A, $f''(x)$ goes from $+$ to $-$.
(c) In B, we have $f''(x) > 0$ for all x.
(d) In D, $f''(x)$ goes from $-$ to $+$.

2. Match each statement with a graph in Figure 14 that represents company profits as a function of time.

(a) The outlook is great: The growth rate keeps increasing.
(b) We're losing money, but not as quickly as before.
(c) We're losing money, and it's getting worse as time goes on.
(d) We're doing well, but our growth rate is leveling off.
(e) Business had been cooling off, but now it's picking up.
(f) Business had been picking up, but now it's cooling off.

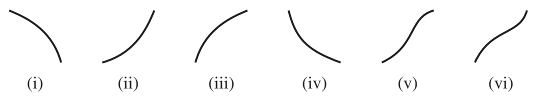


FIGURE 14

SOLUTION

(a) **(ii)** An increasing growth rate implies an increasing f', and so a graph that is concave up.

(b) **(iv)** "Losing money" implies a downward curve. "Not as fast" implies that f' is becoming less negative, so that $f''(x) > 0$.

(c) **(i)** "Losing money" implies a downward curve. "Getting worse" implies that f' is becoming more negative, so the curve is concave down.

(d) **(iii)** "We're doing well" implies that f is increasing, but "the growth rate is leveling off" implies that f' is decreasing, so that the graph is concave down.

(e) **(vi)** "Cooling off" generally means increasing at a decreasing rate. The use of "had" implies that only the beginning of the graph is that way. The phrase "...now it's picking up" implies that the end of the graph is concave up.

(f) **(v)** "Business had been picking up" implies that the graph started out concave up. The phrase "…but now it's cooling off" implies that the graph ends up concave down.

In Exercises 3–18, determine the intervals on which the function is concave up or down and find the points of inflection.

3. $y = x^2 - 4x + 3$

SOLUTION Let $f(x) = x^2 - 4x + 3$. Then $f'(x) = 2x - 4$ and $f''(x) = 2 > 0$ for all x. Therefore, f is concave up everywhere, and there are no points of inflection.

4. $y = t^3 - 6t^2 + 4$

SOLUTION Let $f(t) = t^3 - 6t^2 + 4$. Then $f'(t) = 3t^2 - 12t$ and $f''(t) = 6t - 12 = 0$ at $t = 2$. Now, f is concave up on $(2, \infty)$, since $f''(t) > 0$ there. Moreover, f is concave down on $(-\infty, 2)$, since $f''(t) < 0$ there. Finally, because $f''(t)$ changes sign at $t = 2$, $f(t)$ has a point of inflection at $t = 2$.

5. $y = 10x^3 - x^5$

SOLUTION Let $f(x) = 10x^3 - x^5$. Then $f'(x) = 30x^2 - 5x^4$ and $f''(x) = 60x - 20x^3 = 20x(3 - x^2)$. Now, f is concave up for $x < -\sqrt{3}$ and for $0 < x < \sqrt{3}$ since $f''(x) > 0$ there. Moreover, f is concave down for $-\sqrt{3} < x < 0$ and for $x > \sqrt{3}$ since $f''(x) < 0$ there. Finally, because $f''(x)$ changes sign at $x = 0$ and at $x = \pm\sqrt{3}$, $f(x)$ has a point of inflection at $x = 0$ and at $x = \pm\sqrt{3}$.

6. $y = 5x^2 + x^4$

SOLUTION Let $f(x) = 5x^2 + x^4$. Then $f'(x) = 10x + 4x^3$ and $f''(x) = 10 + 12x^2 > 10$ for all x. Thus, f is concave up for all x and has no points of inflection.

7. $y = \theta - 2\sin\theta, \quad [0, 2\pi]$

SOLUTION Let $f(\theta) = \theta - 2\sin\theta$. Then $f'(\theta) = 1 - 2\cos\theta$ and $f''(\theta) = 2\sin\theta$. Now, f is concave up for $0 < \theta < \pi$ since $f''(\theta) > 0$ there. Moreover, f is concave down for $\pi < \theta < 2\pi$ since $f''(\theta) < 0$ there. Finally, because $f''(\theta)$ changes sign at $\theta = \pi$, $f(\theta)$ has a point of inflection at $\theta = \pi$.

8. $y = \theta + \sin^2\theta, \quad [0, \pi]$

SOLUTION Let $f(\theta) = \theta + \sin^2\theta$. Then $f'(\theta) = 1 + 2\sin\theta\cos\theta = 1 + \sin 2\theta$ and $f''(\theta) = 2\cos 2\theta$. Now, f is concave up for $0 < \theta < \pi/4$ and for $3\pi/4 < \theta < \pi$ since $f''(\theta) > 0$ there. Moreover, f is concave down for $\pi/4 < \theta < 3\pi/4$ since $f''(\theta) < 0$ there. Finally, because $f''(\theta)$ changes sign at $\theta = \pi/4$ and at $\theta = 3\pi/4$, $f(\theta)$ has a point of inflection at $\theta = \pi/4$ and at $\theta = 3\pi/4$.

9. $y = x(x - 8\sqrt{x}) \quad (x \geq 0)$

SOLUTION Let $f(x) = x(x - 8\sqrt{x}) = x^2 - 8x^{3/2}$. Then $f'(x) = 2x - 12x^{1/2}$ and $f''(x) = 2 - 6x^{-1/2}$. Now, f is concave down for $0 < x < 9$ since $f''(x) < 0$ there. Moreover, f is concave up for $x > 9$ since $f''(x) > 0$ there. Finally, because $f''(x)$ changes sign at $x = 9$, $f(x)$ has a point of inflection at $x = 9$.

10. $y = x^{7/2} - 35x^2$

SOLUTION Let $f(x) = x^{7/2} - 35x^2$. Then

$$f'(x) = \frac{7}{2}x^{5/2} - 70x \quad \text{and} \quad f''(x) = \frac{35}{4}x^{3/2} - 70.$$

Now, f is concave down for $0 < x < 4$ since $f''(x) < 0$ there. Moreover, f is concave up for $x > 4$ since $f''(x) > 0$ there. Finally, because $f''(x)$ changes sign at $x = 4$, $f(x)$ has a point of inflection at $x = 4$.

11. $y = (x - 2)(1 - x^3)$

SOLUTION Let $f(x) = (x - 2)\left(1 - x^3\right) = x - x^4 - 2 + 2x^3$. Then $f'(x) = 1 - 4x^3 + 6x^2$ and $f''(x) = 12x - 12x^2 = 12x(1 - x) = 0$ at $x = 0$ and $x = 1$. Now, f is concave up on $(0, 1)$ since $f''(x) > 0$ there. Moreover, f is concave down on $(-\infty, 0) \cup (1, \infty)$ since $f''(x) < 0$ there. Finally, because $f''(x)$ changes sign at both $x = 0$ and $x = 1$, $f(x)$ has a point of inflection at both $x = 0$ and $x = 1$.

12. $y = x^{7/5}$

SOLUTION Let $f(x) = x^{7/5}$. Then $f'(x) = \frac{7}{5}x^{2/5}$ and $f''(x) = \frac{14}{25}x^{-3/5}$. Now, f is concave down for $x < 0$ since $f''(x) < 0$ there. Moreover, f is concave up for $x > 0$ since $f''(x) > 0$ there. Finally, because $f''(x)$ changes sign at $x = 0$, $f(x)$ has a point of inflection at $x = 0$.

13. $y = \dfrac{1}{x^2 + 3}$

SOLUTION Let $f(x) = \dfrac{1}{x^2 + 3}$. Then $f'(x) = -\dfrac{2x}{(x^2 + 3)^2}$ and

$$f''(x) = -\frac{2(x^2 + 3)^2 - 8x^2(x^2 + 3)}{(x^2 + 3)^4} = \frac{6x^2 - 6}{(x^2 + 3)^3}.$$

Now, f is concave up for $|x| > 1$ since $f''(x) > 0$ there. Moreover, f is concave down for $|x| < 1$ since $f''(x) < 0$ there. Finally, because $f''(x)$ changes sign at both $x = -1$ and $x = 1$, $f(x)$ has a point of inflection at both $x = -1$ and $x = 1$.

14. $y = \dfrac{x}{x^2 + 9}$

SOLUTION Let $f(x) = \frac{x}{x^2+9}$. Then

$$f'(x) = \frac{(x^2 + 9)(1) - x(2x)}{(x^2 + 9)^2} = \frac{9 - x^2}{(x^2 + 9)^2}$$

and

$$f''(x) = \frac{(x^2 + 9)^2(-2x) - (9 - x^2)(2)(x^2 + 9)(2x)}{(x^2 + 9)^4} = \frac{2x(x^2 - 27)}{(x^2 + 9)^3}.$$

Now, f is concave up for $-3\sqrt{3} < x < 0$ and for $x > 3\sqrt{3}$ since $f''(x) > 0$ there. Moreover, f is concave down for $x < -3\sqrt{3}$ and for $0 < x < 3\sqrt{3}$ since $f''(x) < 0$ there. Finally, because $f''(x)$ changes sign at $x = 0$ and at $x = \pm 3\sqrt{3}$, $f(x)$ has a point of inflection at $x = 0$ and at $x = \pm 3\sqrt{3}$.

15. $y = xe^{-3x}$

SOLUTION Let $f(x) = xe^{-3x}$. Then $f'(x) = -3xe^{-3x} + e^{-3x} = (1 - 3x)e^{-3x}$ and $f''(x) = -3(1 - 3x)e^{-3x} - 3e^{-3x} = (9x - 6)e^{-3x}$. Now, f is concave down for $x < \frac{2}{3}$ since $f''(x) < 0$ there. Moreover, f is concave up for $x > \frac{2}{3}$ since $f''(x) > 0$ there. Finally, because $f''(x)$ changes sign at $x = \frac{2}{3}$, $x = \frac{2}{3}$ is a point of inflection.

16. $y = (x^2 - 7)e^x$

SOLUTION Let $f(x) = (x^2 - 7)e^x$. Then $f'(x) = (x^2 - 7)e^x + 2xe^x = (x^2 + 2x07)e^x$ and $f''(x) = (x^2 + 2x - 7)e^x + (2x + 2)e^x = (x + 5)(x - 1)e^x$. Now, f is concave up for $x < -5$ and for $x > 1$ since $f''(x) > 0$ there. Moreover, f is concave down for $-5 < x < 1$ since $f''(x) < 0$ there. Finally, because $f''(x)$ changes sign at $x = -5$ and at $x = 1$, f has a point of inflection at $x = -5$ and at $x = 1$.

17. $y = 2x^2 + \ln x \quad (x > 0)$

SOLUTION Let $f(x) = 2x^2 + \ln x$. Then $f'(x) = 4x + x^{-1}$ and $f''(x) = 4 - x^{-2}$. Now, f is concave down for $x < \frac{1}{2}$ since $f''(x) < 0$ there. Moreover, f is concave up for $x > \frac{1}{2}$ since $f''(x) > 0$ there. Finally, because $f''(x)$ changes sign at $x = \frac{1}{2}$, f has a point of inflection at $x = \frac{1}{2}$.

18. $y = x - \ln x \quad (x > 0)$

SOLUTION Let $f(x) = x - \ln x$. Then $f'(x) = 1 - 1/x$ and $f''(x) = x^{-2} > 0$ for all $x > 0$. Thus, f is concave up for all $x > 0$ and has no points of inflection.

19. The growth of a sunflower during the first 100 days after sprouting is modeled well by the *logistic curve* $y = h(t)$ shown in Figure 15. Estimate the growth rate at the point of inflection and explain its significance. Then make a rough sketch of the first and second derivatives of $h(t)$.

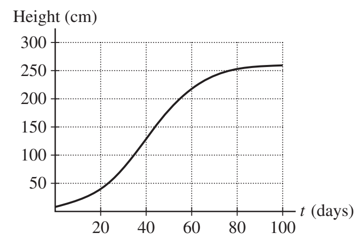
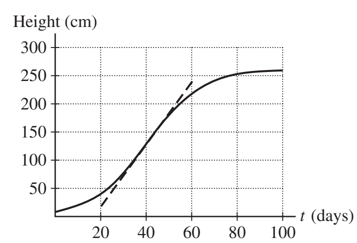


FIGURE 15

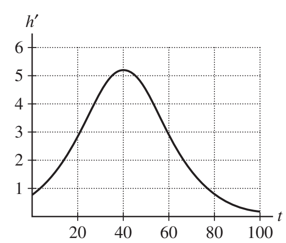
SOLUTION The point of inflection in Figure 15 appears to occur at $t = 40$ days. The graph below shows the logistic curve with an approximate tangent line drawn at $t = 40$. The approximate tangent line passes roughly through the points (20, 20) and (60, 240). The growth rate at the point of inflection is thus

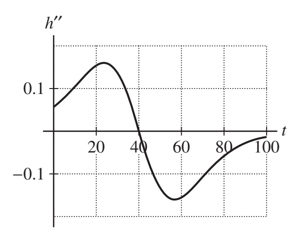
$$\frac{240 - 20}{60 - 20} = \frac{220}{40} = 5.5 \text{ cm/day}.$$

Because the logistic curve changes from concave up to concave down at $t = 40$, the growth rate at this point is the maximum growth rate for the sunflower plant.



Sketches of the first and second derivative of $h(t)$ are shown below at the left and at the right, respectively.





20. Assume that Figure 16 is the graph of $f(x)$. Where do the points of inflection of $f(x)$ occur, and on which interval is $f(x)$ concave down?

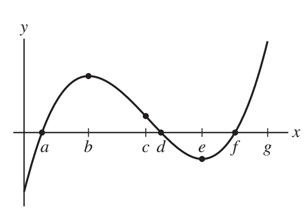


FIGURE 16

SOLUTION The function in Figure 16 changes concavity at $x = c$; therefore, there is a single point of inflection at $x = c$. The graph is concave down for $x < c$.

21. Repeat Exercise 20 but assume that Figure 16 is the graph of the *derivative* $f'(x)$.

SOLUTION Points of inflection occur when $f''(x)$ changes sign. Consequently, points of inflection occur when $f'(x)$ changes from increasing to decreasing or from decreasing to increasing. In Figure 16, this occurs at $x = b$ and at $x = e$; therefore, $f(x)$ has an inflection point at $x = b$ and another at $x = e$. The function $f(x)$ will be concave down when $f''(x) < 0$ or when $f'(x)$ is decreasing. Thus, $f(x)$ is concave down for $b < x < e$.

22. Repeat Exercise 20 but assume that Figure 16 is the graph of the *second derivative* $f''(x)$.

SOLUTION Inflection points occur when $f''(x)$ changes sign; therefore, $f(x)$ has inflection points at $x = a$, $x = d$ and $x = f$. The function $f(x)$ is concave down for $x < a$ and for $d < x < f$.

23. Figure 17 shows the *derivative* $f'(x)$ on [0, 1.2]. Locate the points of inflection of $f(x)$ and the points where the local minima and maxima occur. Determine the intervals on which $f(x)$ has the following properties:

(a) Increasing
(b) Decreasing
(c) Concave up
(d) Concave down

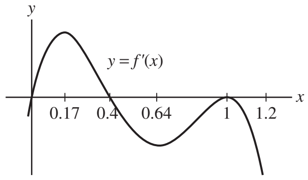


FIGURE 17

SOLUTION Recall that the graph is that of f', *not* f. The inflection points of f occur where f' changes from increasing to decreasing or vice versa because it is at these points that the sign of f'' changes. From the graph we conclude that f has points of inflection at $x = 0.17$, $x = 0.64$, and $x = 1$. The local extrema of f occur where f' changes sign. This occurs at $x = 0.4$. Because the sign of f' changes from $+$ to $-$, $f(0.4)$ is a local maximum. There are no local minima.

(a) f is increasing when f' is positive. Hence, f is increasing on $(0, 0.4)$.

(b) f is decreasing when f' is negative. Hence, f is decreasing on $(0.4, 1) \cup (1, 1.2)$.

(c) Now f is concave up where f' is increasing. This occurs on $(0, 0.17) \cup (0.64, 1)$.

(d) Moreover, f is concave down where f' is decreasing. This occurs on $(0.17, 0.64) \cup (1, 1.2)$.

24. Leticia has been selling solar-powered laptop chargers through her website, with monthly sales as recorded below. In a report to investors, she states, “Sales reached a point of inflection when I started using pay-per-click advertising.” In which month did that occur? Explain.

Month	1	2	3	4	5	6	7	8
Sales	2	30	50	60	90	150	230	340

SOLUTION Note that in successive months, sales increased by 28, 20, 10, 30, 60, 80 and 110. Until month 5, the rate of increase in sales was decreasing. After month 5, the rate of increase in sales increased. Thus, Leticia began using pay-per-click advertising in month 5.

In Exercises 25–38, find the critical points and apply the Second Derivative Test.

25. $f(x) = x^3 - 12x^2 + 45x$

SOLUTION Let $f(x) = x^3 - 12x^2 + 45x$. Then $f'(x) = 3x^2 - 24x + 45 = 3(x-3)(x-5)$, and the critical points are $x = 3$ and $x = 5$. Moreover, $f''(x) = 6x - 24$, so $f''(3) = -6 < 0$ and $f''(5) = 6 > 0$. Therefore, by the Second Derivative Test, $f(3) = 54$ is a local maximum, and $f(5) = 50$ is a local minimum.

26. $f(x) = x^4 - 8x^2 + 1$

SOLUTION Let $f(x) = x^4 - 8x^2 + 1$. Then $f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$, and the critical points are $x = 0$ and $x = \pm 2$. Moreover, $f''(x) = 12x^2 - 16$, so $f''(-2) = f''(2) = 32 > 0$ and $f''(0) = -16 < 0$. Therefore, by the second derivative test, $f(-2) = -15$ and $f(2) = -15$ are local minima, and $f(0) = 1$ is a local maximum.

27. $f(x) = 3x^4 - 8x^3 + 6x^2$

SOLUTION Let $f(x) = 3x^4 - 8x^3 + 6x^2$. Then $f'(x) = 12x^3 - 24x^2 + 12x = 12x(x-1)^2 = 0$ at $x = 0, 1$ and $f''(x) = 36x^2 - 48x + 12$. Thus, $f''(0) > 0$, which implies $f(0)$ is a local minimum; however, $f''(1) = 0$, which is inconclusive.

28. $f(x) = x^5 - x^3$

SOLUTION Let $f(x) = x^5 - x^3$. Then $f'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3) = 0$ at $x = 0$, $x = \pm\sqrt{\frac{3}{5}}$ and $f''(x) = 20x^3 - 6x = x(20x^2 - 6)$. Thus, $f''\left(\sqrt{\frac{3}{5}}\right) > 0$, which implies $f\left(\sqrt{\frac{3}{5}}\right)$ is a local minimum, and $f''\left(-\sqrt{\frac{3}{5}}\right) < 0$, which implies that $f\left(-\sqrt{\frac{3}{5}}\right)$ is a local maximum; however, $f''(0) = 0$, which is inconclusive.

29. $f(x) = \dfrac{x^2 - 8x}{x+1}$

SOLUTION Let $f(x) = \dfrac{x^2 - 8x}{x+1}$. Then

$$f'(x) = \frac{x^2 + 2x - 8}{(x+1)^2} \quad \text{and} \quad f''(x) = \frac{2(x+1)^2 - 2(x^2 + 2x - 8)}{(x+1)^3}.$$

Thus, the critical points are $x = -4$ and $x = 2$. Moreover, $f''(-4) < 0$ and $f''(2) > 0$. Therefore, by the second derivative test, $f(-4) = -16$ is a local maximum and $f(2) = -4$ is a local minimum.

30. $f(x) = \dfrac{1}{x^2 - x + 2}$

SOLUTION Let $f(x) = \dfrac{1}{x^2 - x + 2}$. Then $f'(x) = \dfrac{-2x + 1}{(x^2 - x + 2)^2} = 0$ at $x = \frac{1}{2}$ and

$$f''(x) = \frac{-2(x^2 - x + 2) + 2(2x - 1)^2}{(x^2 - x + 2)^3}.$$

Thus $f''\left(\frac{1}{2}\right) < 0$, which implies that $f\left(\frac{1}{2}\right)$ is a local maximum.

31. $y = 6x^{3/2} - 4x^{1/2}$

SOLUTION Let $f(x) = 6x^{3/2} - 4x^{1/2}$. Then $f'(x) = 9x^{1/2} - 2x^{-1/2} = x^{-1/2}(9x - 2)$, so there are two critical points: $x = 0$ and $x = \frac{2}{9}$. Now,

$$f''(x) = \frac{9}{2}x^{-1/2} + x^{-3/2} = \frac{1}{2}x^{-3/2}(9x + 2).$$

Thus, $f''\left(\frac{2}{9}\right) > 0$, which implies $f\left(\frac{2}{9}\right)$ is a local minimum. $f''(x)$ is undefined at $x = 0$, so the Second Derivative Test cannot be applied there.

32. $y = 9x^{7/3} - 21x^{1/2}$

SOLUTION Let $f(x) = 9x^{7/3} - 21x^{1/2}$. Then $f'(x) = 21x^{4/3} - \frac{21}{2}x^{-1/2} = 0$ when

$$x = \left(\frac{1}{2}\right)^{6/11},$$

and $f''(x) = 28x^{1/3} + \frac{21}{4}x^{-3/2}$. Thus,

$$f''\left(\left(\frac{1}{2}\right)^{6/11}\right) > 0,$$

which implies $f\left(\left(\frac{1}{2}\right)^{6/11}\right)$ is a local minimum.

33. $f(x) = \sin^2 x + \cos x, \quad [0, \pi]$

SOLUTION Let $f(x) = \sin^2 x + \cos x$. Then $f'(x) = 2\sin x \cos x - \sin x = \sin x(2\cos x - 1)$. On the interval $[0, \pi]$, $f'(x) = 0$ at $x = 0$, $x = \frac{\pi}{3}$ and $x = \pi$. Now,

$$f''(x) = 2\cos^2 x - 2\sin^2 x - \cos x.$$

Thus, $f''(0) > 0$, so $f(0)$ is a local minimum. On the other hand, $f''(\frac{\pi}{3}) < 0$, so $f(\frac{\pi}{3})$ is a local maximum. Finally, $f''(\pi) > 0$, so $f(\pi)$ is a local minimum.

34. $y = \dfrac{1}{\sin x + 4}, \quad [0, 2\pi]$

SOLUTION Let $f(x) = (\sin x + 4)^{-1}$. Then

$$f'(x) = -\frac{\cos x}{(\sin x + 4)^2} \quad \text{and} \quad f''(x) = \frac{2\cos^2 x + \sin^2 x + 4\sin x}{(\sin x + 4)^3}.$$

Now, $f'(x) = 0$ when $x = \pi/2$ and when $x = 3\pi/2$. Since $f''(\pi/2) > 0$, it follows that $f(\pi/2)$ is a local minimum. On the other hand, $f''(3\pi/2) < 0$, so $f(3\pi/2)$ is a local maximum.

35. $f(x) = xe^{-x^2}$

SOLUTION Let $f(x) = xe^{-x^2}$. Then $f'(x) = -2x^2e^{-x^2} + e^{-x^2} = (1 - 2x^2)e^{-x^2}$, so there are two critical points: $x = \pm\frac{\sqrt{2}}{2}$. Now,

$$f''(x) = (4x^3 - 2x)e^{-x^2} - 4xe^{-x^2} = (4x^3 - 6x)e^{-x^2}.$$

Thus, $f''\left(\frac{\sqrt{2}}{2}\right) < 0$, so $f\left(\frac{\sqrt{2}}{2}\right)$ is a local maximum. On the other hand, $f''\left(-\frac{\sqrt{2}}{2}\right) > 0$, so $f\left(-\frac{\sqrt{2}}{2}\right)$ is a local minimum.

36. $f(x) = e^{-x} - 4e^{-2x}$

SOLUTION Let $f(x) = e^{-x} - 4e^{-2x}$. Then $f'(x) = -e^{-x} + 8e^{-2x} = 0$ when $x = 3\ln 2$. Now, $f''(x) = e^{-x} - 16e^{-2x}$, so $f''(3\ln 2) < 0$. Thus, $f(3\ln 2)$ is a local maximum.

37. $f(x) = x^3 \ln x \quad (x > 0)$

SOLUTION Let $f(x) = x^3 \ln x$. Then $f'(x) = x^2 + 3x^2 \ln x = x^2(1 + 3\ln x)$, so there is only one critical point: $x = e^{-1/3}$. Now,

$$f''(x) = 3x + 2x(1 + 3\ln x) = x(5 + 6lnx).$$

Thus, $f''\left(e^{-1/3}\right) > 0$, so $f\left(e^{-1/3}\right)$ is a local minimum.

38. $f(x) = \ln x + \ln(4 - x^2), \quad (0, 2)$

SOLUTION Let $f(x) = \ln x + \ln(4 - x^2)$. Then

$$f'(x) = \frac{1}{x} - \frac{2x}{4 - x^2},$$

so there is only one critical point on the interval $0 < x < 2$: $x = \frac{2\sqrt{3}}{3}$. Now,

$$f''(x) = -\frac{1}{x^2} - \frac{(4 - x^2)(2) - 2x(-2x)}{(4 - x^2)^2} = -\frac{1}{x^2} - \frac{8 + 2x^2}{(4 - x^2)^2}.$$

Thus, $f''\left(\frac{2\sqrt{3}}{3}\right) < 0$, so $f\left(\frac{2\sqrt{3}}{3}\right)$ is a local maximum.

In Exercises 39–52, find the intervals on which f is concave up or down, the points of inflection, the critical points, and the local minima and maxima.

SOLUTION Here is a table legend for Exercises 39–49.

SYMBOL	MEANING
−	The entity is negative on the given interval.
0	The entity is zero at the specified point.
+	The entity is positive on the given interval.
U	The entity is undefined at the specified point.
↗	The function (f, g, etc.) is increasing on the given interval.
↘	The function (f, g, etc.) is decreasing on the given interval.
⌣	The function (f, g, etc.) is concave up on the given interval.
⌢	The function (f, g, etc.) is concave down on the given interval.
M	The function (f, g, etc.) has a local maximum at the specified point.
m	The function (f, g, etc.) has a local minimum at the specified point.
I	The function (f, g, etc.) has an inflection point here.
¬	There is no local extremum or inflection point here.

39. $f(x) = x^3 - 2x^2 + x$

SOLUTION Let $f(x) = x^3 - 2x^2 + x$.

- Then $f'(x) = 3x^2 - 4x + 1 = (x-1)(3x-1) = 0$ yields $x = 1$ and $x = \frac{1}{3}$ as candidates for extrema.
- Moreover, $f''(x) = 6x - 4 = 0$ gives a candidate for a point of inflection at $x = \frac{2}{3}$.

x	$\left(-\infty, \frac{1}{3}\right)$	$\frac{1}{3}$	$\left(\frac{1}{3}, 1\right)$	1	$(1, \infty)$
f'	+	0	−	0	+
f	↗	M	↘	m	↗

x	$\left(-\infty, \frac{2}{3}\right)$	$\frac{2}{3}$	$\left(\frac{2}{3}, \infty\right)$
f''	−	0	+
f	⌢	I	⌣

40. $f(x) = x^2(x - 4)$

SOLUTION Let $f(x) = x^2(x-4) = x^3 - 4x^2$.

- Then $f'(x) = 3x^2 - 8x = x(3x - 8) = 0$ yields $x = 0$ and $x = \frac{8}{3}$ as candidates for extrema.
- Moreover, $f''(x) = 6x - 8 = 0$ gives a candidate for a point of inflection at $x = \frac{4}{3}$.

x	$(-\infty, 0)$	0	$\left(0, \frac{8}{3}\right)$	$\frac{8}{3}$	$\left(\frac{8}{3}, \infty\right)$
f'	+	0	−	0	+
f	↗	M	↘	m	↗

x	$\left(-\infty, \frac{4}{3}\right)$	$\frac{4}{3}$	$\left(\frac{4}{3}, \infty\right)$
f''	−	0	+
f	⌢	I	⌣

41. $f(t) = t^2 - t^3$

SOLUTION Let $f(t) = t^2 - t^3$.

- Then $f'(t) = 2t - 3t^2 = t(2 - 3t) = 0$ yields $t = 0$ and $t = \frac{2}{3}$ as candidates for extrema.
- Moreover, $f''(t) = 2 - 6t = 0$ gives a candidate for a point of inflection at $t = \frac{1}{3}$.

t	$(-\infty, 0)$	0	$\left(0, \frac{2}{3}\right)$	$\frac{2}{3}$	$\left(\frac{2}{3}, \infty\right)$
f'	−	0	+	0	−
f	↘	m	↗	M	↘

t	$\left(-\infty, \frac{1}{3}\right)$	$\frac{1}{3}$	$\left(\frac{1}{3}, \infty\right)$
f''	+	0	−
f	⌣	I	⌢

42. $f(x) = 2x^4 - 3x^2 + 2$

SOLUTION Let $f(x) = 2x^4 - 3x^2 + 2$.

- Then $f'(x) = 8x^3 - 6x = 2x(4x^2 - 3) = 0$ yields $x = 0$ and $x = \pm\frac{\sqrt{3}}{2}$ as candidates for extrema.
- Moreover, $f''(x) = 24x^2 - 6 = 6(4x^2 - 1) = 0$ gives candidates for a point of inflection at $x = \pm\frac{1}{2}$.

x	$\left(-\infty, -\frac{\sqrt{3}}{2}\right)$	$-\frac{\sqrt{3}}{2}$	$\left(-\frac{\sqrt{3}}{2}, 0\right)$	0	$\left(0, \frac{\sqrt{3}}{2}\right)$	$\frac{\sqrt{3}}{2}$	$\left(\frac{\sqrt{3}}{2}, \infty\right)$
f'	−	0	+	0	−	0	+
f	↘	m	↗	M	↘	m	↗

x	$\left(-\infty, -\frac{1}{2}\right)$	$-\frac{1}{2}$	$\left(-\frac{1}{2}, \frac{1}{2}\right)$	$\frac{1}{2}$	$\left(\frac{1}{2}, \infty\right)$
f''	+	0	−	0	+
f	⌣	I	⌢	I	⌣

43. $f(x) = x^2 - 8x^{1/2} \quad (x \geq 0)$

SOLUTION Let $f(x) = x^2 - 8x^{1/2}$. Note that the domain of f is $x \geq 0$.

- Then $f'(x) = 2x - 4x^{-1/2} = x^{-1/2}\left(2x^{3/2} - 4\right) = 0$ yields $x = 0$ and $x = (2)^{2/3}$ as candidates for extrema.
- Moreover, $f''(x) = 2 + 2x^{-3/2} > 0$ for all $x \geq 0$, which means there are no inflection points.

x	0	$\left(0, (2)^{2/3}\right)$	$(2)^{2/3}$	$\left((2)^{2/3}, \infty\right)$
f'	U	$-$	0	$+$
f	M	↘	m	↗

44. $f(x) = x^{3/2} - 4x^{-1/2} \quad (x > 0)$

SOLUTION Let $f(x) = x^{3/2} - 4x^{-1/2}$. Then

$$f'(x) = \frac{3}{2}x^{1/2} + 2x^{-3/2} > 0$$

for all $x > 0$. Thus, f is always increasing and there are no local extrema. Now,

$$f''(x) = \frac{3}{4}x^{-1/2} - 3x^{-5/2}$$

so $x = 2$ is a candidate point of inflection.

x	$(0, 2)$	2	$(2, \infty)$
f''	$-$	0	$+$
f	⌢	I	⌣

45. $f(x) = \dfrac{x}{x^2 + 27}$

SOLUTION Let $f(x) = \dfrac{x}{x^2 + 27}$.

- Then $f'(x) = \dfrac{27 - x^2}{\left(x^2 + 27\right)^2} = 0$ yields $x = \pm 3\sqrt{3}$ as candidates for extrema.
- Moreover, $f''(x) = \dfrac{-2x\left(x^2 + 27\right)^2 - (27 - x^2)(2)\left(x^2 + 27\right)(2x)}{(x^2 + 27)^4} = \dfrac{2x\left(x^2 - 81\right)}{\left(x^2 + 27\right)^3} = 0$ gives candidates for a point of inflection at $x = 0$ and at $x = \pm 9$.

x	$\left(-\infty, -3\sqrt{3}\right)$	$-3\sqrt{3}$	$\left(-3\sqrt{3}, 3\sqrt{3}\right)$	$3\sqrt{3}$	$\left(3\sqrt{3}, \infty\right)$
f'	$-$	0	$+$	0	$-$
f	↘	m	↗	M	↘

x	$(-\infty, -9)$	-9	$(-9, 0)$	0	$(0, 9)$	9	$(9, \infty)$
f''	$-$	0	$+$	0	$-$	0	$+$
f	⌢	I	⌣	I	⌢	I	⌣

46. $f(x) = \dfrac{1}{x^4 + 1}$

SOLUTION Let $f(x) = \dfrac{1}{x^4 + 1}$.

- Then $f'(x) = -\dfrac{4x^3}{\left(x^4 + 1\right)^2} = 0$ yields $x = 0$ as a candidate for an extremum.
- Moreover,

$$f''(x) = \frac{\left(x^4 + 1\right)^2\left(-12x^2\right) - (-4x^3) \cdot 2\left(x^4 + 1\right)\left(4x^3\right)}{\left(x^2 + 1\right)^4} = \frac{4x^2\left(5x^4 - 3\right)}{\left(x^4 + 1\right)^3} = 0$$

gives candidates for a point of inflection at $x = 0$ and at $x = \pm\left(\frac{3}{5}\right)^{1/4}$.

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	+	0	−
f	↗	M	↘

x	$\left(-\infty, -\left(\frac{3}{5}\right)^{1/4}\right)$	$-\left(\frac{3}{5}\right)^{1/4}$	$\left(-\left(\frac{3}{5}\right)^{1/4}, 0\right)$	0	$\left(0, \left(\frac{3}{5}\right)^{1/4}\right)$	$\left(\frac{3}{5}\right)^{1/4}$	$\left(\left(\frac{3}{5}\right)^{1/4}, \infty\right)$
f''	+	0	−	0	−	0	+
f	⌣	I	⌢	¬	⌢	I	⌣

47. $f(\theta) = \theta + \sin\theta, \quad [0, 2\pi]$

SOLUTION Let $f(\theta) = \theta + \sin\theta$ on $[0, 2\pi]$.

- Then $f'(\theta) = 1 + \cos\theta = 0$ yields $\theta = \pi$ as a candidate for an extremum.
- Moreover, $f''(\theta) = -\sin\theta = 0$ gives candidates for a point of inflection at $\theta = 0$, at $\theta = \pi$, and at $\theta = 2\pi$.

θ	$(0, \pi)$	π	$(\pi, 2\pi)$
f'	+	0	+
f	↗	¬	↗

θ	0	$(0, \pi)$	π	$(\pi, 2\pi)$	2π
f''	0	−	0	+	0
f	¬	⌢	I	⌣	¬

48. $f(x) = \cos^2 x, \quad [0, \pi]$

SOLUTION Let $f(x) = \cos^2 x$. Then $f'(x) = -2\cos x \sin x = -2\sin 2x = 0$ when $x = 0$, $x = \pi/2$ and $x = \pi$. All three are candidates for extrema. Moreover, $f''(x) = -4\cos 2x = 0$ when $x = \pi/4$ and $x = 3\pi/4$. Both are candidates for a point of inflection.

x	0	$\left(0, \frac{\pi}{2}\right)$	$\frac{\pi}{2}$	$\left(\frac{\pi}{2}, \pi\right)$	π
f'	0	−	0	+	0
f	M	↘	m	↗	M

x	$\left(0, \frac{\pi}{4}\right)$	$\frac{\pi}{4}$	$\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$	$\frac{3\pi}{4}$	$\left(\frac{3\pi}{4}, \pi\right)$
f''	−	0	+	0	−
f	⌢	I	⌣	I	⌢

49. $f(x) = \tan x, \quad \left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$

SOLUTION Let $f(x) = \tan x$ on $\left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$.

- Then $f'(x) = \sec^2 x \geq 1 > 0$ on $\left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$.
- Moreover, $f''(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x = 0$ gives a candidate for a point of inflection at $x = 0$.

x	$\left(-\frac{\pi}{4}, \frac{\pi}{3}\right)$
f'	+
f	↗

x	$\left(-\frac{\pi}{4}, 0\right)$	0	$\left(0, \frac{\pi}{3}\right)$
f''	−	0	+
f	⌢	I	⌣

50. $f(x) = e^{-x}\cos x, \quad \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

SOLUTION Let $f(x) = e^{-x}\cos x$ on $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

- Then, $f'(x) = -e^{-x}\sin x - e^{-x}\cos x = -e^{-x}(\sin x + \cos x) = 0$ gives $x = -\frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ as candidates for extrema.
- Moreover,

$$f''(x) = -e^{-x}(\cos x - \sin x) + e^{-x}(\sin x + \cos x) = 2e^{-x}\sin x = 0$$

gives $x = 0$ and $x = \pi$ as inflection point candidates.

x	$\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$	$-\frac{\pi}{4}$	$\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$	$\frac{3\pi}{4}$	$\left(\frac{3\pi}{4}, \frac{3\pi}{2}\right)$
f'	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗

x	$\left(-\frac{\pi}{2}, 0\right)$	0	$(0, \pi)$	π	$\left(\pi, \frac{3\pi}{2}\right)$
f''	$-$	0	$+$	0	$-$
f	⌢	I	⌣	I	⌢

51. $y = (x^2 - 2)e^{-x} \quad (x > 0)$

SOLUTION Let $f(x) = (x^2 - 2)e^{-x}$.

- Then $f'(x) = -(x^2 - 2x - 2)e^{-x} = 0$ gives $x = 1 + \sqrt{3}$ as a candidate for an extrema.
- Moreover, $f''(x) = (x^2 - 4x)e^{-x} = 0$ gives $x = 4$ as a candidate for a point of inflection.

x	$\left(0, 1 + \sqrt{3}\right)$	$1 + \sqrt{3}$	$\left(1 + \sqrt{3}, \infty\right)$
f'	$+$	0	$-$
f	↗	M	↘

x	$(0, 4)$	4	$(4, \infty)$
f''	$-$	0	$+$
f	⌢	I	⌣

52. $y = \ln(x^2 + 2x + 5)$

SOLUTION Let $f(x) = \ln(x^2 + 2x + 5)$. Then

$$f'(x) = \frac{2x + 2}{x^2 + 2x + 5} = 0$$

when $x = -1$. This is the only critical point. Moreover,

$$f''(x) = -\frac{2(x - 1)(x + 3)}{(x^2 + 2x + 5)^2},$$

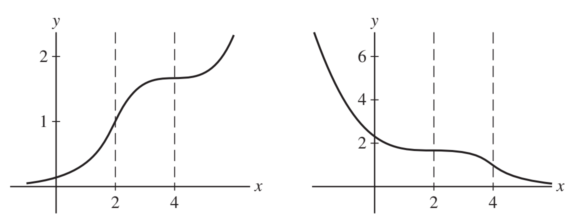
so $x = 1$ and $x = -3$ are candidates for inflection points.

x	$(-\infty, -1)$	-1	$(-1, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

x	$(-\infty, -3)$	-3	$(-3, 1)$	1	$(1, \infty)$
f''	$-$	0	$+$	0	$-$
f	⌢	I	⌣	I	⌢

53. Sketch the graph of an increasing function such that $f''(x)$ changes from $+$ to $-$ at $x = 2$ and from $-$ to $+$ at $x = 4$. Do the same for a decreasing function.

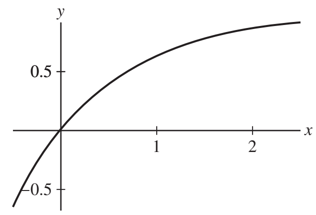
SOLUTION The graph shown below at the left is an increasing function which changes from concave up to concave down at $x = 2$ and from concave down to concave up at $x = 4$. The graph shown below at the right is a decreasing function which changes from concave up to concave down at $x = 2$ and from concave down to concave up at $x = 4$.



In Exercises 54–56, sketch the graph of a function $f(x)$ satisfying all of the given conditions.

54. $f'(x) > 0$ and $f''(x) < 0$ for all x.

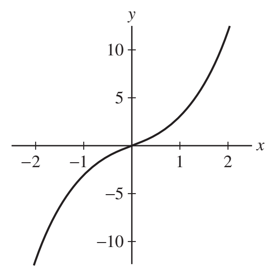
SOLUTION Here is the graph of a function $f(x)$ satisfying $f'(x) > 0$ for all x and $f''(x) < 0$ for all x.



55. **(i)** $f'(x) > 0$ for all x, and

(ii) $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.

SOLUTION Here is the graph of a function $f(x)$ satisfying **(i)** $f'(x) > 0$ for all x and **(ii)** $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.



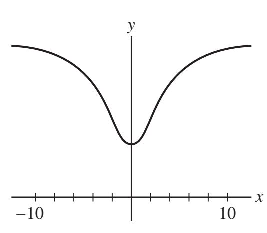
56. **(i)** $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$, and

(ii) $f''(x) < 0$ for $|x| > 2$, and $f''(x) > 0$ for $|x| < 2$.

SOLUTION

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
Direction	↘	↘	↗	↗
Concavity	⌢	⌣	⌣	⌢

One potential graph with this shape is the following:



57. An infectious flu spreads slowly at the beginning of an epidemic. The infection process accelerates until a majority of the susceptible individuals are infected, at which point the process slows down.

(a) If $R(t)$ is the number of individuals infected at time t, describe the concavity of the graph of R near the beginning and end of the epidemic.

(b) Describe the status of the epidemic on the day that $R(t)$ has a point of inflection.

SOLUTION

(a) Near the beginning of the epidemic, the graph of R is concave up. Near the epidemic's end, R is concave down.

(b) "Epidemic subsiding: number of new cases declining."

58. Water is pumped into a sphere at a constant rate (Figure 18). Let $h(t)$ be the water level at time t. Sketch the graph of $h(t)$ (approximately, but with the correct concavity). Where does the point of inflection occur?

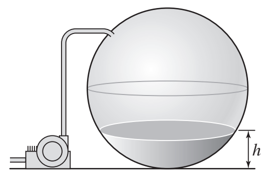
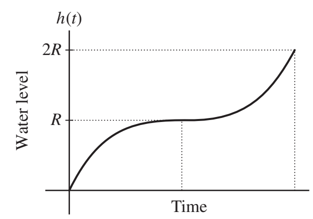


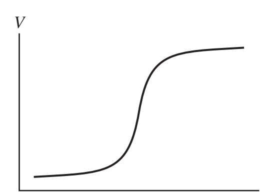
FIGURE 18

SOLUTION Because water is entering the sphere at a constant rate, we expect the water level to rise more rapidly near the bottom and top of the sphere where the sphere is not as "wide" and to rise more slowly near the middle of the sphere. The graph of $h(t)$ should therefore start concave down and end concave up, with an inflection point when the sphere is half full; that is, when the water level is equal to the radius of the sphere. A possible graph of $h(t)$ is shown below.



59. Water is pumped into a sphere of radius R at a variable rate in such a way that the water level rises at a constant rate (Figure 18). Let $V(t)$ be the volume of water in the tank at time t. Sketch the graph $V(t)$ (approximately, but with the correct concavity). Where does the point of inflection occur?

SOLUTION Because water is entering the sphere in such a way that the water level rises at a constant rate, we expect the volume to increase more slowly near the bottom and top of the sphere where the sphere is not as "wide" and to increase more rapidly near the middle of the sphere. The graph of $V(t)$ should therefore start concave up and change to concave down when the sphere is half full; that is, the point of inflection should occur when the water level is equal to the radius of the sphere. A possible graph of $V(t)$ is shown below.



60. (Continuation of Exercise 59) If the sphere has radius R, the volume of water is $V = \pi\left(Rh^2 - \frac{1}{3}h^3\right)$ where h is the water level. Assume the level rises at a constant rate of 1 (that is, $h = t$).

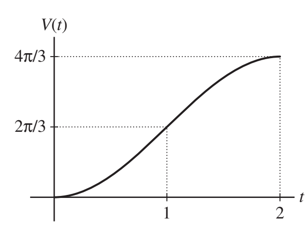
(a) Find the inflection point of $V(t)$. Does this agree with your conclusion in Exercise 59?

(b) GU Plot $V(t)$ for $R = 1$.

SOLUTION

(a) With $h = t$ and $V(t) = \pi(Rt^2 - \frac{1}{3}t^3)$. Then, $V'(t) = \pi(2Rt - t^2)$ and $V''(t) = \pi(2R - 2t)$. Therefore, $V(t)$ is concave up for $t < R$, concave down for $t > R$ and has an inflection point at $t = R$. In other words, $V(t)$ has an inflection point when the water level is equal to the radius of the sphere, in agreement with the conclusion of Exercise 59.

(b) With $h = t$ and $R = 1$, $V(t) = \pi(t^2 - \frac{1}{3}t^3)$. The graph of $V(t)$ is shown below.



61. Image Processing The intensity of a pixel in a digital image is measured by a number u between 0 and 1. Often, images can be enhanced by rescaling intensities (Figure 19), where pixels of intensity u are displayed with intensity $g(u)$ for a suitable function $g(u)$. One common choice is the **sigmoidal correction**, defined for constants a, b by

$$g(u) = \frac{f(u) - f(0)}{f(1) - f(0)} \quad \text{where} \quad f(u) = \left(1 + e^{b(a-u)}\right)^{-1}$$

Figure 20 shows that $g(u)$ reduces the intensity of low-intensity pixels (where $g(u) < u$) and increases the intensity of high-intensity pixels.

(a) Verify that $f'(u) > 0$ and use this to show that $g(u)$ increases from 0 to 1 for $0 \le u \le 1$.
(b) Where does $g'(u)$ have a point of inflection?

Original   Sigmoidal correction

FIGURE 19

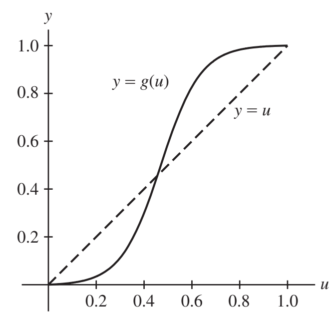


FIGURE 20 Sigmoidal correction with $a = 0.47$, $b = 12$.

SOLUTION

(a) With $f(u) = (1 + e^{b(a-u)})^{-1}$, it follows that

$$f'(u) = -(1 + e^{b(a-u)})^{-2} \cdot -be^{b(a-u)} = \frac{be^{b(a-u)}}{(1 + e^{b(a-u)})^2} > 0$$

for all u. Next, observe that

$$g(0) = \frac{f(0) - f(0)}{f(1) - f(0)} = 0, \quad g(1) = \frac{f(1) - f(0)}{f(1) - f(0)} = 1,$$

and

$$g'(u) = \frac{1}{f(1) - f(0)} f'(u) > 0$$

for all u. Thus, $g(u)$ increases from 0 to 1 for $0 \le u \le 1$.

(b) Working from part (a), we find

$$f''(u) = \frac{b^2 e^{b(a-u)}(2e^{b(a-u)} - 1)}{(1 + e^{b(a-u)})^3}.$$

Because

$$g''(u) = \frac{1}{f(1) - f(0)} f''(u),$$

it follows that $g(u)$ has a point of inflection when

$$2e^{b(a-u)} - 1 = 0 \quad \text{or} \quad u = a + \frac{1}{b}\ln 2.$$

62. Use graphical reasoning to determine whether the following statements are true or false. If false, modify the statement to make it correct.

(a) If $f(x)$ is increasing, then $f^{-1}(x)$ is decreasing.
(b) If $f(x)$ is decreasing, then $f^{-1}(x)$ is decreasing.
(c) If $f(x)$ is concave up, then $f^{-1}(x)$ is concave up.
(d) If $f(x)$ is concave down, then $f^{-1}(x)$ is concave up.

SOLUTION

(a) False. Should be: If $f(x)$ is increasing, then $f^{-1}(x)$ is increasing.
(b) True.
(c) False. Should be: If $f(x)$ is concave up, then $f^{-1}(x)$ is concave down.
(d) True.

Further Insights and Challenges

In Exercises 63–65, assume that $f(x)$ is differentiable.

63. Proof of the Second Derivative Test Let c be a critical point such that $f''(c) > 0$ (the case $f''(c) < 0$ is similar).

(a) Show that $f''(c) = \lim_{h\to 0} \frac{f'(c+h)}{h}$.

(b) Use (a) to show that there exists an open interval (a, b) containing c such that $f'(x) < 0$ if $a < x < c$ and $f'(x) > 0$ if $c < x < b$. Conclude that $f(c)$ is a local minimum.

SOLUTION

(a) Because c is a critical point, either $f'(c) = 0$ or $f'(c)$ does not exist; however, $f''(c)$ exists, so $f'(c)$ must also exist. Therefore, $f'(c) = 0$. Now, from the definition of the derivative, we have

$$f''(c) = \lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \frac{f'(c+h)}{h}.$$

(b) We are given that $f''(c) > 0$. By part (a), it follows that

$$\lim_{h\to 0} \frac{f'(c+h)}{h} > 0;$$

in other words, for sufficiently small h,

$$\frac{f'(c+h)}{h} > 0.$$

Now, if h is sufficiently small but negative, then $f'(c+h)$ must also be negative (so that the ratio $f'(c+h)/h$ will be positive) and $c + h < c$. On the other hand, if h is sufficiently small but positive, then $f'(c+h)$ must also be positive and $c + h > c$. Thus, there exists an open interval (a, b) containing c such that $f'(x) < 0$ for $a < x < c$ and $f'(c) > 0$ for $c < x < b$. Finally, because $f'(x)$ changes from negative to positive at $x = c$, $f(c)$ must be a local minimum.

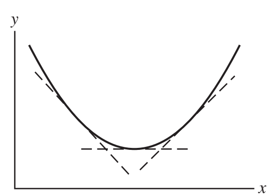
64. Prove that if $f''(x)$ exists and $f''(x) > 0$ for all x, then the graph of $f(x)$ "sits above" its tangent lines.

(a) For any c, set $G(x) = f(x) - f'(c)(x - c) - f(c)$. It is sufficient to prove that $G(x) \geq 0$ for all c. Explain why with a sketch.

(b) Show that $G(c) = G'(c) = 0$ and $G''(x) > 0$ for all x. Conclude that $G'(x) < 0$ for $x < c$ and $G'(x) > 0$ for $x > c$. Then deduce, using the MVT, that $G(x) > G(c)$ for $x \neq c$.

SOLUTION

(a) Let c be any number. Then $y = f'(c)(x - c) + f(c)$ is the equation of the line tangent to the graph of $f(x)$ at $x = c$ and $G(x) = f(x) - f'(c)(x - c) - f(c)$ measures the amount by which the value of the function exceeds the value of the tangent line (see the figure below). Thus, to prove that the graph of $f(x)$ "sits above" its tangent lines, it is sufficient to prove that $G(x) \geq 0$ for all c.



(b) Note that $G(c) = f(c) - f'(c)(c - c) - f(c) = 0$, $G'(x) = f'(x) - f'(c)$ and $G'(c) = f'(c) - f'(c) = 0$. Moreover, $G''(x) = f''(x) > 0$ for all x. Now, because $G'(c) = 0$ and $G'(x)$ is increasing, it must be true that $G'(x) < 0$ for $x < c$ and that $G'(x) > 0$ for $x > c$. Therefore, $G(x)$ is decreasing for $x < c$ and increasing for $x > c$. This implies that $G(c) = 0$ is a minimum; consequently $G(x) > G(c) = 0$ for $x \neq c$.

65. Assume that $f''(x)$ exists and let c be a point of inflection of $f(x)$.

(a) Use the method of Exercise 64 to prove that the tangent line at $x = c$ *crosses the graph* (Figure 21). *Hint:* Show that $G(x)$ changes sign at $x = c$.

(b) GU Verify this conclusion for $f(x) = \dfrac{x}{3x^2+1}$ by graphing $f(x)$ and the tangent line at each inflection point on the same set of axes.

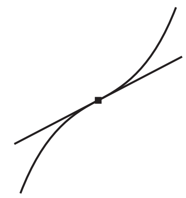


FIGURE 21 Tangent line crosses graph at point of inflection.

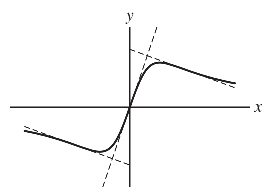
SOLUTION

(a) Let $G(x) = f(x) - f'(c)(x-c) - f(c)$. Then, as in Exercise 63, $G(c) = G'(c) = 0$ and $G''(x) = f''(x)$. If $f''(x)$ changes from positive to negative at $x = c$, then so does $G''(x)$ and $G'(x)$ is increasing for $x < c$ and decreasing for $x > c$. This means that $G'(x) < 0$ for $x < c$ and $G'(x) < 0$ for $x > c$. This in turn implies that $G(x)$ is decreasing, so $G(x) > 0$ for $x < c$ but $G(x) < 0$ for $x > c$. On the other hand, if $f''(x)$ changes from negative to positive at $x = c$, then so does $G''(x)$ and $G'(x)$ is decreasing for $x < c$ and increasing for $x > c$. Thus, $G'(x) > 0$ for $x < c$ and $G'(x) > 0$ for $x > c$. This in turn implies that $G(x)$ is increasing, so $G(x) < 0$ for $x < c$ and $G(x) > 0$ for $x > c$. In either case, $G(x)$ changes sign at $x = c$, and the tangent line at $x = c$ crosses the graph of the function.

(b) Let $f(x) = \dfrac{x}{3x^2+1}$. Then

$$f'(x) = \frac{1-3x^2}{(3x^2+1)^2} \qquad \text{and} \qquad f''(x) = \frac{-18x(1-x^2)}{(3x^2+1)^3}.$$

Therefore $f(x)$ has a point of inflection at $x = 0$ and at $x = \pm 1$. The figure below shows the graph of $y = f(x)$ and its tangent lines at each of the points of inflection. It is clear that each tangent line crosses the graph of $f(x)$ at the inflection point.



66. Let $C(x)$ be the cost of producing x units of a certain good. Assume that the graph of $C(x)$ is concave up.

(a) Show that the average cost $A(x) = C(x)/x$ is minimized at the production level x_0 such that average cost equals marginal cost—that is, $A(x_0) = C'(x_0)$.

(b) Show that the line through $(0, 0)$ and $(x_0, C(x_0))$ is tangent to the graph of $C(x)$.

SOLUTION Let $C(x)$ be the cost of producing x units of a commodity. Assume the graph of C is concave up.

(a) Let $A(x) = C(x)/x$ be the average cost and let x_0 be the production level at which average cost is minimized. Then $A'(x_0) = \dfrac{x_0C'(x_0) - C(x_0)}{x_0^2} = 0$ implies $x_0C'(x_0) - C(x_0) = 0$, whence $C'(x_0) = C(x_0)/x_0 = A(x_0)$. In other words, $A(x_0) = C'(x_0)$ or average cost equals marginal cost at production level x_0. To confirm that x_0 corresponds to a local minimum of A, we use the Second Derivative Test. We find

$$A''(x_0) = \frac{x_0^2C''(x_0) - 2(x_0C'(x_0) - C(x_0))}{x_0^3} = \frac{C''(x_0)}{x_0} > 0$$

because C is concave up. Hence, x_0 corresponds to a local minimum.

(b) The line between $(0, 0)$ and $(x_0, C(x_0))$ is

$$\begin{aligned}\frac{C(x_0) - 0}{x_0 - 0}(x - x_0) + C(x_0) &= \frac{C(x_0)}{x_0}(x - x_0) + C(x_0) = A(x_0)(x - x_0) + C(x_0)\\ &= C'(x_0)(x - x_0) + C(x_0)\end{aligned}$$

which is the tangent line to C at x_0.

67. Let $f(x)$ be a polynomial of degree $n \geq 2$. Show that $f(x)$ has at least one point of inflection if n is odd. Then give an example to show that $f(x)$ need not have a point of inflection if n is even.

SOLUTION Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial of degree n. Then $f'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$ and $f''(x) = n(n-1)a_nx^{n-2} + (n-1)(n-2)a_{n-1}x^{n-3} + \cdots + 6a_3x + 2a_2$. If $n \geq 3$ and is odd, then $n - 2$ is also odd and $f''(x)$ is a polynomial of odd degree. Therefore $f''(x)$ must take on both positive and negative values. It follows that $f''(x)$ has at least one root c such that $f''(x)$ changes sign at c. The function $f(x)$ will then have a point of inflection at $x = c$. On the other hand, the functions $f(x) = x^2, x^4$ and x^8 are polynomials of even degree that do not have any points of inflection.

68. Critical and Inflection Points If $f'(c) = 0$ and $f(c)$ is neither a local min nor a local max, must $x = c$ be a point of inflection? This is true for "reasonable" functions (including the functions studied in this text), but it is not true in general. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

(a) Use the limit definition of the derivative to show that $f'(0)$ exists and $f'(0) = 0$.
(b) Show that $f(0)$ is neither a local min nor a local max.
(c) Show that $f'(x)$ changes sign infinitely often near $x = 0$. Conclude that $x = 0$ is not a point of inflection.

SOLUTION Let $f(x) = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$.

(a) Now $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$ by the Squeeze Theorem: as $x \to 0$ we have

$$\left| x \sin\left(\frac{1}{x}\right) - 0 \right| = |x| \left| \sin\left(\frac{1}{x}\right) \right| \to 0,$$

since $|\sin u| \leq 1$.
(b) Since $\sin(\frac{1}{x})$ oscillates through every value between -1 and 1 with increasing frequency as $x \to 0$, in any open interval $(-\delta, \delta)$ there are points a and b such that $f(a) = a^2 \sin(\frac{1}{a}) < 0$ and $f(b) = b^2 \sin(\frac{1}{b}) > 0$. Accordingly, $f(0) = 0$ can neither be a local minimum value nor a local maximum value of f.
(c) In part (a) it was shown that $f'(0) = 0$. For $x \neq 0$, we have

$$f'(x) = x^2 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 2x \sin\left(\frac{1}{x}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

As $x \to 0$, $f'(x)$ oscillates increasingly rapidly; consequently, $f'(x)$ changes sign infinitely often near $x = 0$. From this we conclude that $f(x)$ does not have a point of inflection at $x = 0$.

4.5 L'Hôpital's Rule

Preliminary Questions

1. What is wrong with applying L'Hôpital's Rule to $\lim_{x\to 0} \frac{x^2 - 2x}{3x - 2}$?

SOLUTION As $x \to 0$,

$$\frac{x^2 - 2x}{3x - 2}$$

is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so L'Hôpital's Rule cannot be used.

2. Does L'Hôpital's Rule apply to $\lim_{x\to a} f(x)g(x)$ if $f(x)$ and $g(x)$ both approach ∞ as $x \to a$?

SOLUTION No. L'Hôpital's Rule only applies to limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Exercises

In Exercises 1–10, use L'Hôpital's Rule to evaluate the limit, or state that L'Hôpital's Rule does not apply.

1. $\lim_{x\to 3} \frac{2x^2 - 5x - 3}{x - 4}$

SOLUTION Because the quotient is not indeterminate at $x = 3$,

$$\left.\frac{2x^2 - 5x - 3}{x - 4}\right|_{x=3} = \frac{18 - 15 - 3}{3 - 4} = \frac{0}{-1},$$

L'Hôpital's Rule does not apply.

2. $\displaystyle\lim_{x\to -5} \frac{x^2-25}{5-4x-x^2}$

SOLUTION The functions $x^2 - 25$ and $5 - 4x - x^2$ are differentiable, but the quotient is indeterminate at $x = -5$,

$$\left.\frac{x^2-25}{5-4x-x^2}\right|_{x=-5} = \frac{25-25}{5+20-25} = \frac{0}{0},$$

so L'Hôpital's Rule applies. We find

$$\lim_{x\to -5} \frac{x^2-25}{5-4x-x^2} = \lim_{x\to -5} \frac{2x}{-4-2x} = \frac{-10}{-4+10} = -\frac{5}{3}.$$

3. $\displaystyle\lim_{x\to 4} \frac{x^3-64}{x^2+16}$

SOLUTION Because the quotient is not indeterminate at $x = 4$,

$$\left.\frac{x^3-64}{x^2+16}\right|_{x=4} = \frac{64-64}{16+16} = \frac{0}{32},$$

L'Hôpital's Rule does not apply.

4. $\displaystyle\lim_{x\to -1} \frac{x^4+2x+1}{x^5-2x-1}$

SOLUTION The functions $x^4 + 2x + 1$ and $x^5 - 2x - 1$ are differentiable, but the quotient is indeterminate at $x = -1$,

$$\left.\frac{x^4+2x+1}{x^5-2x-1}\right|_{x=-1} = \frac{1-2+1}{-1+2-1} = \frac{0}{0},$$

so L'Hôpital's Rule applies. We find

$$\lim_{x\to -1} \frac{x^4+2x+1}{x^5-2x-1} = \lim_{x\to -1} \frac{4x^3+2}{5x^4-2} = \frac{-4+2}{5-2} = -\frac{2}{3}.$$

5. $\displaystyle\lim_{x\to 9} \frac{x^{1/2}+x-6}{x^{3/2}-27}$

SOLUTION Because the quotient is not indeterminate at $x = 9$,

$$\left.\frac{x^{1/2}+x-6}{x^{3/2}-27}\right|_{x=9} = \frac{3+9-6}{27-27} = \frac{6}{0},$$

L'Hôpital's Rule does not apply.

6. $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x^3-7x-6}$

SOLUTION The functions $\sqrt{x+1} - 2$ and $x^3 - 7x - 6$ are differentiable, but the quotient is indeterminate at $x = 3$,

$$\left.\frac{\sqrt{x+1}-2}{x^3-7x-6}\right|_{x=3} = \frac{2-2}{27-21-6} = \frac{0}{0},$$

so L'Hôpital's Rule applies. We find

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x^3-7x-6} = \frac{\frac{1}{2\sqrt{x+1}}}{3x^2-7} = \frac{\frac{1}{4}}{20} = \frac{1}{80}.$$

7. $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{x^2+3x+1}$

SOLUTION Because the quotient is not indeterminate at $x = 0$,

$$\left.\frac{\sin 4x}{x^2+3x+1}\right|_{x=0} = \frac{0}{0+0+1} = \frac{0}{1},$$

L'Hôpital's Rule does not apply.

8. $\lim_{x\to 0} \frac{x^3}{\sin x - x}$

SOLUTION The functions x^3 and $\sin x - x$ are differentiable, but the quotient is indeterminate at $x = 0$,

$$\left.\frac{x^3}{\sin x - x}\right|_{x=0} = \frac{0}{0-0} = \frac{0}{0},$$

so L'Hôpital's Rule applies. Here, we use L'Hôpital's Rule three times to find

$$\lim_{x\to 0} \frac{x^3}{\sin x - x} = \lim_{x\to 0} \frac{3x^2}{\cos x - 1} = \lim_{x\to 0} \frac{6x}{-\sin x} = \lim_{x\to 0} \frac{6}{-\cos x} = -6.$$

9. $\lim_{x\to 0} \frac{\cos 2x - 1}{\sin 5x}$

SOLUTION The functions $\cos 2x - 1$ and $\sin 5x$ are differentiable, but the quotient is indeterminate at $x = 0$,

$$\left.\frac{\cos 2x - 1}{\sin 5x}\right|_{x=0} = \frac{1-1}{0} = \frac{0}{0},$$

so L'Hôpital's Rule applies. We find

$$\lim_{x\to 0} \frac{\cos 2x - 1}{\sin 5x} = \lim_{x\to 0} \frac{-2\sin 2x}{5\cos 5x} = \frac{0}{5} = 0.$$

10. $\lim_{x\to 0} \frac{\cos x - \sin^2 x}{\sin x}$

SOLUTION Because the quotient is not indeterminate at $x = 0$,

$$\left.\frac{\cos x - \sin^2 x}{\sin x}\right|_{x=0} = \frac{1-0}{0} = \frac{1}{0},$$

L'Hôpital's Rule does not apply.

In Exercises 11–16, show that L'Hôpital's Rule is applicable to the limit as $x \to \pm\infty$ and evaluate.

11. $\lim_{x\to \infty} \frac{9x+4}{3-2x}$

SOLUTION As $x \to \infty$, the quotient $\frac{9x+4}{3-2x}$ is of the form $\frac{\infty}{\infty}$, so L'Hôpital's Rule applies. We find

$$\lim_{x\to \infty} \frac{9x+4}{3-2x} = \lim_{x\to \infty} \frac{9}{-2} = -\frac{9}{2}.$$

12. $\lim_{x\to -\infty} x \sin \frac{1}{x}$

SOLUTION As $x \to \infty$, $x \sin \frac{1}{x}$ is of the form $\infty \cdot 0$, so L'Hôpital's Rule does not immediately apply. If we rewrite $x \sin \frac{1}{x}$ as $\frac{\sin(1/x)}{1/x}$, the rewritten expression is of the form $\frac{0}{0}$ as $x \to \infty$, so L'Hôpital's Rule now applies. We find

$$\lim_{x\to \infty} x \cdot \sin\left(\frac{1}{x}\right) = \lim_{x\to \infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to \infty} \frac{\cos(1/x)(-1/x^2)}{-1/x^2} = \lim_{x\to \infty} \cos(1/x) = \cos 0 = 1.$$

13. $\lim_{x\to\infty} \frac{\ln x}{x^{1/2}}$

SOLUTION As $x \to \infty$, the quotient $\frac{\ln x}{x^{1/2}}$ is of the form $\frac{\infty}{\infty}$, so L'Hôpital's Rule applies. We find

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{1}{2x^{1/2}} = 0.$$

14. $\lim_{x\to\infty} \frac{x}{e^x}$

SOLUTION As $x \to \infty$, the quotient $\frac{x}{e^x}$ is of the form $\frac{\infty}{\infty}$, so L'Hôpital's Rule applies. We find

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

15. $\lim_{x\to-\infty} \frac{\ln(x^4+1)}{x}$

SOLUTION As $x \to \infty$, the quotient $\frac{\ln(x^4+1)}{x}$ is of the form $\frac{\infty}{\infty}$, so L'Hôpital's Rule applies. Here, we use L'Hôpital's Rule twice to find

$$\lim_{x\to\infty} \frac{\ln(x^4+1)}{x} = \lim_{x\to\infty} \frac{\frac{4x^3}{x^4+1}}{1} = \lim_{x\to\infty} \frac{12x^2}{4x^3} = \lim_{x\to\infty} \frac{3}{x} = 0.$$

16. $\lim_{x\to\infty} \frac{x^2}{e^x}$

SOLUTION As $x \to \infty$, the quotient $\frac{x^2}{e^x}$ is of the form $\frac{\infty}{\infty}$, so L'Hôpital's Rule applies. Here, we use L'Hôpital's Rule twice to find

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

In Exercises 17–54, evaluate the limit.

17. $\lim_{x\to 1} \frac{\sqrt{8+x} - 3x^{1/3}}{x^2 - 3x + 2}$

SOLUTION $\lim_{x\to 1} \frac{\sqrt{8+x} - 3x^{1/3}}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{\frac{1}{2}(8+x)^{-1/2} - x^{-2/3}}{2x-3} = \frac{\frac{1}{6} - 1}{-1} = \frac{5}{6}.$

18. $\lim_{x\to 4} \left[\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right]$

SOLUTION $\lim_{x\to 4} \left[\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right] = \lim_{x\to 4} \left[\frac{\sqrt{x}+2}{x-4} - \frac{4}{x-4}\right] = \lim_{x\to 4} \frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{4}.$

19. $\lim_{x\to-\infty} \frac{3x-2}{1-5x}$

SOLUTION $\lim_{x\to-\infty} \frac{3x-2}{1-5x} = \lim_{x\to-\infty} \frac{3}{-5} = -\frac{3}{5}.$

20. $\lim_{x\to\infty} \frac{x^{2/3} + 3x}{x^{5/3} - x}$

SOLUTION $\lim_{x\to\infty} \frac{x^{2/3} + 3x}{x^{5/3} - x} = \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{3}{x^{2/3}}}{1 - \frac{1}{x^{2/3}}} = \frac{0+0}{1-0} = 0.$

21. $\lim_{x\to-\infty} \frac{7x^2 + 4x}{9 - 3x^2}$

SOLUTION $\lim_{x\to-\infty} \frac{7x^2 + 4x}{9 - 3x^2} = \lim_{x\to-\infty} \frac{14x + 4}{-6x} = \lim_{x\to-\infty} \frac{14}{-6} = -\frac{7}{3}.$

22. $\displaystyle\lim_{x\to\infty}\frac{3x^3+4x^2}{4x^3-7}$

SOLUTION $\displaystyle\lim_{x\to\infty}\frac{3x^3+4x^2}{4x^3-7}=\lim_{x\to\infty}\frac{9x^2+8x}{12x^2}=\lim_{x\to\infty}\frac{18x+8}{24x}=\frac{18}{24}=\frac{3}{4}.$

23. $\displaystyle\lim_{x\to 1}\frac{(1+3x)^{1/2}-2}{(1+7x)^{1/3}-2}$

SOLUTION Apply L'Hôpital's Rule once:

$$\lim_{x\to 1}\frac{(1+3x)^{1/2}-2}{(1+7x)^{1/3}-2}=\lim_{x\to 1}\frac{\frac{3}{2}(1+3x)^{-1/2}}{\frac{7}{3}(1+7x)^{-2/3}}$$

$$=\frac{(\frac{3}{2})\frac{1}{2}}{(\frac{7}{3})(\frac{1}{4})}=\frac{9}{7}$$

24. $\displaystyle\lim_{x\to 8}\frac{x^{5/3}-2x-16}{x^{1/3}-2}$

SOLUTION $\displaystyle\lim_{x\to 8}\frac{x^{5/3}-2x-16}{x^{1/3}-2}=\lim_{x\to 8}\frac{\frac{5}{3}x^{2/3}-2}{\frac{1}{3}x^{-2/3}}=\frac{\frac{20}{3}-2}{\frac{1}{12}}=56.$

25. $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{\sin 7x}$

SOLUTION $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{\sin 7x}=\lim_{x\to 0}\frac{2\cos 2x}{7\cos 7x}=\frac{2}{7}.$

26. $\displaystyle\lim_{x\to\pi/2}\frac{\tan 4x}{\tan 5x}$

SOLUTION

$$\lim_{x\to\pi/2}\frac{\tan 4x}{\tan 5x}=\lim_{x\to\pi/2}\frac{4\sec^2 4x}{5\sec^2 5x}=\frac{4}{5}\lim_{x\to\pi/2}\frac{\cos^2 5x}{\cos^2 4x}$$

$$=\frac{4}{5}\lim_{x\to\pi/2}\frac{-10\sin 5x\cos 5x}{-8\sin 4x\cos 4x}=\lim_{x\to\pi/2}\frac{\sin 10x}{\sin 8x}$$

$$=\lim_{x\to\pi/2}\frac{10\cos 10x}{8\cos 8x}=-\frac{5}{4}.$$

27. $\displaystyle\lim_{x\to 0}\frac{\tan x}{x}$

SOLUTION $\displaystyle\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sec^2 x}{1}=1.$

28. $\displaystyle\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)$

SOLUTION

$$\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{-x\sin x+\cos x-\cos x}{x\cos x+\sin x}=\lim_{x\to 0}\frac{-x\sin x}{x\cos x+\sin x}$$

$$=\lim_{x\to 0}\frac{-x\cos x-x}{-x\sin x+\cos x+\cos x}=\frac{0}{2}=0.$$

29. $\displaystyle\lim_{x\to 0}\frac{\sin x-x\cos x}{x-\sin x}$

SOLUTION

$$\lim_{x\to 0}\frac{\sin x-x\cos x}{x-\sin x}=\lim_{x\to 0}\frac{x\sin x}{1-\cos x}=\lim_{x\to 0}\frac{\sin x+x\cos x}{\sin x}=\lim_{x\to 0}\frac{\cos x+\cos x-x\sin x}{\cos x}=2.$$

30. $\displaystyle\lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\tan x$

SOLUTION

$$\lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\tan x=\lim_{x\to\pi/2}\frac{x-\pi/2}{1/\tan x}=\lim_{x\to\pi/2}\frac{x-\pi/2}{\cot x}=\lim_{x\to\pi/2}\frac{1}{-\csc^2 x}=\lim_{x\to\pi/2}-\sin^2 x=-1.$$

31. $\lim_{x\to 0} \frac{\cos(x+\frac{\pi}{2})}{\sin x}$

SOLUTION $\lim_{x\to 0} \frac{\cos(x+\frac{\pi}{2})}{\sin x} = \lim_{x\to 0} \frac{-\sin(x+\frac{\pi}{2})}{\cos x} = -1.$

32. $\lim_{x\to 0} \frac{x^2}{1-\cos x}$

SOLUTION $\lim_{x\to 0} \frac{x^2}{1-\cos x} = \lim_{x\to 0} \frac{2x}{\sin x} = \lim_{x\to 0} \frac{2}{\cos x} = 2.$

33. $\lim_{x\to \pi/2} \frac{\cos x}{\sin(2x)}$

SOLUTION $\lim_{x\to \pi/2} \frac{\cos x}{\sin(2x)} = \lim_{x\to \pi/2} \frac{-\sin x}{2\cos(2x)} = \frac{1}{2}.$

34. $\lim_{x\to 0} \left(\frac{1}{x^2} - \csc^2 x\right)$

SOLUTION

$$\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{x^2} - \csc^2 x\right) &= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \\
&= \lim_{x\to 0} \frac{2\sin x\cos x - 2x}{2x^2 \sin x \cos x + 2x\sin^2 x} = \lim_{x\to 0} \frac{\sin 2x - 2x}{x^2 \sin 2x + 2x\sin^2 x} \\
&= \lim_{x\to 0} \frac{2\cos 2x - 2}{2x^2\cos 2x + 2x\sin 2x + 4x\sin x\cos x + 2\sin^2 x} = \lim_{x\to 0} \frac{\cos 2x - 1}{x^2\cos 2x + 2x\sin 2x + \sin^2 x} \\
&= \lim_{x\to 0} \frac{-2\sin 2x}{-2x^2\sin 2x + 2x\cos 2x + 4x\cos 2x + 2\sin 2x + 2\sin x\cos x} \\
&= \lim_{x\to 0} \frac{-2\sin 2x}{(3-2x^2)\sin 2x + 6x\cos 2x} \\
&= \lim_{x\to 0} \frac{-4\cos 2x}{2(3-2x^2)\cos 2x - 4x\sin 2x + -12x\sin 2x + 6\cos 2x} = -\frac{1}{3}.
\end{aligned}$$

35. $\lim_{x\to \pi/2} (\sec x - \tan x)$

SOLUTION

$$\lim_{x\to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x\to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to \frac{\pi}{2}} \left(\frac{1-\sin x}{\cos x}\right) = \lim_{x\to \frac{\pi}{2}} \left(\frac{-\cos x}{-\sin x}\right) = 0.$$

36. $\lim_{x\to 2} \frac{e^{x^2} - e^4}{x-2}$

SOLUTION $\lim_{x\to 2} \frac{e^{x^2} - e^4}{x-2} = \lim_{x\to 2} \frac{2xe^{x^2}}{1} = 4e^4.$

37. $\lim_{x\to 1} \tan\left(\frac{\pi x}{2}\right) \ln x$

SOLUTION $\lim_{x\to 1} \tan\left(\frac{\pi x}{2}\right) \ln x = \lim_{x\to 1} \frac{\ln x}{\cot(\frac{\pi x}{2})} = \lim_{x\to 1} \frac{\frac{1}{x}}{-\frac{\pi}{2}\csc^2(\frac{\pi x}{2})} = \lim_{x\to 1} \frac{-2}{\pi x}\sin^2\left(\frac{\pi}{2}x\right) = -\frac{2}{\pi}.$

38. $\lim_{x\to 1} \frac{x(\ln x - 1) + 1}{(x-1)\ln x}$

SOLUTION

$$\lim_{x\to 1} \frac{x(\ln x - 1) + 1}{(x-1)\ln x} = \lim_{x\to 1} \frac{x(\frac{1}{x}) + (\ln x - 1)}{(x-1)(\frac{1}{x}) + \ln x} = \lim_{x\to 1} \frac{\ln x}{1 - \frac{1}{x} + \ln x} = \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{1}{1+1} = \frac{1}{2}.$$

39. $\lim_{x\to 0} \frac{e^x - 1}{\sin x}$

SOLUTION $\lim_{x\to 0} \frac{e^x - 1}{\sin x} = \lim_{x\to 0} \frac{e^x}{\cos x} = 1.$

40. $\lim_{x\to 1} \dfrac{e^x - e}{\ln x}$

SOLUTION $\lim_{x\to 1} \dfrac{e^x - e}{\ln x} = \lim_{x\to 1} \dfrac{e^x}{x^{-1}} = \dfrac{e}{1} = e.$

41. $\lim_{x\to 0} \dfrac{e^{2x} - 1 - x}{x^2}$

SOLUTION $\lim_{x\to 0} \dfrac{e^{2x} - 1 - x}{x^2} = \lim_{x\to 0} \dfrac{2e^{2x} - 1}{2x}$ which does not exist.

42. $\lim_{x\to \infty} \dfrac{e^{2x} - 1 - x}{x^2}$

SOLUTION

$$\lim_{x\to\infty} \frac{e^{2x} - 1 - x}{x^2} = \lim_{x\to\infty} \frac{2e^{2x} - 1}{2x} = \lim_{x\to\infty} \frac{4e^{2x}}{2} = \infty.$$

43. $\lim_{t\to 0+} (\sin t)(\ln t)$

SOLUTION

$$\lim_{t\to 0+} (\sin t)(\ln t) = \lim_{t\to 0+} \frac{\ln t}{\csc t} = \lim_{t\to 0+} \frac{\frac{1}{t}}{-\csc t \cot t} = \lim_{t\to 0+} \frac{-\sin^2 t}{t\cos t} = \lim_{t\to 0+} \frac{-2\sin t\cos t}{\cos t - t\sin t} = 0.$$

44. $\lim_{x\to\infty} e^{-x}(x^3 - x^2 + 9)$

SOLUTION

$$\lim_{x\to\infty} e^{-x}(x^3 - x^2 + 9) = \lim_{x\to\infty} \frac{x^3 - x^2 + 9}{e^x} = \lim_{x\to\infty} \frac{3x^2 - 2x}{e^x} = \lim_{x\to\infty} \frac{6x - 2}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = 0.$$

45. $\lim_{x\to 0} \dfrac{a^x - 1}{x} \quad (a > 0)$

SOLUTION $\lim_{x\to 0} \dfrac{a^x - 1}{x} = \lim_{x\to 0} \dfrac{\ln a \cdot a^x}{1} = \ln a.$

46. $\lim_{x\to\infty} x^{1/x^2}$

SOLUTION $\lim_{x\to\infty} \ln x^{1/x^2} = \lim_{x\to\infty} \dfrac{\ln x}{x^2} = \lim_{x\to\infty} \dfrac{1}{2x^2} = 0.$ Hence,

$$\lim_{x\to\infty} x^{1/x^2} = \lim_{x\to\infty} e^{\ln x^{1/x^2}} = e^0 = 1.$$

47. $\lim_{x\to 1} (1 + \ln x)^{1/(x-1)}$

SOLUTION $\lim_{x\to 1} \ln(1 + \ln x)^{1/(x-1)} = \lim_{x\to 1} \dfrac{\ln(1 + \ln x)}{x - 1} = \lim_{x\to 1} \dfrac{1}{x(1 + \ln x)} = 1.$ Hence,

$$\lim_{x\to 1} (1 + \ln x)^{1/(x-1)} = \lim_{x\to 1} e^{(1+\ln x)^{1/(x-1)}} = e.$$

48. $\lim_{x\to 0+} x^{\sin x}$

SOLUTION

$$\lim_{x\to 0+} \ln(x^{\sin x}) = \lim_{x\to 0+} \sin x(\ln x) = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\cos x(\sin x)^{-2}}$$

$$= \lim_{x\to 0+} -\frac{\sin^2 x}{x\cos x} = \lim_{x\to 0+} -\frac{2\sin x\cos x}{-x\sin x + \cos x} = 0.$$

Hence, $\lim_{x\to 0+} x^{\sin x} = \lim_{x\to 0+} e^{\ln(x^{\sin x})} = e^0 = 1.$

49. $\lim_{x\to 0} (\cos x)^{3/x^2}$

SOLUTION

$$\lim_{x\to 0} \ln(\cos x)^{3/x^2} = \lim_{x\to 0} \frac{3\ln\cos x}{x^2} = \lim_{x\to 0} -\frac{3\tan x}{2x} = \lim_{x\to 0} -\frac{3\sec^2 x}{2} = -\frac{3}{2}.$$

Hence, $\lim_{x\to 0} (\cos x)^{3/x^2} = e^{-3/2}$.

50. $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x$

SOLUTION

$$\lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+1}\right)}{1/x} = \lim_{x\to\infty} \frac{\left(\frac{x+1}{x}\right)\left(\frac{1}{(x+1)^2}\right)}{-1/x^2} = \lim_{x\to\infty} -\frac{x}{x+1} = -1.$$

Hence,

$$\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x = \frac{1}{e}.$$

51. $\lim_{x\to 0} \frac{\sin^{-1} x}{x}$

SOLUTION $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1.$

52. $\lim_{x\to 0} \frac{\tan^{-1} x}{\sin^{-1} x}$

SOLUTION $\lim_{x\to 0} \frac{\tan^{-1} x}{\sin^{-1} x} = \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{\frac{1}{\sqrt{1-x^2}}} = 1.$

53. $\lim_{x\to 1} \frac{\tan^{-1} x - \frac{\pi}{4}}{\tan\frac{\pi}{4}x - 1}$

SOLUTION $\lim_{x\to 1} \frac{\tan^{-1} x - \frac{\pi}{4}}{\tan(\pi x/4) - 1} = \lim_{x\to 1} \frac{\frac{1}{1+x^2}}{\frac{\pi}{4}\sec^2(\pi x/4)} = \frac{\frac{1}{2}}{\frac{\pi}{2}} = \frac{1}{\pi}.$

54. $\lim_{x\to 0+} \ln x \tan^{-1} x$

SOLUTION Let $h(x) = \ln x \tan^{-1} x$. $\lim_{x\to 0} h(x) = -\infty \cdot 0$, so we apply L'Hôpital's rule to $h(x) = \frac{f(x)}{g(x)}$, where $f(x) = \tan^{-1}(x)$ and $g(x) = \frac{1}{\ln x}$.

$$f'(x) = \frac{1}{1+x^2}$$
$$\lim_{x\to 0} f'(x) = 1$$
$$g'(x) = -\frac{1}{x(\ln x)^2}$$
$$\lim_{x\to 0} g'(x) = -\infty$$

Hence, L'Hôpital's rule yields:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 0} f'(x)}{\lim_{x\to 0} g'(x)} = -\frac{1}{\infty} = 0.$$

55. Evaluate $\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx}$, where $m, n \neq 0$ are integers.

SOLUTION Suppose m and n are even. Then there exist integers k and l such that $m = 2k$ and $n = 2l$ and

$$\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx} = \frac{\cos k\pi}{\cos l\pi} = (-1)^{k-l}.$$

Now, suppose m is even and n is odd. Then

$$\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx}$$

does not exist (from one side the limit tends toward $-\infty$, while from the other side the limit tends toward $+\infty$). Third, suppose m is odd and n is even. Then

$$\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx} = 0.$$

Finally, suppose m and n are odd. This is the only case when the limit is indeterminate. Then there exist integers k and l such that $m = 2k+1$, $n = 2l+1$ and, by L'Hôpital's Rule,

$$\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx} = \lim_{x\to\pi/2}\frac{-m\sin mx}{-n\sin nx} = (-1)^{k-l}\frac{m}{n}.$$

To summarize,

$$\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx} = \begin{cases} (-1)^{(m-n)/2}, & m, n \text{ even} \\ \text{does not exist}, & m \text{ even}, n \text{ odd} \\ 0 & m \text{ odd}, n \text{ even} \\ (-1)^{(m-n)/2}\frac{m}{n}, & m, n \text{ odd} \end{cases}$$

56. Evaluate $\lim_{x\to 1}\frac{x^m-1}{x^n-1}$ for any numbers $m, n \neq 0$.

SOLUTION $\lim_{x\to 1}\frac{x^m-1}{x^n-1} = \lim_{x\to 1}\frac{mx^{m-1}}{nx^{n-1}} = \frac{m}{n}$.

57. Prove the following limit formula for e:

$$e = \lim_{x\to 0}(1+x)^{1/x}$$

Then find a value of x such that $|(1+x)^{1/x} - e| \le 0.001$.

SOLUTION Using L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = 1.$$

Thus,

$$\lim_{x\to 0}\ln\left((1+x)^{1/x}\right) = \lim_{x\to 0}\frac{1}{x}\ln(1+x) = \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1,$$

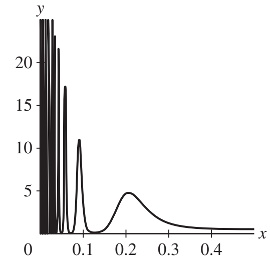
and $\lim_{x\to 0}(1+x)^{1/x} = e^1 = e$. For $x = 0.0005$,

$$\left|(1+x)^{1/x} - e\right| = |(1.0005)^{2000} - e| \approx 6.79\times 10^{-4} < 0.001.$$

58. GU Can L'Hôpital's Rule be applied to $\lim_{x\to 0+} x^{\sin(1/x)}$? Does a graphical or numerical investigation suggest that the limit exists?

SOLUTION Since $\sin(1/x)$ oscillates as $x \to 0+$, L'Hôpital's Rule cannot be applied. Both numerical and graphical investigations suggest that the limit does not exist due to the oscillation.

x	1	0.1	0.01	0.001	0.0001	0.00001
$x^{\sin(1/x)}$	1	3.4996	10.2975	0.003316	16.6900	0.6626



59. Let $f(x) = x^{1/x}$ for $x > 0$.

(a) Calculate $\lim_{x\to 0+} f(x)$ and $\lim_{x\to\infty} f(x)$.

(b) Find the maximum value of $f(x)$, and determine the intervals on which $f(x)$ is increasing or decreasing.

SOLUTION

(a) Let $f(x) = x^{1/x}$. Note that $\lim_{x\to 0+} x^{1/x}$ is not indeterminate. As $x \to 0+$, the base of the function tends toward 0 and the exponent tends toward $+\infty$. Both of these factors force $x^{1/x}$ toward 0. Thus, $\lim_{x\to 0+} f(x) = 0$. On the other hand, $\lim_{x\to\infty} f(x)$ is indeterminate. We calculate this limit as follows:

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0,$$

so $\lim_{x\to\infty} f(x) = e^0 = 1$.

(b) Again, let $f(x) = x^{1/x}$, so that $\ln f(x) = \frac{1}{x}\ln x$. To find the derivative f', we apply the derivative to both sides:

$$\frac{d}{dx}\ln f(x) = \frac{d}{dx}\left(\frac{1}{x}\ln x\right)$$

$$\frac{1}{f(x)} f'(x) = -\frac{\ln x}{x^2} + \frac{1}{x^2}$$

$$f'(x) = f(x)\left(-\frac{\ln x}{x^2} + \frac{1}{x^2}\right) = \frac{x^{1/x}}{x^2}(1 - \ln x)$$

Thus, f is increasing for $0 < x < e$, is decreasing for $x > e$ and has a maximum at $x = e$. The maximum value is $f(e) = e^{1/e} \approx 1.444668$.

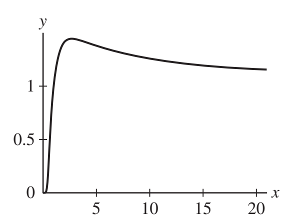
60. (a) Use the results of Exercise 59 to prove that $x^{1/x} = c$ has a unique solution if $0 < c \le 1$ or $c = e^{1/e}$, two solutions if $1 < c < e^{1/e}$, and no solutions if $c > e^{1/e}$.

(b) GU Plot the graph of $f(x) = x^{1/x}$ and verify that it confirms the conclusions of (a).

SOLUTION

(a) Because $(e, e^{1/e})$ is the only maximum, no solution exists for $c > e^{1/e}$ and only one solution exists for $c = e^{1/e}$. Moreover, because $f(x)$ increases from 0 to $e^{1/e}$ as x goes from 0 to e and then decreases from $e^{1/e}$ to 1 as x goes from e to $+\infty$, it follows that there are two solutions for $1 < c < e^{1/e}$, but only one solution for $0 < c \le 1$.

(b) Observe that if we sketch the horizontal line $y = c$, this line will intersect the graph of $y = f(x)$ only once for $0 < c \le 1$ and $c = e^{1/e}$ and will intersect the graph of $y = f(x)$ twice for $1 < c < e^{1/e}$. There are no points of intersection for $c > e^{1/e}$.



61. Determine whether $f << g$ or $g << f$ (or neither) for the functions $f(x) = \log_{10} x$ and $g(x) = \ln x$.

SOLUTION Because

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\log_{10} x}{\ln x} = \lim_{x\to\infty} \frac{\frac{\ln x}{\ln 10}}{\ln x} = \frac{1}{\ln 10},$$

neither $f << g$ or $g << f$ is satisfied.

62. Show that $(\ln x)^2 << \sqrt{x}$ and $(\ln x)^4 << x^{1/10}$.

SOLUTION

- $(\ln x)^2 << \sqrt{x}$:

$$\lim_{x\to\infty} \frac{\sqrt{x}}{(\ln x)^2} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{2}{x}\ln x} = \lim_{x\to\infty} \frac{\sqrt{x}}{4\ln x} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{4}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{8} = \infty.$$

- $(\ln x)^4 << x^{1/10}$:

$$\lim_{x\to\infty} \frac{x^{1/10}}{(\ln x)^4} = \lim_{x\to\infty} \frac{\frac{1}{10x^{9/10}}}{\frac{4}{x}(\ln x)^3} = \lim_{x\to\infty} \frac{x^{1/10}}{40(\ln x)^3} = \lim_{x\to\infty} \frac{\frac{1}{10x^{9/10}}}{\frac{120}{x}(\ln x)^2} = \lim_{x\to\infty} \frac{x^{1/10}}{1200(\ln x)^2}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{10x^{9/10}}}{\frac{2400}{x}(\ln x)} = \lim_{x\to\infty} \frac{x^{1/10}}{24000\ln x} = \lim_{x\to\infty} \frac{\frac{1}{10x^{9/10}}}{\frac{24000}{x}} = \lim_{x\to\infty} \frac{x^{1/10}}{240000} = \infty.$$

63. Just as exponential functions are distinguished by their rapid rate of increase, the logarithm functions grow particularly slowly. Show that $\ln x << x^a$ for all $a > 0$.

SOLUTION Using L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln x}{x^a} = \lim_{x\to\infty} \frac{x^{-1}}{ax^{a-1}} = \lim_{x\to\infty} \frac{1}{a}x^{-a} = 0;$$

hence, $\ln x << (x^a)$.

64. Show that $(\ln x)^N << x^a$ for all N and all $a > 0$.

SOLUTION

$$\lim_{x\to\infty} \frac{x^a}{(\ln x)^N} = \lim_{x\to\infty} \frac{ax^{a-1}}{\frac{N}{x}(\ln x)^{N-1}} = \lim_{x\to\infty} \frac{ax^a}{N(\ln x)^{N-1}} = \cdots$$

If we continue in this manner, L'Hôpital's Rule will give a factor of x^a in the numerator, but the power on $\ln x$ in the denominator will eventually be zero. Thus,

$$\lim_{x\to\infty} \frac{x^a}{(\ln x)^N} = \infty,$$

so $(\ln x)^N << x^a$ for all N and for all $a > 0$.

65. Determine whether $\sqrt{x} << e^{\sqrt{\ln x}}$ or $e^{\sqrt{\ln x}} << \sqrt{x}$. *Hint:* Use the substitution $u = \ln x$ instead of L'Hôpital's Rule.

SOLUTION Let $u = \ln x$, then $x = e^u$, and as $x \to \infty$, $u \to \infty$. So

$$\lim_{x\to\infty} \frac{e^{\sqrt{\ln x}}}{\sqrt{x}} = \lim_{u\to\infty} \frac{e^{\sqrt{u}}}{e^{u/2}} = \lim_{u\to\infty} e^{\sqrt{u}-\frac{u}{2}}.$$

We need to examine $\lim_{u\to\infty}(\sqrt{u} - \frac{u}{2})$. Since

$$\lim_{u\to\infty} \frac{u/2}{\sqrt{u}} = \lim_{u\to\infty} \frac{\frac{1}{2}}{\frac{1}{2\sqrt{u}}} = \lim_{u\to\infty} \sqrt{u} = \infty,$$

$\sqrt{u} = o(u/2)$ and $\lim_{u\to\infty}\left(\sqrt{u} - \dfrac{u}{2}\right) = -\infty$. Thus

$$\lim_{u\to\infty} e^{\sqrt{u}-\frac{u}{2}} = e^{-\infty} = 0 \quad \text{so} \quad \lim_{x\to\infty} \frac{e^{\sqrt{\ln x}}}{\sqrt{x}} = 0$$

and $e^{\sqrt{\ln x}} << \sqrt{x}$.

66. Show that $\lim_{x\to\infty} x^n e^{-x} = 0$ for all whole numbers $n > 0$.

SOLUTION

$$\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x}$$
$$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x}$$
$$\vdots$$
$$= \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

67. Assumptions Matter Let $f(x) = x(2 + \sin x)$ and $g(x) = x^2 + 1$.

(a) Show directly that $\lim_{x\to\infty} f(x)/g(x) = 0$.

(b) Show that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, but $\lim_{x\to\infty} f'(x)/g'(x)$ does not exist.

Do (a) and (b) contradict L'Hôpital's Rule? Explain.

SOLUTION

(a) $1 \le 2 + \sin x \le 3$, so

$$\frac{x}{x^2+1} \le \frac{x(2+\sin x)}{x^2+1} \le \frac{3x}{x^2+1}.$$

Since,

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{3x}{x^2+1} = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x\to\infty} \frac{x(2+\sin x)}{x^2+1} = 0.$$

(b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x(2+\sin x) \ge \lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} (x^2+1) = \infty$, but

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{x(\cos x) + (2+\sin x)}{2x}$$

does not exist since $\cos x$ oscillates. This does not violate L'Hôpital's Rule since the theorem clearly states

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

"provided the limit on the right exists."

68. Let $H(b) = \lim_{x\to\infty} \frac{\ln(1+b^x)}{x}$ for $b > 0$.

(a) Show that $H(b) = \ln b$ if $b \ge 1$

(b) Determine $H(b)$ for $0 < b \le 1$.

SOLUTION

(a) Suppose $b \ge 1$. Then

$$H(b) = \lim_{x\to\infty} \frac{\ln(1+b^x)}{x} = \lim_{x\to\infty} \frac{b^x \ln b}{1+b^x} = \frac{b^x \ln b}{b^x} = \ln b.$$

(b) Now, suppose $0 < b < 1$. Then

$$H(b) = \lim_{x\to\infty} \frac{\ln(1+b^x)}{x} = \lim_{x\to\infty} \frac{b^x \ln b}{1+b^x} = \frac{0}{1} = 0.$$

69. Let $G(b) = \lim_{x\to\infty} (1+b^x)^{1/x}$.

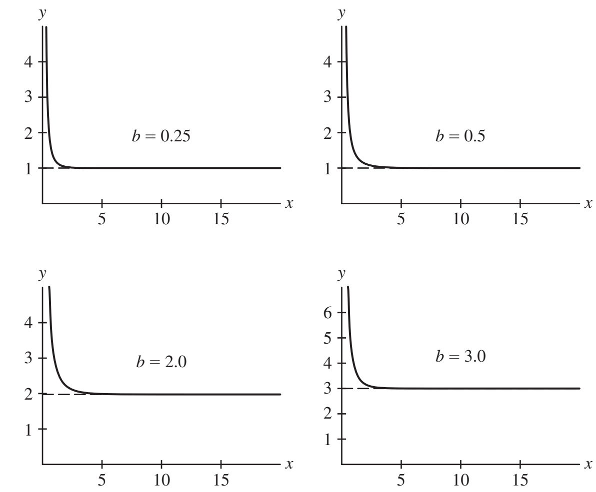
(a) Use the result of Exercise 68 to evaluate $G(b)$ for all $b > 0$.

(b) GU Verify your result graphically by plotting $y = (1+b^x)^{1/x}$ together with the horizontal line $y = G(b)$ for the values $b = 0.25, 0.5, 2, 3$.

SOLUTION

(a) Using Exercise 68, we see that $G(b) = e^{H(b)}$. Thus, $G(b) = 1$ if $0 \le b \le 1$ and $G(b) = b$ if $b > 1$.

(b)



70. Show that $\lim_{t\to\infty} t^k e^{-t^2} = 0$ for all k. *Hint:* Compare with $\lim_{t\to\infty} t^k e^{-t} = 0$.

SOLUTION Because we are interested in the limit as $t \to +\infty$, we will restrict attention to $t > 1$. Then, for all k,

$$0 \le t^k e^{-t^2} \le t^k e^{-t}.$$

As $\lim_{t\to\infty} t^k e^{-t} = 0$, it follows from the Squeeze Theorem that

$$\lim_{t\to\infty} t^k e^{-t^2} = 0.$$

In Exercises 71–73, let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$$

These exercises show that $f(x)$ has an unusual property: All of its derivatives at $x = 0$ exist and are equal to zero.

71. Show that $\lim_{x\to 0} \frac{f(x)}{x^k} = 0$ for all k. *Hint:* Let $t = x^{-1}$ and apply the result of Exercise 70.

SOLUTION $\lim_{x\to 0} \frac{f(x)}{x^k} = \lim_{x\to 0} \frac{1}{x^k e^{1/x^2}}$. Let $t = 1/x$. As $x \to 0$, $t \to \infty$. Thus,

$$\lim_{x\to 0} \frac{1}{x^k e^{1/x^2}} = \lim_{t\to\infty} \frac{t^k}{e^{t^2}} = 0$$

by Exercise 70.

72. Show that $f'(0)$ exists and is equal to zero. Also, verify that $f''(0)$ exists and is equal to zero.

SOLUTION Working from the definition,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = 0$$

by the previous exercise. Thus, $f'(0)$ exists and is equal to 0. Moreover,

$$f'(x) = \begin{cases} e^{-1/x^2}\left(\frac{2}{x^3}\right) & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Now,

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} e^{-1/x^2}\left(\frac{2}{x^4}\right) = 2\lim_{x\to 0} \frac{f(x)}{x^4} = 0$$

by the previous exercise. Thus, $f''(0)$ exists and is equal to 0.

73. Show that for $k \geq 1$ and $x \neq 0$,

$$f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r}$$

for some polynomial $P(x)$ and some exponent $r \geq 1$. Use the result of Exercise 71 to show that $f^{(k)}(0)$ exists and is equal to zero for all $k \geq 1$.

SOLUTION For $x \neq 0$, $f'(x) = e^{-1/x^2}\left(\frac{2}{x^3}\right)$. Here $P(x) = 2$ and $r = 3$. Assume $f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r}$. Then

$$f^{(k+1)}(x) = e^{-1/x^2}\left(\frac{x^3P'(x) + (2 - rx^2)P(x)}{x^{r+3}}\right)$$

which is of the form desired.

Moreover, from Exercise 72, $f'(0) = 0$. Suppose $f^{(k)}(0) = 0$. Then

$$f^{(k+1)}(0) = \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x\to 0} \frac{P(x)e^{-1/x^2}}{x^{r+1}} = P(0) \lim_{x\to 0} \frac{f(x)}{x^{r+1}} = 0.$$

Further Insights and Challenges

74. Show that L'Hôpital's Rule applies to $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$ but that it does not help. Then evaluate the limit directly.

SOLUTION Both the numerator $f(x) = x$ and the denominator $g(x) = \sqrt{x^2+1}$ tend to infinity as $x \to \infty$, and $g'(x) = x/\sqrt{x^2+1}$ is nonzero for $x > 0$. Therefore, L'Hôpital's Rule applies:

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{x(x^2+1)^{-1/2}} = \lim_{x\to\infty} \frac{(x^2+1)^{1/2}}{x}$$

We may apply L'Hôpital's Rule again: $\lim_{x\to\infty} \frac{(x^2+1)^{1/2}}{x} = \lim_{x\to\infty} \frac{x(x^2+1)^{-1/2}}{1} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$. This takes us back to the original limit, so L'Hôpital's Rule is ineffective. However, we can evaluate the limit directly by observing that

$$\frac{x}{\sqrt{x^2+1}} = \frac{x^{-1}(x)}{x^{-1}\sqrt{x^2+1}} = \frac{1}{\sqrt{1+x^{-2}}} \quad \text{and hence} \quad \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+x^{-2}}} = 1.$$

75. The Second Derivative Test for critical points fails if $f''(c) = 0$. This exercise develops a **Higher Derivative Test** based on the sign of the first nonzero derivative. Suppose that

$$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0, \quad \text{but} \quad f^{(n)}(c) \neq 0$$

(a) Show, by applying L'Hôpital's Rule n times, that

$$\lim_{x\to c} \frac{f(x) - f(c)}{(x-c)^n} = \frac{1}{n!} f^{(n)}(c)$$

where $n! = n(n-1)(n-2)\cdots(2)(1)$.

(b) Use (a) to show that if n is even, then $f(c)$ is a local minimum if $f^{(n)}(c) > 0$ and is a local maximum if $f^{(n)}(c) < 0$. *Hint:* If n is even, then $(x-c)^n > 0$ for $x \neq a$, so $f(x) - f(c)$ must be positive for x near c if $f^{(n)}(c) > 0$.

(c) Use (a) to show that if n is odd, then $f(c)$ is neither a local minimum nor a local maximum.

SOLUTION

(a) Repeated application of L'Hôpital's rule yields

$$\begin{aligned}
\lim_{x\to c} \frac{f(x) - f(c)}{(x-c)^n} &= \lim_{x\to c} \frac{f'(x)}{n(x-c)^{n-1}} \\
&= \lim_{x\to c} \frac{f''(x)}{n(n-1)(x-c)^{n-2}} \\
&= \lim_{x\to c} \frac{f'''(x)}{n(n-1)(n-2)(x-c)^{n-3}} \\
&= \cdots \\
&= \frac{1}{n!} f^{(n)}(c)
\end{aligned}$$

(b) Suppose n is even. Then $(x-c)^n > 0$ for all $x \neq c$. If $f^{(n)}(c) > 0$, it follows that $f(x) - f(c)$ must be positive for x near c. In other words, $f(x) > f(c)$ for x near c and $f(c)$ is a local minimum. On the other hand, if $f^{(n)}(c) < 0$, it follows that $f(x) - f(c)$ must be negative for x near c. In other words, $f(x) < f(c)$ for x near c and $f(c)$ is a local maximum.

(c) If n is odd, then $(x-c)^n > 0$ for $x > c$ but $(x-c)^n < 0$ for $x < c$. If $f^{(n)}(c) > 0$, it follows that $f(x) - f(c)$ must be positive for x near c and $x > c$ but is negative for x near c and $x < c$. In other words, $f(x) > f(c)$ for x near c and $x > c$ but $f(x) < f(c)$ for x near c and $x < c$. Thus, $f(c)$ is neither a local minimum nor a local maximum. We obtain a similar result if $f^{(n)}(c) < 0$.

76. When a spring with natural frequency $\lambda/2\pi$ is driven with a sinusoidal force $\sin(\omega t)$ with $\omega \neq \lambda$, it oscillates according to

$$y(t) = \frac{1}{\lambda^2 - \omega^2}\big(\lambda \sin(\omega t) - \omega \sin(\lambda t)\big)$$

Let $y_0(t) = \lim_{\omega\to\lambda} y(t)$.

(a) Use L'Hôpital's Rule to determine $y_0(t)$.

(b) Show that $y_0(t)$ ceases to be periodic and that its amplitude $|y_0(t)|$ tends to ∞ as $t \to \infty$ (the system is said to be in **resonance**; eventually, the spring is stretched beyond its limits).

(c) CAS Plot $y(t)$ for $\lambda = 1$ and $\omega = 0.8, 0.9, 0.99$, and 0.999. Do the graphs confirm your conclusion in (b)?

SOLUTION

(a)

$$\lim_{\omega\to\lambda} y(t) = \lim_{\omega\to\lambda} \frac{\lambda\sin(\omega t) - \omega\sin(\lambda t)}{\lambda^2 - \omega^2} = \lim_{\omega\to\lambda} \frac{\frac{d}{d\omega}(\lambda\sin(\omega t) - \omega\sin(\lambda t))}{\frac{d}{d\omega}(\lambda^2 - \omega^2)}$$

$$= \lim_{\omega\to\lambda} \frac{\lambda t\cos(\omega t) - \sin(\lambda t)}{-2\omega} = \frac{\lambda t\cos(\lambda t) - \sin(\lambda t)}{-2\lambda}$$

(b) From part (a)

$$y_0(t) = \lim_{\omega\to\lambda} y(t) = \frac{\lambda t\cos(\lambda t) - \sin(\lambda t)}{-2\lambda}.$$

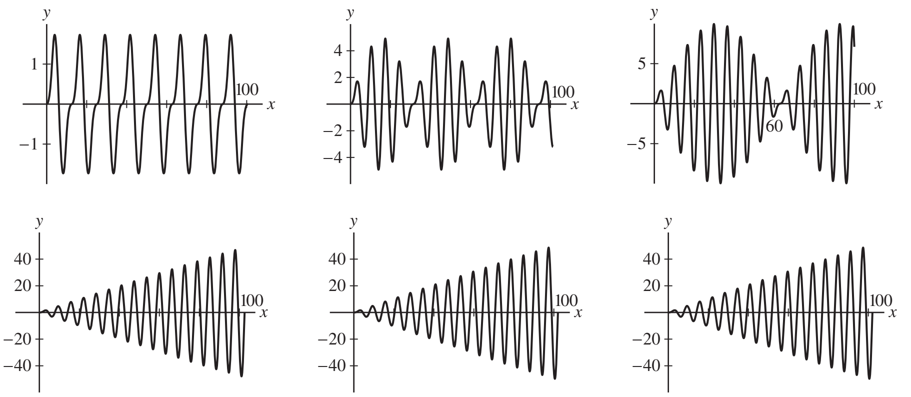
This may be rewritten as

$$y_0(t) = \frac{\sqrt{\lambda^2 t^2 + 1}}{-2\lambda}\cos(\lambda t + \phi),$$

where $\cos\phi = \dfrac{\lambda t}{\sqrt{\lambda^2 t^2 + 1}}$ and $\sin\phi = \dfrac{1}{\sqrt{\lambda^2 t^2 + 1}}$. Since the amplitude varies with t, $y_0(t)$ is not periodic. Also note that

$$\frac{\sqrt{\lambda^2 t^2 + 1}}{-2\lambda} \to \infty \quad \text{as} \quad t \to \infty.$$

(c) The graphs below were produced with $\lambda = 1$. Moving from left to right and from top to bottom, $\omega = 0.5, 0.8, 0.9, 0.99, 0.999, 1$.



77. We expended a lot of effort to evaluate $\lim\limits_{x\to 0} \frac{\sin x}{x}$ in Chapter 2. Show that we could have evaluated it easily using L'Hôpital's Rule. Then explain why this method would involve *circular reasoning*.

SOLUTION $\lim\limits_{x\to 0} \frac{\sin x}{x} = \lim\limits_{x\to 0} \frac{\cos x}{1} = 1$. To use L'Hôpital's Rule to evaluate $\lim\limits_{x\to 0} \frac{\sin x}{x}$, we must know that the derivative of $\sin x$ is $\cos x$, but to determine the derivative of $\sin x$, we must be able to evaluate $\lim\limits_{x\to 0} \frac{\sin x}{x}$.

78. By a fact from algebra, if f, g are polynomials such that $f(a) = g(a) = 0$, then there are polynomials f_1, g_1 such that

$$f(x) = (x-a)f_1(x), \qquad g(x) = (x-a)g_1(x)$$

Use this to verify L'Hôpital's Rule directly for $\lim\limits_{x\to a} f(x)/g(x)$.

SOLUTION As in the problem statement, let $f(x)$ and $g(x)$ be two polynomials such that $f(a) = g(a) = 0$, and let $f_1(x)$ and $g_1(x)$ be the polynomials such that $f(x) = (x-a)f_1(x)$ and $g(x) = (x-a)g_1(x)$. By the product rule, we have the following facts,

$$f'(x) = (x-a)f_1'(x) + f_1(x)$$
$$g'(x) = (x-a)g_1'(x) + g_1(x)$$

so

$$\lim_{x\to a} f'(x) = f_1(a) \quad \text{and} \quad \lim_{x\to a} g'(x) = g_1(a).$$

L'Hôpital's Rule stated for f and g is: if $\lim_{x\to a} g'(x) \neq 0$, so that $g_1(a) \neq 0$,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f_1(a)}{g_1(a)}.$$

Suppose $g_1(a) \neq 0$. Then, by direct computation,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{(x-a)f_1(x)}{(x-a)g_1(x)} = \lim_{x\to a} \frac{f_1(x)}{g_1(x)} = \frac{f_1(a)}{g_1(a)},$$

exactly as predicted by L'Hôpital's Rule.

79. Patience Required Use L'Hôpital's Rule to evaluate and check your answers numerically:

(a) $\lim\limits_{x\to 0+} \left(\frac{\sin x}{x}\right)^{1/x^2}$ **(b)** $\lim\limits_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$

SOLUTION

(a) We start by evaluating

$$\lim_{x\to 0+} \ln\left(\frac{\sin x}{x}\right)^{1/x^2} = \lim_{x\to 0+} \frac{\ln(\sin x) - \ln x}{x^2}.$$

Repeatedly using L'Hôpital's Rule, we find

$$\begin{aligned}\lim_{x\to 0+} \ln\left(\frac{\sin x}{x}\right)^{1/x^2} &= \lim_{x\to 0+} \frac{\cot x - x^{-1}}{2x} = \lim_{x\to 0+} \frac{x\cos x - \sin x}{2x^2 \sin x} = \lim_{x\to 0+} \frac{-x\sin x}{2x^2\cos x + 4x\sin x} \\ &= \lim_{x\to 0+} \frac{-x\cos x - \sin x}{8x\cos x + 4\sin x - 2x^2\sin x} = \lim_{x\to 0+} \frac{-2\cos x + x\sin x}{12\cos x - 2x^2\cos x - 12x\sin x} \\ &= -\frac{2}{12} = -\frac{1}{6}.\end{aligned}$$

Therefore, $\lim\limits_{x\to 0+} \left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$. Numerically we find:

x	1	0.1	0.01
$\left(\frac{\sin x}{x}\right)^{1/x^2}$	0.841471	0.846435	0.846481

Note that $e^{-1/6} \approx 0.846481724$.

(b) Repeatedly using L'Hôpital's Rule and simplifying, we find

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{2x-2\sin x\cos x}{x^2(2\sin x\cos x)+2x\sin^2 x}=\lim_{x\to 0}\frac{2x-2\sin 2x}{x^2\sin 2x+2x\sin^2 x}$$

$$=\lim_{x\to 0}\frac{2-2\cos 2x}{2x^2\cos 2x+2x\sin 2x+4x\sin x\cos x+2\sin^2 x}$$

$$=\lim_{x\to 0}\frac{2-2\cos 2x}{2x^2\cos 2x+4x\sin 2x+2\sin^2 x}$$

$$=\lim_{x\to 0}\frac{4\sin 2x}{-4x^2\sin 2x+4x\cos 2x+8x\cos 2x+4\sin 2x+4\sin x\cos x}$$

$$=\lim_{x\to 0}\frac{4\sin 2x}{(6-4x^2)\sin 2x+12x\cos 2x}$$

$$=\lim_{x\to 0}\frac{8\cos 2x}{(12-8x^2)\cos 2x-8x\sin 2x+12\cos 2x-24x\sin 2x}=\frac{1}{3}.$$

Numerically we find:

x	1	0.1	0.01
$\dfrac{1}{\sin^2 x}-\dfrac{1}{x^2}$	0.412283	0.334001	0.333340

80. In the following cases, check that $x = c$ is a critical point and use Exercise 75 to determine whether $f(c)$ is a local minimum or a local maximum.
(a) $f(x) = x^5 - 6x^4 + 14x^3 - 16x^2 + 9x + 12 \quad (c = 1)$
(b) $f(x) = x^6 - x^3 \quad (c = 0)$

SOLUTION

(a) Let $f(x) = x^5 - 6x^4 + 14x^3 - 16x^2 + 9x + 12$. Then $f'(x) = 5x^4 - 24x^3 + 42x^2 - 32x + 9$, so $f'(1) = 5 - 24 + 42 - 32 + 9 = 0$ and $c = 1$ is a critical point. Now,

$$f''(x) = 20x^3 - 72x^2 + 84x - 32 \text{ so } f''(1) = 0;$$
$$f'''(x) = 60x^2 - 144x + 84 \text{ so } f'''(1) = 0;$$
$$f^{(4)}(x) = 120x - 144 \text{ so } f^{(4)}(1) = -24 \neq 0.$$

Thus, $n = 4$ is even and $f^{(4)} < 0$, so $f(1)$ is a local maximum.
(b) Let $f(x) = x^6 - x^3$. Then, $f'(x) = 6x^5 - 3x^2$, so $f'(0) = 0$ and $c = 0$ is a critical point. Now,

$$f''(x) = 30x^4 - 6x \text{ so } f''(0) = 0;$$
$$f'''(x) = 120x - 6 \text{ so } f'''(0) = -6 \neq 0.$$

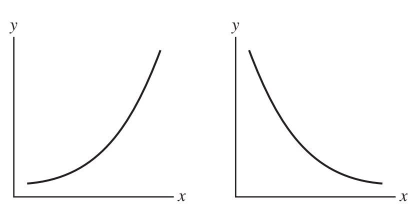
Thus, $n = 3$ is odd, so $f(0)$ is neither a local minimum nor a local maximum.

4.6 Graph Sketching and Asymptotes

Preliminary Questions

1. Sketch an arc where f' and f'' have the sign combination $++$. Do the same for $-+$.

SOLUTION An arc with the sign combination $++$ (increasing, concave up) is shown below at the left. An arc with the sign combination $-+$ (decreasing, concave up) is shown below at the right.



2. If the sign combination of f' and f'' changes from $++$ to $+-$ at $x = c$, then (choose the correct answer):

(a) $f(c)$ is a local min **(b)** $f(c)$ is a local max

(c) c is a point of inflection

SOLUTION Because the sign of the second derivative changes at $x = c$, the correct response is **(c)**: c is a point of inflection.

3. The second derivative of the function $f(x) = (x-4)^{-1}$ is $f''(x) = 2(x-4)^{-3}$. Although $f''(x)$ changes sign at $x = 4$, $f(x)$ does not have a point of inflection at $x = 4$. Why not?

SOLUTION The function f does not have a point of inflection at $x = 4$ because $x = 4$ is not in the domain of f.

Exercises

1. Determine the sign combinations of f' and f'' for each interval A–G in Figure 16.

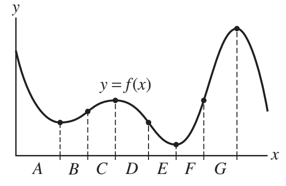


FIGURE 16

SOLUTION

- In A, f is decreasing and concave up, so $f' < 0$ and $f'' > 0$.
- In B, f is increasing and concave up, so $f' > 0$ and $f'' > 0$.
- In C, f is increasing and concave down, so $f' > 0$ and $f'' < 0$.
- In D, f is decreasing and concave down, so $f' < 0$ and $f'' < 0$.
- In E, f is decreasing and concave up, so $f' < 0$ and $f'' > 0$.
- In F, f is increasing and concave up, so $f' > 0$ and $f'' > 0$.
- In G, f is increasing and concave down, so $f' > 0$ and $f'' < 0$.

2. State the sign change at each transition point A–G in Figure 17. Example: $f'(x)$ goes from $+$ to $-$ at A.

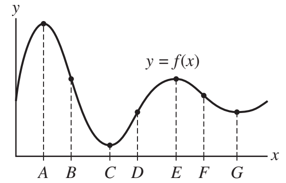


FIGURE 17

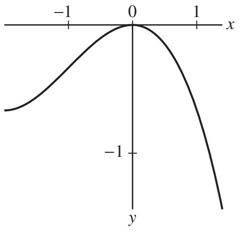
SOLUTION

- At B, the graph changes from concave down to concave up, so f'' goes from $-$ to $+$.
- At C, the graph changes from decreasing to increasing, so f' goes from $-$ to $+$.
- At D, the graph changes from concave up to concave down, so f'' goes from $+$ to $-$.
- At E, the graph changes from increasing to decreasing, so f' goes from $+$ to $-$.
- At F, the graph changes from concave down to concave up, so f'' goes from $-$ to $+$.
- At G, the graph changes from decreasing to increasing, so f' goes from $-$ to $+$.

In Exercises 3–6, draw the graph of a function for which f' and f'' take on the given sign combinations.

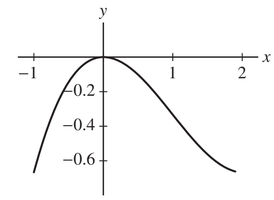
3. $++$, $+-$, $--$

SOLUTION This function changes from concave up to concave down at $x = -1$ and from increasing to decreasing at $x = 0$.



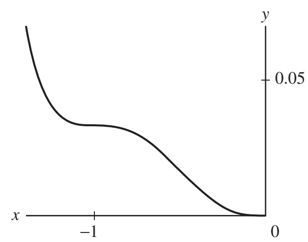
4. $+-$, $--$, $-+$

SOLUTION This function changes from increasing to decreasing at $x = 0$ and from concave down to concave up at $x = 1$.



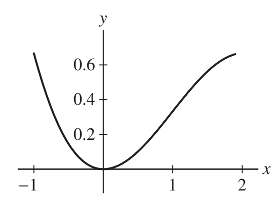
5. $-+$, $--$, $-+$

SOLUTION The function is decreasing everywhere and changes from concave up to concave down at $x = -1$ and from concave down to concave up at $x = -\frac{1}{2}$.



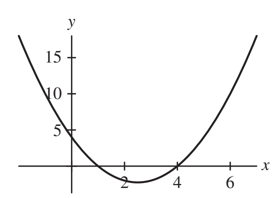
6. $-+$, $++$, $+-$

SOLUTION This function changes from decreasing to increasing at $x = 0$ and from concave up to concave down at $x = 1$.



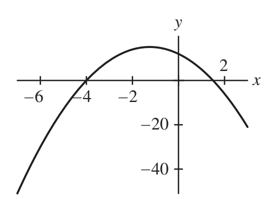
7. Sketch the graph of $y = x^2 - 5x + 4$.

SOLUTION Let $f(x) = x^2 - 5x + 4$. Then $f'(x) = 2x - 5$ and $f''(x) = 2$. Hence f is decreasing for $x < 5/2$, is increasing for $x > 5/2$, has a local minimum at $x = 5/2$ and is concave up everywhere.



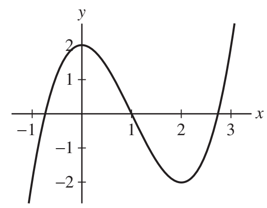
8. Sketch the graph of $y = 12 - 5x - 2x^2$.

SOLUTION Let $f(x) = 12 - 5x - 2x^2$. Then $f'(x) = -5 - 4x$ and $f''(x) = -4$. Hence f is increasing for $x < -5/4$, is decreasing for $x > -5/4$, has a local maximum at $x = -5/4$ and is concave down everywhere.



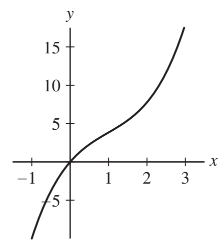
9. Sketch the graph of $f(x) = x^3 - 3x^2 + 2$. Include the zeros of $f(x)$, which are $x = 1$ and $1 \pm \sqrt{3}$ (approximately $-0.73, 2.73$).

SOLUTION Let $f(x) = x^3 - 3x^2 + 2$. Then $f'(x) = 3x^2 - 6x = 3x(x-2) = 0$ yields $x = 0, 2$ and $f''(x) = 6x - 6$. Thus f is concave down for $x < 1$, is concave up for $x > 1$, has an inflection point at $x = 1$, is increasing for $x < 0$ and for $x > 2$, is decreasing for $0 < x < 2$, has a local maximum at $x = 0$, and has a local minimum at $x = 2$.



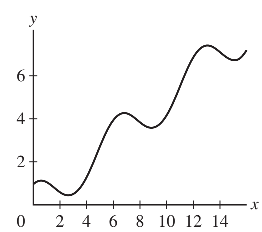
10. Show that $f(x) = x^3 - 3x^2 + 6x$ has a point of inflection but no local extreme values. Sketch the graph.

SOLUTION Let $f(x) = x^3 - 3x^2 + 6x$. Then $f'(x) = 3x^2 - 6x + 6 = 3\left((x-1)^2 + 1\right) > 0$ for all values of x and $f''(x) = 6x - 6$. Hence f is everywhere increasing and has an inflection point at $x = 1$. It is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.



11. Extend the sketch of the graph of $f(x) = \cos x + \frac{1}{2}x$ in Example 4 to the interval $[0, 5\pi]$.

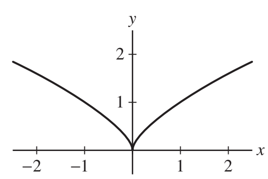
SOLUTION Let $f(x) = \cos x + \frac{1}{2}x$. Then $f'(x) = -\sin x + \frac{1}{2} = 0$ yields critical points at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}$, and $\frac{29\pi}{6}$. Moreover, $f''(x) = -\cos x$ so there are points of inflection at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$, and $\frac{9\pi}{2}$.



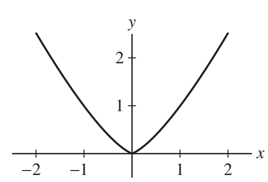
12. Sketch the graphs of $y = x^{2/3}$ and $y = x^{4/3}$.

SOLUTION

- Let $f(x) = x^{2/3}$. Then $f'(x) = \frac{2}{3}x^{-1/3}$ and $f''(x) = -\frac{2}{9}x^{-4/3}$, neither of which exist at $x = 0$. Thus f is decreasing and concave down for $x < 0$ and increasing and concave down for $x > 0$.



- Let $f(x) = x^{4/3}$. Then $f'(x) = \frac{4}{3}x^{1/3}$ and $f''(x) = \frac{4}{9}x^{-2/3}$. Thus f is decreasing and concave up for $x < 0$ and increasing and concave up for $x > 0$.



In Exercises 13–34, find the transition points, intervals of increase/decrease, concavity, and asymptotic behavior. Then sketch the graph, with this information indicated.

13. $y = x^3 + 24x^2$

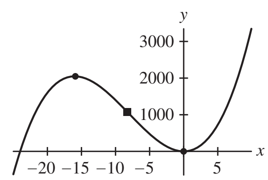
SOLUTION Let $f(x) = x^3 + 24x^2$. Then $f'(x) = 3x^2 + 48x = 3x(x + 16)$ and $f''(x) = 6x + 48$. This shows that f has critical points at $x = 0$ and $x = -16$ and a candidate for an inflection point at $x = -8$.

Interval	$(-\infty, -16)$	$(-16, -8)$	$(-8, 0)$	$(0, \infty)$
Signs of f' and f''	$+-$	$--$	$-+$	$++$

Thus, there is a local maximum at $x = -16$, a local minimum at $x = 0$, and an inflection point at $x = -8$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with these transition points highlighted as in the graphs in the textbook.



14. $y = x^3 - 3x + 5$

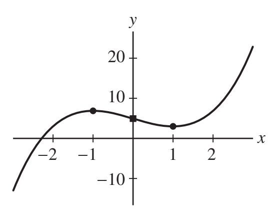
SOLUTION Let $f(x) = x^3 - 3x + 5$. Then $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. Critical points are at $x = \pm 1$ and the sole candidate point of inflection is at $x = 0$.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Signs of f' and f''	$+-$	$--$	$-+$	$++$

Thus, $f(-1)$ is a local maximum, $f(1)$ is a local minimum, and there is a point of inflection at $x = 0$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \lim_{x\to\infty} f(x) = \infty.$$

Here is the graph of f with the transition points highlighted as in the textbook.



15. $y = x^2 - 4x^3$

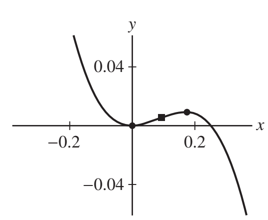
SOLUTION Let $f(x) = x^2 - 4x^3$. Then $f'(x) = 2x - 12x^2 = 2x(1 - 6x)$ and $f''(x) = 2 - 24x$. Critical points are at $x = 0$ and $x = \frac{1}{6}$, and the sole candidate point of inflection is at $x = \frac{1}{12}$.

Interval	$(-\infty, 0)$	$(0, \frac{1}{12})$	$(\frac{1}{12}, \frac{1}{6})$	$(\frac{1}{6}, \infty)$
Signs of f' and f''	$-+$	$++$	$+-$	$--$

Thus, $f(0)$ is a local minimum, $f(\frac{1}{6})$ is a local maximum, and there is a point of inflection at $x = \frac{1}{12}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is the graph of f with transition points highlighted as in the textbook:

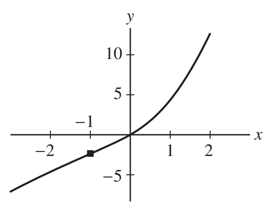


16. $y = \frac{1}{3}x^3 + x^2 + 3x$

SOLUTION Let $f(x) = \frac{1}{3}x^3 + x^2 + 3x$. Then $f'(x) = x^2 + 2x + 3$, and $f''(x) = 2x + 2 = 0$ if $x = -1$. Sign analysis shows that $f'(x) = (x+1)^2 + 2 > 0$ for all x (so that $f(x)$ has no critical points and is always increasing), and that $f''(x)$ changes from negative to positive at $x = -1$, implying that the graph of $f(x)$ has an inflection point at $(-1, f(-1))$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \lim_{x\to\infty} f(x) = \infty.$$

A graph with the inflection point indicated appears below:



17. $y = 4 - 2x^2 + \frac{1}{6}x^4$

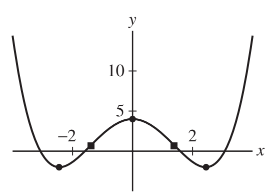
SOLUTION Let $f(x) = \frac{1}{6}x^4 - 2x^2 + 4$. Then $f'(x) = \frac{2}{3}x^3 - 4x = \frac{2}{3}x\left(x^2 - 6\right)$ and $f''(x) = 2x^2 - 4$. This shows that f has critical points at $x = 0$ and $x = \pm\sqrt{6}$ and has candidates for points of inflection at $x = \pm\sqrt{2}$.

Interval	$(-\infty, -\sqrt{6})$	$(-\sqrt{6}, -\sqrt{2})$	$(-\sqrt{2}, 0)$	$(0, \sqrt{2})$	$(\sqrt{2}, \sqrt{6})$	$(\sqrt{6}, \infty)$
Signs of f' and f''	$-+$	$++$	$+-$	$--$	$-+$	$++$

Thus, f has local minima at $x = \pm\sqrt{6}$, a local maximum at $x = 0$, and inflection points at $x = \pm\sqrt{2}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.



18. $y = 7x^4 - 6x^2 + 1$

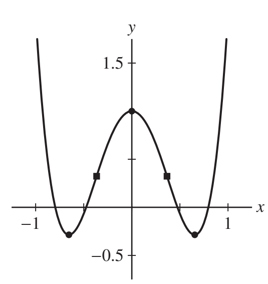
SOLUTION Let $f(x) = 7x^4 - 6x^2 + 1$. Then $f'(x) = 28x^3 - 12x = 4x\left(7x^2 - 3\right)$ and $f''(x) = 84x^2 - 12$. This shows that f has critical points at $x = 0$ and $x = \pm\frac{\sqrt{21}}{7}$ and candidates for points of inflection at $x = \pm\frac{\sqrt{7}}{7}$.

Interval	$(-\infty, -\frac{\sqrt{21}}{7})$	$(-\frac{\sqrt{21}}{7}, -\frac{\sqrt{7}}{7})$	$(-\frac{\sqrt{7}}{7}, 0)$	$(0, \frac{\sqrt{7}}{7})$	$(\frac{\sqrt{7}}{7}, \frac{\sqrt{21}}{7})$	$(\frac{\sqrt{21}}{7}, \infty)$
Signs of f' and f''	$-+$	$++$	$+-$	$--$	$-+$	$++$

Thus, f has local minima at $x = \pm\frac{\sqrt{21}}{7}$, a local maximum at $x = 0$, and inflection points at $x = \pm\frac{\sqrt{7}}{7}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

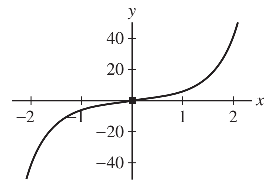


19. $y = x^5 + 5x$

SOLUTION Let $f(x) = x^5 + 5x$. Then $f'(x) = 5x^4 + 5 = 5(x^4 + 1)$ and $f''(x) = 20x^3$. $f'(x) > 0$ for all x, so the graph has no critical points and is always increasing. $f''(x) = 0$ at $x = 0$. Sign analyses reveal that $f''(x)$ changes from negative to positive at $x = 0$, so that the graph of $f(x)$ has an inflection point at $(0, 0)$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

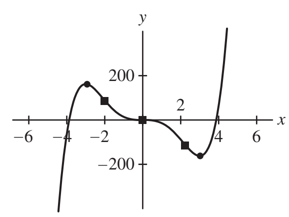


20. $y = x^5 - 15x^3$

SOLUTION Let $f(x) = x^5 - 15x^3$. Then $f'(x) = 5x^4 - 45x^2 = 5x^2(x^2 - 9)$ and $f''(x) = 20x^3 - 90x = 10x(2x^2 - 9)$. This shows that f has critical points at $x = 0$ and $x = \pm 3$ and candidate inflection points at $x = 0$ and $x = \pm 3\sqrt{2}/2$. Sign analyses reveal that $f'(x)$ changes from positive to negative at $x = -3$, is negative on either side of $x = 0$ and changes from negative to positive at $x = 3$. The graph therefore has a local maximum at $x = -3$ and a local minimum at $x = 3$. Further sign analyses show that $f''(x)$ transitions from positive to negative at $x = 0$ and from negative to positive at $x = \pm 3\sqrt{2}/2$. The graph therefore has points of inflection at $x = 0$ and $x = \pm 3\sqrt{2}/2$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

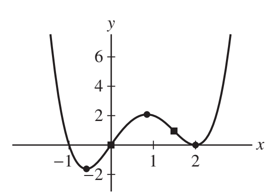


21. $y = x^4 - 3x^3 + 4x$

SOLUTION Let $f(x) = x^4 - 3x^3 + 4x$. Then $f'(x) = 4x^3 - 9x^2 + 4 = (4x^2 - x - 2)(x - 2)$ and $f''(x) = 12x^2 - 18x = 6x(2x - 3)$. This shows that f has critical points at $x = 2$ and $x = \dfrac{1 \pm \sqrt{33}}{8}$ and candidate points of inflection at $x = 0$ and $x = \frac{3}{2}$. Sign analyses reveal that $f'(x)$ changes from negative to positive at $x = \frac{1-\sqrt{33}}{8}$, from positive to negative at $x = \frac{1+\sqrt{33}}{8}$, and again from negative to positive at $x = 2$. Therefore, $f(\frac{1-\sqrt{33}}{8})$ and $f(2)$ are local minima of $f(x)$, and $f(\frac{1+\sqrt{33}}{8})$ is a local maximum. Further sign analyses reveal that $f''(x)$ changes from positive to negative at $x = 0$ and from negative to positive at $x = \frac{3}{2}$, so that there are points of inflection both at $x = 0$ and $x = \frac{3}{2}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of $f(x)$ with transition points highlighted.



22. $y = x^2(x - 4)^2$

SOLUTION Let $f(x) = x^2(x - 4)^2$. Then

$$f'(x) = 2x(x-4)^2 + 2x^2(x-4) = 2x(x-4)(x-4+x) = 4x(x-4)(x-2)$$

and

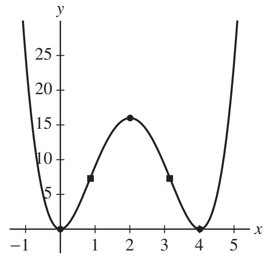
$$f''(x) = 12x^2 - 48x + 32 = 4(3x^2 - 12x + 8).$$

Critical points are therefore at $x = 0, x = 4$, and $x = 2$. Candidate inflection points are at solutions of $4(3x^2 - 12x + 8) = 0$, which, from the quadratic formula, are at $2 \pm \frac{\sqrt{48}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$.

Sign analyses reveal that $f'(x)$ changes from negative to positive at $x = 0$ and $x = 4$, and from positive to negative at $x = 2$. Therefore, $f(0)$ and $f(4)$ are local minima, and $f(2)$ a local maximum, of $f(x)$. Also, $f''(x)$ changes from positive to negative at $2 - \frac{2\sqrt{3}}{3}$ and from negative to positive at $2 + \frac{2\sqrt{3}}{3}$. Therefore there are points of inflection at both $x = 2 \pm \frac{2\sqrt{3}}{3}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of $f(x)$ with transition points highlighted.

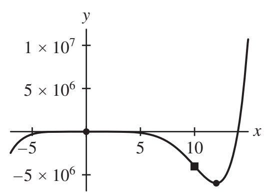


23. $y = x^7 - 14x^6$

SOLUTION Let $f(x) = x^7 - 14x^6$. Then $f'(x) = 7x^6 - 84x^5 = 7x^5(x - 12)$ and $f''(x) = 42x^5 - 420x^4 = 42x^4(x - 10)$. Critical points are at $x = 0$ and $x = 12$, and candidate inflection points are at $x = 0$ and $x = 10$. Sign analyses reveal that $f'(x)$ changes from positive to negative at $x = 0$ and from negative to positive at $x = 12$. Therefore $f(0)$ is a local maximum and $f(12)$ is a local minimum. Also, $f''(x)$ changes from negative to positive at $x = 10$. Therefore, there is a point of inflection at $x = 10$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

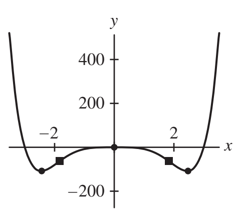


24. $y = x^6 - 9x^4$

SOLUTION Let $f(x) = x^6 - 9x^4$. Then $f'(x) = 6x^5 - 36x^3 = 6x^3(x^2 - 6)$ and $f''(x) = 30x^4 - 108x^2 = 6x^2(5x^2 - 18)$. This shows that f has critical points at $x = 0$ and $x = \pm\sqrt{6}$ and candidate inflection points at $x = 0$ and $x = \pm 3\sqrt{10}/5$. Sign analyses reveal that $f'(x)$ changes from negative to positive at $x = -\sqrt{6}$, from positive to negative at $x = 0$ and from negative to positive at $x = \sqrt{6}$. The graph therefore has a local maximum at $x = 0$ and local minima at $x = \pm\sqrt{6}$. Further sign analyses show that $f''(x)$ transitions from positive to negative at $x = -3\sqrt{10}/5$ and from negative to positive at $x = 3\sqrt{10}/5$. The graph therefore has points of inflection at $x = \pm 3\sqrt{10}/5$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

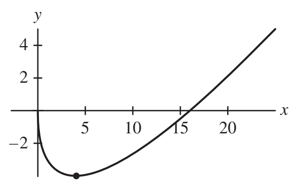


25. $y = x - 4\sqrt{x}$

SOLUTION Let $f(x) = x - 4\sqrt{x} = x - 4x^{1/2}$. Then $f'(x) = 1 - 2x^{-1/2}$. This shows that f has critical points at $x = 0$ (where the derivative does not exist) and at $x = 4$ (where the derivative is zero). Because $f'(x) < 0$ for $0 < x < 4$ and $f'(x) > 0$ for $x > 4$, $f(4)$ is a local minimum. Now $f''(x) = x^{-3/2} > 0$ for all $x > 0$, so the graph is always concave up. Moreover,

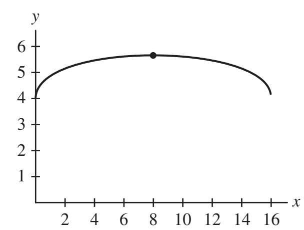
$$\lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.



26. $y = \sqrt{x} + \sqrt{16 - x}$

SOLUTION Let $f(x) = \sqrt{x} + \sqrt{16 - x} = x^{1/2} + (16 - x)^{1/2}$. Note that the domain of f is [0, 16]. Now, $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}(16 - x)^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2} - \frac{1}{4}(16 - x)^{-3/2}$. Thus, the critical points are $x = 0$, $x = 8$ and $x = 16$. Sign analysis reveals that $f'(x) > 0$ for $0 < x < 8$ and $f'(x) < 0$ for $8 < x < 16$, so f has a local maximum at $x = 9$. Further, $f''(x) < 0$ on (0, 16), so the graph is always concave down. Here is a graph of f with the transition point highlighted.



27. $y = x(8 - x)^{1/3}$

SOLUTION Let $f(x) = x(8 - x)^{1/3}$. Then

$$f'(x) = x \cdot \tfrac{1}{3}(8 - x)^{-2/3}(-1) + (8 - x)^{1/3} \cdot 1 = \frac{24 - 4x}{3(8 - x)^{2/3}}$$

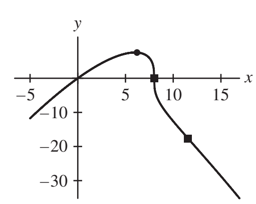
and similarly

$$f''(x) = \frac{4x - 48}{9(8 - x)^{5/3}}.$$

Critical points are at $x = 8$ and $x = 6$, and candidate inflection points are $x = 8$ and $x = 12$. Sign analyses reveal that $f'(x)$ changes from positive to negative at $x = 6$ and $f'(x)$ remains negative on either side of $x = 8$. Moreover, $f''(x)$ changes from negative to positive at $x = 8$ and from positive to negative at $x = 12$. Therefore, f has a local maximum at $x = 6$ and inflection points at $x = 8$ and $x = 12$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = -\infty.$$

Here is a graph of f with the transition points highlighted.



28. $y = (x^2 - 4x)^{1/3}$

SOLUTION Let $f(x) = (x^2 - 4x)^{1/3}$. Then

$$f'(x) = \frac{2}{3}(x-2)(x^2-4x)^{-2/3}$$

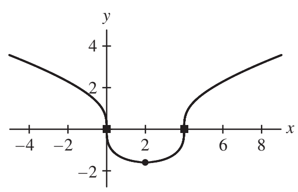
and

$$f''(x) = \frac{2}{3}\left((x^2-4x)^{-2/3} - \frac{4}{3}(x-2)^2(x^2-4x)^{-5/3}\right)$$
$$= \frac{2}{9}(x^2-4x)^{-5/3}\left(3(x^2-4x) - 4(x-2)^2\right) = -\frac{2}{9}(x^2-4x)^{-5/3}(x^2-4x+16).$$

Critical points of $f(x)$ are $x = 2$ (where the derivative is zero) an $x = 0$ and $x = 4$ (where the derivative does not exist); candidate points of inflection are $x = 0$ and $x = 4$. Sign analyses reveal that $f''(x) < 0$ for $x < 0$ and for $x > 4$, while $f''(x) > 0$ for $0 < x < 4$. Therefore, the graph of $f(x)$ has points of inflection at $x = 0$ and $x = 4$. Since $(x^2 - x)^{-2/3}$ is positive wherever it is defined, the sign of $f'(x)$ depends solely on the sign of $x - 2$. Hence, $f'(x)$ does not change sign at $x = 0$ or $x = 4$, and goes from negative to positive at $x = 2$. $f(2)$ is, in that case, a local minimum. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of $f(x)$ with the transition points indicated.



29. $y = xe^{-x^2}$

SOLUTION Let $f(x) = xe^{-x^2}$. Then

$$f'(x) = -2x^2e^{-x^2} + e^{-x^2} = (1-2x^2)e^{-x^2},$$

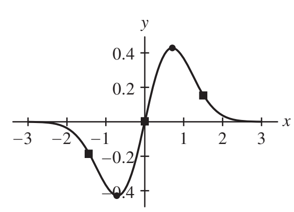
and

$$f''(x) = (4x^3 - 2x)e^{-x^2} - 4xe^{-x^2} = 2x(2x^2-3)e^{-x^2}.$$

There are critical points at $x = \pm\frac{\sqrt{2}}{2}$, and $x = 0$ and $x = \pm\frac{\sqrt{3}}{2}$ are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from negative to positive at $x = -\frac{\sqrt{2}}{2}$ and from positive to negative at $x = \frac{\sqrt{2}}{2}$. Moreover, $f''(x)$ changes from negative to positive at both $x = \pm\frac{\sqrt{3}}{2}$ and from positive to negative at $x = 0$. Therefore, f has a local minimum at $x = -\frac{\sqrt{2}}{2}$, a local maximum at $x = \frac{\sqrt{2}}{2}$ and inflection points at $x = 0$ and at $x = \pm\frac{\sqrt{3}}{2}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = 0,$$

so the graph has a horizontal asymptote at $y = 0$. Here is a graph of f with the transition points highlighted.



30. $y = (2x^2 - 1)e^{-x^2}$

SOLUTION Let $f(x) = (2x^2 - 1)e^{-x^2}$. Then

$$f'(x) = (2x - 4x^3)e^{-x^2} + 4xe^{-x^2} = 2x(3-2x^2)e^{-x^2},$$

and

$$f''(x) = (8x^4 - 12x^2)e^{-x^2} + (6 - 12x^2)e^{-x^2} = 2(4x^4 - 12x^2 + 3)e^{-x^2}.$$

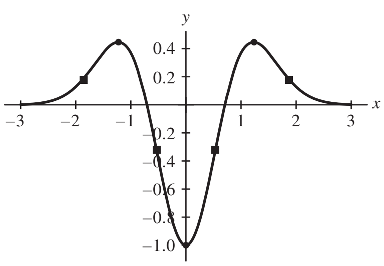
There are critical points at $x = 0$ and $x = \pm\frac{\sqrt{3}}{2}$, and

$$x = -\sqrt{\frac{3+\sqrt{6}}{2}}, \quad x = -\sqrt{\frac{3-\sqrt{6}}{2}}, \quad x = \sqrt{\frac{3-\sqrt{6}}{2}}, \quad x = \sqrt{\frac{3+\sqrt{6}}{2}}$$

are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from positive to negative at $x = \pm\frac{\sqrt{3}}{2}$ and from negative to positive at $x = 0$. Moreover, $f''(x)$ changes from positive to negative at $x = -\sqrt{\frac{3+\sqrt{6}}{2}}$ and at $x = \sqrt{\frac{3-\sqrt{6}}{2}}$ and from negative to positive at $x = -\sqrt{\frac{3-\sqrt{6}}{2}}$ and at $x = \sqrt{\frac{3+\sqrt{6}}{2}}$. Therefore, f has local maxima at $x = \pm\frac{\sqrt{3}}{2}$, a local minimum at $x = 0$ and points of inflection at $x = \pm\sqrt{\frac{3\pm\sqrt{6}}{2}}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = 0,$$

so the graph has a horizontal asymptote at $y = 0$. Here is a graph of f with the transition points highlighted.



31. $y = x - 2\ln x$

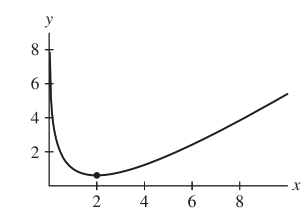
SOLUTION Let $f(x) = x - 2\ln x$. Note that the domain of f is $x > 0$. Now,

$$f'(x) = 1 - \frac{2}{x} \quad \text{and} \quad f''(x) = \frac{2}{x^2}.$$

The only critical point is $x = 2$. Sign analysis shows that $f'(x)$ changes from negative to positive at $x = 2$, so $f(2)$ is a local minimum. Further, $f''(x) > 0$ for $x > 0$, so the graph is always concave up. Moreover,

$$\lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with the transition points highlighted.



32. $y = x(4 - x) - 3\ln x$

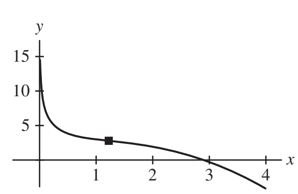
SOLUTION Let $f(x) = x(4 - x) - 3\ln x$. Note that the domain of f is $x > 0$. Now,

$$f'(x) = 4 - 2x - \frac{3}{x} \quad \text{and} \quad f''(x) = -2 + \frac{3}{x^2}.$$

Because $f'(x) < 0$ for all $x > 0$, the graph is always decreasing. On the other hand, $f''(x)$ changes from positive to negative at $x = \sqrt{\frac{3}{2}}$, so there is a point of inflection at $x = \sqrt{\frac{3}{2}}$. Moreover,

$$\lim_{x\to 0+} f(x) = \infty \quad \text{and} \quad \lim_{x\to infty} f(x) = -\infty,$$

so f has a vertical asymptote at $x = 0$. Here is a graph of f with the transition points highlighted.

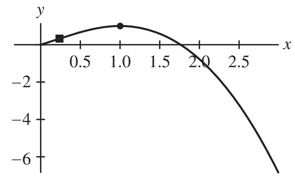


33. $y = x - x^2 \ln x$

SOLUTION Let $f(x) = x - x^2 \ln x$. Then $f'(x) = 1 - x - 2x \ln x$ and $f''(x) = -3 - 2\ln x$. There is a critical point at $x = 1$, and $x = e^{-3/2} \approx 0.223$ is a candidate inflection point. Sign analysis shows that $f'(x)$ changes from positive to negative at $x = 1$ and that $f''(x)$ changes from positive to negative at $x = e^{-3/2}$. Therefore, f has a local maximum at $x = 1$ and a point of inflection at $x = e^{-3/2}$. Moreover,

$$\lim_{x\to\infty} f(x) = -\infty.$$

Here is a graph of f with the transition points highlighted.



34. $y = x - 2\ln(x^2 + 1)$

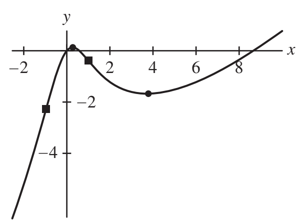
SOLUTION Let $f(x) = x - 2\ln(x^2 + 1)$. Then $f'(x) = 1 - \dfrac{4x}{x^2 + 1}$, and

$$f''(x) = -\frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} = \frac{4(x^2-1)}{(x^2+1)^2}.$$

There are critical points at $x = 2 \pm \sqrt{3}$, and $x = \pm 1$ are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from positive to negative at $x = 2 - \sqrt{3}$ and from negative to positive at $x = 2 + \sqrt{3}$. Moreover, $f''(x)$ changes from positive to negative at $x = -1$ and from negative to positive at $x = 1$. Therefore, f has a local maximum at $x = 2 - \sqrt{3}$, a local minimum at $x = 2 + \sqrt{3}$ and points of inflection at $x = \pm 1$. Finally,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with the transition points highlighted.



35. Sketch the graph of $f(x) = 18(x-3)(x-1)^{2/3}$ using the formulas

$$f'(x) = \frac{30\left(x - \frac{9}{5}\right)}{(x-1)^{1/3}}, \qquad f''(x) = \frac{20\left(x - \frac{3}{5}\right)}{(x-1)^{4/3}}$$

SOLUTION

$$f'(x) = \frac{30(x - \frac{9}{5})}{(x-1)^{1/3}}$$

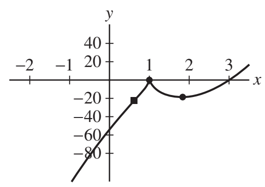
yields critical points at $x = \frac{9}{5}$, $x = 1$.

$$f''(x) = \frac{20(x - \frac{3}{5})}{(x-1)^{4/3}}$$

yields potential inflection points at $x = \frac{3}{5}$, $x = 1$.

Interval	signs of f' and f''
$(-\infty, \frac{3}{5})$	$+-$
$(\frac{3}{5}, 1)$	$++$
$(1, \frac{9}{5})$	$-+$
$(\frac{9}{5}, \infty)$	$++$

The graph has an inflection point at $x = \frac{3}{5}$, a local maximum at $x = 1$ (at which the graph has a cusp), and a local minimum at $x = \frac{9}{5}$. The sketch looks something like this.



36. Sketch the graph of $f(x) = \dfrac{x}{x^2+1}$ using the formulas

$$f'(x) = \frac{1-x^2}{(1+x^2)^2}, \qquad f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

SOLUTION Let $f(x) = \dfrac{x}{x^2+1}$.

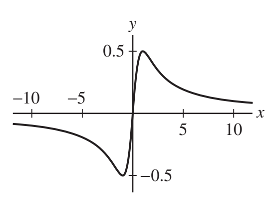
- Because $\lim_{x\to\pm\infty} f(x) = \frac{1}{1} \cdot \lim_{x\to\pm\infty} x^{-1} = 0$, $y = 0$ is a horizontal asymptote for f.
- Now $f'(x) = \dfrac{1-x^2}{\left(x^2+1\right)^2}$ is negative for $x < -1$ and $x > 1$, positive for $-1 < x < 1$, and 0 at $x = \pm 1$. Accordingly, f is decreasing for $x < -1$ and $x > 1$, is increasing for $-1 < x < 1$, has a local minimum value at $x = -1$ and a local maximum value at $x = 1$.
- Moreover,

$$f''(x) = \frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}.$$

Here is a sign chart for the second derivative, similar to those constructed in various exercises in Section 4.4. (The legend is on page 425.)

x	$\left(-\infty, -\sqrt{3}\right)$	$-\sqrt{3}$	$\left(-\sqrt{3}, 0\right)$	0	$\left(0, \sqrt{3}\right)$	$\sqrt{3}$	$\left(\sqrt{3}, \infty\right)$
f''	−	0	+	0	−	0	+
f	⌢	I	⌣	I	⌢	I	⌣

- Here is a graph of $f(x) = \dfrac{x}{x^2+1}$.



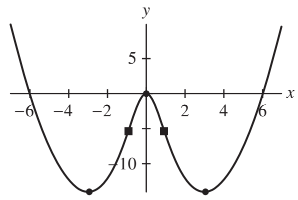
CAS *In Exercises 37–40, sketch the graph of the function, indicating all transition points. If necessary, use a graphing utility or computer algebra system to locate the transition points numerically.*

37. $y = x^2 - 10\ln(x^2+1)$

SOLUTION Let $f(x) = x^2 - 10\ln(x^2+1)$. Then $f'(x) = 2x - \dfrac{20x}{x^2+1}$, and

$$f''(x) = 2 - \frac{(x^2+1)(20) - (20x)(2x)}{(x^2+1)^2} = \frac{x^4+12x^2-9}{(x^2+1)^2}.$$

There are critical points at $x = 0$ and $x = \pm 3$, and $x = \pm\sqrt{-6+3\sqrt{5}}$ are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from negative to positive at $x = \pm 3$ and from positive to negative at $x = 0$. Moreover, $f''(x)$ changes from positive to negative at $x = -\sqrt{-6+3\sqrt{5}}$ and from negative to positive at $x = \sqrt{-6+3\sqrt{5}}$. Therefore, f has a local maximum at $x = 0$, local minima at $x = \pm 3$ and points of inflection at $x = \pm\sqrt{-6+3\sqrt{5}}$. Here is a graph of f with the transition points highlighted.



38. $y = e^{-x/2} \ln x$

SOLUTION Let $f(x) = e^{-x/2} \ln x$. Then

$$f'(x) = \frac{e^{-x/2}}{x} - \frac{1}{2}e^{-x/2} \ln x = e^{-x/2}\left(\frac{1}{x} - \frac{1}{2}\ln x\right)$$

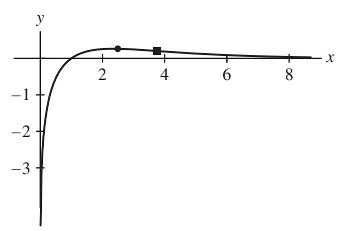
and

$$f''(x) = e^{-x/2}\left(-\frac{1}{x^2} - \frac{1}{2x}\right) - \frac{1}{2}e^{-x/2}\left(\frac{1}{x} - \frac{1}{2}\ln x\right)$$
$$= e^{-x/2}\left(-\frac{1}{x^2} - \frac{1}{x} + \frac{1}{4}\ln x\right).$$

There is a critical point at $x = 2.345751$ and a candidate point of inflection at $x = 3.792199$. Sign analysis reveals that $f'(x)$ changes from positive to negative at $x = 2.345751$ and that $f''(x)$ changes from negative to positive at $x = 3.792199$. Therefore, f has a local maximum at $x = 2.345751$ and a point of inflection at $x = 3.792199$. Moreover,

$$\lim_{x\to 0+} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = 0.$$

Here is a graph of f with the transition points highlighted.

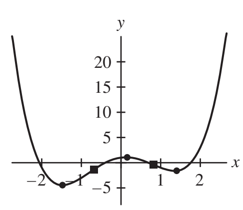


39. $y = x^4 - 4x^2 + x + 1$

SOLUTION Let $f(x) = x^4 - 4x^2 + x + 1$. Then $f'(x) = 4x^3 - 8x + 1$ and $f''(x) = 12x^2 - 8$. The critical points are $x = -1.473$, $x = 0.126$ and $x = 1.347$, while the candidates for points of inflection are $x = \pm\sqrt{\frac{2}{3}}$. Sign analysis reveals that $f'(x)$ changes from negative to positive at $x = -1.473$, from positive to negative at $x = 0.126$ and from negative to positive at $x = 1.347$. For the second derivative, $f''(x)$ changes from positive to negative at $x = -\sqrt{\frac{2}{3}}$ and from negative to positive at $x = \sqrt{\frac{2}{3}}$. Therefore, f has local minima at $x = -1.473$ and $x = 1.347$, a local maximum at $x = 0.126$ and points of inflection at $x = \pm\sqrt{\frac{2}{3}}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of f with the transition points highlighted.

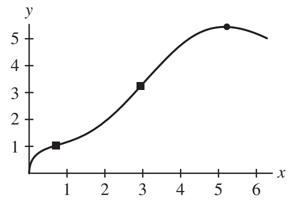


40. $y = 2\sqrt{x} - \sin x, \quad 0 \le x \le 2\pi$

SOLUTION Let $f(x) = 2\sqrt{x} - \sin x$. Then

$$f'(x) = \frac{1}{\sqrt{x}} - \cos x \quad \text{and} \quad f''(x) = -\frac{1}{2}x^{-3/2} + \sin x.$$

On $0 \le x \le 2\pi$, there is a critical point at $x = 5.167866$ and candidate points of inflection at $x = 0.790841$ and $x = 3.047468$. Sign analysis reveals that $f'(x)$ changes from positive to negative at $x = 5.167866$, while $f''(x)$ changes from negative to positive at $x = 0.790841$ and from positive to negative at $x = 3.047468$. Therefore, f has a local maximum at $x = 5.167866$ and points of inflection at $x = 0.790841$ and $x = 3.047468$. Here is a graph of f with the transition points highlighted.



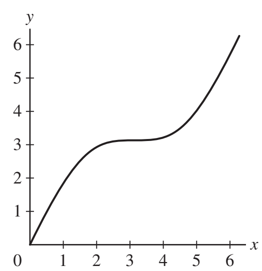
In Exercises 41–46, sketch the graph over the given interval, with all transition points indicated.

41. $y = x + \sin x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = x + \sin x$. Setting $f'(x) = 1 + \cos x = 0$ yields $\cos x = -1$, so that $x = \pi$ is the lone critical point on the interval $[0, 2\pi]$. Setting $f''(x) = -\sin x = 0$ yields potential points of inflection at $x = 0, \pi, 2\pi$ on the interval $[0, 2\pi]$.

Interval	signs of f' and f''
$(0, \pi)$	$+-$
$(\pi, 2\pi)$	$++$

The graph has an inflection point at $x = \pi$, and no local maxima or minima. Here is a sketch of the graph of $f(x)$:

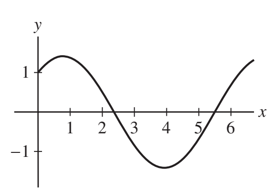


42. $y = \sin x + \cos x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = \sin x + \cos x$. Setting $f'(x) = \cos x - \sin x = 0$ yields $\sin x = \cos x$, so that $\tan x = 1$, and $x = \frac{\pi}{4}, \frac{5\pi}{4}$. Setting $f''(x) = -\sin x - \cos x = 0$ yields $\sin x = -\cos x$, so that $-\tan x = 1$, and $x = \frac{3\pi}{4}, x = \frac{7\pi}{4}$.

Interval	signs of f' and f''
$(0, \frac{\pi}{4})$	$+-$
$(\frac{\pi}{4}, \frac{3\pi}{4})$	$--$
$(\frac{3\pi}{4}, \frac{5\pi}{4})$	$-+$
$(\frac{5\pi}{4}, \frac{7\pi}{4})$	$++$
$(\frac{7\pi}{4}, 2\pi)$	$+-$

The graph has a local maximum at $x = \frac{\pi}{4}$, a local minimum at $x = \frac{5\pi}{4}$, and inflection points at $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$. Here is a sketch of the graph of $f(x)$:

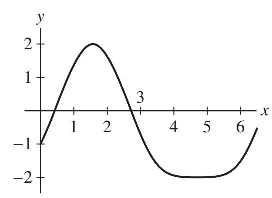


43. $y = 2\sin x - \cos^2 x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = 2\sin x - \cos^2 x$. Then $f'(x) = 2\cos x - 2\cos x\,(-\sin x) = \sin 2x + 2\cos x$ and $f''(x) = 2\cos 2x - 2\sin x$. Setting $f'(x) = 0$ yields $\sin 2x = -2\cos x$, so that $2\sin x\cos x = -2\cos x$. This implies $\cos x = 0$ or $\sin x = -1$, so that $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Setting $f''(x) = 0$ yields $2\cos 2x = 2\sin x$, so that $2\sin(\frac{\pi}{2} - 2x) = 2\sin x$, or $\frac{\pi}{2} - 2x = x \pm 2n\pi$. This yields $3x = \frac{\pi}{2} + 2n\pi$, or $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6} = \frac{3\pi}{2}$.

Interval	signs of f' and f''
$\left(0, \frac{\pi}{6}\right)$	++
$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	+−
$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$	−−
$\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$	−+
$\left(\frac{3\pi}{2}, 2\pi\right)$	++

The graph has a local maximum at $x = \frac{\pi}{2}$, a local minimum at $x = \frac{3\pi}{2}$, and inflection points at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Here is a graph of f *without* transition points highlighted.

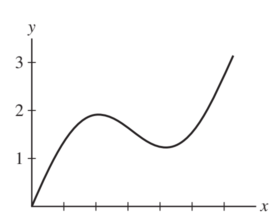


44. $y = \sin x + \frac{1}{2}x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = \sin x + \frac{1}{2}x$. Setting $f'(x) = \cos x + \frac{1}{2} = 0$ yields $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Setting $f''(x) = -\sin x = 0$ yields potential points of inflection at $x = 0, \pi, 2\pi$.

Interval	signs of f' and f''
$\left(0, \frac{2\pi}{3}\right)$	+−
$\left(\frac{2\pi}{3}, \pi\right)$	−−
$\left(\pi, \frac{4\pi}{3}\right)$	−+
$\left(\frac{4\pi}{3}, 2\pi\right)$	++

The graph has a local maximum at $x = \frac{2\pi}{3}$, a local minimum at $x = \frac{4\pi}{3}$, and an inflection point at $x = \pi$. Here is a graph of f *without* transition points highlighted.

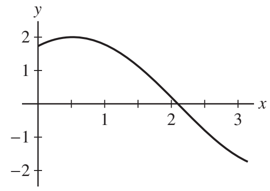


45. $y = \sin x + \sqrt{3}\cos x, \quad [0, \pi]$

SOLUTION Let $f(x) = \sin x + \sqrt{3}\cos x$. Setting $f'(x) = \cos x - \sqrt{3}\sin x = 0$ yields $\tan x = \frac{1}{\sqrt{3}}$. In the interval $[0, \pi]$, the solution is $x = \frac{\pi}{6}$. Setting $f''(x) = -\sin x - \sqrt{3}\cos x = 0$ yields $\tan x = -\sqrt{3}$. In the interval $[0, \pi]$, the lone solution is $x = \frac{2\pi}{3}$.

Interval	signs of f' and f''
$(0, \pi/6)$	$+-$
$(\pi/6, 2\pi/3)$	$--$
$(2\pi/3, \pi)$	$-+$

The graph has a local maximum at $x = \frac{\pi}{6}$ and a point of inflection at $x = \frac{2\pi}{3}$. A plot without the transition points highlighted is given below:



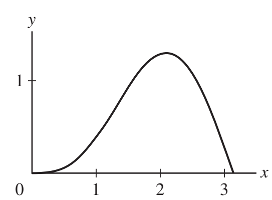
46. $y = \sin x - \frac{1}{2}\sin 2x, \quad [0, \pi]$

SOLUTION Let $f(x) = \sin x - \frac{1}{2}\sin 2x$. Setting $f'(x) = \cos x - \cos 2x = 0$ yields $\cos 2x = \cos x$. Using the double angle formula for cosine, this gives $2\cos^2 x - 1 = \cos x$ or $(2\cos x + 1)(\cos x - 1) = 0$. Solving for $x \in [0, \pi]$, we find $x = 0$ or $\frac{2\pi}{3}$.

Setting $f''(x) = -\sin x + 2\sin 2x = 0$ yields $4\sin x\cos x = \sin x$, so $\sin x = 0$ or $\cos x = \frac{1}{4}$. Hence, there are potential points of inflection at $x = 0$, $x = \pi$ and $x = \cos^{-1}\frac{1}{4} \approx 1.31812$.

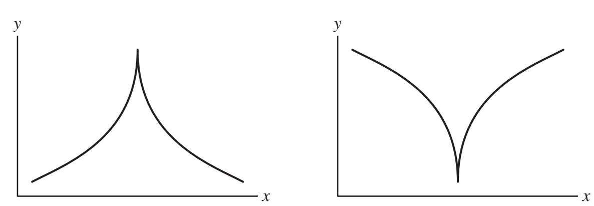
Interval	Sign of f' and f''
$\left(0, \cos^{-1}\frac{1}{4}\right)$	$++$
$\left(\cos^{-1}\frac{1}{4}, \frac{2\pi}{3}\right)$	$+-$
$\left(\frac{2\pi}{3}, \pi\right)$	$--$

The graph of $f(x)$ has a local maximum at $x = \frac{2\pi}{3}$ and a point of inflection at $x = \cos^{-1}\frac{1}{4}$.



47. Are all sign transitions possible? Explain with a sketch why the transitions $++ \to -+$ and $-- \to +-$ do not occur if the function is differentiable. (See Exercise 76 for a proof.)

SOLUTION In both cases, there is a point where f is not differentiable at the transition from increasing to decreasing or decreasing to increasing.



48. Suppose that f is twice differentiable satisfying (i) $f(0) = 1$, (ii) $f'(x) > 0$ for all $x \neq 0$, and (iii) $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$. Let $g(x) = f(x^2)$.

(a) Sketch a possible graph of $f(x)$.

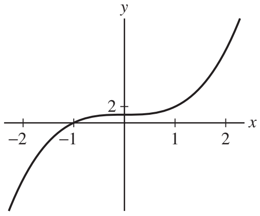
(b) Prove that $g(x)$ has no points of inflection and a unique local extreme value at $x = 0$. Sketch a possible graph of $g(x)$.

SOLUTION

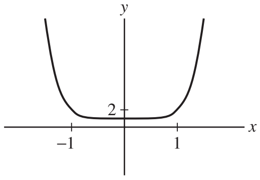
(a) To produce a possible sketch, we give the direction and concavity of the graph over every interval.

Interval	$(-\infty, 0)$	$(0, \infty)$
Direction	↗	↗
Concavity	⌢	⌣

A sketch of one possible such function appears here:



(b) Let $g(x) = f(x^2)$. Then $g'(x) = 2xf'(x^2)$. If $g'(x) = 0$, either $x = 0$ or $f'(x^2) = 0$, which implies that $x = 0$ as well. Since $f'(x^2) > 0$ for all $x \neq 0$, $g'(x) < 0$ for $x < 0$ and $g'(x) > 0$ for $x > 0$. This gives $g(x)$ a unique local extreme value at $x = 0$, a minimum. $g''(x) = 2f'(x^2) + 4x^2 f''(x^2)$. For all $x \neq 0$, $x^2 > 0$, and so $f''(x^2) > 0$ and $f'(x^2) > 0$. Thus $g''(x) > 0$, and so $g''(x)$ does not change sign, and can have no inflection points. A sketch of $g(x)$ based on the sketch we made for $f(x)$ follows: indeed, this sketch shows a unique local minimum at $x = 0$.



49. Which of the graphs in Figure 18 *cannot* be the graph of a polynomial? Explain.

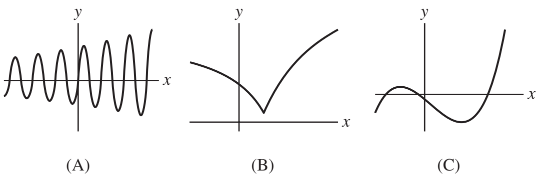


FIGURE 18

SOLUTION Polynomials are everywhere differentiable. Accordingly, graph (B) cannot be the graph of a polynomial, since the function in (B) has a cusp (sharp corner), signifying nondifferentiability at that point.

50. Which curve in Figure 19 is the graph of $f(x) = \dfrac{2x^4 - 1}{1 + x^4}$? Explain on the basis of horizontal asymptotes.

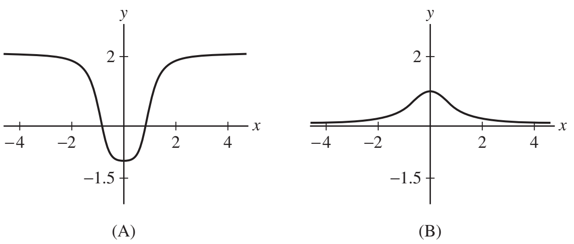


FIGURE 19

SOLUTION Since

$$\lim_{x\to\pm\infty} \frac{2x^4 - 1}{1 + x^4} = \frac{2}{1} \cdot \lim_{x\to\pm\infty} 1 = 2$$

the graph has left and right horizontal asymptotes at $y = 2$, so the left curve is the graph of $f(x) = \dfrac{2x^4 - 1}{1 + x^4}$.

51. Match the graphs in Figure 20 with the two functions $y = \dfrac{3x}{x^2-1}$ and $y = \dfrac{3x^2}{x^2-1}$. Explain.

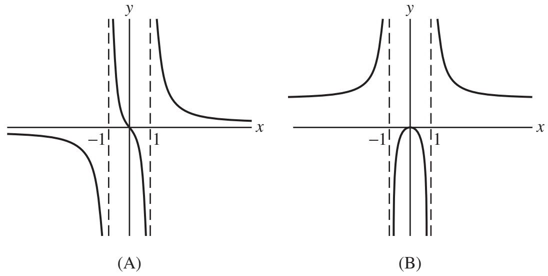


FIGURE 20

SOLUTION Since $\displaystyle\lim_{x\to\pm\infty} \frac{3x^2}{x^2-1} = \frac{3}{1} \cdot \lim_{x\to\pm\infty} 1 = 3$, the graph of $y = \dfrac{3x^2}{x^2-1}$ has a horizontal asymptote of $y = 3$; hence, the right curve is the graph of $f(x) = \dfrac{3x^2}{x^2-1}$. Since

$$\lim_{x\to\pm\infty} \frac{3x}{x^2-1} = \frac{3}{1} \cdot \lim_{x\to\pm\infty} x^{-1} = 0,$$

the graph of $y = \dfrac{3x}{x^2-1}$ has a horizontal asymptote of $y = 0$; hence, the left curve is the graph of $f(x) = \dfrac{3x}{x^2-1}$.

52. Match the functions with their graphs in Figure 21.

(a) $y = \dfrac{1}{x^2-1}$

(b) $y = \dfrac{x^2}{x^2+1}$

(c) $y = \dfrac{1}{x^2+1}$

(d) $y = \dfrac{x}{x^2-1}$

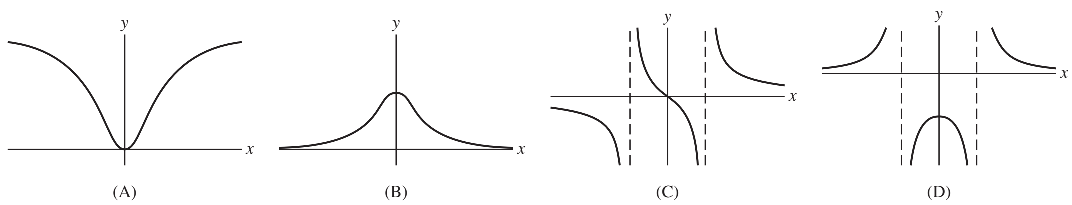


FIGURE 21

SOLUTION

(a) The graph of $\frac{1}{x^2-1}$ should have a horizontal asymptote at $y = 0$ and vertical asymptotes at $x = \pm 1$. Further, the graph should consist of positive values for $|x| > 1$ and negative values for $|x| < 1$. Hence, the graph of $\frac{1}{x^2-1}$ is (D).

(b) The graph of $\frac{x^2}{x^2+1}$ should have a horizontal asymptote at $y = 1$ and no vertical asymptotes. Hence, the graph of $\frac{x^2}{x^2+1}$ is (A).

(c) The graph of $\frac{1}{x^2+1}$ should have a horizontal asymptote at $y = 0$ and no vertical asymptotes. Hence, the graph of $\frac{1}{x^2+1}$ is (B).

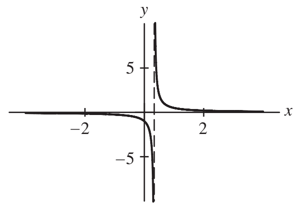
(d) The graph of $\frac{x}{x^2-1}$ should have a horizontal asymptote at $y = 0$ and vertical asymptotes at $x = \pm 1$. Further, the graph should consist of positive values for $-1 < x < 0$ and $x > 1$ and negative values for $x < 1$ and $0 < x < 1$. Hence, the graph of $\frac{x}{x^2-1}$ is (C).

In Exercises 53–70, sketch the graph of the function. Indicate the transition points and asymptotes.

53. $y = \dfrac{1}{3x-1}$

SOLUTION Let $f(x) = \dfrac{1}{3x-1}$. Then $f'(x) = \dfrac{-3}{(3x-1)^2}$, so that f is decreasing for all $x \neq \frac{1}{3}$. Moreover, $f''(x) = \dfrac{18}{(3x-1)^3}$, so that f is concave up for $x > \frac{1}{3}$ and concave down for $x < \frac{1}{3}$. Because $\displaystyle\lim_{x\to\pm\infty} \frac{1}{3x-1} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has a vertical asymptote at $x = \frac{1}{3}$ with

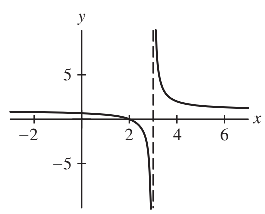
$$\lim_{x\to\frac{1}{3}-} \frac{1}{3x-1} = -\infty \qquad \text{and} \qquad \lim_{x\to\frac{1}{3}+} \frac{1}{3x-1} = \infty.$$



54. $y = \dfrac{x-2}{x-3}$

SOLUTION Let $f(x) = \dfrac{x-2}{x-3}$. Then $f'(x) = \dfrac{-1}{(x-3)^2}$, so that f is decreasing for all $x \neq 3$. Moreover, $f''(x) = \dfrac{2}{(x-3)^3}$, so that f is concave up for $x > 3$ and concave down for $x < 3$. Because $\lim_{x\to\pm\infty} \dfrac{x-2}{x-3} = 1$, f has a horizontal asymptote at $y = 1$. Finally, f has a vertical asymptote at $x = 3$ with

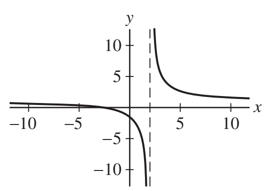
$$\lim_{x\to 3-} \frac{x-2}{x-3} = -\infty \qquad \text{and} \qquad \lim_{x\to 3+} \frac{x-2}{x-3} = \infty.$$



55. $y = \dfrac{x+3}{x-2}$

SOLUTION Let $f(x) = \dfrac{x+3}{x-2}$. Then $f'(x) = \dfrac{-5}{(x-2)^2}$, so that f is decreasing for all $x \neq 2$. Moreover, $f''(x) = \dfrac{10}{(x-2)^3}$, so that f is concave up for $x > 2$ and concave down for $x < 2$. Because $\lim_{x\to\pm\infty} \dfrac{x+3}{x-2} = 1$, f has a horizontal asymptote at $y = 1$. Finally, f has a vertical asymptote at $x = 2$ with

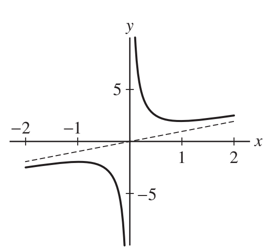
$$\lim_{x\to 2-} \frac{x+3}{x-2} = -\infty \qquad \text{and} \qquad \lim_{x\to 2+} \frac{x+3}{x-2} = \infty.$$



56. $y = x + \dfrac{1}{x}$

SOLUTION Let $f(x) = x + x^{-1}$. Then $f'(x) = 1 - x^{-2}$, so that f is increasing for $x < -1$ and $x > 1$ and decreasing for $-1 < x < 0$ and $0 < x < 1$. Moreover, $f''(x) = 2x^{-3}$, so that f is concave up for $x > 0$ and concave down for $x < 0$. f has no horizontal asymptote and has a vertical asymptote at $x = 0$ with

$$\lim_{x\to 0-} (x + x^{-1}) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+} (x + x^{-1}) = \infty.$$



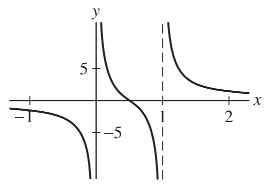
57. $y = \dfrac{1}{x} + \dfrac{1}{x-1}$

SOLUTION Let $f(x) = \dfrac{1}{x} + \dfrac{1}{x-1}$. Then $f'(x) = -\dfrac{2x^2 - 2x + 1}{x^2(x-1)^2}$, so that f is decreasing for all $x \neq 0, 1$. Moreover, $f''(x) = \dfrac{2\left(2x^3 - 3x^2 + 3x - 1\right)}{x^3(x-1)^3}$, so that f is concave up for $0 < x < \frac{1}{2}$ and $x > 1$ and concave down for $x < 0$ and $\frac{1}{2} < x < 1$. Because $\lim_{x\to\pm\infty}\left(\dfrac{1}{x} + \dfrac{1}{x-1}\right) = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 0$ and $x = 1$ with

$$\lim_{x\to 0-}\left(\frac{1}{x} + \frac{1}{x-1}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x} + \frac{1}{x-1}\right) = \infty$$

and

$$\lim_{x\to 1-}\left(\frac{1}{x} + \frac{1}{x-1}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 1+}\left(\frac{1}{x} + \frac{1}{x-1}\right) = \infty.$$



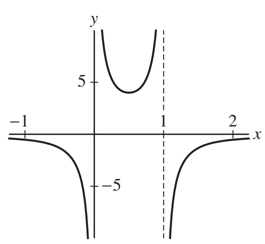
58. $y = \dfrac{1}{x} - \dfrac{1}{x-1}$

SOLUTION Let $f(x) = \dfrac{1}{x} - \dfrac{1}{x-1}$. Then $f'(x) = \dfrac{2x-1}{x^2(x-1)^2}$, so that f is decreasing for $x < 0$ and $0 < x < \frac{1}{2}$ and increasing for $\frac{1}{2} < x < 1$ and $x > 1$. Moreover, $f''(x) = -\dfrac{2\left(3x^2 - 3x + 1\right)}{x^3(x-1)^3}$, so that f is concave up for $0 < x < 1$ and concave down for $x < 0$ and $x > 1$. Because $\lim_{x\to\pm\infty}\left(\dfrac{1}{x} - \dfrac{1}{x-1}\right) = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 0$ and $x = 1$ with

$$\lim_{x\to 0-}\left(\frac{1}{x} - \frac{1}{x-1}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x} - \frac{1}{x-1}\right) = \infty$$

and

$$\lim_{x\to 1-}\left(\frac{1}{x} - \frac{1}{x-1}\right) = \infty \qquad \text{and} \qquad \lim_{x\to 1+}\left(\frac{1}{x} - \frac{1}{x-1}\right) = -\infty.$$



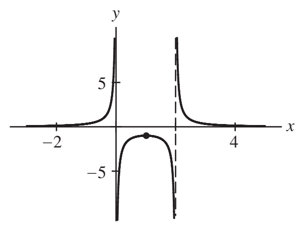
59. $y = \dfrac{1}{x(x-2)}$

SOLUTION Let $f(x) = \dfrac{1}{x(x-2)}$. Then $f'(x) = \dfrac{2(1-x)}{x^2(x-2)^2}$, so that f is increasing for $x < 0$ and $0 < x < 1$ and decreasing for $1 < x < 2$ and $x > 2$. Moreover, $f''(x) = \dfrac{2(3x^2 - 6x + 4)}{x^3(x-2)^3}$, so that f is concave up for $x < 0$ and $x > 2$ and concave down for $0 < x < 2$. Because $\lim_{x\to\pm\infty}\left(\dfrac{1}{x(x-2)}\right) = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 0$ and $x = 2$ with

$$\lim_{x\to 0-}\left(\frac{1}{x(x-2)}\right) = +\infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x(x-2)}\right) = -\infty$$

and

$$\lim_{x\to 2-}\left(\frac{1}{x(x-2)}\right)=-\infty \qquad \text{and} \qquad \lim_{x\to 2+}\left(\frac{1}{x(x-2)}\right)=\infty.$$



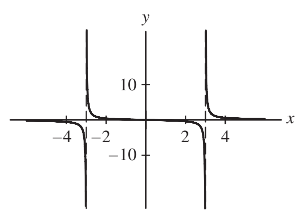
60. $y=\dfrac{x}{x^2-9}$

SOLUTION Let $f(x)=\dfrac{x}{x^2-9}$. Then $f'(x)=-\dfrac{x^2+9}{(x^2-9)^2}$, so that f is decreasing for all $x\neq\pm3$. Moreover, $f''(x)=\dfrac{6x(x^2+6)}{(x^2-9)^3}$, so that f is concave down for $x<-3$ and for $0<x<3$ and is concave up for $-3<x<0$ and for $x>3$. Because $\lim_{x\to\pm\infty}\dfrac{x}{x^2-9}=0$, f has a horizontal asymptote at $y=0$. Finally, f has vertical asymptotes at $x=\pm3$, with

$$\lim_{x\to -3-}\left(\frac{x}{x^2-9}\right)=-\infty \qquad \text{and} \qquad \lim_{x\to -3+}\left(\frac{x}{x^2-9}\right)=\infty$$

and

$$\lim_{x\to 3-}\left(\frac{x}{x^2-9}\right)=-\infty \qquad \text{and} \qquad \lim_{x\to 3+}\left(\frac{x}{x^2-9}\right)=\infty.$$



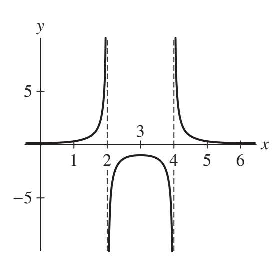
61. $y=\dfrac{1}{x^2-6x+8}$

SOLUTION Let $f(x)=\dfrac{1}{x^2-6x+8}=\dfrac{1}{(x-2)(x-4)}$. Then $f'(x)=\dfrac{6-2x}{\left(x^2-6x+8\right)^2}$, so that f is increasing for $x<2$ and for $2<x<3$, is decreasing for $3<x<4$ and for $x>4$, and has a local maximum at $x=3$. Moreover, $f''(x)=\dfrac{2\left(3x^2-18x+28\right)}{\left(x^2-6x+8\right)^3}$, so that f is concave up for $x<2$ and for $x>4$ and is concave down for $2<x<4$. Because $\lim_{x\to\pm\infty}\dfrac{1}{x^2-6x+8}=0$, f has a horizontal asymptote at $y=0$. Finally, f has vertical asymptotes at $x=2$ and $x=4$, with

$$\lim_{x\to 2-}\left(\frac{1}{x^2-6x+8}\right)=\infty \qquad \text{and} \qquad \lim_{x\to 2+}\left(\frac{1}{x^2-6x+8}\right)=-\infty$$

and

$$\lim_{x\to 4-}\left(\frac{1}{x^2-6x+8}\right)=-\infty \qquad \text{and} \qquad \lim_{x\to 4+}\left(\frac{1}{x^2-6x+8}\right)=\infty.$$



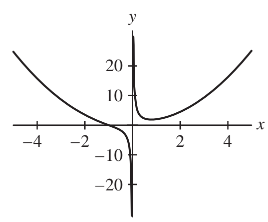
62. $y = \dfrac{x^3+1}{x}$

SOLUTION Let $f(x) = \dfrac{x^3+1}{x} = x^2 + x^{-1}$. Then $f'(x) = 2x - x^{-2}$, so that f is decreasing for $x < 0$ and for $0 < x < \sqrt[3]{1/2}$ and increasing for $x > \sqrt[3]{1/2}$. Moreover, $f''(x) = 2 + 2x^{-3}$, so f is concave up for $x < -1$ and for $x > 0$ and concave down for $-1 < x < 0$. Because

$$\lim_{x\to\pm\infty} \frac{x^3+1}{x} = \infty,$$

f has no horizontal asymptotes. Finally, f has a vertical asymptote at $x = 0$ with

$$\lim_{x\to 0-} \frac{x^3+1}{x} = -\infty \qquad \text{and} \qquad \lim_{x\to 0+} \frac{x^3+1}{x} = \infty.$$



63. $y = 1 - \dfrac{3}{x} + \dfrac{4}{x^3}$

SOLUTION Let $f(x) = 1 - \dfrac{3}{x} + \dfrac{4}{x^3}$. Then

$$f'(x) = \frac{3}{x^2} - \frac{12}{x^4} = \frac{3(x-2)(x+2)}{x^4},$$

so that f is increasing for $|x| > 2$ and decreasing for $-2 < x < 0$ and for $0 < x < 2$. Moreover,

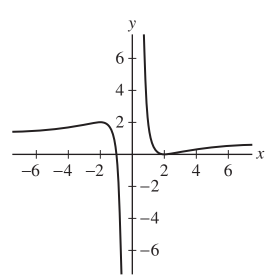
$$f''(x) = -\frac{6}{x^3} + \frac{48}{x^5} = \frac{6(8-x^2)}{x^5},$$

so that f is concave down for $-2\sqrt{2} < x < 0$ and for $x > 2\sqrt{2}$, while f is concave up for $x < -2\sqrt{2}$ and for $0 < x < 2\sqrt{2}$. Because

$$\lim_{x\to\pm\infty} \left(1 - \frac{3}{x} + \frac{4}{x^3}\right) = 1,$$

f has a horizontal asymptote at $y = 1$. Finally, f has a vertical asymptote at $x = 0$ with

$$\lim_{x\to 0-} \left(1 - \frac{3}{x} + \frac{4}{x^3}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+} \left(1 - \frac{3}{x} + \frac{4}{x^3}\right) = \infty.$$



64. $y = \dfrac{1}{x^2} + \dfrac{1}{(x-2)^2}$

SOLUTION Let $f(x) = \dfrac{1}{x^2} + \dfrac{1}{(x-2)^2}$. Then

$$f'(x) = -2x^{-3} - 2(x-2)^{-3} = -\frac{4(x-1)(x^2-2x+4)}{x^3(x-2)^3},$$

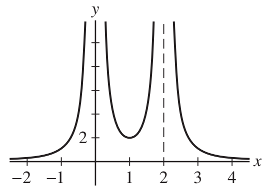
so that f is increasing for $x < 0$ and for $1 < x < 2$, is decreasing for $0 < x < 1$ and for $x > 2$, and has a local minimum at $x = 1$. Moreover, $f''(x) = 6x^{-4} + 6(x-2)^{-4}$, so that f is concave up for all $x \neq 0, 2$. Because

$\lim_{x\to\pm\infty}\left(\frac{1}{x^2}+\frac{1}{(x-2)^2}\right)=0$, f has a horizontal asymptote at $y=0$. Finally, f has vertical asymptotes at $x=0$ and $x=2$ with

$$\lim_{x\to 0-}\left(\frac{1}{x^2}+\frac{1}{(x-2)^2}\right)=\infty \quad\text{and}\quad \lim_{x\to 0+}\left(\frac{1}{x^2}+\frac{1}{(x-2)^2}\right)=\infty$$

and

$$\lim_{x\to 2-}\left(\frac{1}{x^2}+\frac{1}{(x-2)^2}\right)=\infty \quad\text{and}\quad \lim_{x\to 2+}\left(\frac{1}{x^2}+\frac{1}{(x-2)^2}\right)=\infty.$$



65. $y=\frac{1}{x^2}-\frac{1}{(x-2)^2}$

SOLUTION Let $f(x)=\frac{1}{x^2}-\frac{1}{(x-2)^2}$. Then $f'(x)=-2x^{-3}+2(x-2)^{-3}$, so that f is increasing for $x<0$ and for $x>2$ and is decreasing for $0<x<2$. Moreover,

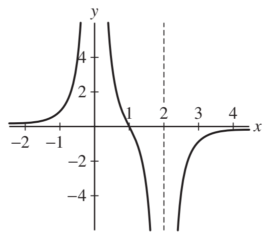
$$f''(x)=6x^{-4}-6(x-2)^{-4}=-\frac{48(x-1)(x^2-2x+2)}{x^4(x-2)^4},$$

so that f is concave up for $x<0$ and for $0<x<1$, is concave down for $1<x<2$ and for $x>2$, and has a point of inflection at $x=1$. Because $\lim_{x\to\pm\infty}\left(\frac{1}{x^2}-\frac{1}{(x-2)^2}\right)=0$, f has a horizontal asymptote at $y=0$. Finally, f has vertical asymptotes at $x=0$ and $x=2$ with

$$\lim_{x\to 0-}\left(\frac{1}{x^2}-\frac{1}{(x-2)^2}\right)=\infty \quad\text{and}\quad \lim_{x\to 0+}\left(\frac{1}{x^2}-\frac{1}{(x-2)^2}\right)=\infty$$

and

$$\lim_{x\to 2-}\left(\frac{1}{x^2}-\frac{1}{(x-2)^2}\right)=-\infty \quad\text{and}\quad \lim_{x\to 2+}\left(\frac{1}{x^2}-\frac{1}{(x-2)^2}\right)=-\infty.$$



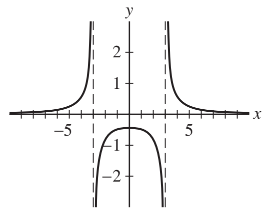
66. $y=\frac{4}{x^2-9}$

SOLUTION Let $f(x)=\frac{4}{x^2-9}$. Then $f'(x)=-\frac{8x}{\left(x^2-9\right)^2}$, so that f is increasing for $x<-3$ and for $-3<x<0$, is decreasing for $0<x<3$ and for $x>3$, and has a local maximum at $x=0$. Moreover, $f''(x)=\frac{24\left(x^2+3\right)}{\left(x^2-9\right)^3}$, so that f is concave up for $x<-3$ and for $x>3$ and is concave down for $-3<x<3$. Because $\lim_{x\to\pm\infty}\frac{4}{x^2-9}=0$, f has a horizontal asymptote at $y=0$. Finally, f has vertical asymptotes at $x=-3$ and $x=3$, with

$$\lim_{x\to -3-}\left(\frac{4}{x^2-9}\right)=\infty \quad\text{and}\quad \lim_{x\to -3+}\left(\frac{4}{x^2-9}\right)=-\infty$$

and

$$\lim_{x\to 3-}\left(\frac{4}{x^2-9}\right)=-\infty \quad\text{and}\quad \lim_{x\to 3+}\left(\frac{4}{x^2-9}\right)=\infty.$$

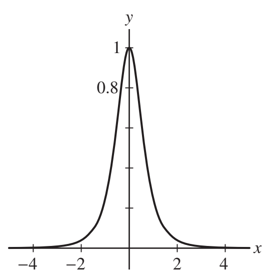


67. $y = \dfrac{1}{(x^2+1)^2}$

SOLUTION Let $f(x) = \dfrac{1}{(x^2+1)^2}$. Then $f'(x) = \dfrac{-4x}{(x^2+1)^3}$, so that f is increasing for $x < 0$, is decreasing for $x > 0$ and has a local maximum at $x = 0$. Moreover,

$$f''(x) = \frac{-4(x^2+1)^3 + 4x \cdot 3(x^2+1)^2 \cdot 2x}{(x^2+1)^6} = \frac{20x^2-4}{(x^2+1)^4},$$

so that f is concave up for $|x| > 1/\sqrt{5}$, is concave down for $|x| < 1/\sqrt{5}$, and has points of inflection at $x = \pm 1/\sqrt{5}$. Because $\lim_{x\to\pm\infty} \dfrac{1}{(x^2+1)^2} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has no vertical asymptotes.



68. $y = \dfrac{x^2}{(x^2-1)(x^2+1)}$

SOLUTION Let

$$f(x) = \frac{x^2}{(x^2-1)(x^2+1)}.$$

Then

$$f'(x) = -\frac{2x(1+x^4)}{(x-1)^2(x+1)^2(x^2+1)^2},$$

so that f is increasing for $x < -1$ and for $-1 < x < 0$, is decreasing for $0 < x < 1$ and for $x > 1$, and has a local maximum at $x = 0$. Moreover,

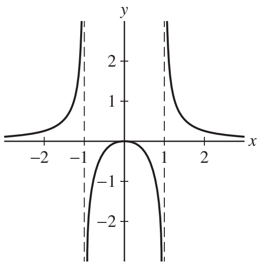
$$f''(x) = \frac{2+24x^4+6x^8}{(x-1)^3(x+1)^3(x^2+1)^3},$$

so that f is concave up for $|x| > 1$ and concave down for $|x| < 1$. Because $\lim_{x\to\pm\infty} \dfrac{x^2}{(x^2-1)(x^2+1)} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = -1$ and $x = 1$, with

$$\lim_{x\to -1-} \frac{x^2}{(x^2-1)(x^2+1)} = \infty \qquad \text{and} \qquad \lim_{x\to -1+} \frac{x^2}{(x^2-1)(x^2+1)} = -\infty$$

and

$$\lim_{x\to 1-} \frac{x^2}{(x^2-1)(x^2+1)} = -\infty \qquad \text{and} \qquad \lim_{x\to 1+} \frac{x^2}{(x^2-1)(x^2+1)} = \infty.$$



69. $y = \dfrac{1}{\sqrt{x^2+1}}$

SOLUTION Let $f(x) = \frac{1}{\sqrt{x^2+1}}$. Then

$$f'(x) = -\frac{x}{\sqrt{(x^2+1)^3}} = -x(x^2+1)^{-3/2},$$

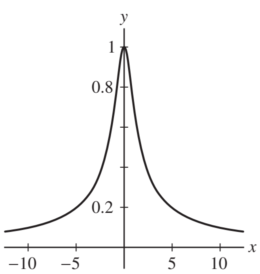
so that f is increasing for $x < 0$ and decreasing for $x > 0$. Moreover,

$$f''(x) = -\frac{3}{2}x(x^2+1)^{-5/2}(-2x) - (x^2+1)^{-3/2} = (2x^2-1)(x^2+1)^{-5/2},$$

so that f is concave down for $|x| < \frac{\sqrt{2}}{2}$ and concave up for $|x| > \frac{\sqrt{2}}{2}$. Because

$$\lim_{x\to\pm\infty} \frac{1}{\sqrt{x^2+1}} = 0,$$

f has a horizontal asymptote at $y = 0$. Finally, f has no vertical asymptotes.



70. $y = \dfrac{x}{\sqrt{x^2+1}}$

SOLUTION Let

$$f(x) = \frac{x}{\sqrt{x^2+1}}.$$

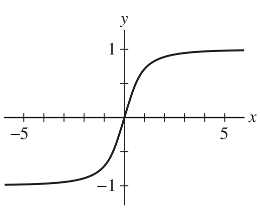
Then

$$f'(x) = (x^2+1)^{-3/2} \quad \text{and} \quad f''(x) = \frac{-3x}{(x^2+1)^{5/2}}.$$

Thus, f is increasing for all x, is concave up for $x < 0$, is concave down for $x > 0$, and has a point of inflection at $x = 0$. Because

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1,$$

f has horizontal asymptotes of $y = -1$ and $y = 1$. There are no vertical asymptotes.



Further Insights and Challenges

In Exercises 71–75, we explore functions whose graphs approach a nonhorizontal line as $x \to \infty$. A line $y = ax + b$ is called a ***slant asymptote*** *if*

$$\lim_{x\to\infty} (f(x) - (ax + b)) = 0$$

or

$$\lim_{x\to-\infty} (f(x) - (ax + b)) = 0$$

71. Let $f(x) = \dfrac{x^2}{x-1}$ (Figure 22). Verify the following:

(a) $f(0)$ is a local max and $f(2)$ a local min.

(b) f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

(c) $\lim_{x\to 1-} f(x) = -\infty$ and $\lim_{x\to 1+} f(x) = \infty$.

(d) $y = x + 1$ is a slant asymptote of $f(x)$ as $x \to \pm\infty$.

(e) The slant asymptote lies above the graph of $f(x)$ for $x < 1$ and below the graph for $x > 1$.

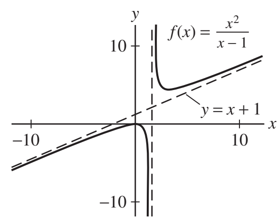


FIGURE 22

SOLUTION Let $f(x) = \dfrac{x^2}{x-1}$. Then $f'(x) = \dfrac{x(x-2)}{(x-1)^2}$ and $f''(x) = \dfrac{2}{(x-1)^3}$.

(a) Sign analysis of $f''(x)$ reveals that $f''(x) < 0$ on $(-\infty, 1)$ and $f''(x) > 0$ on $(1, \infty)$.

(b) Critical points of $f'(x)$ occur at $x = 0$ and $x = 2$. $x = 1$ is not a critical point because it is not in the domain of f. Sign analyses reveal that $x = 2$ is a local minimum of f and $x = 0$ is a local maximum.

(c)

$$\lim_{x\to 1-} f(x) = -1 \lim_{x\to 1-} \frac{1}{1-x} = -\infty \quad \text{and} \quad \lim_{x\to 1+} f(x) = 1 \lim_{x\to 1+} \frac{1}{x-1} = \infty.$$

(d) Note that using polynomial division, $f(x) = \dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1}$. Then $\lim_{x\to\pm\infty} (f(x) - (x+1)) = \lim_{x\to\pm\infty} x + 1 + \dfrac{1}{x-1} - (x+1) = \lim_{x\to\pm\infty} \dfrac{1}{x-1} = 0.$

(e) For $x > 1$, $f(x) - (x+1) = \dfrac{1}{x-1} > 0$, so $f(x)$ approaches $x + 1$ from above. Similarly, for $x < 1$, $f(x) - (x + 1) = \dfrac{1}{x-1} < 0$, so $f(x)$ approaches $x + 1$ from below.

72. If $f(x) = P(x)/Q(x)$, where P and Q are polynomials of degrees $m + 1$ and m, then by long division, we can write

$$f(x) = (ax + b) + P_1(x)/Q(x)$$

where P_1 is a polynomial of degree $< m$. Show that $y = ax + b$ is the slant asymptote of $f(x)$. Use this procedure to find the slant asymptotes of the following functions:

(a) $y = \dfrac{x^2}{x+2}$ **(b)** $y = \dfrac{x^3 + x}{x^2 + x + 1}$

SOLUTION Since $\deg(P_1) < \deg(Q)$,

$$\lim_{x\to\pm\infty} \frac{P_1(x)}{Q(x)} = 0.$$

Thus

$$\lim_{x\to\pm\infty} (f(x) - (ax + b)) = 0$$

and $y = ax + b$ is a slant asymptote of f.

(a) $\dfrac{x^2}{x+2} = x - 2 + \dfrac{4}{x+2}$; hence $y = x - 2$ is a slant asymptote of $\dfrac{x^2}{x+2}$.

(b) $\dfrac{x^3+x}{x^2+x+1} = (x-1) + \dfrac{x+1}{x^2-1}$; hence, $y = x - 1$ is a slant asymptote of $\dfrac{x^3+x}{x^2+x+1}$.

73. Sketch the graph of

$$f(x) = \frac{x^2}{x+1}.$$

Proceed as in the previous exercise to find the slant asymptote.

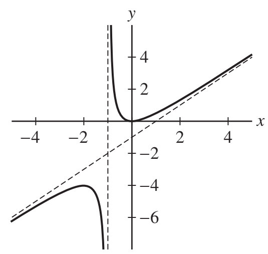
SOLUTION Let $f(x) = \dfrac{x^2}{x+1}$. Then $f'(x) = \dfrac{x(x+2)}{(x+1)^2}$ and $f''(x) = \dfrac{2}{(x+1)^3}$. Thus, f is increasing for $x < -2$ and for $x > 0$, is decreasing for $-2 < x < -1$ and for $-1 < x < 0$, has a local minimum at $x = 0$, has a local maximum at $x = -2$, is concave down on $(-\infty, -1)$ and concave up on $(-1, \infty)$. Limit analyses give a vertical asymptote at $x = -1$, with

$$\lim_{x\to -1-} \frac{x^2}{x+1} = -\infty \qquad \text{and} \qquad \lim_{x\to -1+} \frac{x^2}{x+1} = \infty.$$

By polynomial division, $f(x) = x - 1 + \dfrac{1}{x+1}$ and

$$\lim_{x\to\pm\infty} \left(x - 1 + \frac{1}{x+1} - (x-1) \right) = 0,$$

which implies that the slant asymptote is $y = x - 1$. Notice that f approaches the slant asymptote as in exercise 71.

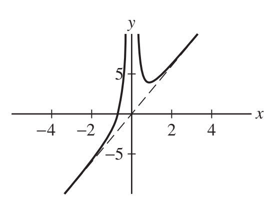


74. Show that $y = 3x$ is a slant asymptote for $f(x) = 3x + x^{-2}$. Determine whether $f(x)$ approaches the slant asymptote from above or below and make a sketch of the graph.

SOLUTION Let $f(x) = 3x + x^{-2}$. Then

$$\lim_{x\to\pm\infty} (f(x) - 3x) = \lim_{x\to\pm\infty} (3x + x^{-2} - 3x) = \lim_{x\to\pm\infty} x^{-2} = 0$$

which implies that $3x$ is the slant asymptote of $f(x)$. Since $f(x) - 3x = x^{-2} > 0$ as $x \to \pm\infty$, $f(x)$ approaches the slant asymptote from above in both directions. Moreover, $f'(x) = 3 - 2x^{-3}$ and $f''(x) = 6x^{-4}$. Sign analyses reveal a local minimum at $x = \left(\frac{3}{2}\right)^{-1/3} \approx 0.87358$ and that f is concave up for all $x \neq 0$. Limit analyses give a vertical asymptote at $x = 0$.

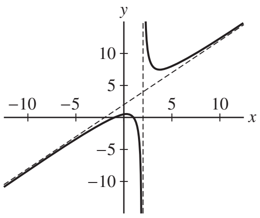


75. Sketch the graph of $f(x) = \dfrac{1-x^2}{2-x}$.

SOLUTION Let $f(x) = \dfrac{1-x^2}{2-x}$. Using polynomial division, $f(x) = x + 2 + \dfrac{3}{x-2}$. Then

$$\lim_{x\to\pm\infty} (f(x) - (x+2)) = \lim_{x\to\pm\infty} \left((x+2) + \frac{3}{x-2} - (x+2) \right) = \lim_{x\to\pm\infty} \frac{3}{x-2} = \frac{3}{1} \cdot \lim_{x\to\pm\infty} x^{-1} = 0$$

which implies that $y = x + 2$ is the slant asymptote of $f(x)$. Since $f(x) - (x+2) = \dfrac{3}{x-2} > 0$ for $x > 2$, $f(x)$ approaches the slant asymptote from above for $x > 2$; similarly, $\dfrac{3}{x-2} < 0$ for $x < 2$ so $f(x)$ approaches the slant asymptote from below for $x < 2$. Moreover, $f'(x) = \dfrac{x^2 - 4x + 1}{(2-x)^2}$ and $f''(x) = \dfrac{-6}{(2-x)^3}$. Sign analyses reveal a local minimum at $x = 2 + \sqrt{3}$, a local maximum at $x = 2 - \sqrt{3}$ and that f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$. Limit analyses give a vertical asymptote at $x = 2$.



76. Assume that $f'(x)$ and $f''(x)$ exist for all x and let c be a critical point of $f(x)$. Show that $f(x)$ cannot make a transition from ++ to −+ at $x = c$. *Hint:* Apply the MVT to $f'(x)$.

SOLUTION Let $f(x)$ be a function such that $f''(x) > 0$ for all x and such that it transitions from ++ to −+ at a critical point c where $f'(c)$ is defined. That is, $f'(c) = 0$, $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$. Let $g(x) = f'(x)$. The previous statements indicate that $g(c) = 0$, $g(x_0) > 0$ for some $x_0 < c$, and $g(x_1) < 0$ for some $x_1 > c$. By the Mean Value Theorem,

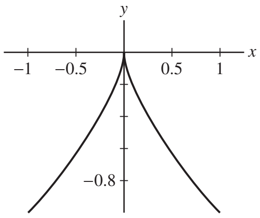
$$\frac{g(x_1) - g(x_0)}{x_1 - x_0} = g'(c_0),$$

for some c_0 between x_0 and x_1. Because $x_1 > c > x_0$ and $g(x_1) < 0 < g(x_0)$,

$$\frac{g(x_1) - g(x_0)}{x_1 - x_0} < 0.$$

But, on the other hand $g'(c_0) = f''(c_0) > 0$, so there is a contradiction. This means that our assumption of the existence of such a function $f(x)$ must be in error, so no function can transition from ++ to −+.

If we drop the requirement that $f'(c)$ exist, such a function can be found. The following is a graph of $f(x) = -x^{2/3}$. $f''(x) > 0$ wherever $f''(x)$ is defined, and $f'(x)$ transitions from positive to negative at $x = 0$.



77. Assume that $f''(x)$ exists and $f''(x) > 0$ for all x. Show that $f(x)$ cannot be negative for all x. *Hint:* Show that $f'(b) \neq 0$ for some b and use the result of Exercise 64 in Section 4.4.

SOLUTION Let $f(x)$ be a function such that $f''(x)$ exists and $f''(x) > 0$ for all x. Since $f''(x) > 0$, there is at least one point $x = b$ such that $f'(b) \neq 0$. If not, $f'(x) = 0$ for all x, so $f''(x) = 0$. By the result of Exercise 64 in Section 4.4, $f(x) \geq f(b) + f'(b)(x - b)$. Now, if $f'(b) > 0$, we find that $f(b) + f'(b)(x - b) > 0$ whenever

$$x > \frac{bf'(b) - f(b)}{f'(b)},$$

a condition that must be met for some x sufficiently large. For such x, $f(x) > f(b) + f'(b)(x - b) > 0$. On the other hand, if $f'(b) < 0$, we find that $f(b) + f'(b)(x - b) > 0$ whenever

$$x < \frac{bf'(b) - f(b)}{f'(b)}.$$

For such an x, $f(x) > f(b) + f'(b)(x - b) > 0$.

4.7 Applied Optimization

Preliminary Questions

1. The problem is to find the right triangle of perimeter 10 whose area is as large as possible. What is the constraint equation relating the base b and height h of the triangle?

SOLUTION The perimeter of a right triangle is the sum of the lengths of the base, the height and the hypotenuse. If the base has length b and the height is h, then the length of the hypotenuse is $\sqrt{b^2+h^2}$ and the perimeter of the triangle is $P = b + h + \sqrt{b^2+h^2}$. The requirement that the perimeter be 10 translates to the constraint equation

$$b + h + \sqrt{b^2+h^2} = 10.$$

2. Describe a way of showing that a continuous function on an open interval (a, b) has a minimum value.

SOLUTION If the function tends to infinity at the endpoints of the interval, then the function must take on a minimum value at a critical point.

3. Is there a rectangle of area 100 of largest perimeter? Explain.

SOLUTION No. Even by fixing the area at 100, we can take one of the dimensions as large as we like thereby allowing the perimeter to become as large as we like.

Exercises

1. Find the dimensions x and y of the rectangle of maximum area that can be formed using 3 meters of wire.

(a) What is the constraint equation relating x and y?

(b) Find a formula for the area in terms of x alone.

(c) What is the interval of optimization? Is it open or closed?

(d) Solve the optimization problem.

SOLUTION

(a) The perimeter of the rectangle is 3 meters, so $3 = 2x + 2y$, which is equivalent to $y = \frac{3}{2} - x$.

(b) Using part (a), $A = xy = x(\frac{3}{2} - x) = \frac{3}{2}x - x^2$.

(c) This problem requires optimization over the closed interval $[0, \frac{3}{2}]$, since both x and y must be non-negative.

(d) $A'(x) = \frac{3}{2} - 2x = 0$, which yields $x = \frac{3}{4}$ and consequently, $y = \frac{3}{4}$. Because $A(0) = A(3/2) = 0$ and $A(\frac{3}{4}) = 0.5625$, the maximum area 0.5625 m^2 is achieved with $x = y = \frac{3}{4}$ m.

2. Wire of length 12 m is divided into two pieces and each piece is bent into a square. How should this be done in order to minimize the sum of the areas of the two squares?

(a) Express the sum of the areas of the squares in terms of the lengths x and y of the two pieces.

(b) What is the constraint equation relating x and y?

(c) What is the interval of optimization? Is it open or closed?

(d) Solve the optimization problem.

SOLUTION Let x and y be the lengths of the pieces.

(a) The perimeter of the first square is x, which implies the length of each side is $\frac{x}{4}$ and the area is $\left(\frac{x}{4}\right)^2$. Similarly, the area of the second square is $\left(\frac{y}{4}\right)^2$. Then the sum of the areas is given by $A = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2$.

(b) $x + y = 12$, so that $y = 12 - x$. Then

$$A(x) = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 = \left(\frac{x}{4}\right)^2 + \left(\frac{12-x}{4}\right)^2 = \frac{1}{8}x^2 - \frac{3}{2}x + 9.$$

(c) Since it is possible for the minimum total area to be realized by not cutting the wire at all, optimization over the closed interval [0, 12] suffices.

(d) Solve $A'(x) = \frac{1}{4}x - \frac{3}{2} = 0$ to obtain $x = 6$ m. Now $A(0) = A(12) = 9$ m^2, whereas $A(4) = \frac{9}{4}$ m^2. Accordingly, the sum of the areas of the squares is minimized if the wire is cut in half.

3. Wire of length 12 m is divided into two pieces and the pieces are bend into a square and a circle. How should this be done in order to minimize the sum of their areas?

SOLUTION Suppose the wire is divided into one piece of length x m that is bent into a circle and a piece of length $12 - x$ m that is bent into a square. Because the circle has circumference x, it follows that the radius of the circle is $x/2\pi$; therefore, the area of the circle is

$$\pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}.$$

As for the square, because the perimeter is $12 - x$, the length of each side is $3 - x/4$ and the area is $(3 - x/4)^2$. Then

$$A(x) = \frac{x^2}{4\pi} + \left(3 - \frac{1}{4}x\right)^2.$$

Now

$$A'(x) = \frac{x}{2\pi} - \frac{1}{2}\left(3 - \frac{1}{4}x\right) = 0$$

when

$$x = \frac{12\pi}{4+\pi} \text{ m} \approx 5.28 \text{ m}.$$

Because $A(0) = 9$ m^2, $A(12) = 36/\pi \approx 11.46$ m^2, and

$$A\left(\frac{12\pi}{4+\pi}\right) \approx 5.04 \text{ m}^2,$$

we see that the sum of the areas is minimized when approximately 5.28 m of the wire is allotted to the circle.

4. Find the positive number x such that the sum of x and its reciprocal is as small as possible. Does this problem require optimization over an open interval or a closed interval?

SOLUTION Let $x > 0$ and $f(x) = x + x^{-1}$. Here we require optimization over the open interval $(0, \infty)$. Solve $f'(x) = 1 - x^{-2} = 0$ for $x > 0$ to obtain $x = 1$. Since $f(x) \to \infty$ as $x \to 0+$ and as $x \to \infty$, we conclude that f has an absolute minimum of $f(1) = 2$ at $x = 1$.

5. A flexible tube of length 4 m is bent into an L-shape. Where should the bend be made to minimize the distance between the two ends?

SOLUTION Let $x, y > 0$ be lengths of the side of the L. Since $x + y = 4$ or $y = 4 - x$, the distance between the ends of L is $h(x) = \sqrt{x^2 + y^2} = \sqrt{x^2 + (4-x)^2}$. We may equivalently minimize the square of the distance,

$$f(x) = x^2 + y^2 = x^2 + (4-x)^2$$

This is easier computationally (when working by hand). Solve $f'(x) = 4x - 8 = 0$ to obtain $x = 2$ m. Now $f(0) = f(4) = 16$, whereas $f(2) = 8$. Hence the distance between the two ends of the L is minimized when the bend is made at the middle of the wire.

6. Find the dimensions of the box with square base with:

(a) Volume 12 and the minimal surface area.

(b) Surface area 20 and maximal volume.

SOLUTION A box has a square base of side x and height y where $x, y > 0$. Its volume is $V = x^2 y$ and its surface area is $S = 2x^2 + 4xy$.

(a) If $V = x^2 y = 12$, then $y = 12/x^2$ and $S(x) = 2x^2 + 4x\left(12/x^2\right) = 2x^2 + 48x^{-1}$. Solve $S'(x) = 4x - 48x^{-2} = 0$ to obtain $x = 12^{1/3}$. Since $S(x) \to \infty$ as $x \to 0+$ and as $x \to \infty$, the minimum surface area is $S(12^{1/3}) = 6\,(12)^{2/3} \approx 31.45$, when $x = 12^{1/3}$ and $y = 12^{1/3}$.

(b) If $S = 2x^2 + 4xy = 20$, then $y = 5x^{-1} - \frac{1}{2}x$ and $V(x) = x^2 y = 5x - \frac{1}{2}x^3$. Note that x must lie on the closed interval $[0, \sqrt{10}]$. Solve $V'(x) = 5 - \frac{3}{2}x^2$ for $x > 0$ to obtain $x = \frac{\sqrt{30}}{3}$. Since $V(0) = V(\sqrt{10}) = 0$ and $V\left(\frac{\sqrt{30}}{3}\right) = \frac{10\sqrt{30}}{9}$, the maximum volume is $V(\frac{\sqrt{30}}{3}) = \frac{10}{9}\sqrt{30} \approx 6.086$, when $x = \frac{\sqrt{30}}{3}$ and $y = \frac{\sqrt{30}}{3}$.

7. A rancher will use 600 m of fencing to build a corral in the shape of a semicircle on top of a rectangle (Figure 9). Find the dimensions that maximize the area of the corral.

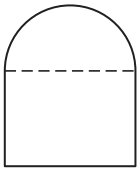


FIGURE 9

SOLUTION Let x be the width of the corral and therefore the diameter of the semicircle, and let y be the height of the rectangular section. Then the perimeter of the corral can be expressed by the equation $2y + x + \frac{\pi}{2}x = 2y + (1 + \frac{\pi}{2})x = 600$ m or equivalently, $y = \frac{1}{2}\left(600 - (1 + \frac{\pi}{2})x\right)$. Since x and y must both be nonnegative, it follows that x must be restricted to the interval $[0, \frac{600}{1+\pi/2}]$. The area of the corral is the sum of the area of the rectangle and semicircle, $A = xy + \frac{\pi}{8}x^2$. Making the substitution for y from the constraint equation,

$$A(x) = \frac{1}{2}x\left(600 - (1 + \frac{\pi}{2})x\right) + \frac{\pi}{8}x^2 = 300x - \frac{1}{2}\left(1 + \frac{\pi}{2}\right)x^2 + \frac{\pi}{8}x^2.$$

Now, $A'(x) = 300 - \left(1 + \frac{\pi}{2}\right)x + \frac{\pi}{4}x = 0$ implies $x = \frac{300}{(1+\frac{\pi}{4})} \approx 168.029746$ m. With $A(0) = 0$ m^2,

$$A\left(\frac{300}{1 + \pi/4}\right) \approx 25204.5 \text{ m}^2 \qquad \text{and} \qquad A\left(\frac{600}{1 + \pi/2}\right) \approx 21390.8 \text{ m}^2,$$

it follows that the corral of maximum area has dimensions

$$x = \frac{300}{1 + \pi/4} \text{ m} \qquad \text{and} \qquad y = \frac{150}{1 + \pi/4} \text{ m}.$$

8. What is the maximum area of a rectangle inscribed in a right triangle with 5 and 8 as in Figure 10. The sides of the rectangle are parallel to the legs of the triangle.

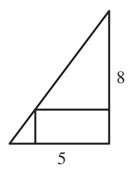


FIGURE 10

SOLUTION Position the triangle with its right angle at the origin, with its side of length 8 along the positive y-axis, and side of length 5 along the positive x-axis. Let $x, y > 0$ be the lengths of sides of the inscribed rectangle along the axes. By similar triangles, we have $\frac{8}{5} = \frac{y}{5-x}$ or $y = 8 - \frac{8}{5}x$. The area of the rectangle is thus $A(x) = xy = 8x - \frac{8}{5}x^2$. To guarantee that both x and y remain nonnegative, we must restrict x to the interval $[0, 5]$. Solve $A'(x) = 8 - \frac{16}{5}x = 0$ to obtain $x = \frac{5}{2}$. Since $A(0) = A(5) = 0$ and $A(\frac{5}{2}) = 10$, the maximum area is $A\left(\frac{5}{2}\right) = 10$ when $x = \frac{5}{2}$ and $y = 4$.

9. Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius $r = 4$ (Figure 11).

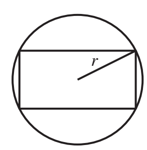


FIGURE 11

SOLUTION Place the center of the circle at the origin with the sides of the rectangle (of lengths $2x > 0$ and $2y > 0$) parallel to the coordinate axes. By the Pythagorean Theorem, $x^2 + y^2 = r^2 = 16$, so that $y = \sqrt{16 - x^2}$. Thus the area of the rectangle is $A(x) = 2x \cdot 2y = 4x\sqrt{16 - x^2}$. To guarantee both x and y are real and nonnegative, we must restrict x to the interval $[0, 4]$. Solve

$$A'(x) = 4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16 - x^2}} = 0$$

for $x > 0$ to obtain $x = \frac{4}{\sqrt{2}} = 2\sqrt{2}$. Since $A(0) = A(4) = 0$ and $A(2\sqrt{2}) = 32$, the rectangle of maximum area has dimensions $2x = 2y = 4\sqrt{2}$.

10. Find the dimensions x and y of the rectangle inscribed in a circle of radius r that maximizes the quantity xy^2.

SOLUTION Place the center of the circle of radius r at the origin with the sides of the rectangle (of lengths $x > 0$ and $y > 0$) parallel to the coordinate axes. By the Pythagorean Theorem, we have $(\frac{x}{2})^2 + (\frac{y}{2})^2 = r^2$, whence $y^2 = 4r^2 - x^2$. Let $f(x) = xy^2 = 4xr^2 - x^3$. Allowing for degenerate rectangles, we have $0 \le x \le 2r$. Solve $f'(x) = 4r^2 - 3x^2$ for $x \ge 0$ to obtain $x = \frac{2r}{\sqrt{3}}$. Since $f(0) = f(2r) = 0$, the maximal value of f is $f(\frac{2r}{\sqrt{3}}) = \frac{16}{9}\sqrt{3}r^3$ when $x = \frac{2r}{\sqrt{3}}$ and $y = 2\sqrt{\frac{2}{3}}r$.

11. Find the point on the line $y = x$ closest to the point $(1, 0)$. *Hint:* It is equivalent and easier to minimize the *square* of the distance.

SOLUTION With $y = x$, let's equivalently minimize the square of the distance, $f(x) = (x - 1)^2 + y^2 = 2x^2 - 2x + 1$, which is computationally easier (when working by hand). Solve $f'(x) = 4x - 2 = 0$ to obtain $x = \frac{1}{2}$. Since $f(x) \to \infty$ as $x \to \pm\infty$, $(\frac{1}{2}, \frac{1}{2})$ is the point on $y = x$ closest to $(1, 0)$.

12. Find the point P on the parabola $y = x^2$ closest to the point $(3, 0)$ (Figure 12).

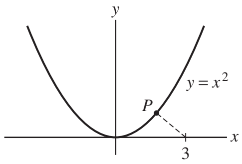


FIGURE 12

SOLUTION With $y = x^2$, let's equivalently minimize the square of the distance,

$$f(x) = (x - 3)^2 + y^2 = x^4 + x^2 - 6x + 9.$$

Then

$$f'(x) = 4x^3 + 2x - 6 = 2(x - 1)(2x^2 + 2x + 3),$$

so that $f'(x) = 0$ when $x = 1$ (plus two complex solutions, which we discard). Since $f(x) \to \infty$ as $x \to \pm\infty$, $P = (1, 1)$ is the point on $y = x^2$ closest to $(3, 0)$.

13. CAS Find a good numerical approximation to the coordinates of the point on the graph of $y = \ln x - x$ closest to the origin (Figure 13).

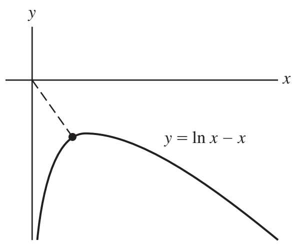


FIGURE 13

SOLUTION The distance from the origin to the point $(x, \ln x - x)$ on the graph of $y = \ln x - x$ is $d = \sqrt{x^2 + (\ln x - x)^2}$. As usual, we will minimize d^2. Let $d^2 = f(x) = x^2 + (\ln x - x)^2$. Then

$$f'(x) = 2x + 2(\ln x - x)\left(\frac{1}{x} - 1\right).$$

To determine x, we need to solve

$$4x + \frac{2 \ln x}{x} - 2 \ln x - 2 = 0.$$

This yields $x \approx .632784$. Thus, the point on the graph of $y = \ln x - x$ that is closest to the origin is approximately $(0.632784, -1.090410)$.

14. Problem of Tartaglia (1500–1557) Among all positive numbers a, b whose sum is 8, find those for which the product of the two numbers and their difference is largest.

SOLUTION The product of a,b and their difference is $ab(a - b)$. Since $a + b = 8$, $b = 8 - a$ and $a - b = 2a - 8$. Thus, let

$$f(a) = a(8 - a)(2a - 8) = -2a^3 + 24a^2 - 64a.$$

where $a \in [0, 8]$. Setting $f'(a) = -6a^2 + 48a - 64 = 0$ yields $a = 4 \pm \frac{4}{3}\sqrt{3}$. Now, $f(0) = f(8) = 0$, while

$$f\left(4 - \frac{4}{3}\sqrt{3}\right) < 0 \quad \text{and} \quad f\left(4 + \frac{4}{3}\sqrt{3}\right) > 0.$$

Hence the numbers a, b maximizing the product are

$$a = 4 + \frac{4\sqrt{3}}{3}, \quad \text{and} \quad b = 8 - a = 4 - \frac{4\sqrt{3}}{3}.$$

15. Find the angle θ that maximizes the area of the isosceles triangle whose legs have length ℓ (Figure 14).

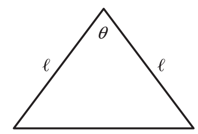


FIGURE 14

SOLUTION The area of the triangle is

$$A(\theta) = \frac{1}{2}\ell^2 \sin\theta,$$

where $0 \le \theta \le \pi$. Setting

$$A'(\theta) = \frac{1}{2}\ell^2 \cos\theta = 0$$

yields $\theta = \frac{\pi}{2}$. Since $A(0) = A(\pi) = 0$ and $A(\frac{\pi}{2}) = \frac{1}{2}\ell^2$, the angle that maximizes the area of the isosceles triangle is $\theta = \frac{\pi}{2}$.

16. A right circular cone (Figure 15) has volume $V = \frac{\pi}{3}r^2 h$ and surface area is $S = \pi r\sqrt{r^2 + h^2}$. Find the dimensions of the cone with surface area 1 and maximal volume.

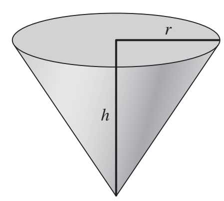


FIGURE 15

SOLUTION We have $\pi r\sqrt{r^2 + h^2} = 1$ so $\pi^2 r^2(r^2 + h^2) = 1$ and hence $h^2 = \frac{1-\pi^2 r^4}{\pi^2 r^2}$ and now we must maximize

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(r^2\frac{\sqrt{1-\pi^2 r^4}}{\pi r}\right) = \frac{1}{3}r\sqrt{1-\pi^2 r^4},$$

where $0 < r \le 1/\sqrt{\pi}$. Because

$$\frac{d}{dr}r\sqrt{1-\pi^2 r^4} = \sqrt{1-\pi^2 r^4} + \frac{1}{2}r\frac{-4\pi^2 r^3}{\sqrt{1-\pi^2 r^4}}$$

the relevant critical point is $r = (3\pi^2)^{-1/4}$.

To find h, we back substitute our solution for r in $h^2 = (1-\pi^2 r^4)/(\pi^2 r^2)$. $r = (3\pi^2)^{-1/4}$, so $r^4 = \frac{1}{3\pi^2}$ and $r^2 = \frac{1}{\sqrt{3}\pi}$; hence, $\pi^2 r^4 = \frac{1}{3}$ and $\pi^2 r^2 = \frac{\pi}{\sqrt{3}}$, and:

$$h^2 = \left(\frac{2}{3}\right) \Big/ \left(\frac{\pi}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}\pi}.$$

From this, $h = \sqrt{2}/\left(3^{1/4}\sqrt{\pi}\right)$. Since

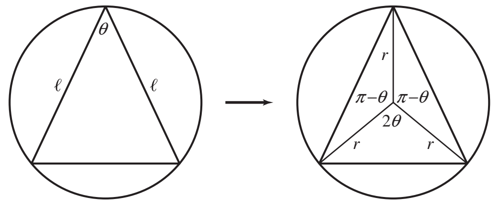
$$\lim_{r\to 0+} V(r) = 0,\ V\left(\frac{1}{\sqrt{\pi}}\right) = 0 \qquad \text{and} \qquad V\left((3\pi^2)^{-1/4}\right) = \frac{1}{3^{7/4}}\sqrt{\frac{2}{\pi}},$$

the cone of surface area 1 with maximal volume has dimensions

$$r = \frac{1}{3^{1/4}\sqrt{\pi}} \qquad \text{and} \qquad h = \frac{\sqrt{2}}{3^{1/4}\sqrt{\pi}}.$$

17. Find the area of the largest isosceles triangle that can be inscribed in a circle of radius r.

SOLUTION Consider the following diagram:



The area of the isosceles triangle is

$$A(\theta) = 2 \cdot \frac{1}{2}r^2 \sin(\pi - \theta) + \frac{1}{2}r^2 \sin(2\theta) = r^2 \sin\theta + \frac{1}{2}r^2 \sin(2\theta),$$

where $0 \le \theta \le \pi$. Solve

$$A'(\theta) = r^2 \cos\theta + r^2 \cos(2\theta) = 0$$

to obtain $\theta = \frac{\pi}{3}, \pi$. Since $A(0) = A(\pi) = 0$ and $A(\frac{\pi}{3}) = \frac{3\sqrt{3}}{4}r^2$, the area of the largest isosceles triangle that can be inscribed in a circle of radius r is $\frac{3\sqrt{3}}{4}r^2$.

18. Find the radius and height of a cylindrical can of total surface area A whose volume is as large as possible. Does there exist a cylinder of surface area A and minimal total volume?

SOLUTION Let a closed cylindrical can be of radius r and height h. Its total surface area is $S = 2\pi r^2 + 2\pi rh = A$, whence $h = \dfrac{A}{2\pi r} - r$. Its volume is thus $V(r) = \pi r^2 h = \frac{1}{2}Ar - \pi r^3$, where $0 < r \le \sqrt{\frac{A}{2\pi}}$. Solve $V'(r) = \frac{1}{2}A - 3\pi r^2$ for $r > 0$ to obtain $r = \sqrt{\dfrac{A}{6\pi}}$. Since $V(0) = V(\sqrt{\frac{A}{2\pi}}) = 0$ and

$$V\left(\sqrt{\frac{A}{6\pi}}\right) = \frac{\sqrt{6}A^{3/2}}{18\sqrt{\pi}},$$

the maximum volume is achieved when

$$r = \sqrt{\frac{A}{6\pi}} \quad \text{and} \quad h = \frac{1}{3}\sqrt{\frac{6A}{\pi}}.$$

For a can of total surface area A, there are cans of arbitrarily small volume since $\lim_{r\to 0+} V(r) = 0$.

19. A poster of area 6000 cm^2 has blank margins of width 10 cm on the top and bottom and 6 cm on the sides. Find the dimensions that maximize the printed area.

SOLUTION Let x be the width of the printed region, and let y be the height. The total printed area is $A = xy$. Because the total area of the poster is 6000 cm^2, we have the constraint $(x + 12)(y + 20) = 6000$, so that $xy + 12y + 20x + 240 = 6000$, or $y = \frac{5760-20x}{x+12}$. Therefore, $A(x) = 20\frac{288x-x^2}{x+12}$, where $0 \le x \le 288$.

$A(0) = A(288) = 0$, so we are looking for a critical point on the interval $[0, 288]$. Setting $A'(x) = 0$ yields

$$20\frac{(x+12)(288-2x) - (288x - x^2)}{(x+12)^2} = 0$$

$$\frac{-x^2 - 24x + 3456}{(x+12)^2} = 0$$

$$x^2 + 24x - 3456 = 0$$

$$(x - 48)(x + 72) = 0$$

Therefore $x = 48$ or $x = -72$. $x = 48$ is the only critical point of $A(x)$ in the interval $[0, 288]$, so $A(48) = 3840$ is the maximum value of $A(x)$ in the interval $[0, 288]$. Now, $y = 20\frac{288-48}{48+12} = 80$ cm, so the poster with maximum printed area is $48 + 12 = 60$ cm. wide by $80 + 20 = 100$ cm. tall.

20. According to postal regulations, a carton is classified as "oversized" if the sum of its height and girth (perimeter of its base) exceeds 108 in. Find the dimensions of a carton with square base that is not oversized and has maximum volume.

SOLUTION Let h denote the height of the carton and s denote the side length of the square base. Clearly the volume will be maximized when the sum of the height and girth equals 108; i.e., $4s + h = 108$, whence $h = 108 - 4s$. Allowing for degenerate cartons, the carton's volume is $V(s) = s^2 h = s^2(108 - 4s)$, where $0 \le s \le 27$. Solve $V'(s) = 216s - 12s^3 = 0$ for s to obtain $s = 0$ or $s = 18$. Since $V(0) = V(27) = 0$, the maximum volume is $V(18) = 11664$ in^3 when $s = 18$ in and $h = 36$ in.

21. Kepler's Wine Barrel Problem In his work *Nova stereometria doliorum vinariorum* (New Solid Geometry of a Wine Barrel), published in 1615, astronomer Johannes Kepler stated and solved the following problem: Find the dimensions of the cylinder of largest volume that can be inscribed in a sphere of radius R. *Hint:* Show that an inscribed cylinder has volume $2\pi x(R^2 - x^2)$, where x is one-half the height of the cylinder.

SOLUTION Place the center of the sphere at the origin in three-dimensional space. Let the cylinder be of radius y and half-height x. The Pythagorean Theorem states, $x^2 + y^2 = R^2$, so that $y^2 = R^2 - x^2$. The volume of the cylinder is $V(x) = \pi y^2 (2x) = 2\pi\left(R^2 - x^2\right)x = 2\pi R^2 x - 2\pi x^3$. Allowing for degenerate cylinders, we have $0 \le x \le R$. Solve $V'(x) = 2\pi R^2 - 6\pi x^2 = 0$ for $x \ge 0$ to obtain $x = \frac{R}{\sqrt{3}}$. Since $V(0) = V(R) = 0$, the largest volume is $V(\frac{R}{\sqrt{3}}) = \frac{4}{9}\pi\sqrt{3}R^3$ when $x = \frac{R}{\sqrt{3}}$ and $y = \sqrt{\frac{2}{3}}R$.

22. Find the angle θ that maximizes the area of the trapezoid with a base of length 4 and sides of length 2, as in Figure 16.

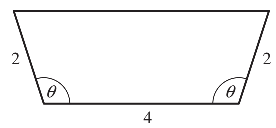


FIGURE 16

SOLUTION Allowing for degenerate trapezoids, we have $0 \le \theta \le \pi$. Via trigonometry and surgery (slice off a right triangle and rearrange the trapezoid into a rectangle), we have that the area of the trapezoid is equivalent to the area of a rectangle of base $4 - 2\cos\theta$ and height $2\sin\theta$; i.e,

$$A(\theta) = (4 - 2\cos\theta)\cdot 2\sin\theta = 8\sin\theta - 4\sin\theta\cos\theta = 8\sin\theta - 2\sin 2\theta,$$

where $0 \le \theta \le \pi$. Solve

$$A'(\theta) = 8\cos\theta - 4\cos 2\theta = 4 + 8\cos\theta - 8\cos^2\theta = 0$$

for $0 \le \theta \le \pi$ to obtain

$$\theta = \theta_0 = \cos^{-1}\left(\frac{1-\sqrt{3}}{2}\right) \approx 1.94553.$$

Since $A(0) = A(\pi) = 0$ and $A(\theta_0) = 3^{1/4}(3+\sqrt{3})\sqrt{2}$, the area of the trapezoid is maximized when $\theta = \cos^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$.

23. A landscape architect wishes to enclose a rectangular garden of area 1,000 m^2 on one side by a brick wall costing \$90/m and on the other three sides by a metal fence costing \$30/m. Which dimensions minimize the total cost?

SOLUTION Let x be the length of the brick wall and y the length of an adjacent side with $x, y > 0$. With $xy = 1000$ or $y = \frac{1000}{x}$, the total cost is

$$C(x) = 90x + 30(x + 2y) = 120x + 60000x^{-1}.$$

Solve $C'(x) = 120 - 60000x^{-2} = 0$ for $x > 0$ to obtain $x = 10\sqrt{5}$. Since $C(x) \to \infty$ as $x \to 0+$ and as $x \to \infty$, the minimum cost is $C(10\sqrt{5}) = 2400\sqrt{5} \approx \$5366.56$ when $x = 10\sqrt{5} \approx 22.36$ m and $y = 20\sqrt{5} \approx 44.72$ m.

24. The amount of light reaching a point at a distance r from a light source A of intensity I_A is I_A/r^2. Suppose that a second light source B of intensity $I_B = 4I_A$ is located 10 m from A. Find the point on the segment joining A and B where the total amount of light is at a minimum.

SOLUTION Place the segment in the xy-plane with A at the origin and B at $(10, 0)$. Let x be the distance from A. Then $10 - x$ is the distance from B. The total amount of light is

$$f(x) = \frac{I_A}{x^2} + \frac{I_B}{(10-x)^2} = I_A\left(\frac{1}{x^2} + \frac{4}{(10-x)^2}\right).$$

Solve

$$f'(x) = I_A\left(\frac{8}{(10-x)^3} - \frac{2}{x^3}\right) = 0$$

for $0 \le x \le 10$ to obtain

$$4 = \frac{(10-x)^3}{x^3} = \left(\frac{10}{x} - 1\right)^3 \quad \text{or} \quad x = \frac{10}{1+\sqrt[3]{4}} \approx 3.86 \text{ m}.$$

Since $f(x) \to \infty$ as $x \to 0+$ and $x \to 10-$ we conclude that the minimal amount of light occurs 3.86 m from A.

25. Find the maximum area of a rectangle inscribed in the region bounded by the graph of $y = \dfrac{4-x}{2+x}$ and the axes (Figure 17).

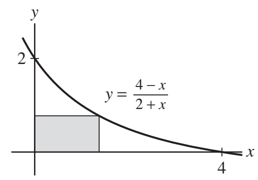


FIGURE 17

SOLUTION Let s be the width of the rectangle. The height of the rectangle is $h = \frac{4-s}{2+s}$, so that the area is

$$A(s) = s\frac{4-s}{2+s} = \frac{4s-s^2}{2+s}.$$

We are maximizing on the closed interval $[0, 4]$. It is obvious from the pictures that $A(0) = A(4) = 0$, so we look for critical points of A.

$$A'(s) = \frac{(2+s)(4-2s)-(4s-s^2)}{(2+s)^2} = -\frac{s^2+4s-8}{(s+2)^2}.$$

The only point where $A'(s)$ doesn't exist is $s = -2$ which isn't under consideration.

Setting $A'(s) = 0$ gives, by the quadratic formula,

$$s = \frac{-4 \pm \sqrt{48}}{2} = -2 \pm 2\sqrt{3}.$$

Of these, only $-2 + 2\sqrt{3}$ is positive, so this is our lone critical point. $A(-2+2\sqrt{3}) \approx 1.0718 > 0$. Since we are finding the maximum over a closed interval and $-2+2\sqrt{3}$ is the only critical point, the maximum area is $A(-2+2\sqrt{3}) \approx 1.0718$.

26. Find the maximum area of a triangle formed by the axes and a tangent line to the graph of $y = (x+1)^{-2}$ with $x > 0$.

SOLUTION Let $P\left(t, \frac{1}{(t+1)^2}\right)$ be a point on the graph of the curve $y = \frac{1}{(x+1)^2}$ in the first quadrant. The tangent line to the curve at P is

$$L(x) = \frac{1}{(t+1)^2} - \frac{2(x-t)}{(t+1)^3},$$

which has x-intercept $a = \frac{3t+1}{2}$ and y-intercept $b = \frac{3t+1}{(t+1)^3}$. The area of the triangle in question is

$$A(t) = \frac{1}{2}ab = \frac{(3t+1)^2}{4(t+1)^3}.$$

Solve

$$A'(t) = \frac{(3t+1)(3-3t)}{4(t+1)^4} = 0$$

for $0 \le t$ to obtain $t = 1$. Because $A(0) = \frac{1}{4}$, $A(1) = \frac{1}{2}$ and $A(t) \to 0$ as $t \to \infty$, it follows that the maximum area is $A(1) = \frac{1}{2}$.

27. Find the maximum area of a rectangle circumscribed around a rectangle of sides L and H. *Hint:* Express the area in terms of the angle θ (Figure 18).

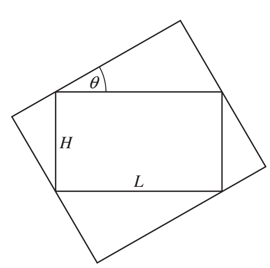


FIGURE 18

SOLUTION Position the $L \times H$ rectangle in the first quadrant of the xy-plane with its "northwest" corner at the origin. Let θ be the angle the base of the circumscribed rectangle makes with the positive x-axis, where $0 \le \theta \le \frac{\pi}{2}$. Then the area of the circumscribed rectangle is $A = LH + 2 \cdot \frac{1}{2}(H\sin\theta)(H\cos\theta) + 2 \cdot \frac{1}{2}(L\sin\theta)(L\cos\theta) = LH + \frac{1}{2}(L^2 + H^2)\sin 2\theta$, which has a maximum value of $LH + \frac{1}{2}(L^2 + H^2)$ when $\theta = \frac{\pi}{4}$ because $\sin 2\theta$ achieves its maximum when $\theta = \frac{\pi}{4}$.

28. A contractor is engaged to build steps up the slope of a hill that has the shape of the graph of $y = x^2(120 - x)/6400$ for $0 \le x \le 80$ with x in meters (Figure 19). What is the maximum vertical rise of a stair if each stair has a horizontal length of one-third meter.

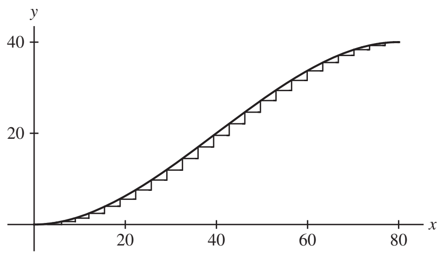


FIGURE 19

SOLUTION Let $f(x) = x^2(120 - x)/6400$. Because the horizontal length of each stair is one-third meter, the vertical rise of each stair is

$$r(x) = f\left(x + \frac{1}{3}\right) - f(x) = \frac{1}{6400}\left(x + \frac{1}{3}\right)^2\left(\frac{359}{3} - x\right) - \frac{1}{6400}x^2(120 - x)$$
$$= \frac{1}{6400}\left(-x^2 + \frac{239}{3}x + \frac{359}{27}\right),$$

where x denotes the location of the beginning of the stair. This is the equation of a downward opening parabola; thus, the maximum occurs when $r'(x) = 0$. Now,

$$r'(x) = \frac{1}{6400}\left(-2x + \frac{239}{3}\right) = 0$$

when $x = 239/6$. Because the stair must start at a location of the form $n/3$ for some integer n, we evaluate $r(x)$ at $x = 119/3$ and $x = 120/3 = 40$. We find

$$r\left(\frac{119}{3}\right) = r(40) = \frac{43199}{172800} \approx 0.249994$$

meters. Thus, the maximum vertical rise of any stair is just below 0.25 meters.

29. Find the equation of the line through $P = (4, 12)$ such that the triangle bounded by this line and the axes in the first quadrant has minimal area.

SOLUTION Let $P = (4, 12)$ be a point in the first quadrant and $y - 12 = m(x - 4)$, $-\infty < m < 0$, be a line through P that cuts the positive x- and y-axes. Then $y = L(x) = m(x - 4) + 12$. The line $L(x)$ intersects the y-axis at $H\,(0, 12 - 4m)$ and the x-axis at $W\left(4 - \frac{12}{m}, 0\right)$. Hence the area of the triangle is

$$A(m) = \frac{1}{2}(12 - 4m)\left(4 - \frac{12}{m}\right) = 48 - 8m - 72m^{-1}.$$

Solve $A'(m) = 72m^{-2} - 8 = 0$ for $m < 0$ to obtain $m = -3$. Since $A \to \infty$ as $m \to -\infty$ or $m \to 0-$, we conclude that the minimal triangular area is obtained when $m = -3$. The equation of the line through $P = (4, 12)$ is $y = -3(x - 4) + 12 = -3x + 24$.

30. Let $P = (a, b)$ lie in the first quadrant. Find the slope of the line through P such that the triangle bounded by this line and the axes in the first quadrant has minimal area. Then show that P is the midpoint of the hypotenuse of this triangle.

SOLUTION Let $P(a, b)$ be a point in the first quadrant (thus $a, b > 0$) and $y - b = m(x - a)$, $-\infty < m < 0$, be a line through P that cuts the positive x- and y-axes. Then $y = L(x) = m(x - a) + b$. The line $L(x)$ intersects the y-axis at $H\,(0, b - am)$ and the x-axis at $W\left(a - \frac{b}{m}, 0\right)$. Hence the area of the triangle is

$$A(m) = \frac{1}{2}(b - am)\left(a - \frac{b}{m}\right) = ab - \frac{1}{2}a^2m - \frac{1}{2}b^2m^{-1}.$$

Solve $A'(m) = \frac{1}{2}b^2m^{-2} - \frac{1}{2}a^2 = 0$ for $m < 0$ to obtain $m = -\frac{b}{a}$. Since $A \to \infty$ as $m \to -\infty$ or $m \to 0-$, we conclude that the minimal triangular area is obtained when $m = -\frac{b}{a}$. For $m = -b/a$, we have $H(0, 2b)$ and $W(2a, 0)$. The midpoint of the line segment connecting H and W is thus $P(a, b)$.

31. Archimedes' Problem A spherical cap (Figure 20) of radius r and height h has volume $V = \pi h^2\left(r - \frac{1}{3}h\right)$ and surface area $S = 2\pi r h$. Prove that the hemisphere encloses the largest volume among all spherical caps of fixed surface area S.

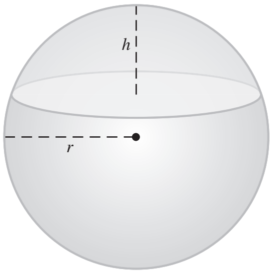


FIGURE 20

SOLUTION Consider all spherical caps of fixed surface area S. Because $S = 2\pi r h$, it follows that

$$r = \frac{S}{2\pi h}$$

and

$$V(h) = \pi h^2\left(\frac{S}{2\pi h} - \frac{1}{3}h\right) = \frac{S}{2}h - \frac{\pi}{3}h^3.$$

Now

$$V'(h) = \frac{S}{2} - \pi h^2 = 0$$

when

$$h^2 = \frac{S}{2\pi} \quad \text{or} \quad h = \frac{S}{2\pi h} = r.$$

Hence, the hemisphere encloses the largest volume among all spherical caps of fixed surface area S.

32. Find the isosceles triangle of smallest area (Figure 21) that circumscribes a circle of radius 1 (from Thomas Simpson's *The Doctrine and Application of Fluxions*, a calculus text that appeared in 1750).

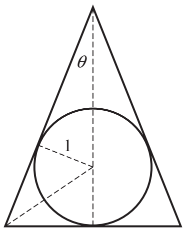


FIGURE 21

SOLUTION From the diagram, we see that the height h and base b of the triangle are $h = 1 + \csc\theta$ and $b = 2h\tan\theta = 2(1 + \csc\theta)\tan\theta$. Thus, the area of the triangle is

$$A(\theta) = \frac{1}{2}hb = (1 + \csc\theta)^2\tan\theta,$$

where $0 < \theta < \pi$. We now set the derivative equal to zero:

$$A'(\theta) = (1 + \csc\theta)(-2\csc\theta + \sec^2\theta(1 + \csc\theta)) = 0.$$

The first factor gives $\theta = 3\pi/2$ which is not in the domain of the problem. To find the roots of the second factor, multiply through by $\cos^2\theta\sin\theta$ to obtain

$$-2\cos^2\theta + \sin\theta + 1 = 0,$$

or

$$2\sin^2\theta + \sin\theta - 1 = 0.$$

This is a quadratic equation in $\sin\theta$ with roots $\sin\theta = -1$ and $\sin\theta = 1/2$. Only the second solution is relevant and gives us $\theta = \pi/6$. Since $A(\theta) \to \infty$ as $\theta \to 0+$ and as $\theta \to \pi-$, we see that the minimum area occurs when the triangle is an equilateral triangle.

33. A box of volume 72 m^3 with square bottom and no top is constructed out of two different materials. The cost of the bottom is \$40/m^2 and the cost of the sides is \$30/m^2. Find the dimensions of the box that minimize total cost.

SOLUTION Let s denote the length of the side of the square bottom of the box and h denote the height of the box. Then

$$V = s^2 h = 72 \quad \text{or} \quad h = \frac{72}{s^2}.$$

The cost of the box is

$$C = 40s^2 + 120sh = 40s^2 + \frac{8640}{s},$$

so

$$C'(s) = 80s - \frac{8640}{s^2} = 0$$

when $s = 3\sqrt[3]{4}$ m and $h = 2\sqrt[3]{4}$ m. Because $C \to \infty$ as $s \to 0-$ and as $s \to \infty$, we conclude that the critical point gives the minimum cost.

34. Find the dimensions of a cylinder of volume 1 m^3 of minimal cost if the top and bottom are made of material that costs twice as much as the material for the side.

SOLUTION Let r be the radius in meters of the top and bottom of the cylinder. Let h be the height in meters of the cylinder. Since $V = \pi r^2 h = 1$, we get $h = \frac{1}{\pi r^2}$. Ignoring the actual cost, and using only the proportion, suppose that the sides cost 1 monetary unit per square meter and the top and the bottom 2. The cost of the top and bottom is $2(2\pi r^2)$ and the cost of the sides is $1(2\pi r h) = 2\pi r(\frac{1}{\pi r^2}) = \frac{2}{r}$. Let $C(r) = 4\pi r^2 + \frac{2}{r}$. Because $C(r) \to \infty$ as $r \to 0+$ and as $r \to \infty$, we are looking for critical points of $C(r)$. Setting $C'(r) = 8\pi r - \frac{2}{r^2} = 0$ yields $8\pi r = \frac{2}{r^2}$, so that $r^3 = \frac{1}{4\pi}$. This yields $r = \frac{1}{(4\pi)^{1/3}} \approx 0.430127$. The dimensions that minimize cost are

$$r = \frac{1}{(4\pi)^{1/3}} \text{ m}, \qquad h = \frac{1}{\pi r^2} = 4^{2/3}\pi^{-1/3} \text{ m}.$$

35. Your task is to design a rectangular industrial warehouse consisting of three separate spaces of equal size as in Figure 22. The wall materials cost \$500 per linear meter and your company allocates \$2,400,000 for the project.

(a) Which dimensions maximize the area of the warehouse?

(b) What is the area of each compartment in this case?

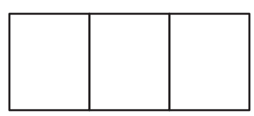


FIGURE 22

SOLUTION Let one compartment have length x and width y. Then total length of the wall of the warehouse is $P = 4x + 6y$ and the constraint equation is cost $= 2{,}400{,}000 = 500(4x + 6y)$, which gives $y = 800 - \frac{2}{3}x$.

(a) Area is given by $A = 3xy = 3x\left(800 - \frac{2}{3}x\right) = 2400x - 2x^2$, where $0 \le x \le 1200$. Then $A'(x) = 2400 - 4x = 0$ yields $x = 600$ and consequently $y = 400$. Since $A(0) = A(1200) = 0$ and $A(600) = 720{,}000$, the area of the warehouse is maximized when each compartment has length of 600 m and width of 400 m.

(b) The area of one compartment is $600 \cdot 400 = 240{,}000$ square meters.

36. Suppose, in the previous exercise, that the warehouse consists of n separate spaces of equal size. Find a formula in terms of n for the maximum possible area of the warehouse.

SOLUTION For n compartments, with x and y as before, cost $= 2{,}400{,}000 = 500((n+1)x + 2ny)$ and $y = \dfrac{4800 - (n+1)x}{2n}$. Then

$$A = nxy = x\frac{4800 - (n+1)x}{2} = 2400x - \frac{n+1}{2}x^2$$

and $A'(x) = 2400 - (n+1)x = 0$ yields $x = \dfrac{2400}{n+1}$ and consequently $y = \dfrac{1200}{n}$. Thus the maximum area is given by

$$A = n\left(\frac{2400}{n+1}\right)\left(\frac{1200}{n}\right) = \frac{28{,}800{,}000}{n+1}.$$

37. According to a model developed by economists E. Heady and J. Pesek, if fertilizer made from N pounds of nitrogen and P pounds of phosphate is used on an acre of farmland, then the yield of corn (in bushels per acre) is

$$Y = 7.5 + 0.6N + 0.7P - 0.001N^2 - 0.002P^2 + 0.001NP$$

A farmer intends to spend \$30 per acre on fertilizer. If nitrogen costs 25 cents/lb and phosphate costs 20 cents/lb, which combination of N and L produces the highest yield of corn?

SOLUTION The farmer's budget for fertilizer is \$30 per acre, so we have the constraint equation

$$0.25N + 0.2P = 30 \qquad \text{or} \qquad P = 150 - 1.25N$$

Substituting for P in the equation for Y, we find

$$\begin{aligned} Y(N) &= 7.5 + 0.6N + 0.7(150 - 1.25N) - 0.001N^2 - 0.002(150 - 1.25N)^2 + 0.001N(150 - 1.25N) \\ &= 67.5 + 0.625N - 0.005375N^2 \end{aligned}$$

Both N and P must be nonnegative. Since $P = 150 - 1.25N \geq 0$, we require that $0 \leq N \leq 120$. Next,

$$\frac{dY}{dN} = 0.625 - 0.01075N = 0 \quad \Rightarrow \quad N = \frac{0.625}{0.01075} \approx 58.14 \text{ pounds.}$$

Now, $Y(0) = 67.5$, $Y(120) = 65.1$ and $Y(58.14) \approx 85.67$, so the maximum yield of corn occurs for $N \approx 58.14$ pounds and $P \approx 77.33$ pounds.

38. Experiments show that the quantities x of corn and y of soybean required to produce a hog of weight Q satisfy $Q = 0.5x^{1/2}y^{1/4}$. The unit of x, y, and Q is the cwt, an agricultural unit equal to 100 lbs. Find the values of x and y that minimize the cost of a hog of weight $Q = 2.5$ cwt if corn costs \$3/cwt and soy costs \$7/cwt.

SOLUTION With $Q = 2.5$, we find that

$$y = \left(\frac{2.5}{0.5x^{1/2}}\right)^4 = \frac{625}{x^2}.$$

The cost is then

$$C = 3x + 7y = 3x + \frac{4375}{x^2}.$$

Solving

$$\frac{dC}{dx} = 3 - \frac{8750}{x^3} = 0$$

yields $x = \sqrt[3]{8750/3} \approx 14.29$. From this, it follows that $y = 625/14.29^2 \approx 3.06$. The overall cost is $C = 3(14.29) + 7(3.06) \approx \$64.29$.

39. All units in a 100-unit apartment building are rented out when the monthly rent is set at $r = \$900$/month. Suppose that one unit becomes vacant with each \$10 increase in rent and that each occupied unit costs \$80/month in maintenance. Which rent r maximizes monthly profit?

SOLUTION Let n denote the number of \$10 increases in rent. Then the monthly profit is given by

$$P(n) = (100 - n)(900 + 10n - 80) = 82000 + 180n - 10n^2,$$

and

$$P'(n) = 180 - 20n = 0$$

when $n = 9$. We know this results in maximum profit because this gives the location of vertex of a downward opening parabola. Thus, monthly profit is maximized with a rent of \$990.

40. An 8-billion-bushel corn crop brings a price of \$2.40/bu. A commodity broker uses the rule of thumb: If the crop is reduced by x percent, then the price increases by $10x$ cents. Which crop size results in maximum revenue and what is the price per bu? *Hint:* Revenue is equal to price times crop size.

SOLUTION Let x denote the percentage reduction in crop size. Then the price for corn is $2.40 + 0.10x$, the crop size is $8(1 - 0.01x)$ and the revenue (in billions of dollars) is

$$R(x) = (2.4 + 0.1x)8(1 - 0.01x) = 8(-0.001x^2 + 0.076x + 2.4),$$

where $0 \leq x \leq 100$. Solve

$$R'(x) = -0.002x + 0.076 = 0$$

to obtain $x = 38$ percent. Since $R(0) = 19.2$, $R(38) = 30.752$, and $R(100) = 0$, revenue is maximized when $x = 38$. So we reduce the crop size to

$$8(1 - 0.38) = 4.96 \text{ billion bushels.}$$

The price would be \$2.40 + 0.10(38) = 2.40 + 3.80 = \$6.20.

41. The monthly output of a Spanish light bulb factory is $P = 2LK^2$ (in millions), where L is the cost of labor and K is the cost of equipment (in millions of euros). The company needs to produce 1.7 million units per month. Which values of L and K would minimize the total cost $L + K$?

SOLUTION Since $P = 1.7$ and $P = 2LK^2$, we have $L = \frac{0.85}{K^2}$. Accordingly, the cost of production is

$$C(K) = L + K = K + \frac{0.85}{K^2}.$$

Solve $C'(K) = 1 - \frac{1.7}{K^3}$ for $K \geq 0$ to obtain $K = \sqrt[3]{1.7}$. Since $C(K) \to \infty$ as $K \to 0+$ and as $K \to \infty$, the minimum cost of production is achieved for $K = \sqrt[3]{1.7} \approx 1.2$ and $L = 0.6$. The company should invest 1.2 million euros in equipment and 600, 000 euros in labor.

42. The rectangular plot in Figure 23 has size 100 m × 200 m. Pipe is to be laid from A to a point P on side BC and from there to C. The cost of laying pipe along the side of the plot is \$45/m and the cost through the plot is \$80/m (since it is underground).

(a) Let $f(x)$ be the total cost, where x is the distance from P to B. Determine $f(x)$, but note that f is discontinuous at $x = 0$ (when $x = 0$, the cost of the entire pipe is \$45/ft).

(b) What is the most economical way to lay the pipe? What if the cost along the sides is \$65/m?

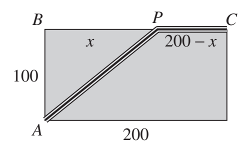


FIGURE 23

SOLUTION

(a) Let x be the distance from P to B. If $x > 0$, then the length of the underground pipe is $\sqrt{100^2 + x^2}$ and the length of the pipe along the side of the plot is $200 - x$. The total cost is

$$f(x) = 80\sqrt{100^2 + x^2} + 45(200 - x).$$

If $x = 0$, all of the pipe is along the side of the plot and $f(0) = 45(200 + 100) = \$13,500$.

(b) To locate the critical points of f, solve

$$f'(x) = \frac{80x}{\sqrt{100^2 + x^2}} - 45 = 0.$$

We find $x = \pm 180/\sqrt{7}$. Note that only the positive value is in the domain of the problem. Because $f(0) = \$13,500$, $f(180/\sqrt{7}) = \$15,614.38$ and $f(200) = \$17,888.54$, the most economical way to lay the pipe is to place the pipe along the side of the plot.

If the cost of laying the pipe along the side of the plot is \$65 per meter, then

$$f(x) = 80\sqrt{100^2 + x^2} + 65(200 - x)$$

and

$$f'(x) = \frac{80x}{\sqrt{100^2 + x^2}} - 65.$$

The only critical point in the domain of the problem is $x = 1300/\sqrt{87} \approx 139.37$. Because $f(0) = \$19,500$, $f(139.37) = \$17,663.69$ and $f(200) = \$17,888.54$, the most economical way to lay the pipe is place the underground pipe from A to a point 139.37 meters to the right of B and continuing to C along the side of the plot.

43. Brandon is on one side of a river that is 50 m wide and wants to reach a point 200 m downstream on the opposite side as quickly as possible by swimming diagonally across the river and then running the rest of the way. Find the best route if Brandon can swim at 1.5 m/s and run at 4 m/s.

SOLUTION Let lengths be in meters, times in seconds, and speeds in m/s. Suppose that Brandon swims diagonally to a point located x meters downstream on the opposite side. Then Brandon then swims a distance $\sqrt{x^2+50^2}$ and runs a distance $200-x$. The total time of the trip is

$$f(x) = \frac{\sqrt{x^2+2500}}{1.5} + \frac{200-x}{4}, \quad 0 \le x \le 200.$$

Solve

$$f'(x) = \frac{2x}{3\sqrt{x^2+2500}} - \frac{1}{4} = 0$$

to obtain $x = 30\frac{5}{11} \approx 20.2$ and $f(20.2) \approx 80.9$. Since $f(0) \approx 83.3$ and $f(200) \approx 137.4$, we conclude that the minimal time is 80.9 s. This occurs when Brandon swims diagonally to a point located 20.2 m downstream and then runs the rest of the way.

44. Snell's Law When a light beam travels from a point A above a swimming pool to a point B below the water (Figure 24), it chooses the path that takes the *least time*. Let v_1 be the velocity of light in air and v_2 the velocity in water (it is known that $v_1 > v_2$). Prove Snell's Law of Refraction:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

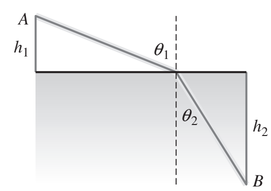


FIGURE 24

SOLUTION The time it takes a beam of light to travel from A to B is

$$f(x) = \frac{a}{v_1} + \frac{b}{v_2} = \frac{\sqrt{x^2+h_1^2}}{v_1} + \frac{\sqrt{(L-x)^2+h_2^2}}{v_2}$$

(See diagram below.) Now

$$f'(x) = \frac{x}{v_1\sqrt{x^2+h_1^2}} - \frac{L-x}{v_2\sqrt{(L-x)^2+h_2^2}} = 0$$

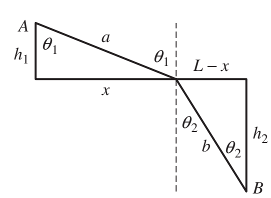
yields

$$\frac{x\Big/\sqrt{x^2+h_1^2}}{v_1} = \frac{(L-x)\Big/\sqrt{(L-x)^2+h_2^2}}{v_2} \qquad \text{or} \qquad \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2},$$

which is Snell's Law. Since

$$f''(x) = \frac{h_1^2}{v_1\left(x^2+h_1^2\right)^{3/2}} + \frac{h_2^2}{v_2\left((L-x)^2+h_2^2\right)^{3/2}} > 0$$

for all x, the minimum time is realized when Snell's Law is satisfied.



In Exercises 45–47, a box (with no top) is to be constructed from a piece of cardboard of sides A and B by cutting out squares of length h from the corners and folding up the sides (Figure 26).

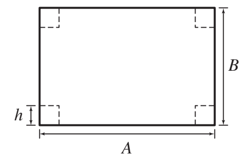


FIGURE 26

45. Find the value of h that maximizes the volume of the box if $A = 15$ and $B = 24$. What are the dimensions of this box?

SOLUTION Once the sides have been folded up, the base of the box will have dimensions $(A - 2h) \times (B - 2h)$ and the height of the box will be h. Thus

$$V(h) = h(A - 2h)(B - 2h) = 4h^3 - 2(A + B)h^2 + ABh.$$

When $A = 15$ and $B = 24$, this gives

$$V(h) = 4h^3 - 78h^2 + 360h,$$

and we need to maximize over $0 \le h \le \frac{15}{2}$. Now,

$$V'(h) = 12h^2 - 156h + 360 = 0$$

yields $h = 3$ and $h = 10$. Because $h = 10$ is not in the domain of the problem and $V(0) = V(15/2) = 0$ and $V(3) = 486$, volume is maximized when $h = 3$. The corresponding dimensions are $9 \times 18 \times 3$.

46. Vascular Branching A small blood vessel of radius r branches off at an angle θ from a larger vessel of radius R to supply blood along a path from A to B. According to Poiseuille's Law, the total resistance to blood flow is proportional to

$$T = \left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right)$$

where a and b are as in Figure 25. Show that the total resistance is minimized when $\cos\theta = (r/R)^4$.

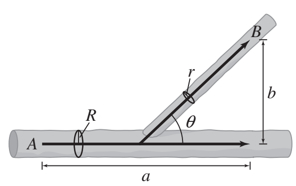


FIGURE 25

SOLUTION With $a, b, r, R > 0$ and $R > r$, let $T(\theta) = \left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right)$. Set

$$T'(\theta) = \left(\frac{b\csc^2\theta}{R^4} - \frac{b\csc\theta\cot\theta}{r^4}\right) = 0.$$

Then

$$\frac{b\left(r^4 - R^4\cos\theta\right)}{R^4 r^4 \sin^2\theta} = 0,$$

so that $\cos\theta = \left(\frac{r}{R}\right)^4$. Since $\lim_{\theta\to 0+} T(\theta) = \infty$ and $\lim_{\theta\to\pi-} T(\theta) = \infty$, the minimum value of $T(\theta)$ occurs when $\cos\theta = \left(\frac{r}{R}\right)^4$.

47. Which values of A and B maximize the volume of the box if $h = 10$ cm and $AB = 900$ cm.

SOLUTION With $h = 10$ and $AB = 900$ (which means that $B = 900/A$), the volume of the box is

$$V(A) = 10(A - 20)\left(\frac{900}{A} - 20\right) = 13{,}000 - 200A - \frac{180{,}000}{A},$$

where $20 \le A \le 45$. Now, solving

$$V'(A) = -200 + \frac{180{,}000}{A^2} = 0$$

yields $A = 30$. Because $V(20) = V(45) = 0$ and $V(30) = 1000\text{ cm}^3$, maximum volume is achieved with $A = B = 30$ cm.

48. Given n numbers $x_1, \ldots, x_n$, find the value of x minimizing the sum of the squares:

$$(x - x_1)^2 + (x - x_2)^2 + \cdots + (x - x_n)^2$$

First solve for $n = 2, 3$ and then try it for arbitrary n.

SOLUTION Note that the sum of squares approaches ∞ as $x \to \pm\infty$, so the minimum must occur at a critical point.

- For $n = 2$: Let $f(x) = (x - x_1)^2 + (x - x_2)^2$. Then setting $f'(x) = 2(x - x_1) + 2(x - x_2) = 0$ yields $x = \frac{1}{2}(x_1 + x_2)$.
- For $n = 3$: Let $f(x) = (x - x_1)^2 + (x - x_2)^2 + (x - x_3)^2$, so that setting $f'(x) = 2(x - x_1) + 2(x - x_2) + 2(x - x_3) = 0$ yields $x = \frac{1}{3}(x_1 + x_2 + x_3)$.
- Let $f(x) = \sum_{k=1}^{n}(x - x_k)^2$. Solve $f'(x) = 2\sum_{k=1}^{n}(x - x_k) = 0$ to obtain $x = \bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k$.

Note that the optimum value for x is the average of $x_1, \ldots, x_n$.

49. A billboard of height b is mounted on the side of a building with its bottom edge at a distance h from the street as in Figure 27. At what distance x should an observer stand from the wall to maximize the angle of observation θ?

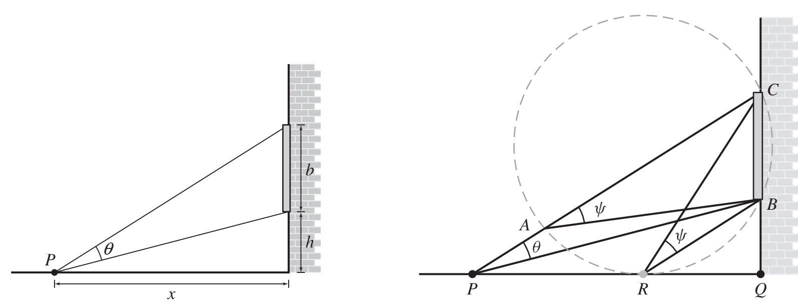


FIGURE 27

SOLUTION From the upper diagram in Figure 27 and the addition formula for the cotangent function, we see that

$$\cot\theta = \frac{1 + \frac{x}{b+h}\frac{x}{h}}{\frac{x}{h} - \frac{x}{b+h}} = \frac{x^2 + h(b+h)}{bx},$$

where b and h are constant. Now, differentiate with respect to x and solve

$$-\csc^2\theta\frac{d\theta}{dx} = \frac{x^2 - h(b+h)}{bx^2} = 0$$

to obtain $x = \sqrt{bh + h^2}$. Since this is the only critical point, and since $\theta \to 0$ as $x \to 0+$ and $\theta \to 0$ as $x \to \infty$, $\theta(x)$ reaches its maximum at $x = \sqrt{bh + h^2}$.

50. Solve Exercise 49 again using geometry rather than calculus. There is a unique circle passing through points B and C which is tangent to the street. Let R be the point of tangency. Note that the two angles labeled ψ in Figure 27 are equal because they subtend equal arcs on the circle.

(a) Show that the maximum value of θ is $\theta = \psi$. *Hint:* Show that $\psi = \theta + \angle PBA$ where A is the intersection of the circle with PC.

(b) Prove that this agrees with the answer to Exercise 49.

(c) Show that $\angle QRB = \angle RCQ$ for the maximal angle ψ.

SOLUTION

(a) We note that $\angle PAB$ is supplementary to both ψ and $\theta + \angle PBA$; hence, $\psi = \theta + \angle PBA$. From here, it is clear that θ is at a maximum when $\angle PBA = 0$; that is, when A coincides with P. This occurs when $P = R$.

(b) To show that the two answers agree, let O be the center of the circle. One observes that if d is the distance from R to the wall, then O has coordinates $(-d, \frac{b}{2} + h)$. This is because the height of the center is equidistant from points B and C and because the center must lie directly above R if the circle is tangent to the floor.

Now we can solve for d. The radius of the circle is clearly $\frac{b}{2} + h$, by the distance formula:

$$\overline{OB}^2 = d^2 + \left(\frac{b}{2} + h - h\right)^2 = \left(\frac{b}{2} + h\right)^2$$

This gives

$$d^2 = \left(\frac{b}{2} + h\right)^2 - \left(\frac{b}{2}\right)^2 = bh + h^2$$

or $d = \sqrt{bh + h^2}$ as claimed.

(c) Observe that the arc RB on the dashed circle is subtended by $\angle QRB$ and also by $\angle RCQ$. Thus, both are equal to one-half the angular measure of the arc.

51. Optimal Delivery Schedule A gas station sells Q gallons of gasoline per year, which is delivered N times per year in equal shipments of Q/N gallons. The cost of each delivery is d dollars and the yearly storage costs are sQT, where T is the length of time (a fraction of a year) between shipments and s is a constant. Show that costs are minimized for $N = \sqrt{sQ/d}$. (*Hint:* $T = 1/N$.) Find the optimal number of deliveries if $Q = 2$ million gal, $d = \$8000$, and $s = 30$ cents/gal-yr. Your answer should be a whole number, so compare costs for the two integer values of N nearest the optimal value.

SOLUTION There are N shipments per year, so the time interval between shipments is $T = 1/N$ years. Hence, the total storage costs per year are sQ/N. The yearly delivery costs are dN and the total costs is $C(N) = dN + sQ/N$. Solving,

$$C'(N) = d - \frac{sQ}{N^2} = 0$$

for N yields $N = \sqrt{sQ/d}$. For the specific case $Q = 2{,}000{,}000$, $d = 8000$ and $s = 0.30$,

$$N = \sqrt{\frac{0.30(2{,}000{,}000)}{8000}} = 8.66.$$

With $C(8) = \$139{,}000$ and $C(9) = \$138{,}667$, the optimal number of deliveries per year is $N = 9$.

52. Victor Klee's Endpoint Maximum Problem Given 40 meters of straight fence, your goal is to build a rectangular enclosure using 80 additional meters of fence that encompasses the greatest area. Let $A(x)$ be the area of the enclosure, with x as in Figure 28.

(a) Find the maximum value of $A(x)$.

(b) Which interval of x values is relevant to our problem? Find the maximum value of $A(x)$ on this interval.

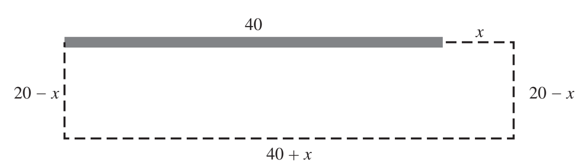


FIGURE 28

SOLUTION

(a) From the diagram, $A(x) = (40 + x)(20 - x) = 800 - 20x - x^2 = 900 - (x + 10)^2$. Thus, the maximum value of $A(x)$ is 900 square meters, occurring when $x = -10$.

(b) For our problem, $x \in [0, 20]$. On this interval, $A(x)$ has no critical points and $A(0) = 800$, while $A(20) = 0$. Thus, on the relevant interval, the maximum enclosed area is 800 square meters.

53. Let (a, b) be a fixed point in the first quadrant and let $S(d)$ be the sum of the distances from $(d, 0)$ to the points $(0, 0)$, (a, b), and $(a, -b)$.

(a) Find the value of d for which $S(d)$ is minimal. The answer depends on whether $b < \sqrt{3}a$ or $b \geq \sqrt{3}a$. *Hint:* Show that $d = 0$ when $b \geq \sqrt{3}a$.

(b) GU Let $a = 1$. Plot $S(d)$ for $b = 0.5$, $\sqrt{3}$, 3 and describe the position of the minimum.

SOLUTION

(a) If $d < 0$, then the distance from $(d, 0)$ to the other three points can all be reduced by increasing the value of d. Similarly, if $d > a$, then the distance from $(d, 0)$ to the other three points can all be reduced by decreasing the value of d. It follows that the minimum of $S(d)$ must occur for $0 \leq d \leq a$. Restricting attention to this interval, we find

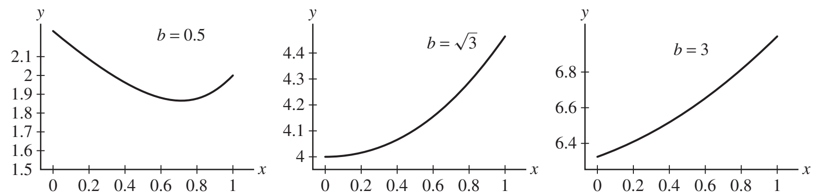
$$S(d) = d + 2\sqrt{(d - a)^2 + b^2}.$$

Solving

$$S'(d) = 1 + \frac{2(d-a)}{\sqrt{(d-a)^2 + b^2}} = 0$$

yields the critical point $d = a - b/\sqrt{3}$. If $b < \sqrt{3}a$, then $d = a - b/\sqrt{3} > 0$ and the minimum occurs at this value of d. On the other hand, if $b \geq \sqrt{3}a$, then the minimum occurs at the endpoint $d = 0$.

(b) Let $a = 1$. Plots of $S(d)$ for $b = 0.5$, $b = \sqrt{3}$ and $b = 3$ are shown below. For $b = 0.5$, the results of (a) indicate the minimum should occur for $d = 1 - 0.5/\sqrt{3} \approx 0.711$, and this is confirmed in the plot. For both $b = \sqrt{3}$ and $b = 3$, the results of (a) indicate that the minimum should occur at $d = 0$, and both of these conclusions are confirmed in the plots.



54. The force F (in Newtons) required to move a box of mass m kg in motion by pulling on an attached rope (Figure 29) is

$$F(\theta) = \frac{fmg}{\cos\theta + f\sin\theta}$$

where θ is the angle between the rope and the horizontal, f is the coefficient of static friction, and $g = 9.8$ m/s^2. Find the angle θ that minimizes the required force F, assuming $f = 0.4$. *Hint:* Find the maximum value of $\cos\theta + f\sin\theta$.

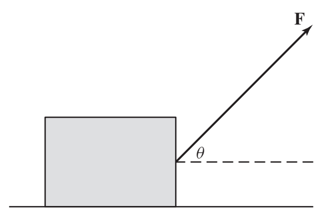


FIGURE 29

SOLUTION Let $F(\alpha) = \dfrac{3.92m}{\sin\alpha + \frac{2}{5}\cos\alpha}$, where $0 \leq \alpha \leq \frac{\pi}{2}$. Solve

$$F'(\alpha) = \frac{3.92m\left(\frac{2}{5}\sin\alpha - \cos\alpha\right)}{\left(\sin\alpha + \frac{2}{5}\cos\alpha\right)^2} = 0$$

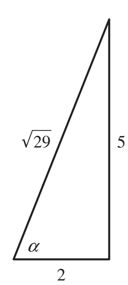
for $0 \leq \alpha \leq \frac{\pi}{2}$ to obtain $\tan\alpha = \frac{5}{2}$. From the diagram below, we note that when $\tan\alpha = \frac{5}{2}$,

$$\sin\alpha = \frac{5}{\sqrt{29}} \qquad \text{and} \qquad \cos\alpha = \frac{2}{\sqrt{29}}.$$

Therefore, at the critical point the force is

$$\frac{3.92m}{\frac{5}{\sqrt{29}} + \frac{2}{5}\frac{2}{\sqrt{29}}} = \frac{5\sqrt{29}}{29}(3.92m) \approx 3.64m.$$

Since $F(0) = \frac{5}{2}(3.92m) = 9.8m$ and $F\left(\frac{\pi}{2}\right) = 3.92m$, we conclude that the minimum force occurs when $\tan\alpha = \frac{5}{2}$.



55. In the setting of Exercise 54, show that for any f the minimal force required is proportional to $1/\sqrt{1+f^2}$.

SOLUTION We minimize $F(\theta)$ by finding the maximum value $g(\theta) = \cos\theta + f\sin\theta$. The angle θ is restricted to the interval $[0, \frac{\pi}{2}]$. We solve for the critical points:

$$g'(\theta) = -\sin\theta + f\cos\theta = 0$$

We obtain

$$f\cos\theta = \sin\theta \Rightarrow \tan\theta = f$$

From the figure below we find that $\cos\theta = 1/\sqrt{1+f^2}$ and $\sin\theta = f/\sqrt{1+f^2}$. Hence

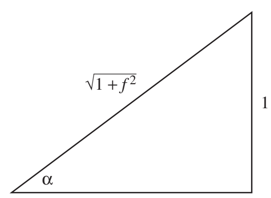
$$g(\theta) = \frac{1}{f} + \frac{f^2}{\sqrt{1+f^2}} = \frac{1+f^2}{\sqrt{1+f^2}} = \sqrt{1+f^2}$$

The values at the endpoints are

$$g(0) = 1, \qquad g\left(\frac{\pi}{2}\right) = f$$

Both of these values are less than $\sqrt{1+f^2}$. Therefore the maximum value of $g(\theta)$ is $\sqrt{1+f^2}$ and the minimum value of $F(\theta)$ is

$$F = \frac{fmg}{g(\theta)} = \frac{fmg}{\sqrt{1+f^2}}$$



56. Bird Migration Ornithologists have found that the power (in joules per second) consumed by a certain pigeon flying at velocity v m/s is described well by the function $P(v) = 17v^{-1} + 10^{-3}v^3$ J/s. Assume that the pigeon can store 5×10^4 J of usable energy as body fat.

(a) Show that at velocity v, a pigeon can fly a total distance of $D(v) = (5 \times 10^4)v/P(v)$ if it uses all of its stored energy.

(b) Find the velocity v_p that *minimizes* $P(v)$.

(c) Migrating birds are smart enough to fly at the velocity that maximizes distance traveled rather than minimizes power consumption. Show that the velocity v_d which maximizes $D(v)$ satisfies $P'(v_d) = P(v_d)/v_d$. Show that v_d is obtained graphically as the velocity coordinate of the point where a line through the origin is tangent to the graph of $P(v)$ (Figure 30).

(d) Find v_d and the maximum distance $D(v_d)$.

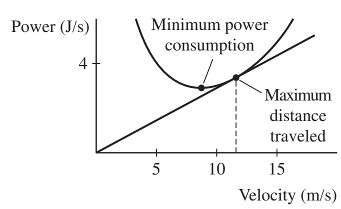


FIGURE 30

SOLUTION

(a) Flying at a velocity v, the birds will exhaust their energy store after $T = \dfrac{5 \cdot 10^4 \text{ joules}}{P(v) \text{ joules/sec}} = \dfrac{5 \cdot 10^4 \text{ sec}}{P(v)}$. The total distance traveled is then $D(v) = vT = \dfrac{5 \cdot 10^4 v}{P(v)}$.

(b) Let $P(v) = 17v^{-1} + 10^{-3}v^3$. Then $P'(v) = -17v^{-2} + 0.003v^2 = 0$ implies $v_p = \left(\frac{17}{0.003}\right)^{1/4} \approx 8.676247$. This critical point is a minimum, because it is the only critical point and $P(v) \to \infty$ both as $v \to 0+$ and as $v \to \infty$.

(c) $D'(v) = \dfrac{P(v) \cdot 5 \cdot 10^4 - 5 \cdot 10^4 v \cdot P'(v)}{(P(v))^2} = 5 \cdot 10^4 \dfrac{P(v) - vP'(v)}{(P(v))^2} = 0$ implies $P(v) - vP'(v) = 0$, or $P'(v) = \dfrac{P(v)}{v}$. Since $D(v) \to 0$ as $v \to 0$ and as $v \to \infty$, the critical point determined by $P'(v) = P(v)/v$ corresponds to a maximum.

Graphically, the expression

$$\frac{P(v)}{v} = \frac{P(v) - 0}{v - 0}$$

is the slope of the line passing through the origin and $(v, P(v))$. The condition $P'(v) = P(v)/v$ which defines v_d therefore indicates that v_d is the velocity component of the point where a line through the origin is tangent to the graph of $P(v)$.

(d) Using $P'(v) = \dfrac{P(v)}{v}$ gives

$$-17v^{-2} + 0.003v^2 = \frac{17v^{-1} + 0.001v^3}{v} = 17v^{-2} + 0.001v^2,$$

which simplifies to $0.002v^4 = 34$ and thus $v_d \approx 11.418583$. The maximum total distance is given by $D(v_d) = \dfrac{5 \cdot 10^4 \cdot v_d}{P(v_d)} = 191.741$ kilometers.

57. The problem is to put a "roof" of side s on an attic room of height h and width b. Find the smallest length s for which this is possible if $b = 27$ and $h = 8$ (Figure 31).

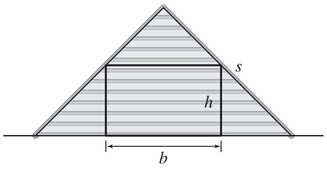
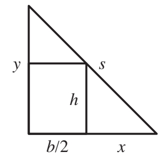


FIGURE 31

SOLUTION Consider the right triangle formed by the right half of the rectangle and its "roof". This triangle has hypotenuse s.



As shown, let y be the height of the roof, and let x be the distance from the right base of the rectangle to the base of the roof. By similar triangles applied to the smaller right triangles at the top and right of the larger triangle, we get:

$$\frac{y - 8}{27/2} = \frac{8}{x} \qquad \text{or} \qquad y = \frac{108}{x} + 8.$$

s, y, and x are related by the Pythagorean Theorem:

$$s^2 = \left(\frac{27}{2} + x\right)^2 + y^2 = \left(\frac{27}{2} + x\right)^2 + \left(\frac{108}{x} + 8\right)^2.$$

Since $s > 0$, s^2 is least whenever s is least, so we can minimize s^2 instead of s. Setting the derivative equal to zero yields

$$2\left(\frac{27}{2} + x\right) + 2\left(\frac{108}{x} + 8\right)\left(-\frac{108}{x^2}\right) = 0$$

$$2\left(\frac{27}{2} + x\right) + 2\frac{8}{x}\left(\frac{27}{2} + x\right)\left(-\frac{108}{x^2}\right) = 0$$

$$2\left(\frac{27}{2} + x\right)\left(1 - \frac{864}{x^3}\right) = 0$$

The zeros are $x = -\frac{27}{2}$ (irrelevant) and $x = 6\sqrt[3]{4}$. Since this is the only critical point of s with $x > 0$, and since $s \to \infty$ as $x \to 0$ and $s \to \infty$ as $x \to \infty$, this is the point where s attains its minimum. For this value of x,

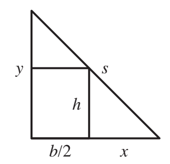
$$s^2 = \left(\frac{27}{2} + 6\sqrt[3]{4}\right)^2 + \left(9\sqrt[3]{2} + 8\right)^2 \approx 904.13,$$

so the smallest roof length is

$$s \approx 30.07.$$

58. Redo Exercise 57 for arbitrary b and h.

SOLUTION Consider the right triangle formed by the right half of the rectangle and its "roof". This triangle has hypotenuse s.



As shown, let y be the height of the roof, and let x be the distance from the right base of the rectangle to the base of the roof. By similar triangles applied to the smaller right triangles at the top and right of the larger triangle, we get:

$$\frac{y-h}{b/2} = \frac{h}{x} \qquad \text{or} \qquad y = \frac{bh}{2x} + h.$$

s, y, and x are related by the Pythagorean Theorem:

$$s^2 = \left(\frac{b}{2}+x\right)^2 + y^2 = \left(\frac{b}{2}+x\right)^2 + \left(\frac{bh}{2x}+h\right)^2.$$

Since $s > 0$, s^2 is least whenever s is least, so we can minimize s^2 instead of s. Setting the derivative equal to zero yields

$$2\left(\frac{b}{2}+x\right) + 2\left(\frac{bh}{2x}+h\right)\left(-\frac{bh}{2x^2}\right) = 0$$
$$2\left(\frac{b}{2}+x\right) + 2\frac{h}{x}\left(\frac{b}{2}+x\right)\left(-\frac{bh}{2x^2}\right) = 0$$
$$2\left(\frac{b}{2}+x\right)\left(1-\frac{bh^2}{2x^3}\right) = 0$$

The zeros are $x = -\frac{b}{2}$ (irrelevant) and

$$x = \frac{b^{1/3}h^{2/3}}{2^{1/3}}.$$

Since this is the only critical point of s with $x > 0$, and since $s \to \infty$ as $x \to 0$ and $s \to \infty$ as $x \to \infty$, this is the point where s attains its minimum. For this value of x,

$$s^2 = \left(\frac{b}{2} + \frac{b^{1/3}h^{2/3}}{2^{1/3}}\right)^2 + \left(\frac{b^{2/3}h^{1/3}}{2^{2/3}} + h\right)^2$$
$$= \frac{b^{2/3}}{2^{2/3}}\left(\frac{b^{2/3}}{2^{2/3}} + h^{2/3}\right)^2 + h^{2/3}\left(\frac{b^{2/3}}{2^{2/3}} + h^{2/3}\right)^2 = \left(\frac{b^{2/3}}{2^{2/3}} + h^{2/3}\right)^3,$$

so the smallest roof length is

$$s = \left(\frac{b^{2/3}}{2^{2/3}} + h^{2/3}\right)^{3/2}.$$

59. Find the maximum length of a pole that can be carried horizontally around a corner joining corridors of widths $a = 24$ and $b = 3$ (Figure 32).

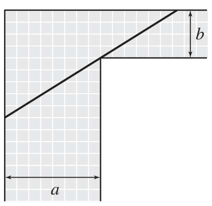


FIGURE 32

SOLUTION In order to find the length of the *longest* pole that can be carried around the corridor, we have to find the *shortest* length from the left wall to the top wall touching the corner of the inside wall. Any pole that does not fit in this shortest space cannot be carried around the corner, so an exact fit represents the longest possible pole.

Let θ be the angle between the pole and a horizontal line to the right. Let c_1 be the length of pole in the corridor of width 24 and let c_2 be the length of pole in the corridor of width 3. By the definitions of sine and cosine,

$$\frac{3}{c_2} = \sin\theta \quad \text{and} \quad \frac{24}{c_1} = \cos\theta,$$

so that $c_1 = \frac{24}{\cos\theta}$, $c_2 = \frac{3}{\sin\theta}$. What must be minimized is the total length, given by

$$f(\theta) = \frac{24}{\cos\theta} + \frac{3}{\sin\theta}.$$

Setting $f'(\theta) = 0$ yields

$$\frac{24\sin\theta}{\cos^2\theta} - \frac{3\cos\theta}{\sin^2\theta} = 0$$
$$\frac{24\sin\theta}{\cos^2\theta} = \frac{3\cos\theta}{\sin^2\theta}$$
$$24\sin^3\theta = 3\cos^3\theta$$

As $\theta < \frac{\pi}{2}$ (the pole is being turned around a corner, after all), we can divide both sides by $\cos^3\theta$, getting $\tan^3\theta = \frac{1}{8}$. This implies that $\tan\theta = \frac{1}{2}$ ($\tan\theta > 0$ as the angle is acute).

Since $f(\theta) \to \infty$ as $\theta \to 0+$ and as $\theta \to \frac{\pi}{2}-$, we can tell that the *minimum* is attained at θ_0 where $\tan\theta_0 = \frac{1}{2}$. Because

$$\tan\theta_0 = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{2},$$

we draw a triangle with opposite side 1 and adjacent side 2. By Pythagoras, $c = \sqrt{5}$, so

$$\sin\theta_0 = \frac{1}{\sqrt{5}} \quad \text{and} \quad \cos\theta_0 = \frac{2}{\sqrt{5}}.$$

From this, we get

$$f(\theta_0) = \frac{24}{\cos\theta_0} + \frac{3}{\sin\theta_0} = \frac{24}{2}\sqrt{5} + 3\sqrt{5} = 15\sqrt{5}.$$

60. Redo Exercise 59 for arbitrary widths a and b.

SOLUTION If the corridors have widths a and b, and if θ is the angle between the beam and the line perpendicular to the corridor of width a, then we have to *minimize*

$$f(\theta) = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}.$$

Setting the derivative equal to zero,

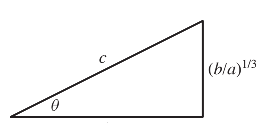
$$a\sec\theta\tan\theta - b\cot\theta\csc\theta = 0,$$

we obtain the critical value θ_0 defined by

$$\tan\theta_0 = \left(\frac{b}{a}\right)^{1/3}$$

and from this we conclude (witness the diagram below) that

$$\cos\theta_0 = \frac{1}{\sqrt{1 + (b/a)^{2/3}}} \quad \text{and} \quad \sin\theta_0 = \frac{(b/a)^{1/3}}{\sqrt{1 + (b/a)^{2/3}}}.$$



This gives the minimum value as

$$\begin{aligned} f(\theta_0) &= a\sqrt{1+(b/a)^{2/3}} + b(b/a)^{-1/3}\sqrt{1+(b/a)^{2/3}} \\ &= a^{2/3}\sqrt{a^{2/3}+b^{2/3}} + b^{2/3}\sqrt{a^{2/3}+b^{2/3}} \\ &= (a^{2/3}+b^{2/3})^{3/2} \end{aligned}$$

61. Find the minimum length ℓ of a beam that can clear a fence of height h and touch a wall located b ft behind the fence (Figure 33).

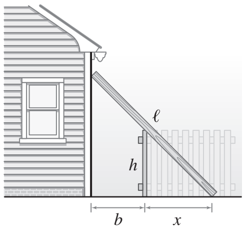


FIGURE 33

SOLUTION Let y be the height of the point where the beam touches the wall in feet. By similar triangles,

$$\frac{y-h}{b} = \frac{h}{x} \quad \text{or} \quad y = \frac{bh}{x} + h$$

and by Pythagoras:

$$\ell^2 = (b+x)^2 + \left(\frac{bh}{x} + h\right)^2.$$

We can minimize ℓ^2 rather than ℓ, so setting the derivative equal to zero gives:

$$2(b+x) + 2\left(\frac{bh}{x} + h\right)\left(-\frac{bh}{x^2}\right) = 2(b+x)\left(1 - \frac{h^2 b}{x^3}\right) = 0.$$

The zeroes are $b = -x$ (irrelevant) and $x = \sqrt[3]{h^2 b}$. Since $\ell^2 \to \infty$ as $x \to 0+$ and as $x \to \infty$, $x = \sqrt[3]{h^2 b}$ corresponds to a minimum for ℓ^2. For this value of x, we have

$$\begin{aligned} \ell^2 &= (b + h^{2/3}b^{1/3})^2 + (h + h^{1/3}b^{2/3})^2 \\ &= b^{2/3}(b^{2/3}+h^{2/3})^2 + h^{2/3}(h^{2/3}+b^{2/3})^2 \\ &= (b^{2/3}+h^{2/3})^3 \end{aligned}$$

and so

$$\ell = (b^{2/3}+h^{2/3})^{3/2}.$$

A beam that clears a fence of height h feet and touches a wall b feet behind the fence must have length at least $\ell = (b^{2/3}+h^{2/3})^{3/2}$ ft.

62. Which value of h maximizes the volume of the box if $A = B$?

SOLUTION When $A = B$, the volume of the box is

$$V(h) = hxy = h(A-2h)^2 = 4h^3 - 4Ah^2 + A^2 h,$$

where $0 \le h \le \frac{A}{2}$ (allowing for degenerate boxes). Solve $V'(h) = 12h^2 - 8Ah + A^2 = 0$ for h to obtain $h = \frac{A}{2}$ or $h = \frac{A}{6}$. Because $V(0) = V(\frac{A}{2}) = 0$ and $V(\frac{A}{6}) = \frac{2}{27}A^3$, volume is maximized when $h = \frac{A}{6}$.

63. A basketball player stands d feet from the basket. Let h and α be as in Figure 34. Using physics, one can show that if the player releases the ball at an angle θ, then the initial velocity required to make the ball go through the basket satisfies

$$v^2 = \frac{16d}{\cos^2\theta(\tan\theta - \tan\alpha)}$$

(a) Explain why this formula is meaningful only for $\alpha < \theta < \frac{\pi}{2}$. Why does v approach infinity at the endpoints of this interval?

(b) GU Take $\alpha = \frac{\pi}{6}$ and plot v^2 as a function of θ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$. Verify that the minimum occurs at $\theta = \frac{\pi}{3}$.

(c) Set $F(\theta) = \cos^2\theta(\tan\theta - \tan\alpha)$. Explain why v is minimized for θ such that $F(\theta)$ is maximized.

(d) Verify that $F'(\theta) = \cos(\alpha - 2\theta)\sec\alpha$ (you will need to use the addition formula for cosine) and show that the maximum value of $F(\theta)$ on $\left[\alpha, \frac{\pi}{2}\right]$ occurs at $\theta_0 = \frac{\alpha}{2} + \frac{\pi}{4}$.

(e) For a given α, the optimal angle for shooting the basket is θ_0 because it minimizes v^2 and therefore minimizes the energy required to make the shot (energy is proportional to v^2). Show that the velocity v_{opt} at the optimal angle θ_0 satisfies

$$v_{\text{opt}}^2 = \frac{32d\cos\alpha}{1-\sin\alpha} = \frac{32\,d^2}{-h+\sqrt{d^2+h^2}}$$

(f) GU Show with a graph that for fixed d (say, $d = 15$ ft, the distance of a free throw), v_{opt}^2 is an increasing function of h. Use this to explain why taller players have an advantage and why it can help to jump while shooting.

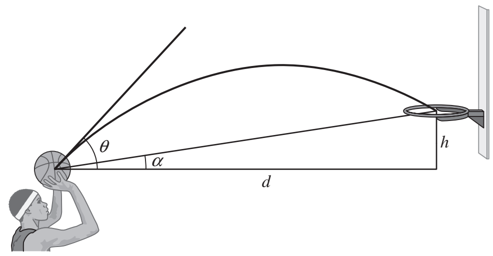


FIGURE 34

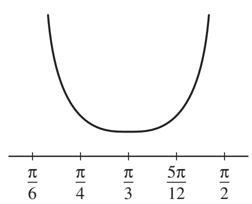
SOLUTION

(a) $\alpha = 0$ corresponds to shooting the ball directly at the basket while $\alpha = \pi/2$ corresponds to shooting the ball directly upward. In neither case is it possible for the ball to go into the basket.

If the angle α is extremely close to 0, the ball is shot almost directly at the basket, so that it must be launched with great speed, as it can only fall an extremely short distance on the way to the basket.

On the other hand, if the angle α is extremely close to $\pi/2$, the ball is launched almost vertically. This requires the ball to travel a great distance upward in order to travel the horizontal distance. In either one of these cases, the ball has to travel at an enormous speed.

(b)



The minimum clearly occurs where $\theta = \pi/3$.

(c) If $F(\theta) = \cos^2\theta\,(\tan\theta - \tan\alpha)$,

$$v^2 = \frac{16d}{\cos^2\theta\,(\tan\theta - \tan\alpha)} = \frac{16d}{F(\theta)}.$$

Since $\alpha \le \theta$, $F(\theta) > 0$, hence v^2 is smallest whenever $F(\theta)$ is greatest.

(d) $F'(\theta) = -2\sin\theta\cos\theta\,(\tan\theta - \tan\alpha) + \cos^2\theta\left(\sec^2\theta\right) = -2\sin\theta\cos\theta\tan\theta + 2\sin\theta\cos\theta\tan\alpha + 1$. We will apply all the double angle formulas:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta;\ \sin 2\theta = 2\sin\theta\cos\theta,$$

getting:

$$\begin{aligned} F'(\theta) &= 2\sin\theta\cos\theta\tan\alpha - 2\sin\theta\cos\theta\tan\theta + 1 \\ &= 2\sin\theta\cos\theta\frac{\sin\alpha}{\cos\alpha} - 2\sin\theta\cos\theta\frac{\sin\theta}{\cos\theta} + 1 \\ &= \sec\alpha\left(-2\sin^2\theta\cos\alpha + 2\sin\theta\cos\theta\sin\alpha + \cos\alpha\right) \\ &= \sec\alpha\left(\cos\alpha\left(1 - 2\sin^2\theta\right) + \sin\alpha\,(2\sin\theta\cos\theta)\right) \\ &= \sec\alpha\,(\cos\alpha(\cos 2\theta) + \sin\alpha(\sin 2\theta)) \\ &= \sec\alpha\cos(\alpha - 2\theta) \end{aligned}$$

A critical point of $F(\theta)$ occurs where $\cos(\alpha - 2\theta) = 0$, so that $\alpha - 2\theta = -\frac{\pi}{2}$ (negative because $2\theta > \theta > \alpha$), and this gives us $\theta = \alpha/2 + \pi/4$. The minimum value $F(\theta_0)$ takes place at $\theta_0 = \alpha/2 + \pi/4$.

(e) Plug in $\theta_0 = \alpha/2 + \pi/4$. To find v_{opt}^2 we must simplify

$$\cos^2\theta_0(\tan\theta_0 - \tan\alpha) = \frac{\cos\theta_0(\sin\theta_0\cos\alpha - \cos\theta_0\sin\alpha)}{\cos\alpha}$$

By the addition law for sine:

$$\sin\theta_0\cos\alpha - \cos\theta_0\sin\alpha = \sin(\theta_0 - \alpha) = \sin(-\alpha/2 + \pi/4)$$

and so

$$\cos\theta_0(\sin\theta_0\cos\alpha - \cos\theta_0\sin\alpha) = \cos(\alpha/2 + \pi/4)\sin(-\alpha/2 + \pi/4)$$

Now use the identity (that follows from the addition law):

$$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$

to get

$$\cos(\alpha/2 + \pi/4)\sin(-\alpha/2 + \pi/4) = (1/2)(1 - \sin\alpha)$$

So we finally get

$$\cos^2\theta_0(\tan\theta_0 - \tan\alpha) = \frac{(1/2)(1 - \sin\alpha)}{\cos\alpha}$$

and therefore

$$v_{\text{opt}}^2 = \frac{32d\cos\alpha}{1 - \sin\alpha}$$

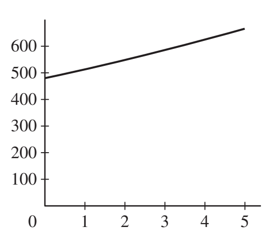
as claimed. From Figure 34 we see that

$$\cos\alpha = \frac{d}{\sqrt{d^2 + h^2}} \qquad \text{and} \qquad \sin\alpha = \frac{h}{\sqrt{d^2 + h^2}}.$$

Substituting these values into the expression for v_{opt}^2 yields

$$v_{\text{opt}}^2 = \frac{32d^2}{-h + \sqrt{d^2 + h^2}}.$$

(f) A sketch of the graph of v_{opt}^2 versus h for $d = 15$ feet is given below: v_{opt}^2 increases with respect to basket height relative to the shooter. This shows that the minimum velocity required to launch the ball to the basket drops as shooter height increases. This shows one of the ways height is an advantage in free throws; a taller shooter need not shoot the ball as hard to reach the basket.



64. Three towns A, B, and C are to be joined by an underground fiber cable as illustrated in Figure 35(A). Assume that C is located directly below the midpoint of $\overline{AB}$. Find the junction point P that minimizes the total amount of cable used.

(a) First show that P must lie directly above C. *Hint:* Use the result of Example 6 to show that if the junction is placed at point Q in Figure 35(B), then we can reduce the cable length by moving Q horizontally over to the point P lying above C.

(b) With x as in Figure 35(A), let $f(x)$ be the total length of cable used. Show that $f(x)$ has a unique critical point c. Compute c and show that $0 \le c \le L$ if and only if $D \le 2\sqrt{3}\,L$.

(c) Find the minimum of $f(x)$ on $[0, L]$ in two cases: $D = 2$, $L = 4$ and $D = 8$, $L = 2$.

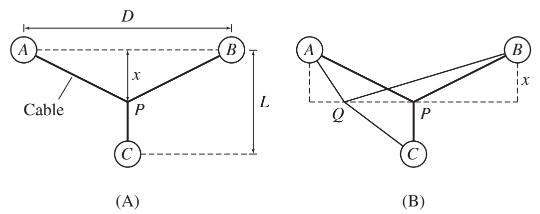


FIGURE 35

SOLUTION

(a) Look at diagram 35(B). Let T be the point directly above Q on $\overline{AB}$. Let $s = AT$ and $D = AB$ so that $TB = D - s$. Let ℓ be the total length of cable from A to Q and B to Q. By the Pythagorean Theorem applied to $\triangle AQT$ and $\triangle BQT$, we get:

$$\ell = \sqrt{s^2 + x^2} + \sqrt{(D - s)^2 + x^2}.$$

From here, it follows that

$$\frac{d\ell}{ds} = \frac{s}{\sqrt{s^2 + x^2}} - \frac{D - s}{\sqrt{(D - s)^2 + x^2}}.$$

Since s and $D - s$ must be non-negative, the only critical point occurs when $s = D/2$. As $\frac{d\ell}{ds}$ changes sign from negative to positive at $s = D/2$, it follows that ℓ is minimized when $s = D/2$, that is, when $Q = P$. Since it is obvious that $PC \le QC$ (QC is the hypotenuse of the triangle $\triangle PQC$), it follows that total cable length is minimized at $Q = P$.

(b) Let $f(x)$ be the total cable length. From diagram 35(A), we get:

$$f(x) = (L - x) + 2\sqrt{x^2 + D^2/4}.$$

Then

$$f'(x) = -1 + \frac{2x}{\sqrt{x^2 + D^2/4}} = 0$$

gives

$$2x = \sqrt{x^2 + D^2/4}$$

or

$$4x^2 = x^2 + D^2/4$$

and the critical point is

$$c = D/2\sqrt{3}.$$

This is the only critical point of f. It lies in the interval $[0, L]$ if and only if $c \le L$, or

$$D \le 2\sqrt{3}L.$$

(c) The minimum of f will depend on whether $D \le 2\sqrt{3}L$.

- $D = 2$, $L = 4$; $2\sqrt{3}L = 8\sqrt{3} > D$, so $c = D/(2\sqrt{3}) = \sqrt{3}/3 \in [0, L]$. $f(0) = L + D = 6$, $f(L) = 2\sqrt{L^2 + D^2/4} = 2\sqrt{17} \approx 8.24621$, and $f(c) = 4 - (\sqrt{3}/3) + 2\sqrt{\frac{1}{3} + 1} = 4 + \sqrt{3} \approx 5.73204$. Therefore, the total length is minimized where $x = c = \sqrt{3}/3$.
- $D = 8$, $L = 2$; $2\sqrt{3}L = 4\sqrt{3} < D$, so c does not lie in the interval $[0, L]$. $f(0) = 2 + 2\sqrt{64/4} = 10$, and $f(L) = 0 + 2\sqrt{4 + 64/4} = 2\sqrt{20} = 4\sqrt{5} \approx 8.94427$. Therefore, the total length is minimized where $x = L$, or where $P = C$.

Further Insights and Challenges

65. Tom and Ali drive along a highway represented by the graph of $f(x)$ in Figure 36. During the trip, Ali views a billboard represented by the segment $\overline{BC}$ along the y-axis. Let Q be the y-intercept of the tangent line to $y = f(x)$. Show that θ is maximized at the value of x for which the angles $\angle QPB$ and $\angle QCP$ are equal. This generalizes Exercise 50 (c) (which corresponds to the case $f(x) = 0$). *Hints:*

(a) Show that $d\theta/dx$ is equal to

$$(b-c)\cdot\frac{(x^2+(xf'(x))^2)-(b-(f(x)-xf'(x)))(c-(f(x)-xf'(x)))}{(x^2+(b-f(x))^2)(x^2+(c-f(x))^2)}$$

(b) Show that the y-coordinate of Q is $f(x) - xf'(x)$.

(c) Show that the condition $d\theta/dx = 0$ is equivalent to

$$PQ^2 = BQ\cdot CQ$$

(d) Conclude that $\triangle QPB$ and $\triangle QCP$ are similar triangles.

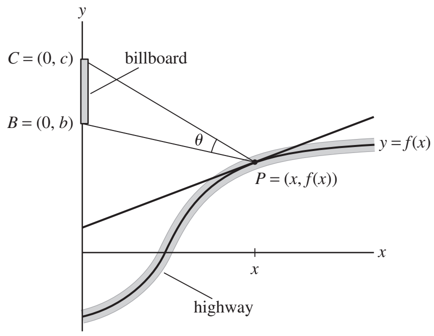


FIGURE 36

SOLUTION

(a) From the figure, we see that

$$\theta(x) = \tan^{-1}\frac{c-f(x)}{x} - \tan^{-1}\frac{b-f(x)}{x}.$$

Then

$$\begin{aligned}\theta'(x) &= \frac{b-(f(x)-xf'(x))}{x^2+(b-f(x))^2} - \frac{c-(f(x)-xf'(x))}{x^2+(c-f(x))^2}\\ &= (b-c)\frac{x^2-bc+(b+c)(f(x)-xf'(x))-(f(x))^2+2xf(x)f'(x)}{(x^2+(b-f(x))^2)(x^2+(c-f(x))^2)}\\ &= (b-c)\frac{(x^2+(xf'(x))^2-(bc-(b+c)(f(x)-xf'(x))+(f(x)-xf'(x))^2)}{(x^2+(b-f(x))^2)(x^2+(c-f(x))^2)}\\ &= (b-c)\frac{(x^2+(xf'(x))^2-(b-(f(x)-xf'(x)))(c-(f(x)-xf'(x)))}{(x^2+(b-f(x))^2)(x^2+(c-f(x))^2)}.\end{aligned}$$

(b) The point Q is the y-intercept of the line tangent to the graph of $f(x)$ at point P. The equation of this tangent line is

$$Y - f(x) = f'(x)(X-x).$$

The y-coordinate of Q is then $f(x) - xf'(x)$.

(c) From the figure, we see that

$$BQ = b-(f(x)-xf'(x)),$$
$$CQ = c-(f(x)-xf'(x))$$

and

$$PQ = \sqrt{x^2+(f(x)-(f(x)-xf'(x)))^2} = \sqrt{x^2+(xf'(x))^2}.$$

Comparing these expressions with the numerator of $d\theta/dx$, it follows that $\frac{d\theta}{dx} = 0$ is equivalent to

$$PQ^2 = BQ\cdot CQ.$$

(d) The equation $PQ^2 = BQ \cdot CQ$ is equivalent to

$$\frac{PQ}{BQ} = \frac{CQ}{PQ}.$$

In other words, the sides CQ and PQ from the triangle ΔQCP are proportional in length to the sides PQ and BQ from the triangle ΔQPB. As $\angle PQB = \angle CQP$, it follows that triangles ΔQCP and ΔQPB are similar.

Seismic Prospecting *Exercises 66–68 are concerned with determining the thickness d of a layer of soil that lies on top of a rock formation. Geologists send two sound pulses from point A to point D separated by a distance s. The first pulse travels directly from A to D along the surface of the earth. The second pulse travels down to the rock formation, then along its surface, and then back up to D (path ABCD), as in Figure 37. The pulse travels with velocity v_1 in the soil and v_2 in the rock.*

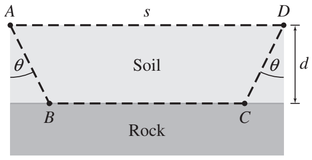


FIGURE 37

66. **(a)** Show that the time required for the first pulse to travel from A to D is $t_1 = s/v_1$.
(b) Show that the time required for the second pulse is

$$t_2 = \frac{2d}{v_1}\sec\theta + \frac{s - 2d\tan\theta}{v_2}$$

provided that

$$\tan\theta \le \frac{s}{2d} \qquad \boxed{2}$$

(*Note:* If this inequality is not satisfied, then point B does not lie to the left of C.)
(c) Show that t_2 is minimized when $\sin\theta = v_1/v_2$.

SOLUTION
(a) We have time $t_1 = \text{distance/velocity} = s/v_1$.
(b) Let p be the length of the base of the right triangle (opposite the angle θ) and h the length of the hypotenuse of this right triangle. Then $\cos\theta = \frac{d}{h}$ and $h = d\sec\theta$. Moreover, $\tan\theta = \frac{p}{d}$ gives $p = d\tan\theta$. Hence

$$t_2 = t_{AB} + t_{CD} + t_{BC} = \frac{h}{v_1} + \frac{h}{v_1} + \frac{s-2p}{v_2} = \frac{2d}{v_1}\sec\theta + \frac{s - 2d\tan\theta}{v_2}$$

(c) Solve $\frac{dt_2}{d\theta} = \frac{2d\sec\theta\tan\theta}{v_1} - \frac{2d\sec^2\theta}{v_2} = 0$ to obtain $\frac{\tan\theta}{v_1} = \frac{\sec\theta}{v_2}$. Therefore $\frac{\sin\theta/\cos\theta}{1/\cos\theta} = \frac{v_1}{v_2}$ or $\sin\theta = \frac{v_1}{v_2}$.

67. In this exercise, assume that $v_2/v_1 \ge \sqrt{1 + 4(d/s)^2}$.
(a) Show that inequality (2) holds if $\sin\theta = v_1/v_2$.
(b) Show that the minimal time for the second pulse is

$$t_2 = \frac{2d}{v_1}(1-k^2)^{1/2} + \frac{s}{v_2}$$

where $k = v_1/v_2$.
(c) Conclude that $\frac{t_2}{t_1} = \frac{2d(1-k^2)^{1/2}}{s} + k$.

SOLUTION
(a) If $\sin\theta = \frac{v_1}{v_2}$, then

$$\tan\theta = \frac{v_1}{\sqrt{v_2^2 - v_1^2}} = \frac{1}{\sqrt{\left(\frac{v_2}{v_1}\right)^2 - 1}}.$$

Because $\frac{v_2}{v_1} \ge \sqrt{1 + 4(\frac{d}{s})^2}$, it follows that

$$\sqrt{\left(\frac{v_2}{v_1}\right)^2 - 1} \ge \sqrt{1 + 4\left(\frac{d}{s}\right)^2 - 1} = \frac{2d}{s}.$$

Hence, $\tan\theta \le \frac{s}{2d}$ as required.

(b) For the time-minimizing choice of θ, we have $\sin\theta = \dfrac{v_1}{v_2}$ from which $\sec\theta = \dfrac{v_2}{\sqrt{v_2^2 - v_1^2}}$ and $\tan\theta = \dfrac{v_1}{\sqrt{v_2^2 - v_1^2}}$.

Thus

$$t_2 = \frac{2d}{v_1}\sec\theta + \frac{s - 2d\tan\theta}{v_2} = \frac{2d}{v_1}\,\frac{v_2}{\sqrt{v_2^2 - v_1^2}} + \frac{s - 2d\frac{v_1}{\sqrt{v_2^2-v_1^2}}}{v_2}$$

$$= \frac{2d}{v_1}\left(\frac{v_2}{\sqrt{v_2^2 - v_1^2}} - \frac{v_1^2}{v_2\sqrt{v_2^2 - v_1^2}}\right) + \frac{s}{v_2}$$

$$= \frac{2d}{v_1}\left(\frac{v_2^2 - v_1^2}{v_2\sqrt{v_2^2 - v_1^2}}\right) + \frac{s}{v_2} = \frac{2d}{v_1}\left(\frac{\sqrt{v_2^2 - v_1^2}}{\sqrt{v_2^2}}\right) + \frac{s}{v_2}$$

$$= \frac{2d}{v_1}\sqrt{1 - \left(\frac{v_1}{v_2}\right)^2} + \frac{s}{v_2} = \frac{2d\left(1 - k^2\right)^{1/2}}{v_1} + \frac{s}{v_2}.$$

(c) Recall that $t_1 = \dfrac{s}{v_1}$. We therefore have

$$\frac{t_2}{t_1} = \frac{\frac{2d\left(1-k^2\right)^{1/2}}{v_1} + \frac{s}{v_2}}{\frac{s}{v_1}}$$

$$= \frac{2d\left(1 - k^2\right)^{1/2}}{s} + \frac{v_1}{v_2} = \frac{2d\left(1 - k^2\right)^{1/2}}{s} + k.$$

68. Continue with the assumption of the previous exercise.
(a) Find the thickness of the soil layer, assuming that $v_1 = 0.7v_2$, $t_2/t_1 = 1.3$, and $s = 400$ m.
(b) The times t_1 and t_2 are measured experimentally. The equation in Exercise 67(c) shows that t_2/t_1 is a linear function of $1/s$. What might you conclude if experiments were formed for several values of s and the points $(1/s, t_2/t_1)$ did *not* lie on a straight line?

SOLUTION

(a) Substituting $k = v_1/v_2 = 0.7$, $t_2/t_1 = 1.3$, and $s = 400$ into the equation for t_2/t_1 in Exercise 67(c) gives $1.3 = \dfrac{2d\sqrt{1 - (0.7)^2}}{400} + 0.7$. Solving for d yields $d \approx 168.03$ m.

(b) If several experiments for different values of s showed that plots of the points $\left(\dfrac{1}{s}, \dfrac{t_2}{t_1}\right)$ did *not* lie on a straight line, then we would suspect that $\dfrac{t_2}{t_1}$ is *not* a linear function of $\dfrac{1}{s}$ and that a different model is required.

69. In this exercise we use Figure 38 to prove Heron's principle of Example 6 without calculus. By definition, C is the reflection of B across the line $\overline{MN}$ (so that $\overline{BC}$ is perpendicular to $\overline{MN}$ and $BN = CN$. Let P be the intersection of $\overline{AC}$ and $\overline{MN}$. Use geometry to justify:
(a) $\triangle PNB$ and $\triangle PNC$ are congruent and $\theta_1 = \theta_2$.
(b) The paths APB and APC have equal length.
(c) Similarly AQB and AQC have equal length.
(d) The path APC is shorter than AQC for all $Q \neq P$.
Conclude that the shortest path AQB occurs for $Q = P$.

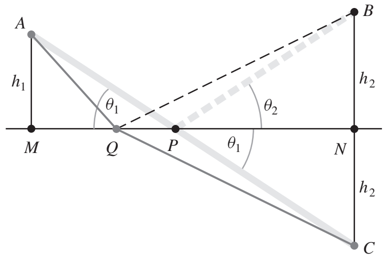


FIGURE 38

SOLUTION

(a) By definition, $\overline{BC}$ is orthogonal to $\overline{QM}$, so triangles $\triangle PNB$ and $\triangle PNC$ are congruent by side–angle–side. Therefore $\theta_1 = \theta_2$

(b) Because $\triangle PNB$ and $\triangle PNC$ are congruent, it follows that $\overline{PB}$ and $\overline{PC}$ are of equal length. Thus, paths APB and APC have equal length.

(c) The same reasoning used in parts (a) and (b) lead us to conclude that $\triangle QNB$ and $\triangle QNC$ are congruent and that $\overline{PB}$ and $\overline{PC}$ are of equal length. Thus, paths AQB and AQC are of equal length.

(d) Consider triangle $\triangle AQC$. By the triangle inequality, the length of side $\overline{AC}$ is less than or equal to the sum of the lengths of the sides $\overline{AQ}$ and $\overline{QC}$. Thus, the path APC is shorter than AQC for all $Q \neq P$.

Finally, the shortest path AQB occurs for $Q = P$.

70. A jewelry designer plans to incorporate a component made of gold in the shape of a frustum of a cone of height 1 cm and fixed lower radius r (Figure 39). The upper radius x can take on any value between 0 and r. Note that $x = 0$ and $x = r$ correspond to a cone and cylinder, respectively. As a function of x, the surface area (not including the top and bottom) is $S(x) = \pi s(r + x)$, where s is the *slant height* as indicated in the figure. Which value of x yields the least expensive design [the minimum value of $S(x)$ for $0 \le x \le r$]?

(a) Show that $S(x) = \pi(r + x)\sqrt{1 + (r - x)^2}$.

(b) Show that if $r < \sqrt{2}$, then $S(x)$ is an increasing function. Conclude that the cone ($x = 0$) has minimal area in this case.

(c) Assume that $r > \sqrt{2}$. Show that $S(x)$ has two critical points $x_1 < x_2$ in $(0, r)$, and that $S(x_1)$ is a local maximum, and $S(x_2)$ is a local minimum.

(d) Conclude that the minimum occurs at $x = 0$ or x_2.

(e) Find the minimum in the cases $r = 1.5$ and $r = 2$.

(f) Challenge: Let $c = \sqrt{(5 + 3\sqrt{3})/4} \approx 1.597$. Prove that the minimum occurs at $x = 0$ (cone) if $\sqrt{2} < r < c$, but the minimum occurs at $x = x_2$ if $r > c$.

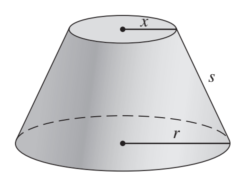


FIGURE 39 Frustrum of height 1 cm.

SOLUTION

(a) Consider a cross-section of the object and notice a triangle can be formed with height 1, hypotenuse s, and base $r - x$. Then, by the Pythagorean Theorem, $s = \sqrt{1 + (r - x)^2}$ and the surface area is $S = \pi(r + x)s = \pi(r + x)\sqrt{1 + (r - x)^2}$.

(b) $S'(x) = \pi\left(\sqrt{1 + (r - x)^2} - (r + x)(1 + (r - x)^2)^{-1/2}(r - x)\right) = \pi\dfrac{2x^2 - 2rx + 1}{\sqrt{1 + (r - x)^2}} = 0$ yields critical points $x = \frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 - 2}$. If $r < \sqrt{2}$ then there are no real critical points and $S'(x) > 0$ for $x > 0$. Hence, $S(x)$ is increasing everywhere and thus the minimum must occur at the left endpoint, $x = 0$.

(c) For $r > \sqrt{2}$, there are two critical points, $x_1 = \frac{1}{2}r - \frac{1}{2}\sqrt{r^2 - 2}$ and $x_2 = \frac{1}{2}r + \frac{1}{2}\sqrt{r^2 - 2}$. Both values are on the interval $[0, r]$ since $r > \sqrt{r^2 - 2}$. Sign analysis reveals that $S(x)$ is increasing for $0 < x < x_1$, decreasing for $x_1 < x < x_2$ and increasing for $x_2 < x < r$. Hence, $S(x_1)$ is a local maximum, and $S(x_2)$ is a local minimum.

(d) The minimum value of S must occur at an endpoint or a critical point. Since $S(x_1)$ is a local maximum and S increases for $x_2 < x < r$, we conclude that the minimum of S must occur either at $x = 0$ or at $x = x_2$.

(e) If $r = 1.5$ cm, $S(x_2) = 8.8357$ cm^2 and $S(0) = 8.4954$ cm^2, so $S(0) = 8.4954$ cm^2 is the minimum (cone). If $r = 2$ cm, $S(x_2) = 12.852$ cm^2 and $S(0) = 14.0496$ cm^2, so $S(x_2) = 12.852$ cm^2 is the minimum.

(f) *Take a deep breath*. Setting $S(x_2) = S(0)$ produces an equation in r (x_2 is given in r, and so is $S(0)$). By means of a great deal of algebraic labor and a clever substitution, we are going to solve for r. $S(0) = \pi r\sqrt{1 + r^2}$, while, since $x_2 = \frac{1}{2}r + \frac{1}{2}\sqrt{r^2 - 2}$,

$$\begin{aligned}S(x_2) &= \pi\left(\frac{3}{2}r + \frac{1}{2}\sqrt{r^2 - 2}\right)\sqrt{1 + (\frac{1}{2}r - \frac{1}{2}\sqrt{r^2 - 2})^2}\\ &= \frac{\pi}{2}\left(3r + \sqrt{r^2 - 2}\right)\sqrt{1 + \frac{1}{4}(r^2 - 2r\sqrt{r^2 - 2} + r^2 - 2)}\\ &= \frac{\pi}{2}\left(3r + \sqrt{r^2 - 2}\right)\sqrt{1 + \frac{1}{2}(r^2 - r\sqrt{r^2 - 2} - 1)}\end{aligned}$$

From this, we simplify by squaring and taking out constants:

$$S(x_2)/\pi = \frac{1}{2}\left(3r + \sqrt{r^2-2}\right)\sqrt{1 + \frac{1}{2}(r^2 - r\sqrt{r^2-2} - 1)}$$

$$(S(x_2)/\pi)^2 = \frac{1}{8}\left(3r + \sqrt{r^2-2}\right)^2\left(2 + (r^2 - r\sqrt{r^2-2} - 1)\right)$$

$$8(S(x_2)/\pi)^2 = \left(3r + \sqrt{r^2-2}\right)^2\left(r^2 - r\sqrt{r^2-2} + 1\right)$$

To solve the equation $S(x_2) = S(0)$, we solve the equivalent equation $8(S(x_2)/\pi)^2 = 8(S(0)/\pi)^2$. $8(S(0)/\pi)^2 = 8r^2(1+r^2) = 8r^2 + 8r^4$. Let $u = r^2 - 2$, so that $\sqrt{r^2-2} = \sqrt{u}$, $r^2 = u + 2$, and $r = \sqrt{u+2}$. The expression for $8(S(x_2)/\pi)^2$ is, then:

$$8(S(x_2)/\pi)^2 = \left(3\sqrt{u+2} + \sqrt{u}\right)^2\left((u+2) - \sqrt{u+2}\sqrt{u} + 1\right)$$

while

$$8(S(0)/\pi)^2 = 8r^2 + 8r^4 = 8(u+2)(u+3) = 8u^2 + 40u + 48.$$

We compute:

$$\begin{aligned}
\left(3\sqrt{u+2} + \sqrt{u}\right)^2 &= 9(u+2) + 6\sqrt{u}\sqrt{u+2} + u \\
&= 10u + 6\sqrt{u}\sqrt{u+2} + 18 \\
\left(10u + 6\sqrt{u}\sqrt{u+2} + 18\right)\left(u - \sqrt{u}\sqrt{u+2} + 3\right) &= 10u^2 + 6u^{3/2}\sqrt{u+2} + 18u - 10u^{3/2}\sqrt{u+2} - 6u^2 - 12u \\
&\quad - 18\sqrt{u+2}\sqrt{u} + 30u + 18\sqrt{u+2}\sqrt{u} + 54 \\
&= 4u^2 - 4u\left(\sqrt{u}\sqrt{u+2}\right) + 36u + 54
\end{aligned}$$

Therefore the equation becomes:

$$\begin{aligned}
8(S(0)/\pi)^2 &= 8(S(x_2)/\pi)^2 \\
8u^2 + 40u + 48 &= 4u^2 - 4u(\sqrt{u}\sqrt{u+2}) + 36u + 54 \\
4u^2 + 4u - 6 &= -4u(\sqrt{u}\sqrt{u+2}) \\
16u^4 + 32u^3 - 32u^2 - 48u + 36 &= 16u^2(u)(u+2) \\
16u^4 + 32u^3 - 32u^2 - 48u + 36 &= 16u^4 + 32u^3 \\
-32u^2 - 48u + 36 &= 0 \\
8u^2 + 12u - 9 &= 0.
\end{aligned}$$

The last quadratic has positive solution:

$$u = \frac{-12 + \sqrt{144 + 4(72)}}{16} = \frac{-12 + 12\sqrt{3}}{16} = \frac{-3 + 3\sqrt{3}}{4}.$$

Therefore

$$r^2 - 2 = \frac{-3 + 3\sqrt{3}}{4},$$

so

$$r^2 = \frac{5 + 3\sqrt{3}}{4}.$$

This gives us that $S(x_2) = S(0)$ when

$$r = c = \sqrt{\frac{5 + 3\sqrt{3}}{4}}.$$

From part (e) we know that for $r = 1.5 < c$, $S(0)$ is the minimum value for S, but for $r = 2 > c$, $S(x_2)$ is the minimum value. Since $r = c$ is the only solution of $S(0) = S(x_2)$ for $r > \sqrt{2}$, it follows that $S(0)$ provides the minimum value for $\sqrt{2} < r < c$ and $S(x_2)$ provides the minimum when $r > c$.

4.8 Newton's Method

Preliminary Questions

1. How many iterations of Newton's Method are required to compute a root if $f(x)$ is a linear function?

SOLUTION Remember that Newton's Method uses the linear approximation of a function to estimate the location of a root. If the original function is linear, then only one iteration of Newton's Method will be required to compute the root.

2. What happens in Newton's Method if your initial guess happens to be a zero of f?

SOLUTION If x_0 happens to be a zero of f, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - 0 = x_0;$$

in other words, every term in the Newton's Method sequence will remain x_0.

3. What happens in Newton's Method if your initial guess happens to be a local min or max of f?

SOLUTION Assuming that the function is differentiable, then the derivative is zero at a local maximum or a local minimum. If Newton's Method is started with an initial guess such that $f'(x_0) = 0$, then Newton's Method will fail in the sense that x_1 will not be defined. That is, the tangent line will be parallel to the x-axis and will never intersect it.

4. Is the following a reasonable description of Newton's Method: "A root of the equation of the tangent line to $f(x)$ is used as an approximation to a root of $f(x)$ itself"? Explain.

SOLUTION Yes, that is a reasonable description. The iteration formula for Newton's Method was derived by solving the equation of the tangent line to $y = f(x)$ at x_0 for its x-intercept.

Exercises

In this exercise set, all approximations should be carried out using Newton's Method.

In Exercises 1–6, apply Newton's Method to $f(x)$ and initial guess x_0 to calculate x_1, x_2, x_3.

1. $f(x) = x^2 - 6, \quad x_0 = 2$

SOLUTION Let $f(x) = x^2 - 6$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 6}{2x_n}.$$

With $x_0 = 2$, we compute

n	1	2	3
x_n	2.5	2.45	2.44948980

2. $f(x) = x^2 - 3x + 1, \quad x_0 = 3$

SOLUTION Let $f(x) = x^2 - 3x + 1$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 3x_n + 1}{2x_n - 3}.$$

With $x_0 = 3$, we compute

n	1	2	3
x_n	2.66666667	2.61904762	2.61803445

3. $f(x) = x^3 - 10, \quad x_0 = 2$

SOLUTION Let $f(x) = x^3 - 10$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 10}{3x_n^2}.$$

With $x_0 = 2$ we compute

n	1	2	3
x_n	2.16666667	2.15450362	2.15443469

4. $f(x) = x^3 + x + 1, \quad x_0 = -1$

SOLUTION Let $f(x) = x^3 + x + 1$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n + 1}{3x_n^2 + 1}.$$

With $x_0 = -1$ we compute

n	1	2	3
x_n	-0.75	-0.68604651	-0.68233958

5. $f(x) = \cos x - 4x, \quad x_0 = 1$

SOLUTION Let $f(x) = \cos x - 4x$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cos x_n - 4x_n}{-\sin x_n - 4}.$$

With $x_0 = 1$ we compute

n	1	2	3
x_n	0.28540361	0.24288009	0.24267469

6. $f(x) = 1 - x\sin x, \quad x_0 = 7$

SOLUTION Let $f(x) = 1 - x\sin x$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{1 - x_n \sin x_n}{-x_n \cos x_n - \sin x_n}.$$

With $x_0 = 7$ we compute

n	1	2	3
x_n	6.39354183	6.43930706	6.43911724

7. Use Figure 6 to choose an initial guess x_0 to the unique real root of $x^3 + 2x + 5 = 0$ and compute the first three Newton iterates.

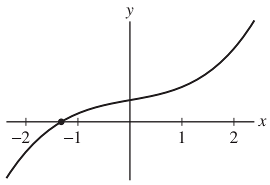


FIGURE 6 Graph of $y = x^3 + 2x + 5$.

SOLUTION Let $f(x) = x^3 + 2x + 5$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 2x_n + 5}{3x_n^2 + 2}.$$

We take $x_0 = -1.4$, based on the figure, and then calculate

n	1	2	3
x_n	-1.330964467	-1.328272820	-1.328268856

8. Approximate a solution of $\sin x = \cos 2x$ in the interval $\left[0, \frac{\pi}{2}\right]$ to three decimal places. Then find the exact solution and compare with your approximation.

SOLUTION Let $f(x) = \sin x - \cos 2x$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\sin x_n - \cos 2x_n}{\cos x_n + 2\sin 2x_n}.$$

With $x_0 = 0.5$ we find

n	1	2
x_n	0.523775116	0.523598785

The root, to three decimal places, is 0.524. The exact root is $\frac{\pi}{6}$, which is equal to 0.524 to three decimal places.

9. Approximate both solutions of $e^x = 5x$ to three decimal places (Figure 7).

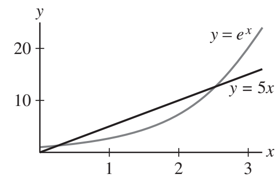


FIGURE 7 Graphs of e^x and $5x$.

SOLUTION We need to solve $e^x - 5x = 0$, so let $f(x) = e^x - 5x$. Then $f'(x) = e^x - 5$. With an initial guess of $x_0 = 0.2$, we calculate

Newton's Method (First root)	$x_0 = 0.2$ (guess)
$x_1 = 0.2 - \frac{f(0.2)}{f'(0.2)}$	$x_1 \approx 0.25859$
$x_2 = 0.25859 - \frac{f(0.25859)}{f'(0.25859)}$	$x_2 \approx 0.25917$
$x_3 = 0.25917 - \frac{f(0.25917)}{f'(0.25917)}$	$x_3 \approx 0.25917$

For the second root, we use an initial guess of $x_0 = 2.5$.

Newton's Method (Second root)	$x_0 = 2.5$ (guess)
$x_1 = 2.5 - \frac{f(2.5)}{f'(2.5)}$	$x_1 \approx 2.54421$
$x_2 = 2.54421 - \frac{f(2.54421)}{f'(2.54421)}$	$x_2 \approx 2.54264$
$x_3 = 2.54264 - \frac{f(2.54264)}{f'(2.54264)}$	$x_3 \approx 2.54264$

Thus the two solutions of $e^x = 5x$ are approximately $r_1 \approx 0.25917$ and $r_2 \approx 2.54264$.

10. The first positive solution of $\sin x = 0$ is $x = \pi$. Use Newton's Method to calculate π to four decimal places.

SOLUTION Let $f(x) = \sin x$. Taking $x_0 = 3$, we have

n	1	2	3
x_n	3.142546543	3.141592653	3.141592654

Hence, $\pi \approx 3.1416$ to four decimal places.

In Exercises 11–14, approximate to three decimal places using Newton's Method and compare with the value from a calculator.

11. $\sqrt{11}$

SOLUTION Let $f(x) = x^2 - 11$, and let $x_0 = 3$. Newton's Method yields:

n	1	2	3
x_n	3.33333333	3.31666667	3.31662479

A calculator yields 3.31662479.

12. $5^{1/3}$

SOLUTION Let $f(x) = x^3 - 5$, and let $x_0 = 2$. Here are approximations to the root of $f(x)$, which is $5^{1/3}$.

n	1	2	3	4
x_n	1.75	1.710884354	1.709976429	1.709975947

A calculator yields 1.709975947.

13. $2^{7/3}$

SOLUTION Note that $2^{7/3} = 4 \cdot 2^{1/3}$. Let $f(x) = x^3 - 2$, and let $x_0 = 1$. Newton's Method yields:

n	1	2	3
x_n	1.33333333	1.26388889	1.25993349

Thus, $2^{7/3} \approx 4 \cdot 1.25993349 = 5.03973397$. A calculator yields 5.0396842.

14. $3^{-1/4}$

SOLUTION Let $f(x) = x^{-4} - 3$, and let $x_0 = 0.8$. Here are approximations to the root of $f(x)$, which is $3^{-1/4}$.

n	1	2	3	4
x_n	0.75424	0.75973342	0.75983565	0.75983569

A calculator yields 0.75983569.

15. Approximate the largest positive root of $f(x) = x^4 - 6x^2 + x + 5$ to within an error of at most 10^{-4}. Refer to Figure 5.

SOLUTION Figure 5 from the text suggests the largest positive root of $f(x) = x^4 - 6x^2 + x + 5$ is near 2. So let $f(x) = x^4 - 6x^2 + x + 5$ and take $x_0 = 2$.

n	1	2	3	4
x_n	2.111111111	2.093568458	2.093064768	2.093064358

The largest positive root of $x^4 - 6x^2 + x + 5$ is approximately 2.093064358.

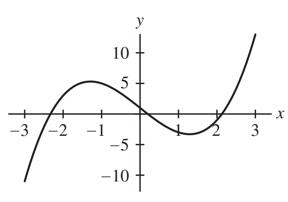
GU *In Exercises 16–19, approximate the root specified to three decimal places using Newton's Method. Use a plot to choose an initial guess.*

16. Largest positive root of $f(x) = x^3 - 5x + 1$.

SOLUTION Let $f(x) = x^3 - 5x + 1$. The graph of $f(x)$ shown below suggests the largest positive root is near $x = 2.2$. Taking $x_0 = 2.2$, Newton's Method gives

n	1	2	3
x_n	2.13193277	2.12842820	2.12841906

The largest positive root of $x^3 - 5x + 1$ is approximately 2.1284.

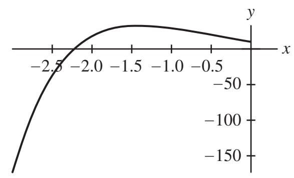


17. Negative root of $f(x) = x^5 - 20x + 10$.

SOLUTION Let $f(x) = x^5 - 20x + 10$. The graph of $f(x)$ shown below suggests taking $x_0 = -2.2$. Starting from $x_0 = -2.2$, the first three iterates of Newton's Method are:

n	1	2	3
x_n	−2.22536529	−2.22468998	−2.22468949

Thus, to three decimal places, the negative root of $f(x) = x^5 - 20x + 10$ is -2.225.



18. Positive solution of $\sin\theta = 0.8\theta$.

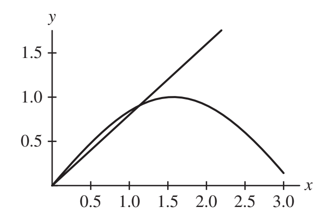
SOLUTION From the graph below, we see that the positive solution to the equation $\sin\theta = 0.8\theta$ is approximately $x = 1.1$. Now, let $f(\theta) = \sin\theta - 0.8\theta$ and define

$$\theta_{n+1} = \theta_n - \frac{f(\theta_n)}{f'(\theta_n)} = \theta_n - \frac{\sin\theta_n - 0.8\theta_n}{\cos\theta_n - 0.8}.$$

With $\theta_0 = 1.1$ we find

n	1	2	3
θ_n	1.13235345	1.13110447	1.13110259

Thus, to three decimal places, the positive solution to the equation $\sin\theta = 0.8\theta$ is 1.131.



19. Solution of $\ln(x + 4) = x$.

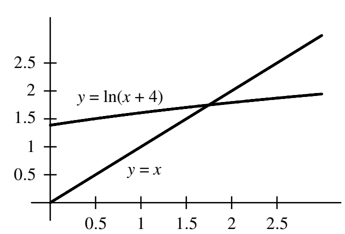
SOLUTION From the graph below, we see that the positive solution to the equation $\ln(x + 4) = x$ is approximately $x = 2$. Now, let $f(x) = \ln(x + 4) - x$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\ln(x_n + 4) - x_n}{\frac{1}{x_n+4} - 1}.$$

With $x_0 = 2$ we find

n	1	2	3
x_n	1.750111363	1.749031407	1.749031386

Thus, to three decimal places, the positive solution to the equation $\ln(x + 4) = x$ is 1.749.



20. Let x_1, x_2 be the estimates to a root obtained by applying Newton's Method with $x_0 = 1$ to the function graphed in Figure 8. Estimate the numerical values of x_1 and x_2, and draw the tangent lines used to obtain them.

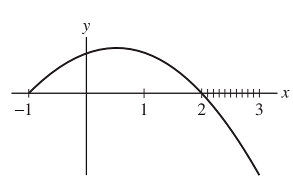
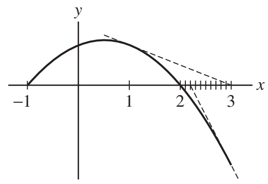


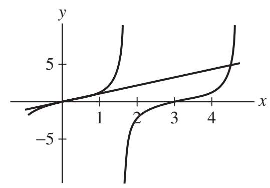
FIGURE 8

SOLUTION The graph with tangent lines drawn on it appears below. The tangent line to the curve at $(x_0, f(x_0))$ has an x-intercept at approximately $x_1 = 3.0$. The tangent line to the curve at $(x_1, f(x_1))$ has an x-intercept at approximately $x_2 = 2.2$.



21. GU Find the smallest positive value of x at which $y = x$ and $y = \tan x$ intersect. *Hint:* Draw a plot.

SOLUTION Here is a plot of $\tan x$ and x on the same axes:



The first intersection with $x > 0$ lies on the second "branch" of $y = \tan x$, between $x = \frac{5\pi}{4}$ and $x = \frac{3\pi}{2}$. Let $f(x) = \tan x - x$. The graph suggests an initial guess $x_0 = \frac{5\pi}{4}$, from which we get the following table:

n	1	2	3	4
x_n	6.85398	21.921	4480.8	7456.27

This is clearly leading nowhere, so we need to try a better initial guess. *Note: This happens with Newton's Method—it is sometimes difficult to choose an initial guess.* We try the point directly between $\frac{5\pi}{4}$ and $\frac{3\pi}{2}$, $x_0 = \frac{11\pi}{8}$:

n	1	2	3	4	5	6	7
x_n	4.64662	4.60091	4.54662	4.50658	4.49422	4.49341	4.49341

The first point where $y = x$ and $y = \tan x$ cross is at approximately $x = 4.49341$, which is approximately 1.4303π.

22. In 1535, the mathematician Antonio Fior challenged his rival Niccolo Tartaglia to solve this problem: A tree stands 12 *braccia* high; it is broken into two parts at such a point that the height of the part left standing is the cube root of the length of the part cut away. What is the height of the part left standing? Show that this is equivalent to solving $x^3 + x = 12$ and find the height to three decimal places. Tartaglia, who had discovered the secret of the cubic equation, was able to determine the exact answer:

$$x = \left(\sqrt[3]{\sqrt{2919} + 54} - \sqrt[3]{\sqrt{2919} - 54}\right) \Big/ \sqrt[3]{9}$$

SOLUTION Suppose that x is the part of the tree left standing, so that x^3 is the part cut away. Since the tree is 12 *braccia* high, this gives that $x + x^3 = 12$. Let $f(x) = x + x^3 - 12$. We are looking for a point where $f(x) = 0$. Using the initial guess $x = 2$ (it seems that most of the tree is cut away), we get the following table:

n	1	2	3	4
x_n	2.15384615385	2.14408201873	2.14404043328	2.14404043253

Hence $x \approx 2.14404043253$. Tartaglia's exact answer is 2.14404043253, so the 4th Newton's Method approximation is accurate to at least 11 decimal places.

23. Find (to two decimal places) the coordinates of the point P in Figure 9 where the tangent line to $y = \cos x$ passes through the origin.

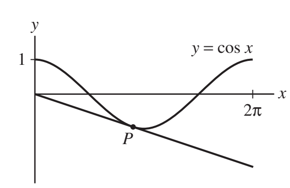


FIGURE 9

SOLUTION Let $(x_r, \cos(x_r))$ be the coordinates of the point P. The slope of the tangent line is $-\sin(x_r)$, so we are looking for a tangent line:

$$y = -\sin(x_r)(x - x_r) + \cos(x_r)$$

such that $y = 0$ when $x = 0$. This gives us the equation:

$$-\sin(x_r)(-x_r) + \cos(x_r) = 0.$$

Let $f(x) = \cos x + x \sin x$. We are looking for the first point $x = r$ where $f(r) = 0$. The sketch given indicates that $x_0 = 3\pi/4$ would be a good initial guess. The following table gives successive Newton's Method approximations:

n	1	2	3	4
x_n	2.931781309	2.803636974	2.798395826	2.798386046

The point P has approximate coordinates $(2.7984, -0.941684)$.

Newton's Method is often used to determine interest rates in financial calculations. In Exercises 24–26, r denotes a yearly interest rate expressed as a decimal (rather than as a percent).

24. If P dollars are deposited every month in an account earning interest at the yearly rate r, then the value S of the account after N years is

$$S = P\left(\frac{b^{12N+1} - b}{b - 1}\right) \qquad \text{where } b = 1 + \frac{r}{12}$$

You have decided to deposit $P = 100$ dollars per month.

(a) Determine S after 5 years if $r = 0.07$ (that is, 7%).

(b) Show that to save \$10,000 after 5 years, you must earn interest at a rate r determined bys the equation $b^{61} - 101b + 100 = 0$. Use Newton's Method to solve for b. Then find r. Note that $b = 1$ is a root, but you want the root satisfying $b > 1$.

SOLUTION

(a) If $r = 0.07$, $b = 1 + r/12 \approx 1.00583$, and :

$$S = 100\frac{(b^{61} - b)}{b - 1} = 7201.05.$$

(b) If our goal is to get \$10,000 after five years, we need $S = 10{,}000$ when $N = 5$.

$$10{,}000 = 100\left(\frac{b^{61} - b}{b - 1}\right),$$

So that:

$$10{,}000(b - 1) = 100\left(b^{61} - b\right)$$
$$100b - 100 = b^{61} - b$$
$$b^{61} - 101b + 100 = 0$$

$b = 1$ is a root, but, since $b - 1$ appears in the denominator of our original equation, it does not satisfy the original equation. Let $f(b) = b^{61} - 101b + 100$. Let's use the initial guess $r = 0.2$, so that $x_0 = 1 + r/12 = 1.016666$.

n	1	2	3
x_n	1.01576	1.01569	1.01569

The solution is approximately $b = 1.01569$. The interest rate r required satisfies $1 + r/12 = 1.01569$, so that $r = 0.01569 \times 12 = 0.18828$. An annual interest rate of 18.828% is required to have \$10,000 after five years.

25. If you borrow L dollars for N years at a yearly interest rate r, your monthly payment of P dollars is calculated using the equation

$$L = P\left(\frac{1-b^{-12N}}{b-1}\right) \qquad \text{where } b = 1 + \frac{r}{12}$$

(a) Find P if $L = \$5000$, $N = 3$, and $r = 0.08$ (8%).
(b) You are offered a loan of $L = \$5000$ to be paid back over 3 years with monthly payments of $P = \$200$. Use Newton's Method to compute b and find the implied interest rate r of this loan. *Hint:* Show that $(L/P)b^{12N+1} - (1 + L/P)b^{12N} + 1 = 0$.

SOLUTION
(a) $b = (1 + 0.08/12) = 1.00667$

$$P = L\left(\frac{b-1}{1-b^{-12N}}\right) = 5000\left(\frac{1.00667-1}{1-1.00667^{-36}}\right) \approx \$156.69$$

(b) Starting from

$$L = P\left(\frac{1-b^{-12N}}{b-1}\right),$$

divide by P, multiply by $b-1$, multiply by b^{12N} and collect like terms to arrive at

$$(L/P)b^{12N+1} - (1 + L/P)b^{12N} + 1 = 0.$$

Since $L/P = 5000/200 = 25$, we must solve

$$25b^{37} - 26b^{36} + 1 = 0.$$

Newton's Method gives $b \approx 1.02121$ and

$$r = 12(b-1) = 12(0.02121) \approx 0.25452$$

So the interest rate is around 25.45%.

26. If you deposit P dollars in a retirement fund every year for N years with the intention of then withdrawing Q dollars per year for M years, you must earn interest at a rate r satisfying $P(b^N - 1) = Q(1 - b^{-M})$, where $b = 1 + r$. Assume that \$2,000 is deposited each year for 30 years and the goal is to withdraw \$10,000 per year for 25 years. Use Newton's Method to compute b and then find r. Note that $b = 1$ is a root, but you want the root satisfying $b > 1$.

SOLUTION Substituting $P = 2000$, $Q = 10{,}000$, $N = 30$ and $M = 25$ into the equation $P(b^N - 1) = Q(1 - b^{-M})$ and then rearranging terms, we find that b must satisfy the equation $b^{55} - 6b^{25} + 5 = 0$. Newton's Method with a starting value of $b_0 = 1.1$ yields $b \approx 1.05217$. Thus, $r \approx 0.05217 = 5.217\%$.

27. There is no simple formula for the position at time t of a planet P in its orbit (an ellipse) around the sun. Introduce the auxiliary circle and angle θ in Figure 10 (note that P determines θ because it is the central angle of point B on the circle). Let $a = OA$ and $e = OS/OA$ (the eccentricity of the orbit).
(a) Show that sector BSA has area $(a^2/2)(\theta - e\sin\theta)$.
(b) By Kepler's Second Law, the area of sector BSA is proportional to the time t elapsed since the planet passed point A, and because the circle has area πa^2, BSA has area $(\pi a^2)(t/T)$, where T is the period of the orbit. Deduce **Kepler's Equation**:

$$\frac{2\pi t}{T} = \theta - e\sin\theta$$

(c) The eccentricity of Mercury's orbit is approximately $e = 0.2$. Use Newton's Method to find θ after a quarter of Mercury's year has elapsed ($t = T/4$). Convert θ to degrees. Has Mercury covered more than a quarter of its orbit at $t = T/4$?

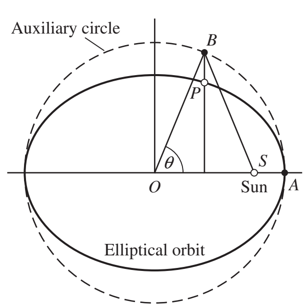


FIGURE 10

SOLUTION

(a) The sector SAB is the slice OAB with the triangle OPS removed. OAB is a central sector with arc θ and radius $\overline{OA} = a$, and therefore has area $\frac{a^2\theta}{2}$. OPS is a triangle with height $a\sin\theta$ and base length $\overline{OS} = ea$. Hence, the area of the sector is

$$\frac{a^2}{2}\theta - \frac{1}{2}ea^2\sin\theta = \frac{a^2}{2}(\theta - e\sin\theta).$$

(b) Since Kepler's second law indicates that the area of the sector is proportional to the time t since the planet passed point A, we get

$$\pi a^2\,(t/T) = a^2/2\,(\theta - e\sin\theta)$$
$$2\pi\frac{t}{T} = \theta - e\sin\theta.$$

(c) If $t = T/4$, the last equation in (b) gives:

$$\frac{\pi}{2} = \theta - e\sin\theta = \theta - .2\sin\theta.$$

Let $f(\theta) = \theta - .2\sin\theta - \frac{\pi}{2}$. We will use Newton's Method to find the point where $f(\theta) = 0$. Since a quarter of the year on Mercury has passed, a good first estimate θ_0 would be $\frac{\pi}{2}$.

n	1	2	3	4
x_n	1.7708	1.76696	1.76696	1.76696

From the point of view of the Sun, Mercury has traversed an angle of approximately 1.76696 radians $= 101.24°$. Mercury has therefore traveled more than one fourth of the way around (from the point of view of central angle) during this time.

28. The roots of $f(x) = \frac{1}{3}x^3 - 4x + 1$ to three decimal places are -3.583, 0.251, and 3.332 (Figure 11). Determine the root to which Newton's Method converges for the initial choices $x_0 = 1.85$, 1.7, and 1.55. The answer shows that a small change in x_0 can have a significant effect on the outcome of Newton's Method.

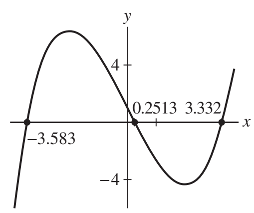


FIGURE 11 Graph of $f(x) = \frac{1}{3}x^3 - 4x + 1$.

SOLUTION Let $f(x) = \frac{1}{3}x^3 - 4x + 1$, and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{3}x_n^3 - 4x_n + 1}{x_n^2 - 4}.$$

- Taking $x_0 = 1.85$, we have

n	1	2	3	4	5	6	7
x_n	−5.58	−4.31	−3.73	−3.59	−3.58294362	−3.582918671	−3.58291867

- Taking $x_0 = 1.7$, we have

n	1	2	3	4	5	6	7	8	9
x_n	−2.05	−33.40	−22.35	−15.02	−10.20	−7.08	−5.15	−4.09	−3.66

n	10	11	12	13
x_n	−3.585312288	−3.582920989	−3.58291867	−3.58291867

- Taking $x_0 = 1.55$, we have

n	1	2	3	4	5	6
x_n	−0.928	0.488	0.245	0.251320515	0.251322863	0.251322863

29. What happens when you apply Newton's Method to find a zero of $f(x) = x^{1/3}$? Note that $x = 0$ is the only zero.

SOLUTION Let $f(x) = x^{1/3}$. Define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

Take $x_0 = 0.5$. Then the sequence of iterates is $-1, 2, -4, 8, -16, 32, -64, \ldots$ That is, for any nonzero starting value, the sequence of iterates diverges spectacularly, since $x_n = (-2)^n x_0$. Thus $\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} 2^n |x_0| = \infty$.

30. What happens when you apply Newton's Method to the equation $x^3 - 20x = 0$ with the unlucky initial guess $x_0 = 2$?

SOLUTION Let $f(x) = x^3 - 20x$. Define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 20x_n}{3x_n^2 - 20}.$$

Take $x_0 = 2$. Then the sequence of iterates is $-2, 2, -2, 2, \ldots$, which diverges by oscillation.

Further Insights and Challenges

31. Newton's Method can be used to compute reciprocals without performing division. Let $c > 0$ and set $f(x) = x^{-1} - c$.

(a) Show that $x - (f(x)/f'(x)) = 2x - cx^2$.

(b) Calculate the first three iterates of Newton's Method with $c = 10.3$ and the two initial guesses $x_0 = 0.1$ and $x_0 = 0.5$.

(c) Explain graphically why $x_0 = 0.5$ does not yield a sequence converging to $1/10.3$.

SOLUTION

(a) Let $f(x) = \frac{1}{x} - c$. Then

$$x - \frac{f(x)}{f'(x)} = x - \frac{\frac{1}{x} - c}{-x^{-2}} = 2x - cx^2.$$

(b) For $c = 10.3$, we have $f(x) = \frac{1}{x} - 10.3$ and thus $x_{n+1} = 2x_n - 10.3x_n^2$.

- Take $x_0 = 0.1$.

n	1	2	3
x_n	0.097	0.0970873	0.09708738

- Take $x_0 = 0.5$.

n	1	2	3
x_n	-1.575	-28.7004375	-8541.66654

(c) The graph is disconnected. If $x_0 = .5$, $(x_1, f(x_1))$ is on the other portion of the graph, which will never converge to any point under Newton's Method.

In Exercises 32 and 33, consider a metal rod of length L fastened at both ends. If you cut the rod and weld on an additional segment of length m, leaving the ends fixed, the rod will bow up into a circular arc of radius R (unknown), as indicated in Figure 12.

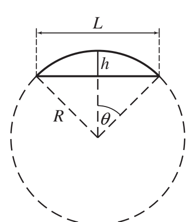


FIGURE 12 The bold circular arc has length $L + m$.

32. Let h be the maximum vertical displacement of the rod.

(a) Show that $L = 2R\sin\theta$ and conclude that

$$h = \frac{L(1 - \cos\theta)}{2\sin\theta}$$

(b) Show that $L + m = 2R\theta$ and then prove

$$\frac{\sin\theta}{\theta} = \frac{L}{L+m} \qquad \boxed{2}$$

SOLUTION

(a) From the figure, we have $\sin\theta = \frac{L/2}{R}$, so that $L = 2R\sin\theta$. Hence

$$h = R - R\cos\theta = R\,(1-\cos\theta) = \frac{\frac{1}{2}L}{\sin\theta}\,(1-\cos\theta) = \frac{L\,(1-\cos\theta)}{2\sin\theta}$$

(b) The arc length $L + m$ is also given by radius $\times$ angle $= R \cdot 2\theta$. Thus, $L + m = 2R\theta$. Dividing $L = 2R\sin\theta$ by $L + m = 2R\theta$ yields

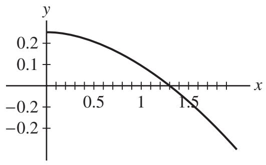
$$\frac{L}{L+m} = \frac{2R\sin\theta}{2R\theta} = \frac{\sin\theta}{\theta}.$$

33. Let $L = 3$ and $m = 1$. Apply Newton's Method to Eq. (2) to estimate θ, and use this to estimate h.

SOLUTION We let $L = 3$ and $m = 1$. We want the solution of:

$$\frac{\sin\theta}{\theta} = \frac{L}{L+m}$$
$$\frac{\sin\theta}{\theta} - \frac{L}{L+m} = 0$$
$$\frac{\sin\theta}{\theta} - \frac{3}{4} = 0.$$

Let $f(\theta) = \frac{\sin\theta}{\theta} - \frac{3}{4}$.



The figure above suggests that $\theta_0 = 1.5$ would be a good initial guess. The Newton's Method approximations for the solution follow:

n	1	2	3	4
θ_n	1.2854388	1.2757223	1.2756981	1.2756981

The angle where $\frac{\sin\theta}{\theta} = \frac{L}{L+m}$ is approximately 1.2757. Hence

$$h = L\frac{1-\cos\theta}{2\sin\theta} \approx 1.11181.$$

34. Quadratic Convergence to Square Roots Let $f(x) = x^2 - c$ and let $e_n = x_n - \sqrt{c}$ be the error in x_n.

(a) Show that $x_{n+1} = \frac{1}{2}(x_n + c/x_n)$ and $e_{n+1} = e_n^2/2x_n$.

(b) Show that if $x_0 > \sqrt{c}$, then $x_n > \sqrt{c}$ for all n. Explain graphically.

(c) Show that if $x_0 > \sqrt{c}$, then $e_{n+1} \le e_n^2/(2\sqrt{c})$.

SOLUTION

(a) Let $f(x) = x^2 - c$. Then

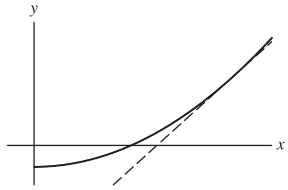
$$xn + 1 = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - c}{2x_n} = \frac{x_n^2 + c}{2x_n} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right),$$

as long as $x_n \ne 0$. Now

$$\frac{e_n^2}{2x_n} = \frac{\left(x_n - \sqrt{c}\right)^2}{2x_n} = \frac{x_n^2 - 2x_n\sqrt{c} + c}{2x_n} = \frac{1}{2}x_n - \sqrt{c} + \frac{c}{2x_n}$$
$$= \frac{1}{2}\left(x_n + \frac{c}{x_n}\right) - \sqrt{c} = x_{n+1} - \sqrt{c} = e_{n+1}.$$

(b) Since $x_0 > \sqrt{c} \geq 0$, we have $e_0 = x_0 - \sqrt{c} > 0$. Now assume that $e_k > 0$ for $k = n$. Then $0 < e_k = e_n = x_n - \sqrt{c}$, whence $x_n > \sqrt{c} \geq 0$; i.e., $x_n > 0$ and $e_n > 0$. By part (a), we have for $k = n + 1$ that $e_k = e_{n+1} = \frac{e_n^2}{2x_n} > 0$ since $x_n > 0$. Thus $e_{n+1} > 0$. Therefore by induction $e_n > 0$ for all $n \geq 0$. Hence $e_n = x_n - \sqrt{c} > 0$ for all $n \geq 0$. Therefore $x_n > \sqrt{c}$ for all $n \geq 0$.

The figure below shows the graph of $f(x) = x^2 - c$. The x-intercept of the graph is, of course, $x = \sqrt{c}$. We see that for any $x_n > \sqrt{c}$, the tangent line to the graph of f intersects the x-axis at a value $x_{n+1} > \sqrt{c}$.



(c) By part (b), if $x_0 > \sqrt{c}$, then $x_n > \sqrt{c}$ for all $n \geq 0$. Accordingly, for all $n \geq 0$ we have $e_{n+1} = \frac{e_n^2}{2x_n} < \frac{e_n^2}{2\sqrt{c}}$. In other words, $e_{n+1} < \frac{e_n^2}{2\sqrt{c}}$ for all $n \geq 0$.

In Exercises 35–37, a flexible chain of length L is suspended between two poles of equal height separated by a distance 2M (Figure 13). By Newton's laws, the chain describes a ***catenary*** $y = a\cosh\left(\frac{x}{a}\right)$*, where a is the number such that* $L = 2a\sinh\left(\frac{M}{a}\right)$*. The sag s is the vertical distance from the highest to the lowest point on the chain.*

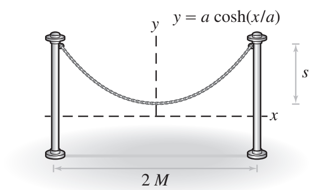


FIGURE 13 Chain hanging between two poles.

35. Suppose that $L = 120$ and $M = 50$.

(a) Use Newton's Method to find a value of a (to two decimal places) satisfying $L = 2a\sinh(M/a)$.

(b) Compute the sag s.

SOLUTION

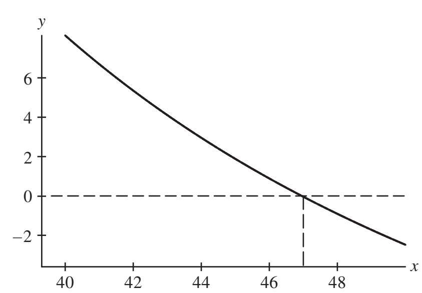
(a) Let

$$f(a) = 2a\sinh\left(\frac{50}{a}\right) - 120.$$

The graph of f shown below suggests $a \approx 47$ is a root of f. Starting with $a_0 = 47$, we find the following approximations using Newton's method:

$$a_1 = 46.95408 \quad \text{and} \quad a_2 = 46.95415$$

Thus, to two decimal places, $a = 46.95$.



(b) The sag is given by

$$s = y(M) - y(0) = \left(a\cosh\frac{M}{a} + C\right) - \left(a\cosh\frac{0}{a} + C\right) = a\cosh\frac{M}{a} - a.$$

Using $M = 50$ and $a = 46.95$, we find $s = 29.24$.

36. Assume that M is fixed.
(a) Calculate $\frac{ds}{da}$. Note that $s = a\cosh\left(\frac{M}{a}\right) - a$.
(b) Calculate $\frac{da}{dL}$ by implicit differentiation using the relation $L = 2a\sinh\left(\frac{M}{a}\right)$.
(c) Use (a) and (b) and the Chain Rule to show that

$$\frac{ds}{dL} = \frac{ds}{da}\frac{da}{dL} = \frac{\cosh(M/a) - (M/a)\sinh(M/a) - 1}{2\sinh(M/a) - (2M/a)\cosh(M/a)} \qquad \boxed{3}$$

SOLUTION The sag in the curve is

$$s = y(M) - y(0) = a\cosh\left(\frac{M}{a}\right) + C - (a\cosh 0 + C) = a\cosh\left(\frac{M}{a}\right) - a.$$

(a) $\dfrac{ds}{da} = \cosh\left(\dfrac{M}{a}\right) - \dfrac{M}{a}\sinh\left(\dfrac{M}{a}\right) - 1$

(b) If we differentiate the relation $L = 2a\sinh\left(\dfrac{M}{a}\right)$ with respect to a, we find

$$0 = 2\frac{da}{dL}\sinh\left(\frac{M}{a}\right) - \frac{2M}{a}\frac{da}{dL}\cosh\left(\frac{M}{a}\right).$$

Solving for da/dL yields

$$\frac{da}{dL} = \left(2\sinh\left(\frac{M}{a}\right) - \frac{2M}{a}\cosh\left(\frac{M}{a}\right)\right)^{-1}.$$

(c) By the Chain Rule,

$$\frac{ds}{dL} = \frac{ds}{da}\cdot\frac{da}{dL}.$$

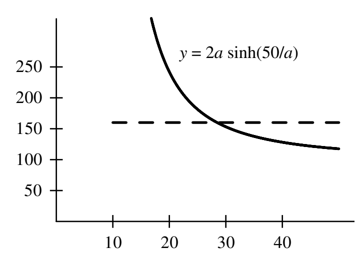
The formula for ds/dL follows upon substituting the results from parts (a) and (b).

37. Suppose that $L = 160$ and $M = 50$.
(a) Use Newton's Method to find a value of a (to two decimal places) satisfying $L = 2a\sinh(M/a)$.
(b) Use Eq. (3) and the Linear Approximation to estimate the increase in sag Δs for changes in length $\Delta L = 1$ and $\Delta L = 5$.
(c) CAS Compute $s(161) - s(160)$ and $s(165) - s(160)$ directly and compare with your estimates in (b).

SOLUTION
(a) Let $f(x) = 2x\sinh(50/x) - 160$. Using the graph below, we select an initial guess of $x_0 = 30$. Newton's Method then yields:

n	1	2	3
x_n	28.30622107	28.45653356	28.45797517

Thus, to two decimal places, $a \approx 28.46$.



(b) With $M = 50$ and $a \approx 28.46$, we find using Eq. (3) that

$$\frac{ds}{dL} = 0.61.$$

By the Linear Approximation,

$$\Delta s \approx \frac{ds}{dL}\cdot\Delta L.$$

If L increases from 160 to 161, then $\Delta L = 1$ and $\Delta s \approx 0.61$; if L increases from 160 to 165, then $\Delta L = 5$ and $\Delta s \approx 3.05$.

(c) When $L = 160$, $a \approx 28.46$ and

$$s(160) = 28.46 \cosh\left(\frac{50}{28.46}\right) - 28.46 \approx 56.45;$$

whereas, when $L = 161$, $a \approx 28.25$ and

$$s(161) = 28.25 \cosh\left(\frac{50}{28.25}\right) - 28.25 \approx 57.07.$$

Therefore, $s(161) - s(160) = 0.62$, very close to the approximation obtained from the Linear Approximation. Moreover, when $L = 165$, $a \approx 27.49$ and

$$s(165) = 27.49 \cosh\left(\frac{50}{27.49}\right) - 27.49 \approx 59.47;$$

thus, $s(165) - s(160) = 3.02$, again very close to the approximation obtained from the Linear Approximation.

4.9 Antiderivatives

Preliminary Questions

1. Find an antiderivative of the function $f(x) = 0$.

SOLUTION Since the derivative of any constant is zero, any constant function is an antiderivative for the function $f(x) = 0$.

2. Is there a difference between finding the general antiderivative of a function $f(x)$ and evaluating $\int f(x)\,dx$?

SOLUTION No difference. The indefinite integral is the symbol for denoting the general antiderivative.

3. Jacques was told that $f(x)$ and $g(x)$ have the same derivative, and he wonders whether $f(x) = g(x)$. Does Jacques have sufficient information to answer his question?

SOLUTION No. Knowing that the two functions have the same derivative is only good enough to tell Jacques that the functions may differ by at most an additive constant. To determine whether the functions are equal for all x, Jacques needs to know the value of each function for a single value of x. If the two functions produce the same output value for a single input value, they must take the same value for all input values.

4. Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$. Which of the following statements are true? Explain.

(a) If $f = g$, then $F = G$.
(b) If F and G differ by a constant, then $f = g$.
(c) If f and g differ by a constant, then $F = G$.

SOLUTION

(a) False. Even if $f(x) = g(x)$, the antiderivatives F and G may differ by an additive constant.
(b) True. This follows from the fact that the derivative of any constant is 0.
(c) False. If the functions f and g are different, then the antiderivatives F and G differ by a linear function: $F(x) - G(x) = ax + b$ for some constants a and b.

5. Is $y = x$ a solution of the following Initial Value Problem?

$$\frac{dy}{dx} = 1, \qquad y(0) = 1$$

SOLUTION Although $\frac{d}{dx}x = 1$, the function $f(x) = x$ takes the value 0 when $x = 0$, so $y = x$ is *not* a solution of the indicated initial value problem.

Exercises

In Exercises 1–8, find the general antiderivative of $f(x)$ and check your answer by differentiating.

1. $f(x) = 18x^2$

SOLUTION

$$\int 18x^2\,dx = 18\int x^2\,dx = 18 \cdot \frac{1}{3}x^3 + C = 6x^3 + C.$$

As a check, we have

$$\frac{d}{dx}(6x^3 + C) = 18x^2$$

as needed.

2. $f(x) = x^{-3/5}$

SOLUTION

$$\int x^{-3/5}\,dx = \frac{x^{2/5}}{2/5} + C = \frac{5}{2}x^{2/5} + C.$$

As a check, we have

$$\frac{d}{dx}\left(\frac{5}{2}x^{2/5} + C\right) = x^{-3/5}$$

as needed.

3. $f(x) = 2x^4 - 24x^2 + 12x^{-1}$

SOLUTION

$$\begin{aligned}\int (2x^4 - 24x^2 + 12x^{-1})\,dx &= 2\int x^4\,dx - 24\int x^2\,dx + 12\int \frac{1}{x}\,dx \\ &= 2\cdot\frac{1}{5}x^5 - 24\cdot\frac{1}{3}x^3 + 12\ln|x| + C \\ &= \frac{2}{5}x^5 - 8x^3 + 12\ln|x| + C.\end{aligned}$$

As a check, we have

$$\frac{d}{dx}\left(\frac{2}{5}x^5 - 8x^3 + 12\ln|x| + C\right) = 2x^4 - 24x^2 + 12x^{-1}$$

as needed.

4. $f(x) = 9x + 15x^{-2}$

SOLUTION

$$\begin{aligned}\int (9x + 15x^{-2})\,dx &= 9\int x\,dx + 15\int x^{-2}\,dx \\ &= 9\cdot\frac{1}{2}x^2 + 15\cdot\frac{x^{-1}}{-1} + C \\ &= \frac{9}{2}x^2 - 15x^{-1} + C.\end{aligned}$$

As a check, we have

$$\frac{d}{dx}\left(\frac{9}{2}x^2 - 15x^{-1} + C\right) = 9x + 15x^{-2}$$

as needed.

5. $f(x) = 2\cos x - 9\sin x$

SOLUTION

$$\begin{aligned}\int (2\cos x - 9\sin x)\,dx &= 2\int \cos x\,dx - 9\int \sin x\,dx \\ &= 2\sin x - 9(-\cos x) + C = 2\sin x + 9\cos x + C\end{aligned}$$

As a check, we have

$$\frac{d}{dx}(2\sin x + 9\cos x + C) = 2\cos x + 9(-\sin x) = 2\cos x - 9\sin x$$

as needed.

6. $f(x) = 4x^7 - 3\cos x$

SOLUTION

$$\begin{aligned}\int (4x^7 - 3\cos x)\,dx &= 4\int x^7\,dx - 3\int \cos x\,dx \\ &= 4\cdot\frac{1}{8}x^8 - 3\sin x + C = \frac{1}{2}x^8 - 3\sin x + C.\end{aligned}$$

As a check, we have

$$\frac{d}{dx}\left(\frac{1}{2}x^8 - 3\sin x + C\right) = 4x^7 - 3\cos x,$$

as needed.

7. $f(x) = 12e^x - 5x^{-2}$

SOLUTION

$$\int (12e^x - 5x^{-2})\,dx = 12\int e^x\,dx - 5\int x^{-2}\,dx = 12e^x - 5(-x^{-1}) + C = 12e^x + 5x^{-1} + C.$$

As a check, we have

$$\frac{d}{dx}\left(12e^x + 5x^{-1} + C\right) = 12e^x + 5(-x^{-2}) = 12e^x - 5x^{-2}$$

as needed.

8. $f(x) = e^x - 4\sin x$

SOLUTION

$$\int (e^x - 4\sin x)\,dx = e^x - 4\int \sin x\,dx$$
$$= e^x - 4(-\cos x) + C = e^x + 4\cos x + C.$$

As a check, we have

$$\frac{d}{dx}\left(e^x + 4\cos x + C\right) = e^x - 4\sin x$$

as needed.

9. Match functions (a)–(d) with their antiderivatives (i)–(iv).

(a) $f(x) = \sin x$	(i) $F(x) = \cos(1 - x)$
(b) $f(x) = x\sin(x^2)$	(ii) $F(x) = -\cos x$
(c) $f(x) = \sin(1 - x)$	(iii) $F(x) = -\frac{1}{2}\cos(x^2)$
(d) $f(x) = x\sin x$	(iv) $F(x) = \sin x - x\cos x$

SOLUTION

(a) An antiderivative of $\sin x$ is $-\cos x$, which is **(ii)**. As a check, we have $\frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x$.

(b) An antiderivative of $x\sin(x^2)$ is $-\frac{1}{2}\cos(x^2)$, which is **(iii)**. This is because, by the Chain Rule, we have $\frac{d}{dx}\left(-\frac{1}{2}\cos(x^2)\right) = -\frac{1}{2}\left(-\sin(x^2)\right)\cdot 2x = x\sin(x^2)$.

(c) An antiderivative of $\sin(1 - x)$ is $\cos(1 - x)$ or **(i)**. As a check, we have $\frac{d}{dx}\cos(1 - x) = -\sin(1 - x)\cdot(-1) = \sin(1 - x)$.

(d) An antiderivative of $x\sin x$ is $\sin x - x\cos x$, which is **(iv)**. This is because

$$\frac{d}{dx}(\sin x - x\cos x) = \cos x - (x(-\sin x) + \cos x \cdot 1) = x\sin x$$

In Exercises 10–39, evaluate the indefinite integral.

10. $\int (9x + 2)\,dx$

SOLUTION $\int (9x + 2)\,dx = \frac{9}{2}x^2 + 2x + C.$

11. $\int (4 - 18x)\,dx$

SOLUTION $\int (4 - 18x)\,dx = 4x - 9x^2 + C.$

12. $\int x^{-3}\,dx$

SOLUTION $\int x^{-3}\,dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2}x^{-2} + C.$

13. $\int t^{-6/11}\,dt$

SOLUTION $\int t^{-6/11}\,dt = \frac{t^{5/11}}{5/11} + C = \frac{11}{5}t^{5/11} + C.$

14. $\int (5t^3 - t^{-3})\,dt$

SOLUTION $\int (5t^3 - t^{-3})\,dt = \frac{5}{4}t^4 - \frac{t^{-2}}{-2} + C = \frac{5}{4}t^4 + \frac{1}{2}t^{-2} + C.$

15. $\int (18t^5 - 10t^4 - 28t)\,dt$

SOLUTION $\int (18t^5 - 10t^4 - 28t)\,dt = 3t^6 - 2t^5 - 14t^2 + C.$

16. $\int 14s^{9/5}\,ds$

SOLUTION $\int 14s^{9/5}\,ds = 14 \cdot \frac{s^{14/5}}{14/5} + C = 5s^{14/5} + C.$

17. $\int (z^{-4/5} - z^{2/3} + z^{5/4})\,dz$

SOLUTION $\int ((z^{-4/5} - z^{2/3} + z^{5/4})\,dz = \frac{z^{1/5}}{1/5} - \frac{z^{5/3}}{5/3} + \frac{z^{9/4}}{9/4} + C = 5z^{1/5} - \frac{3}{5}z^{5/3} + \frac{4}{9}z^{9/4} + C.$

18. $\int \frac{3}{2}\,dx$

SOLUTION $\int \frac{3}{2}\,dx = \frac{3}{2}x + C.$

19. $\int \frac{1}{\sqrt[3]{x}}\,dx$

SOLUTION $\int \frac{1}{\sqrt[3]{x}}\,dx = \int x^{-1/3}\,dx = \frac{x^{2/3}}{2/3} + C = \frac{3}{2}x^{2/3} + C.$

20. $\int \frac{dx}{x^{4/3}}$

SOLUTION $\int \frac{dx}{x^{4/3}} = \int x^{-4/3}\,dx = \frac{x^{-1/3}}{-1/3} + C = -\frac{3}{x^{1/3}} + C.$

21. $\int \frac{36\,dt}{t^3}$

SOLUTION $\int \frac{36}{t^3}\,dt = \int 36t^{-3}\,dt = 36\frac{t^{-2}}{-2} + C = -\frac{18}{t^2} + C.$

22. $\int x(x^2 - 4)\,dx$

SOLUTION $\int x(x^2 - 4)\,dx = \int (x^3 - 4x)\,dx = \frac{1}{4}x^4 - 2x^2 + C.$

23. $\int (t^{1/2}+1)(t+1)\,dt$

SOLUTION

$$\begin{aligned}\int (t^{1/2}+1)(t+1)\,dt &= \int (t^{3/2}+t+t^{1/2}+1)\,dt\\ &= \frac{t^{5/2}}{5/2}+\frac{1}{2}t^2+\frac{t^{3/2}}{3/2}+t+C\\ &= \frac{2}{5}t^{5/2}+\frac{1}{2}t^2+\frac{2}{3}t^{3/2}+t+C\end{aligned}$$

24. $\int \frac{12-z}{\sqrt{z}}\,dz$

SOLUTION $\int \frac{12-z}{\sqrt{z}}\,dz = \int (12z^{-1/2}-z^{1/2})\,dz = 24z^{1/2}-\frac{2}{3}z^{3/2}+C.$

25. $\int \frac{x^3+3x-4}{x^2}\,dx$

SOLUTION

$$\begin{aligned}\int \frac{x^3+3x-4}{x^2}\,dx &= \int (x+3x^{-1}-4x^{-2})\,dx\\ &= \frac{1}{2}x^2+3\ln|x|+4x^{-1}+C\end{aligned}$$

26. $\int \left(\frac{1}{3}\sin x-\frac{1}{4}\cos x\right)dx$

SOLUTION $\int \left(\frac{1}{3}\sin x-\frac{1}{4}\cos x\right)dx = -\frac{1}{3}\cos x-\frac{1}{4}\sin x+C.$

27. $\int 12\sec x\tan x\,dx$

SOLUTION $\int 12\sec x\tan x\,dx = 12\sec x+C.$

28. $\int (\theta+\sec^2\theta)\,d\theta$

SOLUTION $\int (\theta+\sec^2\theta)\,d\theta = \frac{1}{2}\theta^2+\tan\theta+C.$

29. $\int (\csc t\cot t)\,dt$

SOLUTION $\int (\csc t\cot t)\,dt = -\csc t+C.$

30. $\int \sin(7x-5)\,dx$

SOLUTION $\int \sin(7x-5)\,dx = -\frac{1}{7}\cos(7x-5)+C.$

31. $\int \sec^2(7-3\theta)\,d\theta$

SOLUTION $\int \sec^2(7-3\theta)\,d\theta = -\frac{1}{3}\tan(7-3\theta)+C.$

32. $\int (\theta-\cos(1-\theta))\,d\theta$

SOLUTION $\int (\theta-\cos(1-\theta))\,d\theta = \frac{1}{2}\theta^2+\sin(1-\theta)+C.$

33. $\int 25\sec^2(3z+1)\,dz$

SOLUTION $\int 25\sec^2(3z+1)\,dz = \frac{25}{3}\tan(3z+1)+C.$

34. $\int \sec(x+5)\tan(x+5)\,dx$

SOLUTION $\int \sec(x+5)\tan(x+5)\,dx = \sec(x+5)+C.$

35. $\int \left(\cos(3\theta) - \frac{1}{2}\sec^2\left(\frac{\theta}{4}\right)\right) d\theta$

SOLUTION $\int \left(\cos(3\theta) - \frac{1}{2}\sec^2\left(\frac{\theta}{4}\right)\right) d\theta = \frac{1}{3}\sin(3\theta) - 2\tan\left(\frac{\theta}{4}\right) + C.$

36. $\int \left(\frac{4}{x} - e^x\right) dx$

SOLUTION $\int \left(\frac{4}{x} - e^x\right) dx = 4\ln|x| - e^x + C.$

37. $\int (3e^{5x})\, dx$

SOLUTION $\int (3e^{5x})\, dx = \frac{3}{5}e^{5x} + C.$

38. $\int e^{3t-4}\, dt$

SOLUTION $\int e^{3t-4}\, dt = \frac{1}{3}e^{3t-4} + C.$

39. $\int (8x - 4e^{5-2x})\, dx$

SOLUTION $\int (8x - 4e^{5-2x})\, dx = 4x^2 + 2e^{5-2x} + C.$

40. In Figure 3, is graph (A) or graph (B) the graph of an antiderivative of $f(x)$?

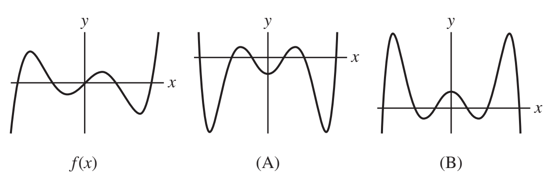


FIGURE 3

SOLUTION Let $F(x)$ be an antiderivative of $f(x)$. By definition, this means $F'(x) = f(x)$. In other words, $f(x)$ provides information as to the increasing/decreasing behavior of $F(x)$. Since, moving left to right, $f(x)$ transitions from $-$ to $+$ to $-$ to $+$ to $-$ to $+$, it follows that $F(x)$ must transition from decreasing to increasing to decreasing to increasing to decreasing to increasing. This describes the graph in (A)!

41. In Figure 4, which of graphs (A), (B), and (C) is *not* the graph of an antiderivative of $f(x)$? Explain.

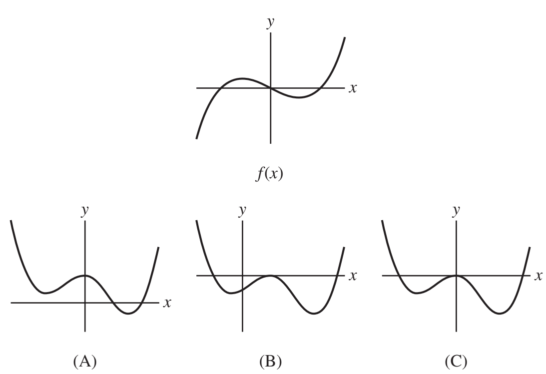


FIGURE 4

SOLUTION Let $F(x)$ be an antiderivative of $f(x)$. Notice that $f(x) = F'(x)$ changes sign from $-$ to $+$ to $-$ to $+$. Hence, $F(x)$ must transition from decreasing to increasing to decreasing to increasing.

- Both graph (A) and graph (C) meet the criteria discussed above and only differ by an additive constant. Thus either could be an antiderivative of $f(x)$.
- Graph (B) does not have the same local extrema as indicated by $f(x)$ and therefore is *not* an antiderivative of $f(x)$.

42. Show that $F(x) = \frac{1}{3}(x+13)^3$ is an antiderivative of $f(x) = (x+13)^2$.

SOLUTION Note that

$$\frac{d}{dx}F(x) = \frac{d}{dx}\frac{1}{3}(x+13)^3 = (x+13)^2.$$

Thus, $F(x) = \frac{1}{3}(x+13)^3$ is an antiderivative of $f(x) = (x+13)^2$.

In Exercises 43–46, verify by differentiation.

43. $\int (x+13)^6\,dx = \frac{1}{7}(x+13)^7 + C$

SOLUTION $\frac{d}{dx}\left(\frac{1}{7}(x+13)^7 + C\right) = (x+13)^6$ as required.

44. $\int (x+13)^{-5}\,dx = -\frac{1}{4}(x+13)^{-4} + C$

SOLUTION $\frac{d}{dx}\left(-\frac{1}{4}(x+13)^{-4} + C\right) = (x+13)^{-5}$ as required.

45. $\int (4x+13)^2\,dx = \frac{1}{12}(4x+13)^3 + C$

SOLUTION $\frac{d}{dx}\left(\frac{1}{12}(4x+13)^3 + C\right) = \frac{1}{4}(4x+13)^2(4) = (4x+13)^2$ as required.

46. $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C$

SOLUTION $\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1} + C\right) = (ax+b)^n$ as required.

In Exercises 47–62, solve the initial value problem.

47. $\frac{dy}{dx} = x^3,\ \ y(0) = 4$

SOLUTION Since $\frac{dy}{dx} = x^3$, we have

$$y = \int x^3\,dx = \frac{1}{4}x^4 + C.$$

Thus,

$$4 = y(0) = \frac{1}{4}0^4 + C = C,$$

so that $C = 4$. Therefore, $y = \frac{1}{4}x^4 + 4$.

48. $\frac{dy}{dt} = 3 - 2t,\ \ y(0) = -5$

SOLUTION Since $\frac{dy}{dt} = 3 - 2t$, we have

$$y = \int (3 - 2t)\,dt = 3t - t^2 + C.$$

Thus,

$$-5 = y(0) = 3(0) - (0)^2 + C = C,$$

so that $C = -5$. Therefore, $y = 3t - t^2 - 5$.

49. $\frac{dy}{dt} = 2t + 9t^2,\ \ y(1) = 2$

SOLUTION Since $\frac{dy}{dt} = 2t + 9t^2$, we have

$$y = \int (2t + 9t^2)\,dt = t^2 + 3t^3 + C.$$

Thus,

$$2 = y(1) = 1^2 + 3(1)^3 + C,$$

so that $C = -2$. Therefore $y = t^2 + 3t^3 - 2$.

50. $\dfrac{dy}{dx} = 8x^3 + 3x^2, \;\; y(2) = 0$

SOLUTION Since $\frac{dy}{dx} = 8x^3 + 3x^2$, we have

$$y = \int (8x^3 + 3x^2)\,dx = 2x^4 + x^3 + C.$$

Thus

$$0 = y(2) = 2(2)^4 + 2^3 + C,$$

so that $C = -40$. Therefore, $y = 2x^4 + x^3 - 40$.

51. $\dfrac{dy}{dt} = \sqrt{t}, \;\; y(1) = 1$

SOLUTION Since $\frac{dy}{dt} = \sqrt{t} = t^{1/2}$, we have

$$y = \int t^{1/2}\,dt = \frac{2}{3}t^{3/2} + C.$$

Thus

$$1 = y(1) = \frac{2}{3} + C,$$

so that $C = \frac{1}{3}$. Therefore, $y = \frac{2}{3}t^{3/2} + \frac{1}{3}$.

52. $\dfrac{dz}{dt} = t^{-3/2}, \;\; z(4) = -1$

SOLUTION Since $\frac{dz}{dt} = t^{-3/2}$, we have

$$z = \int t^{-3/2}\,dt = -2t^{-1/2} + C.$$

Thus

$$-1 = z(4) = -2(4)^{-1/2} + C,$$

so that $C = 0$. Therefore, $z = -2t^{-1/2}$.

53. $\dfrac{dy}{dx} = (3x+2)^3, \;\; y(0) = 1$

SOLUTION Since $\frac{dy}{dx} = (3x+2)^3$, we have

$$y = \int (3x+2)^3\,dx = \frac{1}{4}\cdot\frac{1}{3}(3x+2)^4 + C = \frac{1}{12}(3x+2)^4 + C.$$

Thus,

$$1 = y(0) = \frac{1}{12}(2)^4 + C,$$

so that $C = 1 - \frac{4}{3} = -\frac{1}{3}$. Therefore, $y = \frac{1}{12}(3x+2)^4 - \frac{1}{3}$.

54. $\dfrac{dy}{dt} = (4t+3)^{-2}, \;\; y(1) = 0$

SOLUTION Since $\frac{dy}{dt} = (4t+3)^{-2}$, we have

$$y = \int (4t+3)^{-2}\,dt = \frac{1}{-1}\cdot\frac{1}{4}(4t+3)^{-1} + C = -\frac{1}{4}(4t+3)^{-1} + C.$$

Thus,

$$0 = y(1) = -\frac{1}{4}(7)^{-1} + C,$$

so that $C = \frac{1}{28}$. Therefore, $y = -\frac{1}{4}(4t+3)^{-1} + \frac{1}{28}$.

55. $\dfrac{dy}{dx} = \sin x, \quad y\left(\dfrac{\pi}{2}\right) = 1$

SOLUTION Since $\frac{dy}{dx} = \sin x$, we have

$$y = \int \sin x \, dx = -\cos x + C.$$

Thus

$$1 = y\left(\frac{\pi}{2}\right) = 0 + C,$$

so that $C = 1$. Therefore, $y = 1 - \cos x$.

56. $\dfrac{dy}{dz} = \sin 2z, \quad y\left(\dfrac{\pi}{4}\right) = 4$

SOLUTION Since $\frac{dy}{dz} = \sin 2z$, we have

$$y = \int \sin 2z \, dz = -\frac{1}{2}\cos 2z + C.$$

Thus

$$4 = y\left(\frac{\pi}{4}\right) = 0 + C,$$

so that $C = 4$. Therefore, $y = 4 - \frac{1}{2}\cos 2z$.

57. $\dfrac{dy}{dx} = \cos 5x, \quad y(\pi) = 3$

SOLUTION Since $\frac{dy}{dx} = \cos 5x$, we have

$$y = \int \cos 5x \, dx = \frac{1}{5}\sin 5x + C.$$

Thus $3 = y(\pi) = 0 + C$, so that $C = 3$. Therefore, $y = 3 + \frac{1}{5}\sin 5x$.

58. $\dfrac{dy}{dx} = \sec^2 3x, \quad y\left(\dfrac{\pi}{4}\right) = 2$

SOLUTION Since $\frac{dy}{dx} = \sec^2 3x$, we have

$$y = \int \sec^2(3x) \, dx = \frac{1}{3}\tan(3x) + C.$$

Since $y\left(\frac{\pi}{4}\right) = 2$, we get:

$$2 = \frac{1}{3}\tan\left(3\frac{\pi}{4}\right) + C$$
$$2 = \frac{1}{3}(-1) + C$$
$$\frac{7}{3} = C.$$

Therefore, $y = \frac{1}{3}\tan(3x) + \frac{7}{3}$.

59. $\dfrac{dy}{dx} = e^x, \quad y(2) = 0$

SOLUTION Since $\frac{dy}{dx} = e^x$, we have

$$y = \int e^x \, dx = e^x + C.$$

Thus,

$$0 = y(2) = e^2 + C,$$

so that $C = -e^2$. Therefore, $y = e^x - e^2$.

60. $\dfrac{dy}{dt} = e^{-t}, \; y(0) = 0$

SOLUTION Since $\frac{dy}{dt} = e^{-t}$, we have

$$y = \int e^{-t}\,dt = -e^{-t} + C.$$

Thus,

$$0 = y(0) = -e^{0} + C,$$

so that $C = 1$. Therefore, $y = -e^{-t} + 1$.

61. $\dfrac{dy}{dt} = 9e^{12-3t}, \; y(4) = 7$

SOLUTION Since $\frac{dy}{dt} = 9e^{12-3t}$, we have

$$y = \int 9e^{12-3t}\,dt = -3e^{12-3t} + C.$$

Thus,

$$7 = y(4) = -3e^{0} + C,$$

so that $C = 10$. Therefore, $y = -3e^{12-3t} + 10$.

62. $\dfrac{dy}{dt} = t + 2e^{t-9}, \; y(9) = 4$

SOLUTION Since $\frac{dy}{dt} = t + 2e^{t-9}$, we have

$$y = \int (t + 2e^{t-9})\,dt = \frac{1}{2}t^2 + 2e^{t-9} + C.$$

Thus,

$$4 = y(9) = \frac{1}{2}(9)^2 + 2e^{0} + C,$$

so that $C = -\frac{77}{2}$. Therefore, $y = \frac{1}{2}t^2 + 2e^{t-9} - \frac{77}{2}$.

In Exercises 63–69, first find f' and then find f.

63. $f''(x) = 12x, \quad f'(0) = 1, \quad f(0) = 2$

SOLUTION Let $f''(x) = 12x$. Then $f'(x) = 6x^2 + C$. Given $f'(0) = 1$, it follows that $1 = 6(0)^2 + C$ and $C = 1$. Thus, $f'(x) = 6x^2 + 1$. Next, $f(x) = 2x^3 + x + C$. Given $f(0) = 2$, it follows that $2 = 2(0)^3 + 0 + C$ and $C = 2$. Finally, $f(x) = 2x^3 + x + 2$.

64. $f''(x) = x^3 - 2x, \quad f'(1) = 0, \quad f(1) = 2$

SOLUTION Let $f''(x) = x^3 - 2x$. Then $f'(x) = \frac{1}{4}x^4 - x^2 + C$. Given $f'(1) = 0$, it follows that $0 = \frac{1}{4}(1)^4 - (1)^2 + C$ and $C = \frac{3}{4}$. Thus, $f'(x) = \frac{1}{4}x^4 - x^2 + \frac{3}{4}$. Next, $f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{3}{4}x + C$. Given $f(1) = 2$, it follows that $2 = \frac{1}{20}(1)^5 - \frac{1}{3}(1)^3 + \frac{3}{4} + C$ and $C = \frac{23}{15}$. Finally, $f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{3}{4}x + \frac{23}{15}$.

65. $f''(x) = x^3 - 2x + 1, \quad f'(0) = 1, \quad f(0) = 0$

SOLUTION Let $g(x) = f'(x)$. The statement gives us $g'(x) = x^3 - 2x + 1$, $g(0) = 1$. From this, we get $g(x) = \frac{1}{4}x^4 - x^2 + x + C$. $g(0) = 1$ gives us $1 = C$, so $f'(x) = g(x) = \frac{1}{4}x^4 - x^2 + x + 1$. $f'(x) = \frac{1}{4}x^4 - x^2 + x + 1$, so $f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C$. $f(0) = 0$ gives $C = 0$, so

$$f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x.$$

66. $f''(x) = x^3 - 2x + 1, \quad f'(1) = 0, \quad f(1) = 4$

SOLUTION Let $g(x) = f'(x)$. The problem statement gives us $g'(x) = x^3 - 2x + 1$, $g(0) = 0$. From $g'(x)$, we get $g(x) = \frac{1}{4}x^4 - x^2 + x + C$, and from $g(1) = 0$, we get $0 = \frac{1}{4} - 1 + 1 + C$, so that $C = -\frac{1}{4}$. This gives $f'(x) = g(x) = \frac{1}{4}x^4 - x^2 + x - \frac{1}{4}$. From $f'(x)$, we get $f(x) = \frac{1}{4}(\frac{1}{5}x^5) - \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{4}x + C = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{4}x + C$. From $f(1) = 4$, we get

$$\frac{1}{20} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + C = 4,$$

so that $C = \frac{121}{30}$. Hence,

$$f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{4}x + \frac{121}{30}.$$

67. $f''(t) = t^{-3/2}, \quad f'(4) = 1, \quad f(4) = 4$

SOLUTION Let $g(t) = f'(t)$. The problem statement is $g'(t) = t^{-3/2}$, $g(4) = 1$. From $g'(t)$ we get $g(t) = \frac{1}{-1/2}t^{-1/2} + C = -2t^{-1/2} + C$. From $g(4) = 1$ we get $-1 + C = 1$ so that $C = 2$. Hence $f'(t) = g(t) = -2t^{-1/2} + 2$. From $f'(t)$ we get $f(t) = -2\frac{1}{1/2}t^{1/2} + 2t + C = -4t^{1/2} + 2t + C$. From $f(4) = 4$ we get $-8 + 8 + C = 4$, so that $C = 4$. Hence, $f(t) = -4t^{1/2} + 2t + 4$.

68. $f''(\theta) = \cos\theta, \quad f'\left(\frac{\pi}{2}\right) = 1, \quad f\left(\frac{\pi}{2}\right) = 6$

SOLUTION Let $g(\theta) = f'(\theta)$. The problem statement gives

$$g'(\theta) = \cos\theta, \qquad g\left(\frac{\pi}{2}\right) = 1.$$

From $g'(\theta)$ we get $g(\theta) = \sin\theta + C$. From $g(\frac{\pi}{2}) = 1$ we get $1 + C = 1$, so $C = 0$. Hence $f'(\theta) = g(\theta) = \sin\theta$. From $f'(\theta)$ we get $f(\theta) = -\cos\theta + C$. From $f(\frac{\pi}{2}) = 6$ we get $C = 6$, so

$$f(\theta) = -\cos\theta + 6.$$

69. $f''(t) = t - \cos t, \quad f'(0) = 2, \quad f(0) = -2$

SOLUTION Let $g(t) = f'(t)$. The problem statement gives

$$g'(t) = t - \cos t, \qquad g(0) = 2.$$

From $g'(t)$, we get $g(t) = \frac{1}{2}t^2 - \sin t + C$. From $g(0) = 2$, we get $C = 2$. Hence $f'(t) = g(t) = \frac{1}{2}t^2 - \sin t + 2$. From $f'(t)$, we get $f(t) = \frac{1}{2}(\frac{1}{3}t^3) + \cos t + 2t + C$. From $f(0) = -2$, we get $1 + C = -2$, hence $C = -3$, and

$$f(t) = \frac{1}{6}t^3 + \cos t + 2t - 3.$$

70. Show that $F(x) = \tan^2 x$ and $G(x) = \sec^2 x$ have the same derivative. What can you conclude about the relation between F and G? Verify this conclusion directly.

SOLUTION Let $f(x) = \tan^2 x$ and $g(x) = \sec^2 x$. Then $f'(x) = 2\tan x \sec^2 x$ and $g'(x) = 2\sec x \cdot \sec x \tan x = 2\tan x \sec^2 x$; hence $f'(x) = g'(x)$. Accordingly, $f(x)$ and $g(x)$ must differ by a constant; i.e., $f(x) - g(x) = \tan^2 x - \sec^2 x = C$ for some constant C. To see that this is true directly, divide the identity $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$. This yields $\tan^2 x + 1 = \sec^2 x$, so that $\tan^2 x - \sec^2 x = -1$.

71. A particle located at the origin at $t = 1$ s moves along the x-axis with velocity $v(t) = (6t^2 - t)$ m/s. State the differential equation with initial condition satisfied by the position $s(t)$ of the particle, and find $s(t)$.

SOLUTION The differential equation satisfied by $s(t)$ is

$$\frac{ds}{dt} = v(t) = 6t^2 - t,$$

and the associated initial condition is $s(1) = 0$. From the differential equation, we find

$$s(t) = \int (6t^2 - t)\, dt = 2t^3 - \frac{1}{2}t^2 + C.$$

Using the initial condition, it follows that

$$0 = s(1) = 2 - \frac{1}{2} + C \quad \text{so} \quad C = -\frac{3}{2}.$$

Finally,

$$s(t) = 2t^3 - \frac{1}{2}t^2 - \frac{3}{2}.$$

72. A particle moves along the x-axis with velocity $v(t) = (6t^2 - t)$ m/s. Find the particle's position $s(t)$ assuming that $s(2) = 4$.

SOLUTION The differential equation satisfied by $s(t)$ is

$$\frac{ds}{dt} = v(t) = 6t^2 - t,$$

and the associated initial condition is $s(2) = 4$. From the differential equation, we find

$$s(t) = \int (6t^2 - t)\,dt = 2t^3 - \frac{1}{2}t^2 + C.$$

Using the initial condition, it follows that

$$4 = s(2) = 16 - 2 + C \quad \text{so} \quad C = -10.$$

Finally,

$$s(t) = 2t^3 - \frac{1}{2}t^2 - 10.$$

73. A mass oscillates at the end of a spring. Let $s(t)$ be the displacement of the mass from the equilibrium position at time t. Assuming that the mass is located at the origin at $t = 0$ and has velocity $v(t) = \sin(\pi t/2)$ m/s, state the differential equation with initial condition satisfied by $s(t)$, and find $s(t)$.

SOLUTION The differential equation satisfied by $s(t)$ is

$$\frac{ds}{dt} = v(t) = \sin(\pi t/2),$$

and the associated initial condition is $s(0) = 0$. From the differential equation, we find

$$s(t) = \int \sin(\pi t/2)\,dt = -\frac{2}{\pi}\cos(\pi t/2) + C.$$

Using the initial condition, it follows that

$$0 = s(0) = -\frac{2}{\pi} + C \quad \text{so} \quad C = \frac{2}{\pi}.$$

Finally,

$$s(t) = \frac{2}{\pi}(1 - \cos(\pi t/2)).$$

74. Beginning at $t = 0$ with initial velocity 4 m/s, a particle moves in a straight line with acceleration $a(t) = 3t^{1/2}$ m/s^2. Find the distance traveled after 25 seconds.

SOLUTION Given $a(t) = 3t^{1/2}$ and an initial velocity of 4 m/s, it follows that $v(t)$ satisfies

$$\frac{dv}{dt} = 3t^{1/2}, \quad v(0) = 4.$$

Thus,

$$v(t) = \int 3t^{1/2}\,dt = 2t^{3/2} + C.$$

Using the initial condition, we find

$$4 = v(0) = 2(0)^{3/2} + C \quad \text{so} \quad C = 4$$

and $v(t) = 2t^{3/2} + 4$. Next,

$$s = \int v(t)\,dt = \int (2t^{3/2} + 4)\,dt = \frac{4}{5}t^{5/2} + 4t + C.$$

Finally, the distance traveled after 25 seconds is

$$s(25) - s(0) = \frac{4}{5}(25)^{5/2} + 4(25) = 2600$$

meters.

75. A car traveling 25 m/s begins to decelerate at a constant rate of 4 m/s^2. After how many seconds does the car come to a stop and how far will the car have traveled before stopping?

SOLUTION Since the acceleration of the car is a constant -4m/s^2, v is given by the differential equation:

$$\frac{dv}{dt} = -4, \qquad v(0) = 25.$$

From $\frac{dv}{dt}$, we get $v(t) = \int -4\,dt = -4t + C$. Since $v(0)25$, $C = 25$. From this, $v(t) = -4t + 25\ \frac{\text{m}}{\text{s}}$. To find the time until the car stops, we must solve $v(t) = 0$:

$$\begin{aligned} -4t + 25 &= 0 \\ 4t &= 25 \\ t &= 25/4 = 6.25 \text{ s.} \end{aligned}$$

Now we have a differential equation for $s(t)$. Since we want to know how far the car has traveled from the beginning of its deceleration at time $t = 0$, we have $s(0) = 0$ by definition, so:

$$\frac{ds}{dt} = v(t) = -4t + 25, \qquad s(0) = 0.$$

From this, $s(t) = \int (-4t + 25)\,dt = -2t^2 + 25t + C$. Since $s(0) = 0$, we have $C = 0$, and

$$s(t) = -2t^2 + 25t.$$

At stopping time $t = 0.25$ s, the car has traveled

$$s(6.25) = -2(6.25)^2 + 25(6.25) = 78.125 \text{ m.}$$

76. At time $t = 1$ s, a particle is traveling at 72 m/s and begins to decelerate at the rate $a(t) = -t^{-1/2}$ until it stops. How far does the particle travel before stopping?

SOLUTION With $a(t) = -t^{-1/2}$ and a velocity of 72 m/s at $t = 1$ s, it follows that $v(t)$ satisfies

$$\frac{dv}{dt} = -t^{-1/2}, \quad v(1) = 72.$$

Thus,

$$v(t) = \int -t^{-1/2}\,dt = -2t^{1/2} + C.$$

Using the initial condition, we find

$$72 = v(1) = -2 + C \quad \text{so} \quad C = 74,$$

and $v(t) = 74 - 2t^{1/2}$. The particle comes to rest when

$$74 - 2t^{1/2} = 0 \quad \text{or when} \quad t = 37^2 = 1369$$

seconds. Now,

$$s(t) = \int v(t)\,dt = \int (74 - 2t^{1/2})\,dt = 74t - \frac{4}{3}t^{3/2} + C.$$

The distance traveled by the particle before it comes to rest is then

$$s(1369) - s(1) = 74(1369) - \frac{202612}{3} - 74 + \frac{4}{3} = 33696$$

meters.

77. A 900-kg rocket is released from a space station. As it burns fuel, the rocket's mass decreases and its velocity increases. Let $v(m)$ be the velocity (in meters per second) as a function of mass m. Find the velocity when $m = 729$ if $dv/dm = -50m^{-1/2}$. Assume that $v(900) = 0$.

SOLUTION Since $\frac{dv}{dm} = -50m^{-1/2}$, we have $v(m) = \int -50m^{-1/2}\,dm = -100m^{1/2} + C$. Thus $0 = v(900) = -100\sqrt{900} + C = -3000 + C$, and $C = 3000$. Therefore, $v(m) = 3000 - 100\sqrt{m}$. Accordingly,

$$v(729) = 3000 - 100\sqrt{729} = 3000 - 100(27) = 300 \text{ meters/sec.}$$

78. As water flows through a tube of radius $R = 10$ cm, the velocity v of an individual water particle depends only on its distance r from the center of the tube. The particles at the walls of the tube have zero velocity and $dv/dr = -0.06r$. Determine $v(r)$.

SOLUTION The statement amounts to the differential equation and initial condition:

$$\frac{dv}{dr} = -0.06r, \qquad v(R) = 0.$$

From $\frac{dv}{dr} = -0.06r$, we get

$$v(r) = \int -0.06r\,dr = -0.06\frac{r^2}{2} + C = -0.03r^2 + C.$$

Plugging in $v(R) = 0$, we get $-0.03R^2 + C = 0$, so that $C = 0.03R^2$. Therefore,

$$v(r) = -0.03r^2 + 0.03R^2 = 0.03(R^2 - r^2) \text{ cm/s}.$$

If $R = 10$ centimeters, we get:

$$v(r) = 0.03(10^2 - r^2).$$

79. Verify the linearity properties of the indefinite integral stated in Theorem 4.

SOLUTION To verify the Sum Rule, let $F(x)$ and $G(x)$ be any antiderivatives of $f(x)$ and $g(x)$, respectively. Because

$$\frac{d}{dx}(F(x) + G(x)) = \frac{d}{dx}F(x) + \frac{d}{dx}G(x) = f(x) + g(x),$$

it follows that $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$; i.e.,

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

To verify the Multiples Rule, again let $F(x)$ be any antiderivative of $f(x)$ and let c be a constant. Because

$$\frac{d}{dx}(cF(x)) = c\frac{d}{dx}F(x) = cf(x),$$

it follows that $cF(x)$ is and antiderivative of $cf(x)$; i.e.,

$$\int (cf(x))\,dx = c\int f(x)\,dx.$$

Further Insights and Challenges

80. Find constants c_1 and c_2 such that $F(x) = c_1 x \sin x + c_2 \cos x$ is an antiderivative of $f(x) = x \cos x$.

SOLUTION Let $F(x) = c_1 x \sin x + c_2 \cos x$. If $F(x)$ is to be an antiderivative of $f(x) = x \cos x$, we must have $F'(x) = f(x)$ for all x. Hence $c_1\,(x \cos x + \sin x) - c_2 \sin x = x \cos x$ for all x. Equating coefficients on the left- and right-hand sides, we have $c_1 = 1$ (i.e., the coefficients of $x \cos x$ are equal) and $c_1 - c_2 = 0$ (i.e., the coefficients of $\sin x$ are equal). Thus $c_1 = c_2 = 1$ and hence $F(x) = x \sin x + \cos x$. As a check, we have $F'(x) = x \cos x + \sin x - \sin x = x \cos x = f(x)$, as required.

81. Find constants c_1 and c_2 such that $F(x) = c_1 xe^x + c_2 e^x$ is an antiderivative of $f(x) = xe^x$.

SOLUTION Let $F(x) = c_1 xe^x + c_2 e^x$. If $F(x)$ is to be an antiderivative of $f(x) = xe^x$, we must have $F'(x) = f(x)$ for all x. Hence,

$$c_1 xe^x + (c_1 + c_2)e^x = xe^x = 1 \cdot xe^x + 0 \cdot e^x.$$

Equating coefficients of like terms we have $c_1 = 1$ and $c_1 + c_2 = 0$. Thus, $c_1 = 1$ and $c_2 = -1$.

82. Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$. Is it true that $F(x)G(x)$ is an antiderivative of $f(x)g(x)$? Confirm or provide a counterexample.

SOLUTION Let $f(x) = x^2$ and $g(x) = x^3$. Then $F(x) = \frac{1}{3}x^3$ and $G(x) = \frac{1}{4}x^4$ are antiderivatives for $f(x)$ and $g(x)$, respectively. Let $h(x) = f(x)g(x) = x^5$, the general antiderivative of which is $H(x) = \frac{1}{6}x^6 + C$. There is no value of the constant C for which $F(x)G(x) = \frac{1}{12}x^7$ equals $H(x)$. Accordingly, $F(x)G(x)$ is *not* an antiderivative of $h(x) = f(x)g(x)$.

83. Suppose that $F'(x) = f(x)$.

(a) Show that $\frac{1}{2}F(2x)$ is an antiderivative of $f(2x)$.

(b) Find the general antiderivative of $f(kx)$ for $k \neq 0$.

SOLUTION Let $F'(x) = f(x)$.

(a) By the Chain Rule, we have

$$\frac{d}{dx}\left(\frac{1}{2}F(2x)\right) = \frac{1}{2}F'(2x) \cdot 2 = F'(2x) = f(2x).$$

Thus $\frac{1}{2}F(2x)$ is an antiderivative of $f(2x)$.

(b) For nonzero constant k, the Chain Rules gives

$$\frac{d}{dx}\left(\frac{1}{k}F(kx)\right) = \frac{1}{k}F'(kx) \cdot k = F'(kx) = f(kx).$$

Thus $\frac{1}{k}F(kx)$ is an antiderivative of $f(kx)$. Hence the general antiderivative of $f(kx)$ is $\frac{1}{k}F(kx) + C$, where C is a constant.

84. Find an antiderivative for $f(x) = |x|$.

SOLUTION Let $f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$. Then the general antiderivative of $f(x)$ is

$$F(x) = \int f(x)\,dx = \begin{cases} \int x\,dx & \text{for } x \geq 0 \\ \int -x\,dx & \text{for } x < 0 \end{cases} = \begin{cases} \frac{1}{2}x^2 + C & \text{for } x \geq 0 \\ -\frac{1}{2}x^2 + C & \text{for } x < 0 \end{cases}.$$

85. Using Theorem 1, prove that $F'(x) = f(x)$ where $f(x)$ is a polynomial of degree $n-1$, then $F(x)$ is a polynomial of degree n. Then prove that if $g(x)$ is any function such that $g^{(n)}(x) = 0$, then $g(x)$ is a polynomial of degree at most n.

SOLUTION Suppose $F'(x) = f(x)$ where $f(x)$ is a polynomial of degree $n-1$. Now, we know that the derivative of a polynomial of degree n is a polynomial of degree $n-1$, and hence an antiderivative of a polynomial of degree $n-1$ is a polynomial of degree n. Thus, by Theorem 1, $F(x)$ can differ from a polynomial of degree n by at most a constant term, which is still a polynomial of degree n. Now, suppose that $g(x)$ is any function such that $g^{(n+1)}(x) = 0$. We know that the $n+1$-st derivative of any polynomial of degree at most n is zero, so by repeated application of Theorem 1, $g(x)$ can differ from a polynomial of degree at most n by at most a constant term. Hence, $g(x)$ is a polynomial of degree at most n.

86. Show that $F(x) = \dfrac{x^{n+1}-1}{n+1}$ is an antiderivative of $y = x^n$ for $n \neq -1$. Then use L'Hôpital's Rule to prove that

$$\lim_{n \to -1} F(x) = \ln x$$

In this limit, x is fixed and n is the variable. This result shows that, although the Power Rule breaks down for $n = -1$, the antiderivative of $y = x^{-1}$ is a limit of antiderivatives of x^n as $n \to -1$.

SOLUTION If $n \neq -1$, then

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{x^{n+1}-1}{n+1}\right) = x^n.$$

Therefore, $F(x)$ is an antiderivative of $y = x^n$. Using L'Hôpital's Rule,

$$\lim_{n \to -1} F(x) = \lim_{n \to -1} \frac{x^{n+1}-1}{n+1} = \lim_{n \to -1} \frac{x^{n+1}\ln x}{1} = \ln x.$$

CHAPTER REVIEW EXERCISES

In Exercises 1–6, estimate using the Linear Approximation or linearization, and use a calculator to estimate the error.

1. $8.1^{1/3} - 2$

SOLUTION Let $f(x) = x^{1/3}$, $a = 8$ and $\Delta x = 0.1$. Then $f'(x) = \frac{1}{3}x^{-2/3}$, $f'(a) = \frac{1}{12}$ and, by the Linear Approximation,

$$\Delta f = 8.1^{1/3} - 2 \approx f'(a)\Delta x = \frac{1}{12}(0.1) = 0.00833333.$$

Using a calculator, $8.1^{1/3} - 2 = 0.00829885$. The error in the Linear Approximation is therefore

$$|0.00829885 - 0.00833333| = 3.445 \times 10^{-5}.$$

2. $\dfrac{1}{\sqrt{4.1}} - \dfrac{1}{2}$

SOLUTION Let $f(x) = x^{-1/2}$, $a = 4$ and $\Delta x = 0.1$. Then $f'(x) = -\frac{1}{2}x^{-3/2}$, $f'(a) = -\frac{1}{16}$ and, by the Linear Approximation,

$$\Delta f = \frac{1}{\sqrt{4.1}} - \frac{1}{2} \approx f'(a)\Delta x = -\frac{1}{16}(0.1) = -0.00625.$$

Using a calculator,

$$\frac{1}{\sqrt{4.1}} - \frac{1}{2} = -0.00613520.$$

The error in the Linear Approximation is therefore

$$|-0.00613520 - (-0.00625)| = 1.148 \times 10^{-4}.$$

3. $625^{1/4} - 624^{1/4}$

SOLUTION Let $f(x) = x^{1/4}$, $a = 625$ and $\Delta x = -1$. Then $f'(x) = \frac{1}{4}x^{-3/4}$, $f'(a) = \frac{1}{500}$ and, by the Linear Approximation,

$$\Delta f = 624^{1/4} - 625^{1/4} \approx f'(a)\Delta x = \frac{1}{500}(-1) = -0.002.$$

Thus $625^{1/4} - 624^{1/4} \approx 0.002$. Using a calculator,

$$625^{1/4} - 624^{1/4} = 0.00200120.$$

The error in the Linear Approximation is therefore

$$|0.00200120 - (0.002)| = 1.201 \times 10^{-6}.$$

4. $\sqrt{101}$

SOLUTION Let $f(x) = \sqrt{x}$ and $a = 100$. Then $f(a) = 10$, $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(a) = \frac{1}{20}$. The linearization of $f(x)$ at $a = 100$ is therefore

$$L(x) = f(a) + f'(a)(x - a) = 10 + \frac{1}{20}(x - 100),$$

and $\sqrt{101} \approx L(101) = 10.05$. Using a calculator, $\sqrt{101} = 10.049876$, so the error in the Linear Approximation is

$$|10.049876 - 10.05| = 1.244 \times 10^{-4}.$$

5. $\dfrac{1}{1.02}$

SOLUTION Let $f(x) = x^{-1}$ and $a = 1$. Then $f(a) = 1$, $f'(x) = -x^{-2}$ and $f'(a) = -1$. The linearization of $f(x)$ at $a = 1$ is therefore

$$L(x) = f(a) + f'(a)(x - a) = 1 - (x - 1) = 2 - x,$$

and $\frac{1}{1.02} \approx L(1.02) = 0.98$. Using a calculator, $\frac{1}{1.02} = 0.980392$, so the error in the Linear Approximation is

$$|0.980392 - 0.98| = 3.922 \times 10^{-4}.$$

6. $\sqrt[5]{33}$

SOLUTION Let $f(x) = x^{1/5}$ and $a = 32$. Then $f(a) = 2$, $f'(x) = \frac{1}{5}x^{-4/5}$ and $f'(a) = \frac{1}{80}$. The linearization of $f(x)$ at $a = 32$ is therefore

$$L(x) = f(a) + f'(a)(x - a) = 2 + \frac{1}{80}(x - 32),$$

and $\sqrt[5]{33} \approx L(33) = 2.0125$. Using a calculator, $\sqrt[5]{33} = 2.012347$, so the error in the Linear Approximation is

$$|2.012347 - 2.0125| = 1.534 \times 10^{-4}.$$

In Exercises 7–12, find the linearization at the point indicated.

7. $y = \sqrt{x}, \quad a = 25$

SOLUTION Let $y = \sqrt{x}$ and $a = 25$. Then $y(a) = 5$, $y' = \frac{1}{2}x^{-1/2}$ and $y'(a) = \frac{1}{10}$. The linearization of y at $a = 25$ is therefore

$$L(x) = y(a) + y'(a)(x - 25) = 5 + \frac{1}{10}(x - 25).$$

8. $v(t) = 32t - 4t^2, \quad a = 2$

SOLUTION Let $v(t) = 32t - 4t^2$ and $a = 2$. Then $v(a) = 48$, $v'(t) = 32 - 8t$ and $v'(a) = 16$. The linearization of $v(t)$ at $a = 2$ is therefore

$$L(t) = v(a) + v'(a)(t - a) = 48 + 16(t - 2) = 16t + 16.$$

9. $A(r) = \frac{4}{3}\pi r^3, \quad a = 3$

SOLUTION Let $A(r) = \frac{4}{3}\pi r^3$ and $a = 3$. Then $A(a) = 36\pi$, $A'(r) = 4\pi r^2$ and $A'(a) = 36\pi$. The linearization of $A(r)$ at $a = 3$ is therefore

$$L(r) = A(a) + A'(a)(r - a) = 36\pi + 36\pi(r - 3) = 36\pi(r - 2).$$

10. $V(h) = 4h(2 - h)(4 - 2h), \quad a = 1$

SOLUTION Let $V(h) = 4h(2 - h)(4 - 2h) = 32h - 32h^2 + 8h^3$ and $a = 1$. Then $V(a) = 8$, $V'(h) = 32 - 64h + 24h^2$ and $V'(a) = -8$. The linearization of $V(h)$ at $a = 1$ is therefore

$$L(h) = V(a) + V'(a)(h - a) = 8 - 8(h - 1) = 16 - 8h.$$

11. $P(x) = e^{-x^2/2}, \quad a = 1$

SOLUTION Let $P(x) = e^{-x^2/2}$ and $a = 1$. Then $P(a) = e^{-1/2}$, $P'(x) = -xe^{-x^2/2}$, and $P'(a) = -e^{-1/2}$. The linearization of $P(x)$ at $a = 1$ is therefore

$$L(x) = P(a) + P'(a)(x - a) = e^{-1/2} - e^{-1/2}(x - 1) = \frac{1}{\sqrt{e}}(2 - x).$$

12. $f(x) = \ln(x + e), \quad a = e$

SOLUTION Let $f(x) = \ln(x + e)$ and $a = e$. Then $f(a) = \ln(2e) = 1 + \ln 2$, $P'(x) = \frac{1}{x+e}$, and $P'(a) = \frac{1}{2e}$. The linearization of $f(x)$ at $a = e$ is therefore

$$L(x) = f(a) + f'(a)(x - a) = 1 + \ln 2 + \frac{1}{2e}(x - e).$$

In Exercises 13–18, use the Linear Approximation.

13. The position of an object in linear motion at time t is $s(t) = 0.4t^2 + (t + 1)^{-1}$. Estimate the distance traveled over the time interval [4, 4.2].

SOLUTION Let $s(t) = 0.4t^2 + (t + 1)^{-1}$, $a = 4$ and $\Delta t = 0.2$. Then $s'(t) = 0.8t - (t + 1)^{-2}$ and $s'(a) = 3.16$. Using the Linear Approximation, the distance traveled over the time interval [4, 4.2] is approximately

$$\Delta s = s(4.2) - s(4) \approx s'(a)\Delta t = 3.16(0.2) = 0.632.$$

14. A bond that pays \$10,000 in 6 years is offered for sale at a price P. The percentage yield Y of the bond is

$$Y = 100\left(\left(\frac{10{,}000}{P}\right)^{1/6} - 1\right)$$

Verify that if $P = \$7500$, then $Y = 4.91\%$. Estimate the drop in yield if the price rises to \$7700.

SOLUTION Let $P = \$7500$. Then

$$Y = 100\left(\left(\frac{10{,}000}{7500}\right)^{1/6} - 1\right) = 4.91\%.$$

If the price is raised to \$7700, then $\Delta P = 200$. With

$$\frac{dY}{dP} = -\frac{1}{6}100(10{,}000)^{1/6}P^{-7/6} = -\frac{10^{8/3}}{6}P^{-7/6},$$

we estimate using the Linear Approximation that

$$\Delta Y \approx Y'(7500)\Delta P = -0.46\%.$$

15. When a bus pass from Albuquerque to Los Alamos is priced at p dollars, a bus company takes in a monthly revenue of $R(p) = 1.5p - 0.01p^2$ (in thousands of dollars).

(a) Estimate ΔR if the price rises from \$50 to \$53.

(b) If $p = 80$, how will revenue be affected by a small increase in price? Explain using the Linear Approximation.

SOLUTION

(a) If the price is raised from \$50 to \$53, then $\Delta p = 3$ and

$$\Delta R \approx R'(50)\Delta p = (1.5 - 0.02(50))(3) = 1.5$$

We therefore estimate an increase of \$1500 in revenue.

(b) Because $R'(80) = 1.5 - 0.02(80) = -0.1$, the Linear Approximation gives $\Delta R \approx -0.1\Delta p$. A small increase in price would thus result in a decrease in revenue.

16. A store sells 80 MP4 players per week when the players are priced at $P = \$75$. Estimate the number N sold if P is raised to \$80, assuming that $dN/dP = -4$. Estimate N if the price is lowered to \$69.

SOLUTION If P is raised to \$80, then $\Delta P = 5$. With the assumption that $dN/dP = -4$, we estimate, using the Linear Approximation, that

$$\Delta N \approx \frac{dN}{dP}\Delta P = (-4)(5) = -20;$$

therefore, we estimate that only 60 MP4 players will be sold per week when the price is \$80. On the other hand, if the price is lowered to \$69, then $\Delta P = -6$ and $\Delta N \approx (-4)(-6) = 24$. We therefore estimate that 104 MP4 players will be sold per week when the price is \$69.

17. The circumference of a sphere is measured at $C = 100$ cm. Estimate the maximum percentage error in V if the error in C is at most 3 cm.

SOLUTION The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the circumference is $C = 2\pi r$, where r is the radius of the sphere. Thus, $r = \frac{1}{2\pi}C$ and

$$V = \frac{4}{3}\pi\left(\frac{C}{2\pi}\right)^3 = \frac{1}{6\pi^2}C^3.$$

Using the Linear Approximation,

$$\Delta V \approx \frac{dV}{dC}\Delta C = \frac{1}{2\pi^2}C^2\Delta C,$$

so

$$\frac{\Delta V}{V} \approx \frac{\frac{1}{2\pi^2}C^2\Delta C}{\frac{1}{6\pi^2}C^3} = 3\frac{\Delta C}{C}.$$

With $C = 100$ cm and ΔC at most 3 cm, we estimate that the maximum percentage error in V is $3\frac{3}{100} = 0.09$, or 9%.

18. Show that $\sqrt{a^2+b} \approx a + \frac{b}{2a}$ if b is small. Use this to estimate $\sqrt{26}$ and find the error using a calculator.

SOLUTION Let $a > 0$ and let $f(b) = \sqrt{a^2+b}$. Then

$$f'(b) = \frac{1}{2\sqrt{a^2+b}}.$$

By the Linear Approximation, $f(b) \approx f(0) + f'(0)b$, so

$$\sqrt{a^2+b} \approx a + \frac{b}{2a}.$$

To estimate $\sqrt{26}$, let $a = 5$ and $b = 1$. Then

$$\sqrt{26} = \sqrt{5^2+1} \approx 5 + \frac{1}{10} = 5.1.$$

The error in this estimate is $|\sqrt{26} - 5.1| = 9.80 \times 10^{-4}$.

19. Use the Intermediate Value Theorem to prove that $\sin x - \cos x = 3x$ has a solution, and use Rolle's Theorem to show that this solution is unique.

SOLUTION Let $f(x) = \sin x - \cos x - 3x$, and observe that each root of this function corresponds to a solution of the equation $\sin x - \cos x = 3x$. Now,

$$f\left(-\frac{\pi}{2}\right) = -1 + \frac{3\pi}{2} > 0 \qquad \text{and} \qquad f(0) = -1 < 0.$$

Because f is continuous on $(-\frac{\pi}{2}, 0)$ and $f(-\frac{\pi}{2})$ and $f(0)$ are of opposite sign, the Intermediate Value Theorem guarantees there exists a $c \in (-\frac{\pi}{2}, 0)$ such that $f(c) = 0$. Thus, the equation $\sin x - \cos x = 3x$ has at least one solution.

Next, suppose that the equation $\sin x - \cos x = 3x$ has two solutions, and therefore $f(x)$ has two roots, say a and b. Because f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b) = 0$, Rolle's Theorem guarantees there exists $c \in (a, b)$ such that $f'(c) = 0$. However,

$$f'(x) = \cos x + \sin x - 3 \le -1$$

for all x. We have reached a contradiction. Consequently, $f(x)$ has a unique root and the equation $\sin x - \cos x = 3x$ has a unique solution.

20. Show that $f(x) = 2x^3 + 2x + \sin x + 1$ has precisely one real root.

SOLUTION We have $f(0) = 1$ and $f(-1) = -3 + \sin(-1) = -3.84 < 0$. Therefore $f(x)$ has a root in the interval $[-1, 0]$. Now, suppose that $f(x)$ has two real roots, say a and b. Because $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b) = 0$, Rolle's Theorem guarantees that there exists $c \in (a, b)$ such that $f'(c) = 0$. However

$$f'(x) = 6x^2 + 2 + \cos x > 0$$

for all x (since $2 + \cos x \ge 0$). We have reached a contradiction. Consequently, $f(x)$ must have precisely one real root.

21. Verify the MVT for $f(x) = \ln x$ on $[1, 4]$.

SOLUTION Let $f(x) = \ln x$. On the interval $[1, 4]$, this function is continuous and differentiable, so the MVT applies. Now, $f'(x) = \frac{1}{x}$, so

$$\frac{1}{c} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\ln 4 - \ln 1}{4 - 1} = \frac{1}{3} \ln 4,$$

or

$$c = \frac{3}{\ln 4} \approx 2.164 \in (1, 4).$$

22. Suppose that $f(1) = 5$ and $f'(x) \ge 2$ for $x \ge 1$. Use the MVT to show that $f(8) \ge 19$.

SOLUTION Because f is continuous on $[1, 8]$ and differentiable on $(1, 8)$, the Mean Value Theorem guarantees there exists a $c \in (1, 8)$ such that

$$f'(c) = \frac{f(8) - f(1)}{8 - 1} \qquad \text{or} \qquad f(8) = f(1) + 7f'(c).$$

Now, we are given that $f(1) = 5$ and that $f'(x) \ge 2$ for $x \ge 1$. Therefore,

$$f(8) \ge 5 + 7(2) = 19.$$

23. Use the MVT to prove that if $f'(x) \le 2$ for $x > 0$ and $f(0) = 4$, then $f(x) \le 2x + 4$ for all $x \ge 0$.

SOLUTION Let $x > 0$. Because f is continuous on $[0, x]$ and differentiable on $(0, x)$, the Mean Value Theorem guarantees there exists a $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \qquad \text{or} \qquad f(x) = f(0) + xf'(c).$$

Now, we are given that $f(0) = 4$ and that $f'(x) \le 2$ for $x > 0$. Therefore, for all $x \ge 0$,

$$f(x) \le 4 + x(2) = 2x + 4.$$

24. A function $f(x)$ has derivative $f'(x) = \dfrac{1}{x^4+1}$. Where on the interval $[1, 4]$ does $f(x)$ take on its maximum value?

SOLUTION Let

$$f'(x) = \frac{1}{x^4+1}.$$

Because $f'(x)$ is never 0 and exists for all x, the function f has no critical points on the interval $[1, 4]$ and so must take its maximum value at one of the interval endpoints. Moveover, as $f'(x) > 0$ for all x, the function f is increasing for all x. Consequently, on the interval $[1, 4]$, the function f must take its maximum value at $x = 4$.

In Exercises 25–30, find the critical points and determine whether they are minima, maxima, or neither.

25. $f(x) = x^3 - 4x^2 + 4x$

SOLUTION Let $f(x) = x^3 - 4x^2 + 4x$. Then $f'(x) = 3x^2 - 8x + 4 = (3x-2)(x-2)$, so that $x = \frac{2}{3}$ and $x = 2$ are critical points. Next, $f''(x) = 6x - 8$, so $f''(\frac{2}{3}) = -4 < 0$ and $f''(2) = 4 > 0$. Therefore, by the Second Derivative Test, $f(\frac{2}{3})$ is a local maximum while $f(2)$ is a local minimum.

26. $s(t) = t^4 - 8t^2$

SOLUTION Let $s(t) = t^4 - 8t^2$. Then $s'(t) = 4t^3 - 16t = 4t(t-2)(t+2)$, so that $t = 0$, $t = -2$ and $t = 2$ are critical points. Next, $s''(t) = 12t^2 - 16$, so $s''(-2) = 32 > 0$, $s''(0) = -16 < 0$ and $s''(2) = 32 > 0$. Therefore, by the Second Derivative Test, $s(0)$ is a local maximum while $s(-2)$ and $s(2)$ are local minima.

27. $f(x) = x^2(x+2)^3$

SOLUTION Let $f(x) = x^2(x+2)^3$. Then

$$f'(x) = 3x^2(x+2)^2 + 2x(x+2)^3 = x(x+2)^2(3x+2x+4) = x(x+2)^2(5x+4),$$

so that $x = 0$, $x = -2$ and $x = -\frac{4}{5}$ are critical points. The sign of the first derivative on the intervals surrounding the critical points is indicated in the table below. Based on this information, $f(-2)$ is neither a local maximum nor a local minimum, $f(-\frac{4}{5})$ is a local maximum and $f(0)$ is a local minimum.

Interval	$(-\infty, -2)$	$(-2, -\frac{4}{5})$	$(-\frac{4}{5}, 0)$	$(0, \infty)$
Sign of f'	+	+	−	+

28. $f(x) = x^{2/3}(1-x)$

SOLUTION Let $f(x) = x^{2/3}(1-x) = x^{2/3} - x^{5/3}$. Then

$$f'(x) = \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{2-5x}{3x^{1/3}},$$

so that $x = 0$ and $x = \frac{2}{5}$ are critical points. The sign of the first derivative on the intervals surrounding the critical points is indicated in the table below. Based on this information, $f(0)$ is a local minimum and $f(\frac{2}{5})$ is a local maximum.

Interval	$(-\infty, 0)$	$(0, \frac{2}{5})$	$(\frac{2}{5}, \infty)$
Sign of f'	−	+	−

29. $g(\theta) = \sin^2\theta + \theta$

SOLUTION Let $g(\theta) = \sin^2\theta + \theta$. Then

$$g'(\theta) = 2\sin\theta\cos\theta + 1 = 2\sin 2\theta + 1,$$

so the critical points are

$$\theta = \frac{3\pi}{4} + n\pi$$

for all integers n. Because $g'(\theta) \geq 0$ for all θ, it follows that $g\left(\dfrac{3\pi}{4} + n\pi\right)$ is neither a local maximum nor a local minimum for all integers n.

30. $h(\theta) = 2\cos 2\theta + \cos 4\theta$

SOLUTION Let $h(\theta) = 2\cos 2\theta + \cos 4\theta$. Then

$$h'(\theta) = -4\sin 2\theta - 4\sin 4\theta = -4\sin 2\theta(1 + 2\cos 2\theta),$$

so the critical points are

$$\theta = \frac{n\pi}{2}, \quad \theta = \frac{\pi}{3} + \pi n \quad \text{and} \quad \theta = \frac{2\pi}{3} + \pi n$$

for all integers n. Now,

$$h''(\theta) = -8\cos 2\theta - 16\cos 4\theta,$$

so

$$h''\left(\frac{n\pi}{2}\right) = -8\cos n\pi - 16\cos 2n\pi = -8(-1)^n - 16 < 0;$$

$$h''\left(\frac{\pi}{3} + n\pi\right) = -8\cos\frac{2\pi}{3} - 16\cos\frac{4\pi}{3} = 12 > 0; \text{ and}$$

$$h''\left(\frac{2\pi}{3} + n\pi\right) = -8\cos\frac{4\pi}{3} - 16\cos\frac{8\pi}{3} = 12 > 0,$$

for all integers n. Therefore, by the Second Derivative Test, $h\left(\frac{n\pi}{2}\right)$ is a local maximum, and $h\left(\frac{\pi}{3} + n\pi\right)$ and $h\left(\frac{2\pi}{3} + n\pi\right)$ are local minima for all integers n.

In Exercises 31–38, find the extreme values on the interval.

31. $f(x) = x(10 - x), \quad [-1, 3]$

SOLUTION Let $f(x) = x(10 - x) = 10x - x^2$. Then $f'(x) = 10 - 2x$, so that $x = 5$ is the only critical point. As this critical point is not in the interval $[-1, 3]$, we only need to check the value of f at the endpoints to determine the extreme values. Because $f(-1) = -11$ and $f(3) = 21$, the maximum value of $f(x) = x(10 - x)$ on the interval $[-1, 3]$ is 21 while the minimum value is -11.

32. $f(x) = 6x^4 - 4x^6, \quad [-2, 2]$

SOLUTION Let $f(x) = 6x^4 - 4x^6$. Then $f'(x) = 24x^3 - 24x^5 = 24x^3(1 - x^2)$, so that the critical points are $x = -1$, $x = 0$ and $x = 1$. The table below lists the value of f at each of the critical points and the endpoints of the interval $[-2, 2]$. Based on this information, the minimum value of $f(x) = 6x^4 - 4x^6$ on the interval $[-2, 2]$ is -170 and the maximum value is 2.

x	-2	-1	0	1	2
$f(x)$	-170	2	0	2	-170

33. $g(\theta) = \sin^2\theta - \cos\theta, \quad [0, 2\pi]$

SOLUTION Let $g(\theta) = \sin^2\theta - \cos\theta$. Then

$$g'(\theta) = 2\sin\theta\cos\theta + \sin\theta = \sin\theta(2\cos\theta + 1) = 0$$

when $\theta = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$. The table below lists the value of g at each of the critical points and the endpoints of the interval $[0, 2\pi]$. Based on this information, the minimum value of $g(\theta)$ on the interval $[0, 2\pi]$ is -1 and the maximum value is $\frac{5}{4}$.

θ	0	$2\pi/3$	π	$4\pi/3$	2π
$g(\theta)$	-1	5/4	1	5/4	-1

34. $R(t) = \dfrac{t}{t^2 + t + 1}, \quad [0, 3]$

SOLUTION Let $R(t) = \frac{t}{t^2+t+1}$. Then

$$R'(t) = \frac{t^2 + t + 1 - t(2t + 1)}{(t^2 + t + 1)^2} = \frac{1 - t^2}{(t^2 + t + 1)^2},$$

so that the critical points are $t = \pm 1$. Note that only $t = 1$ is on the interval $[0, 3]$. With $R(0) = 0$, $R(1) = \frac{1}{3}$ and $R(3) = \frac{3}{13}$, it follows that the minimum value of $R(t)$ on the interval $[0, 3]$ is 0 and the maximum value is $\frac{1}{3}$.

35. $f(x) = x^{2/3} - 2x^{1/3}$, $[-1, 3]$

SOLUTION Let $f(x) = x^{2/3} - 2x^{1/3}$. Then $f'(x) = \frac{2}{3}x^{-1/3} - \frac{2}{3}x^{-2/3} = \frac{2}{3}x^{-2/3}(x^{1/3} - 1)$, so that the critical points are $x = 0$ and $x = 1$. With $f(-1) = 3$, $f(0) = 0$, $f(1) = -1$ and $f(3) = \sqrt[3]{9} - 2\sqrt[3]{3} \approx -0.804$, it follows that the minimum value of $f(x)$ on the interval $[-1, 3]$ is -1 and the maximum value is 3.

36. $f(x) = x - \tan x$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

SOLUTION Let $f(x) = x - \tan x$. Then $f'(x) = 1 - \sec^2 x$, so that $x = 0$ is the only critical point on $[-1, 1]$. With $f(-1) = -1 - \tan(-1) > 0$, $f(0) = 0$ and $f(1) = 1 - \tan 1 < 0$, it follows that the minimum value of $f(x)$ on the interval $[-1, 1]$ is $1 - \tan 1 \approx -0.557$ and the maximum value is $-1 - \tan(-1) = -1 + \tan 1 \approx 0.557$.

37. $f(x) = x - 12\ln x$, $[5, 40]$

SOLUTION Let $f(x) = x - 12\ln x$. Then $f'(x) = 1 - \frac{12}{x}$, whence $x = 12$ is the only critical point. The minimum value of f is then $12 - 12\ln 12 \approx -17.818880$, and the maximum value is $40 - 12\ln 40 \approx -4.266553$. Note that $f(5) = 5 - 12\ln 5 \approx -14.313255$.

38. $f(x) = e^x - 20x - 1$, $[0, 5]$

SOLUTION Let $f(x) = e^x - 20x - 1$. Then $f'(x) = e^x - 20$, whence $x = \ln 20$ is the only critical point. The minimum value of f is then $20 - 20\ln 20 - 1 \approx - - 40.914645$, and the maximum value is $e^5 - 101 \approx 47.413159$. Note that $f(0) = 0$.

39. Find the critical points and extreme values of $f(x) = |x - 1| + |2x - 6|$ in $[0, 8]$.

SOLUTION Let

$$f(x) = |x - 1| + |2x - 6| = \begin{cases} 7 - 3x, & x < 1 \\ 5 - x, & 1 \le x < 3 \\ 3x - 7, & x \ge 3 \end{cases}.$$

The derivative of $f(x)$ is never zero but does not exist at the transition points $x = 1$ and $x = 3$. Thus, the critical points of f are $x = 1$ and $x = 3$. With $f(0) = 7$, $f(1) = 4$, $f(3) = 2$ and $f(8) = 17$, it follows that the minimum value of $f(x)$ on the interval $[0, 8]$ is 2 and the maximum value is 17.

40. Match the description of $f(x)$ with the graph of its *derivative* $f'(x)$ in Figure 1.

(a) $f(x)$ is increasing and concave up.

(b) $f(x)$ is decreasing and concave up.

(c) $f(x)$ is increasing and concave down.

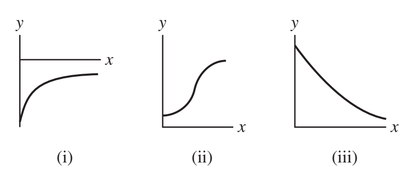


FIGURE 1 Graphs of the derivative.

SOLUTION

(a) If $f(x)$ is increasing and concave up, then $f'(x)$ is positive and increasing. This matches the graph in (ii).

(b) If $f(x)$ is decreasing and concave up, then $f'(x)$ is negative and increasing. This matches the graph in (i).

(c) If $f(x)$ is increasing and concave down, then $f'(x)$ is positive and decreasing. This matches the graph in (iii).

In Exercises 41–46, find the points of inflection.

41. $y = x^3 - 4x^2 + 4x$

SOLUTION Let $y = x^3 - 4x^2 + 4x$. Then $y' = 3x^2 - 8x + 4$ and $y'' = 6x - 8$. Thus, $y'' > 0$ and y is concave up for $x > \frac{4}{3}$, while $y'' < 0$ and y is concave down for $x < \frac{4}{3}$. Hence, there is a point of inflection at $x = \frac{4}{3}$.

42. $y = x - 2\cos x$

SOLUTION Let $y = x - 2\cos x$. Then $y' = 1 + 2\sin x$ and $y'' = 2\cos x$. Thus, $y'' > 0$ and y is concave up on each interval of the form

$$\left(\frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2}\right),$$

while $y'' < 0$ and y is concave down on each interval of the form

$$\left(\frac{(4n+1)\pi}{2}, \frac{(4n+3)\pi}{2}\right),$$

where n is any integer. Hence, there is a point of inflection at

$$x = \frac{(2n+1)\pi}{2}$$

for each integer n.

43. $y = \dfrac{x^2}{x^2+4}$

SOLUTION Let $y = \dfrac{x^2}{x^2+4} = 1 - \dfrac{4}{x^2+4}$. Then $y' = \dfrac{8x}{(x^2+4)^2}$ and

$$y'' = \frac{(x^2+4)^2(8) - 8x(2)(2x)(x^2+4)}{(x^2+4)^4} = \frac{8(4-3x^2)}{(x^2+4)^3}.$$

Thus, $y'' > 0$ and y is concave up for

$$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}},$$

while $y'' < 0$ and y is concave down for

$$|x| \geq \frac{2}{\sqrt{3}}.$$

Hence, there are points of inflection at

$$x = \pm\frac{2}{\sqrt{3}}.$$

44. $y = \dfrac{x}{(x^2-4)^{1/3}}$

SOLUTION Let $y = \dfrac{x}{(x^2-4)^{1/3}}$. Then

$$y' = \frac{(x^2-4)^{1/3} - \frac{1}{3}x(x^2-4)^{-2/3}(2x)}{(x^2-4)^{2/3}} = \frac{1}{3}\frac{x^2-12}{(x^2-4)^{4/3}}$$

and

$$y'' = \frac{1}{3}\frac{(x^2-4)^{4/3}(2x) - (x^2-12)\frac{4}{3}(x^2-4)^{1/3}(2x)}{(x^2-4)^{8/3}} = \frac{2x(36-x^2)}{9(x^2-4)^{7/3}}.$$

Thus, $y'' > 0$ and y is concave up for $x < -6, -2 < x < 0, 2 < x < 6$, while $y'' < 0$ and y is concave down for $-6 < x < -2, 0 < x < 2, x > 6$. Hence, there are points of inflection at $x = \pm 6$ and $x = 0$. Note that $x = \pm 2$ are not points of inflection because these points are not in the domain of the function.

45. $f(x) = (x^2 - x)e^{-x}$

SOLUTION Let $f(x) = (x^2 - x)e^{-x}$. Then

$$y' = -(x^2 - x)e^{-x} + (2x-1)e^{-x} = -(x^2 - 3x + 1)e^{-x},$$

and

$$y'' = (x^2 - 3x + 1)e^{-x} - (2x-3)e^{-x} = e^{-x}(x^2 - 5x + 4) = e^{-x}(x-1)(x-4).$$

Thus, $y'' > 0$ and y is concave up for $x < 1$ and for $x > 4$, while $y'' < 0$ and y is concave down for $1 < x < 4$. Hence, there are points of inflection at $x = 1$ and $x = 4$.

46. $f(x) = x(\ln x)^2$

SOLUTION Let $f(x) = x(\ln x)^2$. Then

$$y' = x \cdot 2\ln x \cdot \frac{1}{x} + (\ln x)^2 = 2\ln x + (\ln x)^2,$$

and

$$y'' = \frac{2}{x} + \frac{2}{x}\ln x = \frac{2}{x}(1 + \ln x).$$

Thus, $y'' > 0$ and y is concave up for $x > \frac{1}{e}$, while $y'' < 0$ and y is concave down for $0 < x < \frac{1}{e}$. Hence, there is a point of inflection at $x = \frac{1}{e}$.

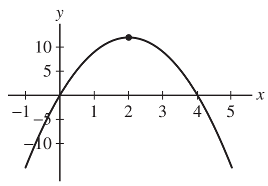
In Exercises 47–56, sketch the graph, noting the transition points and asymptotic behavior.

47. $y = 12x - 3x^2$

SOLUTION Let $y = 12x - 3x^2$. Then $y' = 12 - 6x$ and $y'' = -6$. It follows that the graph of $y = 12x - 3x^2$ is increasing for $x < 2$, decreasing for $x > 2$, has a local maximum at $x = 2$ and is concave down for all x. Because

$$\lim_{x\to\pm\infty}(12x - 3x^2) = -\infty,$$

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

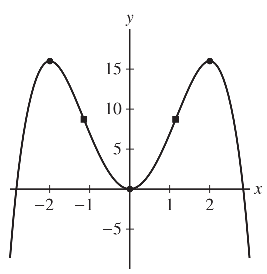


48. $y = 8x^2 - x^4$

SOLUTION Let $y = 8x^2 - x^4$. Then $y' = 16x - 4x^3 = 4x(4 - x^2)$ and $y'' = 16 - 12x^2 = 4(4 - 3x^2)$. It follows that the graph of $y = 8x^2 - x^4$ is increasing for $x < -2$ and $0 < x < 2$, decreasing for $-2 < x < 0$ and $x > 2$, has local maxima at $x = \pm 2$, has a local minimum at $x = 0$, is concave down for $|x| > 2/\sqrt{3}$, is concave up for $|x| < 2/\sqrt{3}$ and has inflection points at $x = \pm 2/\sqrt{3}$. Because

$$\lim_{x\to\pm\infty}(8x^2 - x^4) = -\infty,$$

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

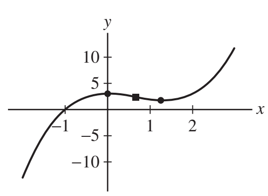


49. $y = x^3 - 2x^2 + 3$

SOLUTION Let $y = x^3 - 2x^2 + 3$. Then $y' = 3x^2 - 4x$ and $y'' = 6x - 4$. It follows that the graph of $y = x^3 - 2x^2 + 3$ is increasing for $x < 0$ and $x > \frac{4}{3}$, is decreasing for $0 < x < \frac{4}{3}$, has a local maximum at $x = 0$, has a local minimum at $x = \frac{4}{3}$, is concave up for $x > \frac{2}{3}$, is concave down for $x < \frac{2}{3}$ and has a point of inflection at $x = \frac{2}{3}$. Because

$$\lim_{x\to-\infty}(x^3 - 2x^2 + 3) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty}(x^3 - 2x^2 + 3) = \infty,$$

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

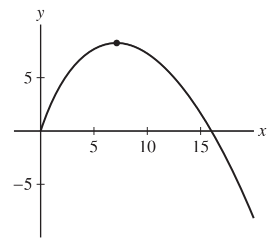


50. $y = 4x - x^{3/2}$

SOLUTION Let $y = 4x - x^{3/2}$. First note that the domain of this function is $x \ge 0$. Now, $y' = 4 - \frac{3}{2}x^{1/2}$ and $y'' = -\frac{3}{4}x^{-1/2}$. It follows that the graph of $y = 4x - x^{3/2}$ is increasing for $0 < x < \frac{64}{9}$, is decreasing for $x > \frac{64}{9}$, has a local maximum at $x = \frac{64}{9}$ and is concave down for all $x > 0$. Because

$$\lim_{x\to\infty}(4x - x^{3/2}) = -\infty,$$

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.



51. $y = \dfrac{x}{x^3 + 1}$

SOLUTION Let $y = \dfrac{x}{x^3+1}$. Then

$$y' = \frac{x^3 + 1 - x(3x^2)}{(x^3+1)^2} = \frac{1 - 2x^3}{(x^3+1)^2}$$

and

$$y'' = \frac{(x^3+1)^2(-6x^2) - (1 - 2x^3)(2)(x^3+1)(3x^2)}{(x^3+1)^4} = -\frac{6x^2(2 - x^3)}{(x^3+1)^3}.$$

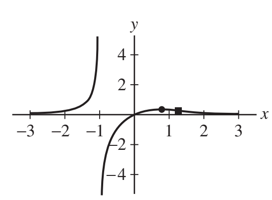
It follows that the graph of $y = \dfrac{x}{x^3+1}$ is increasing for $x < -1$ and $-1 < x < \sqrt[3]{\frac{1}{2}}$, is decreasing for $x > \sqrt[3]{\frac{1}{2}}$, has a local maximum at $x = \sqrt[3]{\frac{1}{2}}$, is concave up for $x < -1$ and $x > \sqrt[3]{2}$, is concave down for $-1 < x < 0$ and $0 < x < \sqrt[3]{2}$ and has a point of inflection at $x = \sqrt[3]{2}$. Note that $x = -1$ is not an inflection point because $x = -1$ is not in the domain of the function. Now,

$$\lim_{x\to\pm\infty} \frac{x}{x^3+1} = 0,$$

so $y = 0$ is a horizontal asymptote. Moreover,

$$\lim_{x\to -1-} \frac{x}{x^3+1} = \infty \qquad \text{and} \qquad \lim_{x\to -1+} \frac{x}{x^3+1} = -\infty,$$

so $x = -1$ is a vertical asymptote. The graph is shown below.



52. $y = \dfrac{x}{(x^2 - 4)^{2/3}}$

SOLUTION Let $y = \dfrac{x}{(x^2-4)^{2/3}}$. Then

$$y' = \frac{(x^2-4)^{2/3} - \frac{2}{3}x(x^2-4)^{-1/3}(2x)}{(x^2-4)^{4/3}} = -\frac{1}{3}\frac{x^2+12}{(x^2-4)^{5/3}}$$

and

$$y'' = -\frac{1}{3}\frac{(x^2-4)^{5/3}(2x) - (x^2+12)\frac{5}{3}(x^2-4)^{2/3}(2x)}{(x^2-4)^{10/3}} = \frac{4x(x^2+36)}{9(x^2-4)^{8/3}}.$$

It follows that the graph of $y = \frac{x}{(x^2-4)^{2/3}}$ is increasing for $-2 < x < 2$, is decreasing for $|x| > 2$, has no local extreme values, is concave up for $0 < x < 2$, $x > 2$, is concave down for $x < -2$, $-2 < x < 0$ and has a point of inflection at $x = 0$. Note that $x = \pm 2$ are neither local extreme values nor inflection points because $x = \pm 2$ are not in the domain of the function. Now,

$$\lim_{x\to\pm\infty} \frac{x}{(x^2-4)^{2/3}} = 0,$$

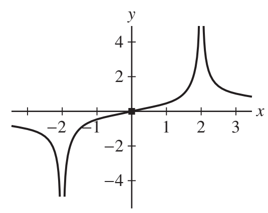
so $y = 0$ is a horizontal asymptote. Moreover,

$$\lim_{x\to -2-} \frac{x}{(x^2-4)^{2/3}} = -\infty \qquad \text{and} \qquad \lim_{x\to -2+} \frac{x}{(x^2-4)^{2/3}} = -\infty$$

while

$$\lim_{x\to 2-} \frac{x}{(x^2-4)^{2/3}} = \infty \qquad \text{and} \qquad \lim_{x\to 2+} \frac{x}{(x^2-4)^{2/3}} = \infty,$$

so $x = \pm 2$ are vertical asymptotes. The graph is shown below.

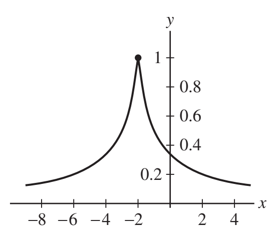


53. $y = \frac{1}{|x+2|+1}$

SOLUTION Let $y = \frac{1}{|x+2|+1}$. Because

$$\lim_{x\to\pm\infty} \frac{1}{|x+2|+1} = 0,$$

the graph of this function has a horizontal asymptote of $y = 0$. The graph has no vertical asymptotes as $|x+2|+1 \ge 1$ for all x. The graph is shown below. From this graph we see there is a local maximum at $x = -2$.



54. $y = \sqrt{2-x^3}$

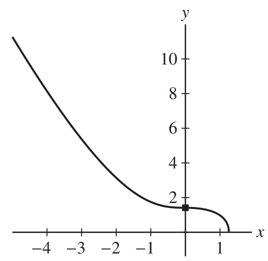
SOLUTION Let $y = \sqrt{2-x^3}$. Note that the domain of this function is $x \le \sqrt[3]{2}$. Moreover, the graph has no vertical and no horizontal asymptotes. With

$$y' = \frac{1}{2}(2-x^3)^{-1/2}(-3x^2) = -\frac{3x^2}{2\sqrt{2-x^3}}$$

and

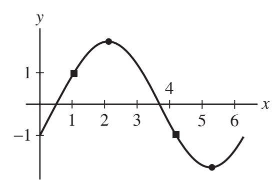
$$y'' = \frac{1}{2}(2-x^3)^{-1/2}(-6x) - \frac{3}{4}x^2(2-x^3)^{-3/2}(3x^2) = \frac{3x(x^3-8)}{4(2-x^3)^{3/2}},$$

it follows that the graph of $y = \sqrt{2-x^3}$ is decreasing over its entire domain, is concave up for $x < 0$, is concave down for $0 < x < \sqrt[3]{2}$ and has a point of inflection at $x = 0$. The graph is shown below.



55. $y = \sqrt{3}\sin x - \cos x$ on $[0, 2\pi]$

SOLUTION Let $y = \sqrt{3}\sin x - \cos x$. Then $y' = \sqrt{3}\cos x + \sin x$ and $y'' = -\sqrt{3}\sin x + \cos x$. It follows that the graph of $y = \sqrt{3}\sin x - \cos x$ is increasing for $0 < x < 5\pi/6$ and $11\pi/6 < x < 2\pi$, is decreasing for $5\pi/6 < x < 11\pi/6$, has a local maximum at $x = 5\pi/6$, has a local minimum at $x = 11\pi/6$, is concave up for $0 < x < \pi/3$ and $4\pi/3 < x < 2\pi$, is concave down for $\pi/3 < x < 4\pi/3$ and has points of inflection at $x = \pi/3$ and $x = 4\pi/3$. The graph is shown below.



56. $y = 2x - \tan x$ on $[0, 2\pi]$

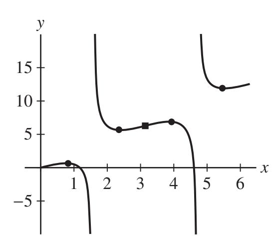
SOLUTION Let $y = 2x - \tan x$. Then $y' = 2 - \sec^2 x$ and $y'' = -2\sec^2 x \tan x$. It follows that the graph of $y = 2x - \tan x$ is increasing for $0 < x < \pi/4, 3\pi/4 < x < 5\pi/4, 7\pi/4 < x < 2\pi$, is decreasing for $\pi/4 < x < \pi/2, \pi/2 < x < 3\pi/4, 5\pi/4 < x < 3\pi/2, 3\pi/2 < x < 7\pi/4$, has local minima at $x = 3\pi/4$ and $x = 7\pi/4$, has local maxima at $x = \pi/4$ and $x = 5\pi/4$, is concave up for $\pi/2 < x < \pi$ and $3\pi/2 < x < 2\pi$, is concave down for $0 < x < \pi/2$ and $\pi < x < 3\pi/2$ and has an inflection point at $x = \pi$. Moreover, because

$$\lim_{x\to\pi/2-}(2x - \tan x) = -\infty \qquad \text{and} \qquad \lim_{x\to\pi/2+}(2x - \tan x) = \infty,$$

while

$$\lim_{x\to 3\pi/2-}(2x - \tan x) = -\infty \qquad \text{and} \qquad \lim_{x\to 3\pi/2+}(2x - \tan x) = \infty,$$

the graph has vertical asymptotes at $x = \pi/2$ and $x = 3\pi/2$. The graph is shown below.



57. Draw a curve $y = f(x)$ for which f' and f'' have signs as indicated in Figure 2.

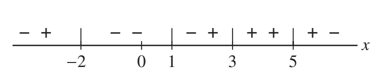
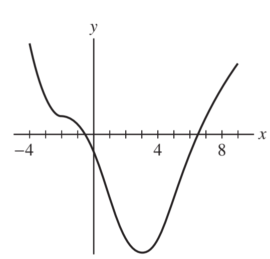


FIGURE 2

SOLUTION The figure below depicts a curve for which $f'(x)$ and $f''(x)$ have the required signs.



58. Find the dimensions of a cylindrical can with a bottom but no top of volume 4 m^3 that uses the least amount of metal.

SOLUTION Let the cylindrical can have height h and radius r. Then

$$V = \pi r^2 h = 4 \quad \text{so} \quad h = \frac{4}{\pi r^2}.$$

The amount of metal needed to make the can is then

$$M = 2\pi r h + \pi r^2 = \frac{8}{r} + \pi r^2.$$

Now,

$$M'(r) = -\frac{8}{r^2} + 2\pi r = 0 \quad \text{when} \quad r = \sqrt[3]{\frac{4}{\pi}}.$$

Because $M \to \infty$ as $r \to 0+$ and as $r \to \infty$, M must achieve its minimum for

$$r = \sqrt[3]{\frac{4}{\pi}} \text{ m}.$$

The height of the can is

$$h = \frac{4}{\pi r^2} = \sqrt[3]{\frac{4}{\pi}} \text{ m}.$$

59. A rectangular box of height h with square base of side b has volume $V = 4$ m^3. Two of the side faces are made of material costing \$40/m^2. The remaining sides cost \$20/m^2. Which values of b and h minimize the cost of the box?

SOLUTION Because the volume of the box is

$$V = b^2 h = 4 \quad \text{it follows that} \quad h = \frac{4}{b^2}.$$

Now, the cost of the box is

$$C = 40(2bh) + 20(2bh) + 20b^2 = 120bh + 20b^2 = \frac{480}{b} + 20b^2.$$

Thus,

$$C'(b) = -\frac{480}{b^2} + 40b = 0$$

when $b = \sqrt[3]{12}$ meters. Because $C(b) \to \infty$ as $b \to 0+$ and as $b \to \infty$, it follows that cost is minimized when $b = \sqrt[3]{12}$ meters and $h = \frac{1}{3}\sqrt[3]{12}$ meters.

60. The corn yield on a certain farm is

$$Y = -0.118x^2 + 8.5x + 12.9 \quad \text{(bushels per acre)}$$

where x is the number of corn plants per acre (in thousands). Assume that corn seed costs \$1.25 (per thousand seeds) and that corn can be sold for \$1.50/bushel. Let $P(x)$ be the profit (revenue minus the cost of seeds) at planting level x.

(a) Compute $P(x_0)$ for the value x_0 that maximizes yield Y.

(b) Find the maximum value of $P(x)$. Does maximum yield lead to maximum profit?

SOLUTION

(a) Let $Y = -0.118x^2 + 8.5x + 12.9$. Then $Y' = -0.236x + 8.5 = 0$ when

$$x_0 = \frac{8.5}{0.236} = 36.017 \text{ thousand corn plants/acre}.$$

Because $Y'' = -0.236 < 0$ for all x, x_0 corresponds to a maximum value for Y. Thus, yield is maximized for a planting level of 36,017 corn plants per acre. At this planting level, the profit is

$$1.5Y(x_0) - 1.25x_0 = 1.5(165.972) - 1.25(36.017) = \$203.94/\text{acre}.$$

(b) As a function of planting level x, the profit is

$$P(x) = 1.5Y(x) - 1.25x = -0.177x^2 + 11.5x + 19.35.$$

Then, $P'(x) = -0.354x + 11.5 = 0$ when

$$x_1 = \frac{11.5}{0.354} = 32.486 \text{ thousand corn plants/acre.}$$

Because $P''(x) = -0.354 < 0$ for all x, x_1 corresponds to a maximum value for P. Thus, profit is maximized for a planting level of 32,486 corn plants per acre.

(c) Note the planting levels obtained in parts (a) and (b) are different. Thus, a maximum yield does not lead to maximum profit.

61. Let $N(t)$ be the size of a tumor (in units of 10^6 cells) at time t (in days). According to the **Gompertz Model**, $dN/dt = N(a - b\ln N)$ where a, b are positive constants. Show that the maximum value of N is $e^{\frac{a}{b}}$ and that the tumor increases most rapidly when $N = e^{\frac{a}{b}-1}$.

SOLUTION Given $dN/dt = N(a - b\ln N)$, the critical points of N occur when $N = 0$ and when $N = e^{a/b}$. The sign of $N'(t)$ changes from positive to negative at $N = e^{a/b}$ so the maximum value of N is $e^{a/b}$. To determine when N changes most rapidly, we calculate

$$N''(t) = N\left(-\frac{b}{N}\right) + a - b\ln N = (a - b) - b\ln N.$$

Thus, $N'(t)$ is increasing for $N < e^{a/b-1}$, is decreasing for $N > e^{a/b-1}$ and is therefore maximum when $N = e^{a/b-1}$. Therefore, the tumor increases most rapidly when $N = e^{\frac{a}{b}-1}$.

62. A truck gets 10 miles per gallon of diesel fuel traveling along an interstate highway at 50 mph. This mileage decreases by 0.15 mpg for each mile per hour increase above 50 mph.

(a) If the truck driver is paid \$30/hour and diesel fuel costs $P = \$3$/gal, which speed v between 50 and 70 mph will minimize the cost of a trip along the highway? Notice that the actual cost depends on the length of the trip, but the optimal speed does not.

(b) GU Plot cost as a function of v (choose the length arbitrarily) and verify your answer to part (a).

(c) GU Do you expect the optimal speed v to increase or decrease if fuel costs go down to $P = \$2$/gal? Plot the graphs of cost as a function of v for $P = 2$ and $P = 3$ on the same axis and verify your conclusion.

SOLUTION

(a) If the truck travels L miles at a speed of v mph, then the time required is L/v, and the wages paid to the driver are $30L/v$. The cost of the fuel is

$$\frac{3L}{10 - 0.15(v - 50)} = \frac{3L}{17.5 - 0.15v};$$

the total cost is therefore

$$C(v) = \frac{30L}{v} + \frac{3L}{17.5 - 0.15v}.$$

Solving

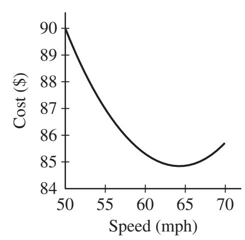
$$C'(v) = L\left(-\frac{30}{v^2} + \frac{0.45}{(17.5 - 0.15v)^2}\right) = 0$$

yields

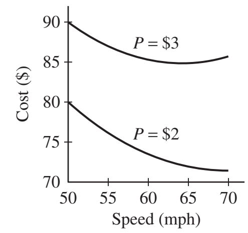
$$v = \frac{175\sqrt{6}}{3 + 1.5\sqrt{6}} \approx 64.2 \text{ mph.}$$

Because $C(50) = 0.9L$, $C(64.2) \approx 0.848L$ and $C(70) \approx 0.857L$, we see that the optimal speed is $v \approx 64.2$ mph.

(b) The cost as a function of speed is shown below for $L = 100$. The optimal speed is clearly around 64 mph.



(c) We expect v to increase if P goes down to \$2 per gallon. When gas is cheaper, it is better to drive faster and thereby save on the driver's wages. The cost as a function of speed for $P = 2$ and $P = 3$ is shown below (with $L = 100$). When $P = 2$, the optimal speed is $v = 70$ mph, which is an increase over the optimal speed when $P = 3$.



63. Find the maximum volume of a right-circular cone placed upside-down in a right-circular cone of radius $R = 3$ and height $H = 4$ as in Figure 3. A cone of radius r and height h has volume $\frac{1}{3}\pi r^2 h$.

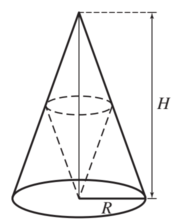


FIGURE 3

SOLUTION Let r denote the radius and h the height of the upside down cone. By similar triangles, we obtain the relation

$$\frac{4-h}{r} = \frac{4}{3} \qquad \text{so} \qquad h = 4\left(1 - \frac{r}{3}\right)$$

and the volume of the upside down cone is

$$V(r) = \frac{1}{3}\pi r^2 h = \frac{4}{3}\pi\left(r^2 - \frac{r^3}{3}\right)$$

for $0 \le r \le 3$. Thus,

$$\frac{dV}{dr} = \frac{4}{3}\pi\left(2r - r^2\right),$$

and the critical points are $r = 0$ and $r = 2$. Because $V(0) = V(3) = 0$ and

$$V(2) = \frac{4}{3}\pi\left(4 - \frac{8}{3}\right) = \frac{16}{9}\pi,$$

the maximum volume of a right-circular cone placed upside down in a right-circular cone of radius 3 and height 4 is

$$\frac{16}{9}\pi.$$

64. Redo Exercise 63 for arbitrary R and H.

SOLUTION Let r denote the radius and h the height of the upside down cone. By similar triangles, we obtain the relation

$$\frac{H-h}{r} = \frac{H}{R} \qquad \text{so} \qquad h = H\left(1 - \frac{r}{R}\right)$$

and the volume of the upside down cone is

$$V(r) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi H\left(r^2 - \frac{r^3}{R}\right)$$

for $0 \le r \le R$. Thus,

$$\frac{dV}{dr} = \frac{1}{3}\pi H\left(2r - \frac{3r^2}{R}\right),$$

and the critical points are $r = 0$ and $r = 2R/3$. Because $V(0) = V(R) = 0$ and

$$V\left(\frac{2R}{3}\right) = \frac{1}{3}\pi H\left(\frac{4R^2}{9} - \frac{8R^2}{27}\right) = \frac{4}{81}\pi R^2 H,$$

the maximum volume of a right-circular cone placed upside down in a right-circular cone of radius R and height H is

$$\frac{4}{81}\pi R^2 H.$$

65. Show that the maximum area of a parallelogram $ADEF$ that is inscribed in a triangle ABC, as in Figure 4, is equal to one-half the area of $\triangle ABC$.

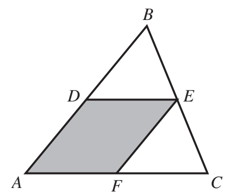


FIGURE 4

SOLUTION Let θ denote the measure of angle BAC. Then the area of the parallelogram is given by $\overline{AD} \cdot \overline{AF} \sin\theta$. Now, suppose that

$$\overline{BE}/\overline{BC} = x.$$

Then, by similar triangles, $\overline{AD} = (1-x)\overline{AB}$, $\overline{AF} = \overline{DE} = x\overline{AC}$, and the area of the parallelogram becomes $\overline{AB} \cdot \overline{AC} x(1-x)\sin\theta$. The function $x(1-x)$ achieves its maximum value of $\frac{1}{4}$ when $x = \frac{1}{2}$. Thus, the maximum area of a parallelogram inscribed in a triangle ΔABC is

$$\frac{1}{4}\overline{AB} \cdot \overline{AC} \sin\theta = \frac{1}{2}\left(\frac{1}{2}\overline{AB} \cdot \overline{AC} \sin\theta\right) = \frac{1}{2}\,(\text{area of } \Delta ABC)\,.$$

66. A box of volume 8 m^3 with a square top and bottom is constructed out of two types of metal. The metal for the top and bottom costs \$50/m^2 and the metal for the sides costs \$30/m^2. Find the dimensions of the box that minimize total cost.

SOLUTION Let the square base have side length s and the box have height h. Then

$$V = s^2 h = 8 \quad \text{so} \quad h = \frac{8}{s^2}.$$

The cost of the box is then

$$C = 100s^2 + 120sh = 100s^2 + \frac{960}{s}.$$

Now,

$$C'(s) = 200s - \frac{960}{s^2} = 0 \quad \text{when} \quad s = \sqrt[3]{4.8}.$$

Because $C(s) \to \infty$ as $s \to 0+$ and as $s \to \infty$, it follows that total cost is minimized when $s = \sqrt[3]{4.8} \approx 1.69$ meters. The height of the box is

$$h = \frac{8}{s^2} \approx 2.81 \text{ meters}.$$

67. Let $f(x)$ be a function whose graph does not pass through the x-axis and let $Q = (a, 0)$. Let $P = (x_0, f(x_0))$ be the point on the graph closest to Q (Figure 5). Prove that $\overline{PQ}$ is perpendicular to the tangent line to the graph of x_0. *Hint:* Find the minimum value of the *square* of the distance from $(x, f(x))$ to $(a, 0)$.

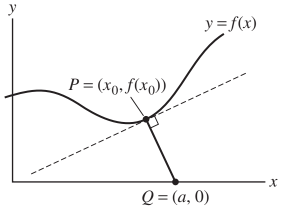


FIGURE 5

SOLUTION Let $P = (a, 0)$ and let $Q = (x_0, f(x_0))$ be the point on the graph of $y = f(x)$ closest to P. The slope of the segment joining P and Q is then

$$\frac{f(x_0)}{x_0 - a}.$$

Now, let

$$q(x) = \sqrt{(x-a)^2 + (f(x))^2},$$

the distance from the arbitrary point $(x, f(x))$ on the graph of $y = f(x)$ to the point P. As $(x_0, f(x_0))$ is the point closest to P, we must have

$$q'(x_0) = \frac{2(x_0 - a) + 2f(x_0)f'(x_0)}{\sqrt{(x_0-a)^2 + (f(x_0))^2}} = 0.$$

Thus,

$$f'(x_0) = -\frac{x_0 - a}{f(x_0)} = -\left(\frac{f(x_0)}{x_0 - a}\right)^{-1}.$$

In other words, the slope of the segment joining P and Q is the negative reciprocal of the slope of the line tangent to the graph of $y = f(x)$ at $x = x_0$; hence; the two lines are perpendicular.

68. Take a circular piece of paper of radius R, remove a sector of angle θ (Figure 6), and fold the remaining piece into a cone-shaped cup. Which angle θ produces the cup of largest volume?

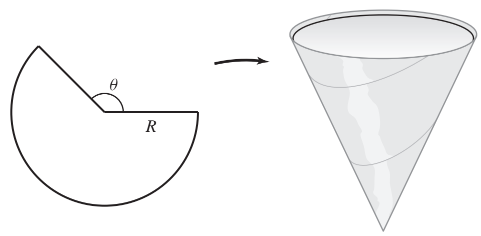


FIGURE 6

SOLUTION Let r denote the radius and h denote the height of the cone-shaped cup. Having removed an angle of θ from the paper, there is an arc of length $(2\pi - \theta)R$ remaining to form the circumference of the cup; hence

$$r = \frac{(2\pi - \theta)R}{2\pi} = \left(1 - \frac{\theta}{2\pi}\right)R.$$

The height of the cup is then

$$h = \sqrt{R^2 - \left(1 - \frac{\theta}{2\pi}\right)^2 R^2} = R\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2},$$

and the volume of the cup is

$$V(\theta) = \frac{1}{3}\pi R^3 \left(1 - \frac{\theta}{2\pi}\right)^2 \sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2}$$

for $0 \le \theta \le 2\pi$. Now,

$$\frac{dV}{d\theta} = 2\left(1 - \frac{\theta}{2\pi}\right)\left(-\frac{1}{2\pi}\right)\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2} + \left(1 - \frac{\theta}{2\pi}\right)^2 \frac{(-2)\left(1 - \frac{\theta}{2\pi}\right)\left(-\frac{1}{2\pi}\right)}{\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2}}$$

$$= \left(1 - \frac{\theta}{2\pi}\right)\left(-\frac{1}{2\pi}\right)\frac{2\left(1 - \left(1 - \frac{\theta}{2\pi}\right)^2\right) - \left(1 - \frac{\theta}{2\pi}\right)^2}{\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2}},$$

so that $\theta = 2\pi$ and $\theta = 2\pi \pm \frac{2\pi\sqrt{6}}{3}$ are critical points. With $V(0) = V(2\pi) = 0$ and

$$V\left(2\pi - \frac{2\pi\sqrt{6}}{3}\right) = \frac{2\sqrt{3}}{27}\pi R^3,$$

the volume of the cup is maximized when $\theta = 2\pi - \frac{2\pi\sqrt{6}}{3}$.

69. Use Newton's Method to estimate $\sqrt[3]{25}$ to four decimal places.

SOLUTION Let $f(x) = x^3 - 25$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 25}{3x_n^2}.$$

With $x_0 = 3$, we find

n	1	2	3
x_n	2.925925926	2.924018982	2.924017738

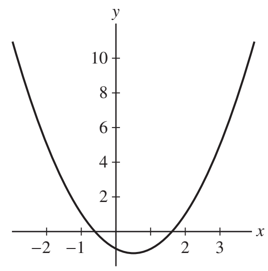
Thus, to four decimal places $\sqrt[3]{25} = 2.9240$.

70. Use Newton's Method to find a root of $f(x) = x^2 - x - 1$ to four decimal places.

SOLUTION Let $f(x) = x^2 - x - 1$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - x_n - 1}{2x_n - 1}.$$

The graph below suggests the two roots of $f(x)$ are located near $x = -1$ and $x = 2$.



With $x_0 = -1$, we find

n	1	2	3	4
x_n	−0.6666666667	−0.6190476191	−0.6180344477	−0.6180339889

On the other hand, with $x_0 = 2$, we find

n	1	2	3	4
x_n	1.666666667	1.619047619	1.618034448	1.618033989

Thus, to four decimal places, the roots of $f(x) = x^2 - x - 1$ are -0.6180 and 1.6180.

In Exercises 71–84, calculate the indefinite integral.

71. $\int (4x^3 - 2x^2)\,dx$

SOLUTION $\int (4x^3 - 2x^2)\,dx = x^4 - \frac{2}{3}x^3 + C.$

72. $\int x^{9/4}\,dx$

SOLUTION $\int x^{9/4}\,dx = \frac{4}{13}x^{13/4} + C.$

73. $\int \sin(\theta - 8)\,d\theta$

SOLUTION $\int \sin(\theta - 8)\,d\theta = -\cos(\theta - 8) + C.$

74. $\int \cos(5 - 7\theta)\,d\theta$

SOLUTION $\int \cos(5 - 7\theta)\,d\theta = -\frac{1}{7}\sin(5 - 7\theta) + C.$

75. $\int (4t^{-3} - 12t^{-4})\,dt$

SOLUTION $\int (4t^{-3} - 12t^{-4})\,dt = -2t^{-2} + 4t^{-3} + C.$

76. $\int (9t^{-2/3} + 4t^{7/3})\,dt$

SOLUTION $\int (9t^{-2/3} + 4t^{7/3})\,dt = 27t^{1/3} + \frac{6}{5}t^{10/3} + C.$

77. $\int \sec^2 x\,dx$

SOLUTION $\int \sec^2 x\,dx = \tan x + C.$

78. $\int \tan 3\theta \sec 3\theta\,d\theta$

SOLUTION $\int \tan 3\theta \sec 3\theta\,d\theta = \frac{1}{3}\sec 3\theta + C.$

79. $\int (y+2)^4\,dy$

SOLUTION $\int (y+2)^4\,dy = \frac{1}{5}(y+2)^5 + C.$

80. $\int \frac{3x^3 - 9}{x^2}\,dx$

SOLUTION $\int \frac{3x^3 - 9}{x^2}\,dx = \int (3x - 9x^{-2})\,dx = \frac{3}{2}x^2 + 9x^{-1} + C.$

81. $\int (e^x - x)\,dx$

SOLUTION $\int (e^x - x)\,dx = e^x - \frac{1}{2}x^2 + C.$

82. $\int e^{-4x}\,dx$

SOLUTION $\int e^{-4x}\,dx = -\frac{1}{4}e^{-4x} + C.$

83. $\int 4x^{-1}\,dx$

SOLUTION $\int 4x^{-1}\,dx = 4\ln|x| + C.$

84. $\int \sin(4x - 9)\,dx$

SOLUTION $\int \sin(4x - 9)\,dx = -\frac{1}{4}\cos(4x - 9) + C.$

In Exercises 85–90, solve the differential equation with the given initial condition.

85. $\frac{dy}{dx} = 4x^3, \quad y(1) = 4$

SOLUTION Let $\frac{dy}{dx} = 4x^3$. Then

$$y(x) = \int 4x^3\,dx = x^4 + C.$$

Using the initial condition $y(1) = 4$, we find $y(1) = 1^4 + C = 4$, so $C = 3$. Thus, $y(x) = x^4 + 3$.

86. $\dfrac{dy}{dt} = 3t^2 + \cos t, \quad y(0) = 12$

SOLUTION Let $\frac{dy}{dt} = 3t^2 + \cos t$. Then

$$y(t) = \int (3t^2 + \cos t)\, dt = t^3 + \sin t + C.$$

Using the initial condition $y(0) = 12$, we find $y(0) = 0^3 + \sin 0 + C = 12$, so $C = 12$. Thus, $y(t) = t^3 + \sin t + 12$.

87. $\dfrac{dy}{dx} = x^{-1/2}, \quad y(1) = 1$

SOLUTION Let $\frac{dy}{dx} = x^{-1/2}$. Then

$$y(x) = \int x^{-1/2}\, dx = 2x^{1/2} + C.$$

Using the initial condition $y(1) = 1$, we find $y(1) = 2\sqrt{1} + C = 1$, so $C = -1$. Thus, $y(x) = 2x^{1/2} - 1$.

88. $\dfrac{dy}{dx} = \sec^2 x, \quad y\left(\frac{\pi}{4}\right) = 2$

SOLUTION Let $\frac{dy}{dx} = \sec^2 x$. Then

$$y(x) = \int \sec^2 x\, dx = \tan x + C.$$

Using the initial condition $y(\frac{\pi}{4}) = 2$, we find $y(\frac{\pi}{4}) = \tan\frac{\pi}{4} + C = 2$, so $C = 1$. Thus, $y(x) = \tan x + 1$.

89. $\dfrac{dy}{dx} = e^{-x}, \quad y(0) = 3$

SOLUTION Let $\frac{dy}{dx} = e^{-x}$. Then

$$y(x) = \int e^{-x}\, dx = -e^{-x} + C.$$

Using the initial condition $y(0) = 3$, we find $y(0) = -e^0 + C = 3$, so $C = 4$. Thus, $y(x) = 4 - e^{-x}$.

90. $\dfrac{dy}{dx} = e^{4x}, \quad y(1) = 1$

SOLUTION Let $\frac{dy}{dx} = e^{4x}$. Then

$$y(x) = \int e^{4x}\, dx = \frac{1}{4}e^{4x} + C.$$

Using the initial condition $y(1) = 1$, we find $y(1) = \frac{1}{4}e^4 + C = 1$, so $C = 1 - \frac{1}{4}e^4$. Thus, $y(x) = \frac{1}{4}e^{4x} + 1 - \frac{1}{4}e^4$.

91. Find $f(t)$ if $f''(t) = 1 - 2t$, $f(0) = 2$, and $f'(0) = -1$.

SOLUTION Suppose $f''(t) = 1 - 2t$. Then

$$f'(t) = \int f''(t)\, dt = \int (1 - 2t)\, dt = t - t^2 + C.$$

Using the initial condition $f'(0) = -1$, we find $f'(0) = 0 - 0^2 + C = -1$, so $C = -1$. Thus, $f'(t) = t - t^2 - 1$. Now,

$$f(t) = \int f'(t)\, dt = \int (t - t^2 - 1)\, dt = \frac{1}{2}t^2 - \frac{1}{3}t^3 - t + C.$$

Using the initial condition $f(0) = 2$, we find $f(0) = \frac{1}{2}0^2 - \frac{1}{3}0^3 - 0 + C = 2$, so $C = 2$. Thus,

$$f(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3 - t + 2.$$

92. At time $t = 0$, a driver begins decelerating at a constant rate of -10 m/s^2 and comes to a halt after traveling 500 m. Find the velocity at $t = 0$.

SOLUTION From the constant deceleration of -10 m/s^2, we determine

$$v(t) = \int (-10)\, dt = -10t + v_0,$$

where v_0 is the velocity of the automobile at $t = 0$. Note the automobile comes to a halt when $v(t) = 0$, which occurs at

$$t = \frac{v_0}{10} \text{ s}.$$

The distance traveled during the braking process is

$$s(t) = \int v(t)\,dt = -5t^2 + v_0 t + C,$$

for some arbitrary constant C. We are given that the braking distance is 500 meters, so

$$s\left(\frac{v_0}{10}\right) - s(0) = -5\left(\frac{v_0}{10}\right)^2 + v_0\left(\frac{v_0}{10}\right) + C - C = 500,$$

leading to

$$v_0 = 100 \text{ m/s}.$$

93. Find the local extrema of $f(x) = \dfrac{e^{2x}+1}{e^{x+1}}$.

SOLUTION To simplify the differentiation, we first rewrite $f(x) = \frac{e^{2x}+1}{e^{x+1}}$ using the Laws of Exponents:

$$f(x) = \frac{e^{2x}}{e^{x+1}} + \frac{1}{e^{x+1}} = e^{2x-(x+1)} + e^{-(x+1)} = e^{x-1} + e^{-x-1}.$$

Now,

$$f'(x) = e^{x-1} - e^{-x-1}.$$

Setting the derivative equal to zero yields

$$e^{x-1} - e^{-x-1} = 0 \quad \text{or} \quad e^{x-1} = e^{-x-1}.$$

Thus,

$$x - 1 = -x - 1 \quad \text{or} \quad x = 0.$$

Next, we use the Second Derivative Test. With $f''(x) = e^{x-1} + e^{-x-1}$, it follows that

$$f''(0) = e^{-1} + e^{-1} = \frac{2}{e} > 0.$$

Hence, $x = 0$ is a local minimum. Since $f(0) = e^{0-1} + e^{-0-1} = \frac{2}{e}$, we conclude that the point $(0, \frac{2}{e})$ is a local minimum.

94. Find the points of inflection of $f(x) = \ln(x^2+1)$, and at each point, determine whether the concavity changes from up to down or from down to up.

SOLUTION With $f(x) = \ln(x^2+1)$, we find

$$f'(x) = \frac{2x}{x^2+1}; \text{ and}$$

$$f''(x) = \frac{2\left(x^2+1\right) - 2x \cdot 2x}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2}$$

Thus, $f''(x) > 0$ for $-1 < x < 1$, whereas $f''(x) < 0$ for $x < -1$ and for $x > 1$. It follows that there are points of inflection at $x = \pm 1$, and that the concavity of f changes from down to up at $x = -1$ and from up to down at $x = 1$.

In Exercises 95–98, find the local extrema and points of inflection, and sketch the graph. Use L'Hôpital's Rule to determine the limits as $x \to 0+$ or $x \to \pm\infty$ if necessary.

95. $y = x \ln x \quad (x > 0)$

SOLUTION Let $y = x \ln x$. Then

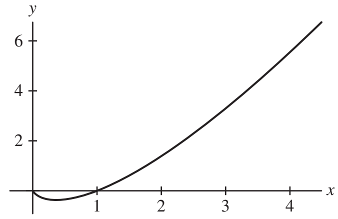
$$y' = \ln x + x\left(\frac{1}{x}\right) = 1 + \ln x,$$

and $y'' = \frac{1}{x}$. Solving $y' = 0$ yields the critical point $x = e^{-1}$. Since $y''(e^{-1}) = e > 0$, the function has a local minimum at $x = e^{-1}$. y'' is positive for $x > 0$, hence the function is concave up for $x > 0$ and there are no points of inflection. As $x \to 0+$ and as $x \to \infty$, we find

$$\lim_{x\to 0+} x \ln x = \lim_{x\to 0+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0+} (-x) = 0;$$

$$\lim_{x\to\infty} x \ln x = \infty.$$

The graph is shown below:



96. $y = e^{x-x^2}$

SOLUTION Let $y = e^{x-x^2}$. Then $y' = (1-2x)e^{x-x^2}$ and

$$y'' = (1-2x)^2 e^{x-x^2} - 2e^{x-x^2} = (4x^2 - 4x - 1)e^{x-x^2}.$$

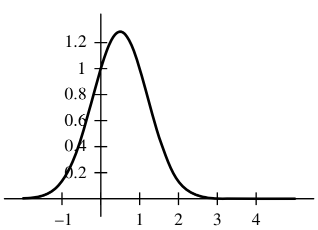
Solving $y' = 0$ yields the critical point $x = \frac{1}{2}$. Since

$$y''\left(\frac{1}{2}\right) = -2e^{1/4} < 0,$$

the function has a local maximum at $x = \frac{1}{2}$. Using the quadratic formula, we find that $y'' = 0$ when $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{2}$. $y'' > 0$ and the function is concave up for $x < \frac{1}{2} - \frac{1}{2}\sqrt{2}$ and for $x > \frac{1}{2} + \frac{1}{2}\sqrt{2}$, whereas $y'' < 0$ and the function is concave down for $\frac{1}{2} - \frac{1}{2}\sqrt{2} < x < \frac{1}{2} + \frac{1}{2}\sqrt{2}$; hence, there are inflection points at $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{2}$. As $x \to \pm\infty$, $x - x^2 \to -\infty$ so

$$\lim_{x\to\pm\infty} e^{x-x^2} = 0.$$

The graph is shown below.



97. $y = x(\ln x)^2 \quad (x > 0)$

SOLUTION Let $y = x(\ln x)^2$. Then

$$y' = x\frac{2\ln x}{x} + (\ln x)^2 = 2\ln x + (\ln x)^2 = \ln x(2 + \ln x),$$

and

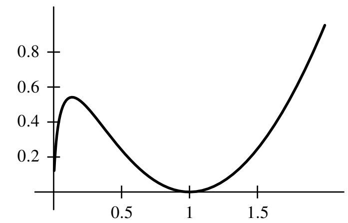
$$y'' = \frac{2}{x} + \frac{2\ln x}{x} = \frac{2}{x}(1 + \ln x).$$

Solving $y' = 0$ yields the critical points $x = e^{-2}$ and $x = 1$. Since $y''(e^{-2}) = -2e^2 < 0$ and $y''(1) = 2 > 0$, the function has a local maximum at $x = e^{-2}$ and a local minimum at $x = 1$. $y'' < 0$ and the function is concave down for $x < e^{-1}$, whereas $y'' > 0$ and the function is concave up for $x > e^{-1}$; hence, there is a point of inflection at $x = e^{-1}$. As $x \to 0+$ and as $x \to \infty$, we find

$$\lim_{x\to 0+} x(\ln x)^2 = \lim_{x\to 0+} \frac{(\ln x)^2}{x^{-1}} = \lim_{x\to 0+} \frac{2\ln x \cdot x^{-1}}{-x^{-2}} = \lim_{x\to 0+} \frac{2\ln x}{-x^{-1}} = \lim_{x\to 0+} \frac{2x^{-1}}{x^{-2}} = \lim_{x\to 0+} 2x = 0;$$

$$\lim_{x\to\infty} x(\ln x)^2 = \infty.$$

The graph is shown below:



98. $y = \tan^{-1}\left(\dfrac{x^2}{4}\right)$

SOLUTION Let $y = \tan^{-1}\left(\frac{x^2}{4}\right)$. Then

$$y' = \frac{1}{1+\left(\frac{x^2}{4}\right)^2}\frac{x}{2} = \frac{8x}{x^4+16},$$

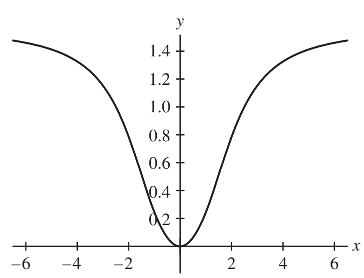
and

$$y'' = \frac{8(x^4+16) - 8x \cdot 4x^3}{(x^4+16)^2} = \frac{128 - 24x^4}{(x^4+16)^2}.$$

Solving $y' = 0$ yields $x = 0$ as the only critical point. Because $y''(0) = \frac{1}{2} > 0$, we conclude the function has a local minimum at $x = 0$. Moreover, $y'' < 0$ for $x < -2 \cdot 3^{-1/4}$ and for $x > 2 \cdot 3^{-1/4}$, whereas $y'' > 0$ for $-2 \cdot 3^{-1/4} < x < 2 \cdot 3^{-1/4}$. Therefore, there are points of inflection at $x = \pm 2 \cdot 3^{-1/4}$. As $x \to \pm\infty$, we find

$$\lim_{x\to\pm\infty} \tan^{-1}\left(\frac{x^2}{4}\right) = \frac{\pi}{2}.$$

The graph is shown below:



99. Explain why L'Hôpital's Rule gives no information about $\displaystyle\lim_{x\to\infty} \frac{2x - \sin x}{3x + \cos 2x}$. Evaluate the limit by another method.

SOLUTION As $x \to \infty$, both $2x - \sin x$ and $3x + \cos 2x$ tend toward infinity, so L'Hôpital's Rule applies to $\displaystyle\lim_{x\to\infty} \frac{2x - \sin x}{3x + \cos 2x}$; however, the resulting limit, $\displaystyle\lim_{x\to\infty} \frac{2 - \cos x}{3 - 2\sin 2x}$, does not exist due to the oscillation of $\sin x$ and $\cos x$ and further applications of L'Hôpital's rule will not change this situation.

To evaluate the limit, we note

$$\lim_{x\to\infty} \frac{2x - \sin x}{3x + \cos 2x} = \lim_{x\to\infty} \frac{2 - \frac{\sin x}{x}}{3 + \frac{\cos 2x}{x}} = \frac{2}{3}.$$

100. Let $f(x)$ be a differentiable function with inverse $g(x)$ such that $f(0) = 0$ and $f'(0) \neq 0$. Prove that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = f'(0)^2$$

SOLUTION Since g and f are inverse functions, we have $g(f(x)) = x$ for all x in the domain of f. In particular, for $x = 0$ we have

$$g(0) = g(f(0)) = 0.$$

Therefore, the limit is an indeterminate form of type $\frac{0}{0}$, so we may apply L'Hôpital's Rule. By the Theorem on the derivative of the inverse function, we have

$$g'(x) = \frac{1}{f'(g(x))}.$$

Therefore,

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{\frac{1}{f'(g(x))}} = \lim_{x\to 0} f'(x)f'(g(x)) = f'(0)f'(g(0)) = f'(0)\cdot f'(0) = f'(0)^2.$$

In Exercises 101–112, verify that L'Hôpital's Rule applies and evaluate the limit.

101. $\displaystyle\lim_{x\to 3}\frac{4x-12}{x^2-5x+6}$

SOLUTION The given expression is an indeterminate form of type $\frac{0}{0}$, therefore L'Hôpital's Rule applies. We find

$$\lim_{x\to 3}\frac{4x-12}{x^2-5x+6} = \lim_{x\to 3}\frac{4}{2x-5} = 4.$$

102. $\displaystyle\lim_{x\to -2}\frac{x^3+2x^2-x-2}{x^4+2x^3-4x-8}$

SOLUTION The given expression is an indeterminate form of type $\frac{0}{0}$, therefore L'Hôpital's Rule applies. We find

$$\lim_{x\to -2}\frac{x^3+2x^2-x-2}{x^4+2x^3-4x-8} = \lim_{x\to -2}\frac{3x^2+4x-1}{4x^3+6x^2-4} = -\frac{3}{12} = -\frac{1}{4}.$$

103. $\displaystyle\lim_{x\to 0+} x^{1/2}\ln x$

SOLUTION First rewrite

$$x^{1/2}\ln x \quad \text{as} \quad \frac{\ln x}{x^{-1/2}}.$$

The rewritten expression is an indeterminate form of type $\frac{\infty}{\infty}$, therefore L'Hôpital's Rule applies. We find

$$\lim_{x\to 0+} x^{1/2}\ln x = \lim_{x\to 0+}\frac{\ln x}{x^{-1/2}} = \lim_{x\to 0+}\frac{1/x}{-\frac{1/2}{x}^{-3/2}} = \lim_{x\to 0+} -\frac{x^{1/2}}{2} = 0.$$

104. $\displaystyle\lim_{t\to\infty}\frac{\ln(e^t+1)}{t}$

SOLUTION The given expression is an indeterminate form of type $\frac{\infty}{\infty}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{t\to\infty}\frac{\ln(e^t+1)}{t} = \lim_{t\to\infty}\frac{\frac{e^t}{e^t+1}}{1} = \lim_{t\to\infty}\frac{1}{1+e^{-t}} = 1.$$

105. $\displaystyle\lim_{\theta\to 0}\frac{2\sin\theta-\sin 2\theta}{\sin\theta-\theta\cos\theta}$

SOLUTION The given expression is an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{\theta\to 0}\frac{2\sin\theta-\sin 2\theta}{\sin\theta-\theta\cos\theta} = \lim_{\theta\to 0}\frac{2\cos\theta-2\cos 2\theta}{\cos\theta-(\cos\theta-\theta\sin\theta)} = \lim_{\theta\to 0}\frac{2\cos\theta-2\cos 2\theta}{\theta\sin\theta}$$

$$= \lim_{\theta\to 0}\frac{-2\sin\theta+4\sin 2\theta}{\sin\theta+\theta\cos\theta} = \lim_{\theta\to 0}\frac{-2\cos\theta+8\cos 2\theta}{\cos\theta+\cos\theta-\theta\sin\theta} = \frac{-2+8}{1+1-0} = 3.$$

106. $\displaystyle\lim_{x\to 0}\frac{\sqrt{4+x}-2\sqrt[8]{1+x}}{x^2}$

SOLUTION The given expression is an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{x\to 0}\frac{\sqrt{4+x}-2\sqrt[8]{1+x}}{x^2} = \lim_{x\to 0}\frac{\frac{1}{2}(4+x)^{-1/2}-\frac{1}{4}(1+x)^{-7/8}}{2x}$$

$$= \lim_{x\to 0}\frac{-\frac{1}{4}(4+x)^{-3/2}+\frac{7}{32}(1+x)^{-15/8}}{2} = \frac{-\frac{1}{4}\cdot\frac{1}{8}+\frac{7}{32}}{2} = \frac{3}{32}.$$

107. $\lim_{t\to\infty} \frac{\ln(t+2)}{\log_2 t}$

SOLUTION The limit is an indeterminate form of type $\frac{\infty}{\infty}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{t\to\infty} \frac{\ln(t+2)}{\log_2 t} = \lim_{t\to\infty} \frac{\frac{1}{t+2}}{\frac{1}{t\ln 2}} = \lim_{t\to\infty} \frac{t\ln 2}{t+2} = \lim_{t\to\infty} \frac{\ln 2}{1} = \ln 2.$$

108. $\lim_{x\to 0} \left(\frac{e^x}{e^x-1} - \frac{1}{x}\right)$

SOLUTION First rewrite the function as a quotient:

$$\frac{e^x}{e^x-1} - \frac{1}{x} = \frac{xe^x - e^x + 1}{x(e^x-1)}.$$

The limit is now an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{x\to 0} \left(\frac{e^x}{e^x-1} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{xe^x + e^x - e^x}{xe^x + e^x - 1} = \lim_{x\to 0} \frac{xe^x}{xe^x + e^x - 1}$$
$$= \lim_{x\to 0} \frac{xe^x + e^x}{xe^x + e^x + e^x} = \frac{1}{1+1} = \frac{1}{2}.$$

109. $\lim_{y\to 0} \frac{\sin^{-1} y - y}{y^3}$

SOLUTION The limit is an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{y\to 0} \frac{\sin^{-1} y - y}{y^3} = \lim_{y\to 0} \frac{\frac{1}{\sqrt{1-y^2}} - 1}{3y^2} = \lim_{y\to 0} \frac{y(1-y^2)^{-3/2}}{6y} = \lim_{y\to 0} \frac{(1-y^2)^{-3/2}}{6} = \frac{1}{6}.$$

110. $\lim_{x\to 1} \frac{\sqrt{1-x^2}}{\cos^{-1} x}$

SOLUTION The limit is an indeterminate form $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{x\to 1} \frac{\sqrt{1-x^2}}{\cos^{-1} x} = \lim_{x\to 1} \frac{-\frac{x}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}} = \lim_{x\to 1} x = 1.$$

111. $\lim_{x\to 0} \frac{\sinh(x^2)}{\cosh x - 1}$

SOLUTION The limit is an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{x\to 0} \frac{\sinh(x^2)}{\cosh x - 1} = \lim_{x\to 0} \frac{2x\cosh(x^2)}{\sinh x} = \lim_{x\to 0} \frac{2\cosh(x^2) + 4x^2\sinh(x^2)}{\cosh x} = \frac{2+0}{1} = 2.$$

112. $\lim_{x\to 0} \frac{\tanh x - \sinh x}{\sin x - x}$

SOLUTION The limit is an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{x\to 0} \frac{\tanh x - \sinh x}{\sin x - x} = \lim_{x\to 0} \frac{\operatorname{sech}^2 x - \cosh x}{\cos x - 1} = \lim_{x\to 0} \frac{2\operatorname{sech} x(-\operatorname{sech} x \tanh x) - \sinh x}{-\sin x}$$
$$= \lim_{x\to 0} \frac{2\operatorname{sech}^2 x\tanh x + \sinh x}{\sin x} = \lim_{x\to 0} \frac{-4\operatorname{sech}^2 x\tanh^2 x + 2\operatorname{sech}^4 x + \cosh x}{\cos x}$$
$$= \frac{-4\cdot 1\cdot 0 + 2\cdot 1 + 1}{1} = 3.$$

113. Let $f(x) = e^{-Ax^2/2}$, where $A > 0$. Given any n numbers $a_1, a_2, \dots, a_n$, set

$$\Phi(x) = f(x-a_1)f(x-a_2)\cdots f(x-a_n)$$

(a) Assume $n = 2$ and prove that $\Phi(x)$ attains its maximum value at the average $x = \frac{1}{2}(a_1 + a_2)$. *Hint:* Calculate $\Phi'(x)$ using logarithmic differentiation.

(b) Show that for any n, $\Phi(x)$ attains its maximum value at $x = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$. This fact is related to the role of $f(x)$ (whose graph is a bell-shaped curve) in statistics.

SOLUTION

(a) For $n = 2$ we have,

$$\Phi(x) = f(x - a_1)\, f(x - a_2) = e^{-\frac{A}{2}(x-a_1)^2} \cdot e^{-\frac{A}{2}(x-a_2)^2} = e^{-\frac{A}{2}\left((x-a_1)^2+(x-a_2)^2\right)}.$$

Since $e^{-\frac{A}{2}y}$ is a decreasing function of y, it attains its maximum value where y is minimum. Therefore, we must find the minimum value of

$$y = (x - a_1)^2 + (x - a_2)^2 = 2x^2 - 2(a_1 + a_2)x + a_1^2 + a_2^2.$$

Now, $y' = 4x - 2(a_1 + a_2) = 0$ when

$$x = \frac{a_1 + a_2}{2}.$$

We conclude that $\Phi(x)$ attains a maximum value at this point.

(b) We have

$$\Phi(x) = e^{-\frac{A}{2}(x-a_1)^2} \cdot e^{-\frac{A}{2}(x-a_2)^2} \cdot \cdots \cdot e^{-\frac{A}{2}(x-a_n)^2} = e^{-\frac{A}{2}\left((x-a_1)^2+\cdots+(x-a_n)^2\right)}.$$

Since the function $e^{-\frac{A}{2}y}$ is a decreasing function of y, it attains a maximum value where y is minimum. Therefore we must minimize the function

$$y = (x - a_1)^2 + (x - a_2)^2 + \cdots + (x - a_n)^2.$$

We find the critical points by solving:

$$y' = 2(x - a_1) + 2(x - a_2) + \cdots + 2(x - a_n) = 0$$
$$2nx = 2(a_1 + a_2 + \cdots + a_n)$$
$$x = \frac{a_1 + \cdots + a_n}{n}.$$

We verify that this point corresponds the minimum value of y by examining the sign of y'' at this point: $y'' = 2n > 0$. We conclude that y attains a minimum value at the point $x = \frac{a_1+\cdots+a_n}{n}$, hence $\Phi(x)$ attains a maximum value at this point.