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# AP CALCULUS

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## OVERVIEW

**Calculus** is the study of smoothly changing functions.

- **Differential calculus** studies the way functions change and curve.
- **Integral calculus** studies areas enclosed by curves, through continuous, as opposed to discrete, summations. Integration has many applications, including finding the length of an arc and the volume of a solid.
- **Differential equations** express a relationship between a function and its dervatives.
- **Infinite series** are a useful way to express certain functions. In particular, expressing an otherwise intractible function as an infinite **Taylor polynomial** makes difficult tasks like value approximation, differentiation, and integration easy.

### SUMMARY OF BASIC TERMS

- A **function** is a rule that assigns to each value of the **domain** a unique value of the **range**.
- Function $f$ is **continuous** on some interval if whenever $x_1$ is close to $x_2$, $f(x_1)$ is close to $f(x_2)$.
- Function $f$ is **increasing** on some interval if whenever $x_1 < x_2$, $f(x_1) < f(x_2)$ (so $f'(x)$ is positive). It is **decreasing** on an interval if whenever $x_1 < x_2$, $f(x_1) > f(x_2)$ (so $f'(x)$ is negative). A nonincreasing or nondecreasing function is called **monotonic**.
- Function $f$ is **differentiable** on an open interval if its derivative exists everywhere on that interval. A differentiable function must be continuous and must not have any vertical tangents.
- Function $f$ is **concave up** on some interval if its second derivative $f''$ is positive everywhere on that interval; its graph cups up. Function $f$ is **concave down** on an interval if $f''$ is negative there; its graph cups down.
- The line $x = a$ is a **vertical asymptote** for $f$ if $f$ blows up to (positive or negative) infinity as $x$ gets close to $a$, from either or both sides. Formally, $x = a$ is a vertical asymptote if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$ or both.
- The line $y = b$ is a **horizontal asymptote** for $f$ if $f(x)$ gets close to $b$ as $|x|$ becomes very large, for $x$ positive or negative or both. Formally, $y = b$ is a horizontal asymptote if $\lim_{x\to+\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$ or both.

## FUNCTIONS, LIMITS, CONTINUITY

### FUNCTIONS

A **function** is a set of ordered pairs $(x, y)$ so that there is no more than one $y$-value for each $x$-value. Plotted on the Cartesian plane, a function must pass the **vertical line test**: Every vertical line cuts the graph of the function at most once.

- The **domain** of a function is the set of all allowable values that can be plugged in for the **independent variable**, often $x$.
- The **range** is the set of all possible outputs (values of the **dependent variable**, often $y$).

Many functions are transformations of the so-called basic functions—polynomial functions, exponential function, logarithmic functions, and trigonometric functions.

#### VERTICAL TRANSLATION

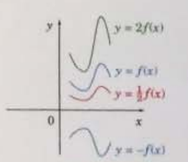
- Adding a constant $c$ to a function $y = f(x)$ shifts it vertically $c$ units (up if $c$ is positive, down if $c$ is negative). The new function $y = f(x) + c$ has the same shape and the same domain as the original function.

#### HORIZONTAL TRANSLATION

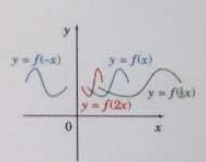
- The function $y = f(x - c)$ is a shift of the original function $c$ units horizontally (to the right if $c$ is positive, to the left if $c$ is negative). The new function has the same shape and the same range as the original function.

#### VERTICAL STRETCHING AND COMPRESSING

- For positve $c$, the function $y = cf(x)$ is a vertical stretch or compression of the original function. If $c > 1$, then the function $y = cf(x)$ is stretch by a factor of $c$. If $c < 1$, then $y = cf(x)$ is a compression by a factor of $\frac{1}{c}$. Horizontal distances remain unchanged.




Vertical stretch




Horizontal stretch

#### HORIZONTAL STRETCHING AND COMPRESSING

- For positive $c$, the function $y = f\left(\frac{x}{c}\right)$ is a horizontal stretch of $y = f(x)$ by a factor of $c$ if $c < 1$ (a compression by $\frac{1}{c}$ if $c > 1$). Vertical distances don't change.

#### REFLECTION OVER THE $x$-AXIS

- The function $y = -f(x)$ is a reflection of the original function over the $x$-axis. The new function has the same domain as the original; the range is the negative of the original range.

#### REFLECTION OVER THE $y$-AXIS

- The function $y = f(-x)$ is a reflection of $y = f(x)$ over the $y$-axis. The new function has the same range as the original; the domain is the negative of the original domain.
- If $f(x) = f(-x)$ for all $x$, then $f$ is called **even**: it remains unchanged when reflected over the $x$-axis.
  **Ex:** $\cos x$ is an even function.
- If $f(x) = -f(-x)$ for all $x$, then $f$ is called **odd**: reflecting $f$ over the $x$-axis is the same reflecting $f$ over the $y$-axis. Equivalently, a $180°$ rotation around the origin leaves $f$ unchanged. **Ex:** $\sin x$ is an odd function.

#### INVERSE FUNCTIONS

If the function $f$ passes the "horizontal line test" in its domain—$f(x)$ never takes the same value twice—then $f(x)$ has a unique **inverse** $f^{-1}(x)$ whose domain is the range of $f(x)$ and vice versa.

- To find the inverse function, switch the roles of $x$ and $y$ in the equation, effectively writing $x = f(y)$. Then solve for $y$. If you can solve for $y$ and each step is reversible, then the function has an inverse.
- **Ex:** $y = mx + b$. The inverse function is $y = \frac{1}{m}(x - b)$.
- **Ex:** $y = a^x$. The inverse function is $y = \log_a x$.
- Graphically, $y = f^{-1}(x)$ has the same shape as the orginal function, but is reflected over the diagonal $y = x$. **Ex:** $y = e^x$ and $y = \log x$ are inverse functions.

### LIMITS

If the function $f$ comes infinitely close to some value $L$ as $x$ gets close to $a$, we say that $L$ is the **limit** of $f(x)$ as $x$ approaches $a$ and write $\lim_{x\to a} f(x) = L$.

- The existence or the value of $\lim_{x\to a} f(x)$ by itself says nothing at all about the existence or the value of $f(a)$. Rather, comparing the $\lim_{x\to a} f(x)$ and $f(a)$ tells about the continuity or the type of discontinuity of $f(x)$ at $x = a$.

#### FORMAL DEFINITION OF A LIMIT

Suppose that there exists a real number $L$ that satisfied the following: for every $\varepsilon > 0$ there exists some $\delta > 0$ such that whenever $x$ is within $\delta$ of $a$, $f(x)$ is within $\varepsilon$ of $L$ (that is, $|x - a| < \delta$ implies $|f(x) - L| < \varepsilon$).

Then $\lim_{x\to a} f(x) = L$.

If no such $L$ exists then $\lim_{x\to a} f(x)$ does not exist.

#### ONE-SIDED LIMITS

We can consider the limit of $f(x)$ as $x$ approaches $a$ from one side only.

- The **left-hand limit** $\lim_{x\to a^-} f(x)$ is the value that $f(x)$ approaches when $x$ is close to and smaller than $a$.
- The **right-hand limit** $\lim_{x\to a^+} f(x)$ is the value that $f(x)$ approaches when $x$ is close to and larger than $a$.

If the limit of $f(x)$ as $x \to a$ exists, then so do both one-sided limits, and the three limits have the same value. Similarly, if both the right-hand and the left-hand limits of $f(x)$ as $x \to a^{\pm}$ exist and are equal, then $\lim_{x\to a} f(x)$ exists and is equal to the value of the one-sided limits.

#### INFINITE LIMITS

If, as $x$ approaches $a$, the value of $f(x)$ grows without bound, then we say that $\lim_{x\to a} f(x) = \infty$.

- We can distinguish between a $+\infty$ limit and a $-\infty$ limit.
- We can similarly discuss one-sided limits that tend to infinity. For example, $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.
- If at least one of $\lim_{x\to a^\pm} f(x) = \pm\infty$ is true, then $f$ has a **vertical asymptote** at $x = a$.

#### LIMITS AT INFINITY

We can also look at the limit of $f(x)$ at $+\infty$ or $-\infty$. A limit at infinity, if it exists, is what $f(x)$ tends toward as $|x|$ gets very large, positively or negatively. Limits at infinity are one-sided.

- If $\lim_{x\to\pm\infty} f(x) = L$ exists and is finite, the line $y = L$ is a **horizontal asymptote** to the graph of $f(x)$.

**Ex:** $\lim_{x\to-\infty} e^x = 0$. The line $y = 0$ is a horizontal asymptote to the function $y = e^x$.

#### PROPERTIES OF LIMITS

Let $f$ and $g$ be two functions and $a$ a point (possibly $\pm\infty$) near which both $f(x)$ and $g(x)$ are defined. Then if *both* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ exist and at least one of them is finite, the following properties hold.

1. $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
3. For any real number $c$, $\lim_{x\to a} (cf(x)) = c \lim_{x\to a} f(x)$.
4. If $\lim_{x\to a} g(x) \neq 0$, then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

**Squeeze Theorem:** If $f(x) \le g(x) \le h(x)$ near $x = a$, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x)$ exists and is equal to $L$.

#### LIMITS TO MEMORIZE

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

$\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n$

### CONTINUITY

If $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$, we say that $f$ is **continuous** at $x = a$.

- If $f$ is continuous at every $a \le x \le b$, we say that $f$ is **continuous on the interval** $[a, b]$.
- If $f$ is continuous at every real $x$, we say that $f$ is continuous on the whole real line, or simply continuous.

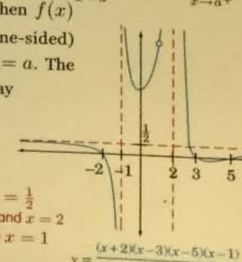
#### REMOVABLE DISCONTINUITY

If $\lim_{x\to a} f(x)$ exists and is finite but is not equal to $f(a)$ (which may or may not exist), we say that $f(x)$ has a **removable discontinuity** at $x = a$.

**Ex:** $f(x) = \frac{x^2 - 4}{x - 2}$ has a removable discontinuity at $x = 2$. Indeed, $f$ is indistinguishable from $x + 2$ everywhere except at $x = 2$, where $f$ is undefined. But the discontinuity at $x = 2$ could easily be "removed" by inserting the point $(2, 4)$ to make the graph continuous.

#### INFINITE DISCONTINUITY

If either of the one-sided limits $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ exists and is infinite, then $f(x)$ has a (possibly one-sided) **vertical asymptote** at $x = a$. The two-sided $\lim_{x\to a} f(x)$ may or may not exist.




Horizontal asymptote: $y = \frac{1}{2}$
Vertical asymptote: $x = -1$ and $x = 2$
Removable discontinuity at $x = 1$

$$y = \frac{(x+2)(x-3)(x-5)(x-1)}{2(x+1)(x-2)^2(x-1)}$$

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## DIFFERENTIATION

### INTUITION

If $a$ and $b$ are two points in the domain of $f(x)$ then the **average rate of change** of $f(x)$ on the interval $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$, a measure of how fast $f(x)$ has increased or decreased over the interval. This is also the slope of the line through the points $(a, f(a))$ and $(b, f(b))$ on the graph of $f(x)$.

The derivative of $f(x)$ at a point $x = a$ is the **instantaneous rate of change**, a measure of how fast $f(x)$ is increasing or decreasing at $a$. Equivalently, the derivative is the slope of the tangent line to the graph of $f(x)$ at the point $x = a$, where the tangent is the unique line through $(a, f(a))$ that touches the graph at only that point near $x = a$.

We compute the derivative $f'(a)$ by looking at the average rate of change of $f(x)$ on the interval $[a, a+h]$ and taking the limit as $h$ goes to 0. Equivalently, $f'(a)$ is the limit as $h \to 0$ of the slope of the line through $(a, f(a))$ and $(a+h, f(a+h))$.

### DEFINITION OF DERIVATIVE

If the limit

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

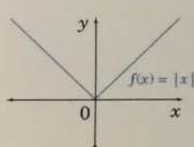
exists, we say that $f$ is **differentiable at** $x = a$ and the limit is the **derivative of** $f(x)$ **at** $x = a$, denoted by $f'(a)$.

The function

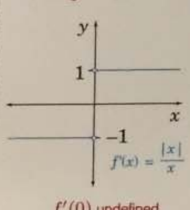
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

is the derivative function of $f(x)$. If it is defined whenever $f(x)$ is defined, then $f$ is a **differentiable** function.

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $a$. The converse is not true: a function can be continuous but not differentiable. There are two cases where this occurs.




No tangent at $x = 0$

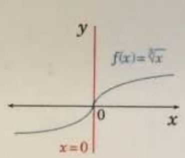



$f'(0)$ undefined

**1. No tangent**

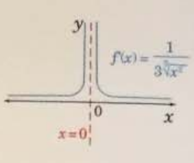
**Ex:** $f(x) = |x|$. The function is continuous at $x = 0$ since $\lim_{x\to 0^+}|x| = \lim_{x\to 0^-}|x| = 0$, but the derivative $f'(0)$ is undefined since the left-hand slope limit, $\lim_{h\to 0^-}\frac{|h|}{h} = -1$, does not equal the right-hand slope limit, $\lim_{h\to 0^+}\frac{|h|}{h} = 1$.

**2. Vertical tangent:** The slope of a vertical line is undefined. If $f(x)$ has a vertical tangent at $x = a$, then the derivative $f'(a)$ is undefined and the graph of $f'(x)$ will have a vertical asymptote at $x = a$.




Vertical tangent at $x = 0$

**Ex:** $f(x) = \sqrt[3]{x}$ has a vertical tangent at the point $(0, 0)$. The derivative function, $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$, goes to infinity at 0.




$f'(0)$ undefined

### NOTATION

There are several notations for the derivative; some are more useful than others in certain contexts. The most common notations used by mathematicians are

$$f'(x),\ y',\ \frac{d}{dx}f(x),\ \text{and}\ \frac{dy}{dx}.$$

The last two are in **Leibniz notation**: $\frac{dy}{dx}$ evolved from $\frac{\triangle y}{\triangle x}$, the expression used to compute slope. The expressions $dy$ and $dx$ represent infinitesimal changes in $y$ and $x$.

- Higher-order derivatives can be written in "prime" notation: $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, $f^{(5)}(x)$, etc., or in Leibniz notation: $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, $\frac{d^5y}{dx^5}$.
- The derivative at a particular point $a$ is most often expressed as $f'(a)$ or $\left.\frac{dy}{dx}\right|_{x=a}$.

### PROPERTIES OF DERIVATIVES

Assume that $f$ and $g$ are two differentiable functions.

**Sum and difference**

$$\frac{d}{dx}(f(x)\pm g(x)) = f'(x)\pm g'(x)$$

**Scalar multiple**

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

**Product**

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

If $f$ is HI and $g$ is HO, then the Product Rule is 'HO d HI plus HI d HO.'

**Quotient**

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

The Quotient Rule is 'HO d HI minus HI d HO over HO HO.'

The **Chain Rule** takes the derivative of composite functions. There are two common ways of expressing it.

1. $(f\circ g)'(x) = f'(g(x))\,g'(x)$.
2. If $u = g(x)$ and $y = f(u) = f(g(x))$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

### IMPLICIT DIFFERENTIATION

**Implicit differentiation** uses the product and chain rules to find slopes of curves when it is difficult or impossible to express $y$ as a function of $x$. Leibniz notation may be easiest when differentiating implicitly. Take the derivative of each term in the equation with respect to $x$. Then rewrite $\frac{dy}{dx} = y'$ and $\frac{dx}{dx} = 1$ and solve for $y'$.

**Ex 1:** $x^2 + y^2 = 1$

Implicitly differentiating with respect to $x$ gives the expression $2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$, which simplifies to $2x + 2yy' = 0$ or $y' = -\frac{x}{y}$. The derivative can now be found for any point on the curve. You will get the same result if you first solve for $y = \pm\sqrt{1-x^2}$ and keep track of the $\pm$ signs in different quadrants.

**Ex 2:** $x\cos y - y^2 = 3x$

Differentiate to obtain first

$$\frac{dx}{dx}\cos y + x\frac{d(\cos y)}{dx} - 2y\frac{dy}{dx} = 3\frac{dx}{dx},$$

and then $\cos y - x\sin y y' - 2yy' = 3$.

Finally, solve for $y' = \frac{\cos y - 3}{x\sin y + 2y}$.

### DERIVATIVES OF BASIC FUNCTIONS

**Constant:** $$\frac{d(c)}{dx} = 0$$

A constant function is always flat.

**Linear:** $$\frac{d(mx+b)}{dx} = m$$

The line $y = mx + b$ has slope $m$.

**Power Rule:** $$\frac{d(x^n)}{dx} = nx^{n-1}$$

True for all real $n \neq 0$.

**Polynomial:**

$$\frac{d(a_nx^n + \cdots + a_2x^2 + a_1x + a_0)}{dx} = na_nx^{n-1} + \cdots + 2a_2x + a_1$$

**Exponential:** $$\frac{d(e^x)}{dx} = e^x$$

This is why $e$ is called the "natural" logarithm base: $Ae^x$ are the only functions that are their own derivatives.

$$\frac{d(a^x)}{dx} = a^x\ln a$$

When in doubt, convert $a^x$ to $e^{x\ln a}$.

**Logarithmic:** $$\frac{d(\ln x)}{dx} = \frac{1}{x}$$

Found using implicit differentiation.

$$\frac{d(\log_a x)}{dx} = \frac{1}{x\ln a}$$

When in doubt, convert $\log_a x$ to $\frac{\ln x}{\ln a}$.

**Trigonometric:** Found using the definition of derivative and the Squeeze Theorem. If you know the derivatives of $\sin x$ and $\cos x$, you can find all the rest using the definitions of the trigonometric functions and the quotient rule.

$$\frac{d(\sin x)}{dx} = \cos x \qquad \frac{d(\cos x)}{dx} = -\sin x$$

$$\frac{d(\tan x)}{dx} = \sec^2 x \qquad \frac{d(\cot x)}{dx} = -\csc^2 x$$

$$\frac{d(\sec x)}{dx} = \sec x\tan x \qquad \frac{d(\csc x)}{dx} = -\csc x\cot x$$

**Inverse trigonometric:** Found by implicit differentiation.

$$\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d(\cos^{-1}x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2} \qquad \frac{d(\cot^{-1}x)}{dx} = -\frac{1}{1+x^2}$$

$$\frac{d(\sec^{-1}x)}{dx} = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d(\csc^{-1}x)}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

### L'HOSPITAL'S RULE

Used to evaluate indeterminate form limits: $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$. Suppose both $f(x)$ and $g(x)$ are differentiable around $a$ and $g'(x) \neq 0$ on an interval near $a$ (except perhaps at $a$).

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

OR

If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$,

then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

- L'Hospital's Rule may be used with one-sided limits.
- L'Hospital's Rule can also be applied if the limit is taken as $x$ approaches infinity.
- If $f'(x)$ and $g'(x)$ also satisfy the conditions for l'Hospital's Rule, higher derivatives can be taken until the limit is well-defined.
- L'Hospital's Rule cannot be applied to a fraction if the top limit is infinite and the bottom is zero, or vice versa.

**Ex:** $\lim_{x\to 1}\frac{\ln x}{x-1}$. Since $\lim_{x\to 1}\ln x = 0$ and $\lim_{x\to 1} x - 1 = 0$, use l'Hospital's Rule:

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{d(\ln x)}{dx}}{\frac{d(x-1)}{dx}} = \lim_{x\to 1}\frac{1/x}{1} = 1.$$

- L'Hospital's Rule can be used to evaluate other indeterminate forms, such as $\pm\infty\cdot 0$. The key is to convert the expression to $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

**Ex:** $\lim_{x\to-\infty} xe^x$. Convert to the expression $\lim_{x\to-\infty}\frac{x}{e^{-x}}$, which is an indeterminate form $\frac{-\infty}{\infty}$. Applying l'Hospital's Rule, convert to $\lim_{x\to-\infty}\frac{1}{-e^{-x}} = 0$.

# ANALYZING CURVES

Playing with a function's derivative gives a lot of information about the function's shape—not only the direction it is traveling in at particular points, but also its overall shape and behavior.

## TANGENT LINES

The **tangent line** to a curve $y = f(x)$ at the point $(a, f(a))$ has the property that, near the point of tangency, it touches the curve exactly once. That is, the tangent line touches the curve at the point of tangency and nowhere else nearby.

- A tangent line has the same slope as the function at the point of tangency. So the line tangent to $f$ at the point where $x = a$ has slope $f'(a)$.

### EQUATION OF THE TANGENT LINE

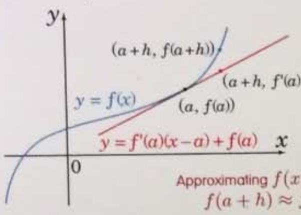
The line

$$y = f(a) + f'(a)(x - a)$$

is tangent to $y = f(x)$ at $(a, f(a))$. The equation comes form the point-slope formula.

### LINEAR APPROXIMATION

The tangent line gives a crude way to approximate values of $f$ near $x = a$: if $h$ is small, then $f(a + h)$ is close to $f'(a)h + f(a)$. This approximation is linear, and therefore easy to work with.




Approximating $f(x)$ near $a$: For small $h$, $f(a + h) \approx f'(a)h + f(a)$

## INCREASING AND DECREASING

The sign of the derivative at a point tells us whether the function is increasing or decreasing at that point.

### INCREASING FUNCTION

The function $f$ is **increasing** at $x = a$ if $f'(a) > 0$.

- This means that for $b$ close to but a little bigger than $a$, we have $f(b) > f(a)$.

### DECREASING FUNCTION

The function $f$ is **decreasing** at $x = a$ if $f'(a) < 0$.

- This means that for $b$ close to but a little bigger than $a$, we have $f(b) < f(a)$.

### NOT CHANGING

If $f'(a) = 0$ then $f$ is neither increasing nor decreasing at $x = a$. The value $f(a)$ may be maximal near $a$, minimal near $a$, or neither.

**Ex:** $f(x) = x^2$ is increasing for $x > 0$, decreasing for $x < 0$, neither at $x = 0$; $f(0)$ is a minimal value for $f$.
**Ex:** $g(x) = x^3$ is increasing for $x \neq 0$ and not changing at $x = 0$. However, $g(0)$ is neither a maximal nor a minimal value of $g$.

## MINIMA AND MAXIMA

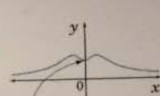
### LOCAL MINIMA AND MAXIMA

A **local minimum** (or **maximum**) of $f$ is a point $(c, f(c))$ such that $f(c)$ is the least (or greatest) value of the function in some interval around $c$.

- Intuitively, a local minimum is a trough (or *valley*), and a local maximum is a peak.
- Local minima and maxima are also called **relative** minima and maxima.

### GLOBAL MINIMA AND MAXIMA

The **global minimum** (or **maximum**) is the point where $f(x)$ assumes its least (or greatest) value in its domain. Global minima and maxima are also called **absolute** minima and maxima.




local minimum
No global minumum

- Every global minimum (or maximum) is also a local minimum (or maximum).
- But a function may have local minima (or maxima) but no global minima (or maxima). **Ex:** $f(x) = (x^2 + 1)e^{-x^2}$; see image above right.
- A continuous function whose domain is a closed interval (an interval that includes its endpoints) will always have both a global minimum and a global maximum, possibly at one of the endpoints (**Extreme Value Theorem**).

### EXTREMA

An **extremum** is either a minimum or a maximum.

### CRITICAL POINTS

A **critical point** is a point $x = a$ in the domain of $f$ where $f'(x)$ is zero or undefined.

- Critical points are useful when searching for minima or maxima: every extremum happens either at an domain endpoint or at a critical point.

## EXTREMA HUNTING

The key facts in this game are short and sweet.

**Important Concept**

- $f$ has a local minimum at $x = a$ if and only if at if the sign of $f'$ changes from $-$ to $+$ as $x$ passes $a$.
- $f$ has a local maximum at $x = a$ if and only if at if the sign of $f'$ changes from $+$ to $-$ as $x$ passes $a$.

**Important Shortcut**

- When seeking global extrema for $f$ defined on a closed interval, find all critical points and compare the values of $f$ at the critical points to the values of $f$ at the interval endpoints. The greatest value is the global maximum; the least value is the global minimum.

The rest is all details.

### 1. CHECK CRITICAL POINTS WHERE $f(x)$ IS UNDEFINED.

- Such a point may be an extremum (**Ex:** $f(x) = |x|$ at $x = 0$).
- It may be a discontinuity (**Ex:** $f(x) = \frac{x^3}{x}$ at $x = 0$).
- Or it may be neither (**Ex:** $f(x) = \frac{1}{\sqrt[3]{x}}$ at $x = 0$).

### 2. CHECK CRITICAL POINTS WHERE $f(x)$ IS IS ZERO.

- If the sign of $f'$ switches from $+$ to $-$ at $x = a$, then $f(a)$ is a local maximum.
- If the sign of $f'$ switches from $-$ to $+$ at $x = a$, then $f(a)$ is a local minimum.
- If the sign of $f'$ does not switch around $x = a$, then $f(a)$ is neither a maximum nor a minimum.

### 3. IF SEEKING GLOBAL EXTREMA, CHECK DOMAIN ENDPOINTS.

This includes checking the behavior of the function at infinity if the domain is unbounded.

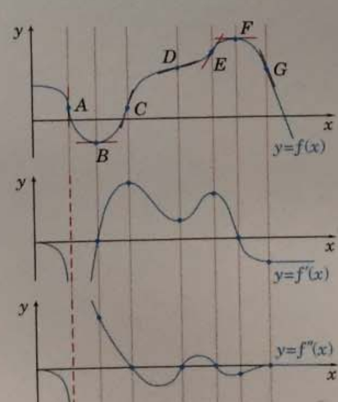
### SECOND DERIVATIVE TEST

The **Second Derivative Test** checks whether $f(a)$ is a local extremum given that $f'(a) = 0$:

- If $f''(a) < 0$, then $f(a)$ is a local maximum.
- If $f''(a) > 0$, then $f(a)$ is a local minimum.
- If $f''(a) = 0$, then the test is inconclusive.

It is usually safer and easier to check how the sign of $f'$ changes near $a$ than to bother with the SDT.

A FUNCTION AND ITS FIRST TWO DERIVATIVES




- At A, $f$ has a vertical tangent, which corresponds to an infinite discontinuity for $f'$.
- At B, $f$ has a local minimum, which corresponds to a zero for $f'$ and a positive sign for $f''$.
- At C, D, and E, $f$ has points of inflection, which correspond to local extrema for $f'$ and zeros for $f''$.
- At F, $f$ has a local maximum, which corresponds to a zero for $f'$ and a negative sign for $f''$.
- At G, $f$ begins to look linear, which corresponds to constant $f'$ and a zero $f''$.

## INFLECTION AND CONCAVITY

### CONCAVITY

A function $f$ is **concave up** at $x = a$ if $f''(a) > 0$. It is **concave down** at $x = a$ if $f''(a) < 0$.

### INFLECTION

A **point of inflection** on the graph of $f$ is a point when the concavity of $f$ changes, from up to down or vice versa.

- $f$ has a point of inflection at $x = a$ if an only if $f'$ has a local extremum at $a$.

## CURVE ANALYSIS SUMMARY

1. **Endpoints:** If the domain is an interval, evaluate the function at the endpoints. If the domain is the whole real line, establish what happens at $\pm\infty$. **Horizontal asymptotes** will appear if $\lim_{x \to \pm\infty} f(x)$ is finite. Evaluate $f(0)$ to find the $y$-intercept.
2. **Gaps:** Find all isolated points $x = a$ where $f(a)$ is not defined. For each point $a$, look at $\lim_{x \to a^\pm} f(x)$. A vertical asymptote will appear if $\lim_{x \to a^\pm} f(x) = \pm\infty$. A removable discontinuity (hole in the graph) will appear if $\lim_{x \to a} f(x)$ exists and is finite.
3. **$x$-intercepts:** If it is easy to determine when $f(x) = 0$, do so. If not, evaluating the function at the critical points and the endpoints will indicate where to look for zeroes.
4. **Rise and fall:** Determine the intervals where the function is increasing and decreasing by looking at the sign of $f'(x)$. If $f'(x) > 0$, then $f(x)$ is increasing. If $f'(x) < 0$, then $f(x)$ is decreasing.
5. **Local extrema:** Find all local extrema by looking at the critical points where $f'(x) = 0$ or where $f'(x)$ is not defined.
6. **Concavity:** Determine when the function cups up or down by looking at the sign of $f''(x)$. If $f''(x) > 0$, the function is concave up; if $f''(x) < 0$, then $f(x)$ is concave down. If $f''(a) = 0$, then the function is temporarily not curving at $x = a$; if $f''(x)$ is changing sign near $x = a$, then this is a point of inflection.

## THEOREMS

### INTERMEDIATE VALUE THEOREM

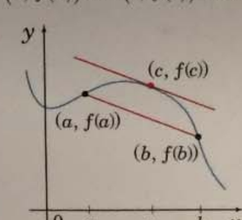
If $f(x)$ is continuous in an interval $[a, b]$, then somewhere on the interval it will achieve every value between $f(a)$ and $f(b)$: if $f(a) \leq M \leq f(b)$, then there exists some $c$ in the interval $[a, b]$ such that $f(c) = M$.

### ROLLE'S THEOREM

If $f(x)$ is continuous on the closed interval $[a, b]$, differentiable on the open interval $(a, b)$, and satisfies $f(a) = f(b)$, then for some $c$ in the interval $(a, b)$, we have $f'(c) = 0$.

### MEAN VALUE THEOREM (MVT)

A generalization of Rolle's Theorem. If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$, then there exists a point $c$ on $(a, b)$ such that the slope of the tangent to $f(x)$ at $x = c$ is the same as the slope of the secant line through the two points $(a, f(a))$ and $(b, f(b))$: that is,




$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Two functions having the same derivative differ by a constant:** If $f'(x) = g'(x)$, then $f(x) = g(x) + C$ for some real $C$. Equivalently, a function has only one family of antiderivatives. This theorem follows from the MVT.

### EXTREME VALUE THEOREM

A function $f$ continuous on the closed inteval $[a, b]$ will assume a global maximum and a global minimum somewhere on $[a, b]$.

CONTINUED ON OTHER SIDE

# INTEGRATION

Given a function $y = f(x)$ on the interval $[a, b]$, what is the area enclosed by this curve, the $x$-axis, and the two vertical lines $x = a$ and $x = b$?

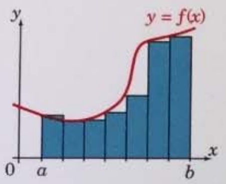
For simplicity of computation, we always speak of signed area: a curve above the $x$-axis is said to enclose "positive" area, while a curve below the $x$-axis encloses "negative" area.

## RIEMANN SUM APPROXIMATIONS

We can approximate the are under a curve with different kinds of **Riemann sums**. The exact area is the limit of these Riemann sums as the approximations get more and more fine.

### LEFT-HAND RIEMANN SUMS

We approximate the area under the curve by the sum of the areas of $n$ rectangles, each of width $\Delta x = \frac{b-a}{n}$, with their bottom corners at $x_0 = a$, $x_1 = a + \Delta x, \ldots,$ $x_n = a + n\Delta x = b$ along the $x$-axis. Number these rectangles 0 to $n-1$ and let the height of each rectangle be the value of $f(x)$ at the *left* $x$-axis corner. The $k^{\text{th}}$ rectangle has height $f(x_k)$ and area $\Delta x f(x_k)$. The total area of the $n$ rectangles, then, is




$$L_n = \Delta x(f(x_0) + f(x_1) + \cdots + f(x_{n-1}))$$
$$= \Delta x \sum_{k=0}^{n-1} f(x_k).$$

The larger $n$ gets, the better the approximation.

**Ex:** We approximate the area under the curve $y = x^2$ on the interval $[0, 1]$ with 4 rectangles of width $\Delta x = \frac{1}{4}$ and heights $0$, $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, for a total area of

$$L_4 = \tfrac{1}{4}\left((0)^2 + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{3}{4}\right)^2\right) = 0.21875.$$

### RIGHT-HAND RIEMANN SUMS

Using the same $n$ subintervals along the $x$-axis, this time we take the height of each rectangle to be the value of $f(x)$ at the *right* endpoint. The height of the $k^{\text{th}}$ rectangle is now $f(x_{k+1})$, for a total area of

$$R_n = \Delta x(f(x_1) + f(x_2) + \cdots + f(x_n)) = \Delta x \sum_{k=1}^{n} f(x_k).$$

Again, the larger $n$ gets, the better the approximation.

**Ex:** For $f(x) = x^2$ on the interval $[0, 1]$,

$$R_4 = \tfrac{1}{4}\left(\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{3}{4}\right)^2 + (1)^2\right) = 0.46875.$$

### MIDPOINT RULE

This time, take the height of each rectangle to be $f$ evaluated at the midpoint of the rectangle. The height of the $k^{\text{th}}$ rectangle is now

$$f\left(a + \Delta x\left(k + \tfrac{1}{2}\right)\right) = f\left(\tfrac{x_k + x_{k+1}}{2}\right),$$

for a total area of

$$M_n = \Delta x\left(f\left(\tfrac{x_0+x_1}{2}\right) + \cdots f\left(\tfrac{x_{n-1}+x_n}{2}\right)\right)$$
$$= \Delta x \sum_{k=0}^{n-1} f\left(\tfrac{x_k + x_{k+1}}{2}\right).$$

### GENERAL RIEMANN SUMS

In general, we can pick *any* **sample point** $x_k^*$ in the $k^{\text{th}}$ subinterval. The area approximation, then, is

$$\Delta x \sum_{k=0}^{n-1} f\left(x_k^*\right).$$

This general area approximation is called a **Riemann sum**, and its limit as $n$ increases gives the area of the region.

## THE DEFINITE INTEGRAL

The definite integral of $f(x)$ from $x = a$ to $b$ is defined as the limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{k=0}^{n-1} f\left(x_k^*\right).$$

If this limit exists, then we say that $f$ is **integrable** on the interval $[a, b]$. This integral represents the area under $f$.

- In this notation, $\int$ is the **integral sign**, $f(x)$ is the **integrand**, and $b$ and $a$ are the **upper** and **lower limits of integration**.
- The marker $dx$ keeps track of the variable of integration and evokes a very small $\Delta x$.
- Just because a function is integrable does not mean that its integral function is easy to write down in closed form.
- All **piecewise continuous** functions (those made up of finitely many pieces of continuous functions) are integrable. In practice, every function encountered in a Calculus class will be integrable except at points where it blows up.

## THE INDEFINITE INTEGRAL

**Antidifferentiation** is the reverse of differentiation: an **antiderivative** of $f(x)$ is any function $F(x)$ whose derivative is equal to the original function: $F'(x) = f(x)$. Functions that differ by constants have the same derivative; therefore, we look for a **family of antiderivatives** $F(x) + C$, where $C$ is any real constant.

The family of the antiderivatives of $f(x)$ is denoted by the **indefinite integral**:

$$\int f(x)\,dx = F(x) + C$$

if and only if $F'(x) = f(x)$.

## FUNDAMENTAL THEOREM OF CALCULUS

The **Fundamental Theorem of Calculus (FTC)** brings together differential and integral calculus. The main point is that differentiation and integration are reverse processes: finding antiderivatives is a lot like calculating areas under curves.

### PART 1

Let $f(x)$ be a function continuous on the interval $[a, b]$. Then the area function

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$ and differentiable on $(a\, b)$, and

$$F'(x) = f(x).$$

### PART 2

If $f(x)$ is a function continuous on the interval $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The total change in the antiderivative function over an interval is the same as the area under the curve.

# TECHNIQUES of INTEGRATION

Unlike differentiation, integration is difficult—there are easy-to-write-down functions that don't have easy antiderivatives. The art of integration requires a bag of tricks. These are them.

## BASIC INTEGRALS

$$\int k\,dx = kx + C \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ if } n \neq -1$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \sec x \tan x\,dx = \sec x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

## INTEGRAL PROPERTIES

Properties written for indefinite integrals also hold for definite integrals. Let $f$ and $g$ be integrable on $[a, b]$.

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

For any real number $c$,

$$\int cf(x)\,dx = c\int f(x)\,dx.$$

For any $p$ with $a < p < b$,

$$\int_a^p f(x)\,dx + \int_p^b f(x)\,dx = \int_a^b f(x)\,dx.$$

Betweenness: If $f(x) \le g(x)$ on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

## u-SUBSTITUTION

If $u = g(x)$ is continuously differentiable on some interval and $f(x)$ is integrable on the range of $g(x)$, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du,$$

or, with limits,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} u\,du.$$

- This is the analog of the **Chain Rule** for integrals. It is useful for composite functions and products.

**Ex:** $\int x^2\sqrt{x^3+8}\,dx$. Since $x^2$ is a lot like $\frac{d(x^3+8)}{dx}$, let $u = x^3 + 8$. Then $du = 3x^2\,dx$, so $x^2\,dx = \frac{du}{3}$. Substituting, we transform the original integral into $\int \frac{1}{3}\sqrt{u}\,du$, or $\int \frac{1}{3}u^{\frac{1}{2}}\,du = \frac{2}{9}u^{\frac{3}{2}} + C = \frac{2}{9}\left(x^3+8\right)^{\frac{3}{2}} + C.$

## INTEGRATION BY PARTS

Some function products and quotients cannot be integrated by substitution alone. Integration by parts works when one piece of the product has a simpler derivative and the other piece is easy to integrate.

- This is the integral analog of the **Product Rule**:

$$\frac{d(fg)}{dx} = f'g + fg'.$$

### INDEFINITE INTEGRALS

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx, \text{ or}$$
$$\int u\,dv = uv - \int v\,du$$

### DEFNITIE INTEGRALS

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$$

**Ex:** $\int (2x+1)e^{-x}\,dx$

Let $u = 2x + 1$ (so $du = 2\,dx$), and $dv = e^{-x}dx$ (so $v = -e^{-x}$). The integral becomes

$$(2x+1)\left(-e^{-x}\right) - \int -2e^{-x}\,dx,$$

which simplifies to $-(2x+1)e^{-x} - 2e^{-x} + C$, or $(-2x-3)e^{-x} + C$.

## PARTIAL FRACTIONS

To integrate a **rational function**—a quotient of two polynomials—first decompose it as a sum of simpler **"partial" fractions.** If the denominator factors as a product of linear factors (and it always will on the AP), the partial fractions will be constants over the linear factors. Solve for the constants; then use $\int \frac{du}{u} = \ln|u| + C$ to integrate each term.

### PARTIAL FRACTIONS STEP BY STEP

To integrate $f(x) = \frac{p(x)}{q(x)} \ldots$

1. If necessary, **use long division** to make sure that the degree of the numerator is less than the degree of the denominator. Integrate the polynomial part separately.
2. **Factor the denominator** into a product of linear factors. If $q(x)$ is quadratic, then $q(x) = (ax+b)(cx+d)$.
3. **Decompose** $\frac{p(x)}{q(x)}$ **into a sum of partial fractions.** If $q(x)$ is quadratic, then

$$\frac{p(x)}{q(x)} = \frac{A}{ax+b} + \frac{B}{cx+d}.$$

Solve for $A$ and $B$ by multiplying the equation by $q(x)$ and equating coefficients.

4. **Integrate** each partial fraction individually. For example, $\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C.$

**Ex:** To integrate $\frac{2x^3-x^2-18x-7}{2x^2-5x-12}$ use long division to reexpress

$$\frac{2x^3 - x^2 - 18x - 7}{2x^2 - 5x - 12} = (x+2) + \frac{4x+17}{2x^2-5x-12}.$$

Factor the denominator and set up the partial fraction:

$$\frac{4x+17}{2x^2-5x-12} = \frac{4x+17}{(2x+3)(x-4)} = \frac{A}{2x+3} + \frac{B}{x-4}.$$

Cross-multiply to get

$$4x + 17 = A(x-4) + B(2x+3)$$
$$= (A + 2B)x - 4A + 3B.$$

Equate coefficients to get $A + 2B = 4$ and $-4A + 3B = 17$. Add four times the first equation to the second to get $11B = 33$, or $B = 3$, which makes $A = -2$. Finally, integrate:

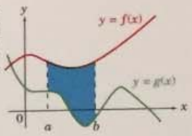
$$\int \frac{4x+17}{2x^2-5x-12}\,dx = \int \frac{-2}{2x+3} + \frac{3}{x-4}\,dx$$
$$= -\ln|2x+3| + 3\ln|x-4| + C.$$

# APPLICATIONS of INTEGRATION

## AREA

Suppose that $f(x) \geq g(x)$ on the interval $[a, b]$ and both functions are continuous. Then the area bounded by the two curves $y = f(x)$, $y = g(x)$ and the two vertical lines $x = a$ and $x = b$ is

$$\int_a^b (f(x) - g(x))\,dx.$$




- In general, the positive area between two (potentially intersecting) continuous functions $f$ and $g$ from $x = a$ to $x = b$ is $\int_a^b |f(x) - g(x)|\,dx$.

Shaded area is $\int_a^b (f(x) - g(x))\,dx$.

## PHYSICAL MOTION

If the position of a particle moving along a straight line is given by $s(t)$, then its velocity is given by $v(t) = s'(t)$ and its acceleration is given by $a(t) = v'(t) = s''(t)$. Conversely,

$$v(t) = v_0 + \int_0^t a(k)\,dk \text{ and } s(t) = s_0 + \int_0^t v(k)\,dk.$$

The total distance that the particle has traveled from $t = a$ to $t = b$ is given by

$$\int_a^b |v(t)|\,dt.$$

## VOLUME

If a solid in space is oriented so that the area of a **cross-section** (the slice of solid created by intersection with a plane) perpendicular to the $x$-axis is given by $A(x)$. Then the volume of the solid bounded by the planes $x = a$ and $x = b$ is

$$V = \int_a^b A(x)\,dx.$$

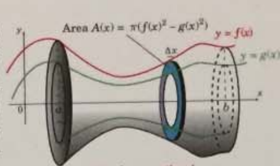
**DISK METHOD**

The volume of the solid swept out by the curve $y = f(x)$ as it revolves around the $x$-axis between $x = a$ and $x = b$ is given by

$$\int \pi(\text{radius})^2\,dx \text{ or } \pi \int_a^b (f(x))^2\,dx.$$

**WASHER METHOD**

If $f(x) \geq g(x)$ between $a$ and $b$, then the volume of the solid swept out between the two curves $y = f(x)$ and $y = g(x)$ as they revolve around the $x$-axis between $x = a$ and $x = b$ is



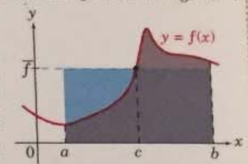

Washer method

$$\int_a^b \pi(\text{outer } r)^2 - \pi(\text{inner } r)^2\,dx = \pi \int_a^b (f(x))^2 - (g(x))^2\,dx.$$

## AVERAGE VALUE

For a discrete set of values, their average multiplied by their number gives their sum. The analog of an average for a continuous function $f(x)$ on the interval $[a, b]$ is the **average value** $\overline{f}$, which has the property that the rectangle of height $\overline{f}$ and width $b - a$ has the same area as is enclosed under the curve $y = f(x)$. Thus

$$\overline{f} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$




$\overline{f}$ is the average value of $f$ on the interval $[a, b]$. Blue area = gray area.

## ARC LENGTH

If $f(x)$ has a continuous derivative on the interval $(a, b)$, then the length of the curve from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

In Leibniz notation this becomes $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$

# DIFFERENTIAL EQUATIONS

A **differential equation** (diffeq) is any equation involving derivatives. The AP only uses only diffeqs that can be solved into the form $\frac{dy}{dx} = F(x, y)$, where $F$ is a function of $x$ and $y$.

- A **solution** (or a **particular solution**) to a diffeq is any curve that satisfies the diff-eq. **Ex:** $y = \sin 2x + 3\cos 2x$ is a solution to the diffeq $y'' + 4y = 0$.
- The **general solution** to a diffeq is the complete family of curves that satisfy the diffeq. **Ex:** $x^2 + y^2 = C$ where $C > 0$, is the general solution to the diffeq $y' = -\frac{x}{y}$.
- An **initial condition** is a point on the curve that chooses a particular solution out of the family of general solutions. **Ex:** The unique curve that both satisfies $y' = -\frac{x}{y}$ and passes through the point $(5, 0)$ is $x^2 + y^2 = 25$.

Although the vast majority of diffeqs are difficult to solve exactly, we can find approximate solutions—graphically, by sketching slope fields, or numerically, by using Euler's method.

## SEPARABLE DIFFEQS

A diffeq is called **separable** if it can be expressed in the form $\frac{dy}{dx} = f(x)g(y)$ where $f$ and $g$ depend on one variable only—that is, if it's possible to *separate* the variables completely.

**Ex:** $\frac{dy}{dx} = x\,e^y \sin^2 x$ is separable, but $\frac{dy}{dx} = x + y$ is not.

- To solve a separable diffeq, abuse Leibniz notation and rewrite as $\frac{1}{g(y)}\,dy = f(x)\,dx$. Integrate each side separately. Only one constant $C$ is necessary.
- Both exponential growth and the logistic equation are separable.

## EXPONENTIAL GROWTH AND DECAY

The diffeq $\frac{dy}{dt} = ky$ is a common separable diffeq.

- **Solution:** General solution is $y = Ae^{kt}$.
- **Solving:** Separate and rewrite as $\int \frac{dy}{y} = \int k\,dt$. Integrate to get $\ln|y| = kt + C$, or $\pm y = e^{kt+C}$. Since $e^C$ is a positive multiplicative factor, replace $\pm e^C$ by the constant $A$. Rewrite as $y = Ae^{kt}$.
- **Growth or decay?** If $k > 0$, the solution represents exponential growth; if $k < 0$, exponential decay.
- $A = y(0)$ is the initial value of the function.

**Word problems that reduce to $\frac{dy}{dt} = ky$**

- **Unlimited population growth:** $y$ is the population size at time $t$; $k$, often given as a percentage, is the **relative growth rate**.
- **Radioactive decay:** $y$ is the mass of the radioactive element present at time $t$; $k$ is negative, sometimes conveyed in terms of the element's **half-life** $h$—the amount of time it takes for half of the remaining mass of the element to decay; $k = -\frac{\ln 2}{h}$.

## SLOPE FIELDS

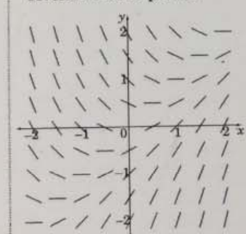
A **slope field** (or **direction field**) graphically shows the family of solutions to a diffeq in $\frac{dy}{dx} = F(x, y)$ form.

**Drawing a slope field**

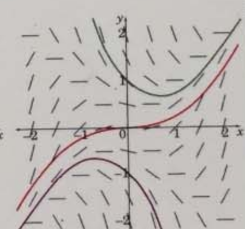
For each lattice point $(x_0, y_0)$, draw a short line segment with slope $F(x_0, y_0)$. This line mimics the shape of the particular solution to the diffeq that passes through $(x_0, y_0)$.

**Interpreting a slope field**

Find the particular solution that goes through a given point by following the shape of the slope field through that point. Find the family of solutions by tracing particular solutions through several different points.




Slope field for $\frac{dy}{dx} = x - y$




Slope field and three solutions to $\frac{dy}{dx} = x^2 - y^2$

**Features of a slope field**

- A slope field's dashes are horizontal whenever $F(x, y) = 0$. **Ex:** The slope field for $\frac{dy}{dx} = x^2 - y^2$ is horizontal whenever $x^2 - y^2 = 0$, that is, along the lines $x = y$ and $x = -y$.

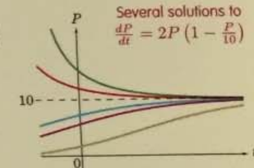
## LOGISTIC GROWTH

The **logistic equation** is a separable diffeq that models population growth taking into account limited natural resources.

- **Equation:** A population $P(t)$ with natural growth rate $k$ and maxiumum **carrying capacity** $P_m$ will satisfy the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{P_m}\right).$$

Several solutions to $\frac{dP}{dt} = 2P\left(1 - \frac{P}{10}\right)$




- **Solving:** Rewrite as $k\,dt = \frac{P_m}{P_m P - P^2}\,dP$ and integrate using partial fractions.
- **General solutuion:** $P(t) = \frac{P_m}{1 - Ae^{-kt}}$, where $A = \frac{P_0 - P_m}{P_0}$ is determined by **initial condition** $P_0 = P(0)$.

**Features of logistic growth**

- If $P_0 = 0$ or $P_0 = P_m$, then $\frac{dP}{dt} = 0$: the population is stable.
- If $0 < P_0 < P_m$, then $P(t)$ is always increasing and $\lim_{t \to \infty} P(t) = P_m$. Inflection point when $P(t) = \frac{P_m}{2}$.
- If $P_0 > P_m$, then $P(t)$ is always decreasing and concave up, and $\lim_{t \to \infty} P(t) = P_m$.

## EULER'S METHOD

**Euler's method** uses iterative approximations to find numerical solutions to a diffeq through a particular point.

- Given diffeq $\frac{dy}{dx} = F(x, y)$, initial point $(x_0, y_0)$, step size $\Delta x$, and number of steps $n$, Euler's method approximates $y_n = y(x_0 + n\Delta x)$:

$$x_1 = x_0 + \Delta x \qquad y_1 = y_0 + \Delta x F(x_0, y_0)$$
$$x_2 = x_1 + \Delta x \qquad y_2 = y_1 + \Delta x F(x_1, y_1)$$
$$\cdots \qquad \cdots$$
$$x_n = x_0 + n\Delta x \qquad y_n = y_{n-1} + \Delta x F(x_{n-1}, y_{n-1})$$

- Increasing $n$ or decreasing $\Delta x$ improves the approximation.

# IMPROPER INTEGRALS

**Improper integrals** come in two types.

1. Integrals over an infinite discontinuity. **Ex:** $\int_0^1 \frac{1}{x}\,dx$
2. Integrals over an infinite interval. **Ex:** $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

Some improper integrals **converge**, that is, represent a finite area; others **diverge**. To check convergence of an improper integral, reinterpret it as a limit expression and check whether the limit exists and is finite.

## INFINITE DISCONTINUITY

The integral $\int_a^b f(x)\,dx$ is improper if at some point $c$, with $a \leq c \leq b$, the function blows up; that is, if $\lim_{x \to c^\pm} f(x) = \pm\infty$.

**LEFT ENDPOINT** ($c = a$)

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

**RIGHT ENDPOINT** ($c = b$)

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

**INTERIOR** ($a < c < b$)

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The original integral converges only if *both* endpoint-improper integrals converge.

**Ex:** $\int_{-1}^1 \frac{dx}{x}$ diverges, but $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}$ converges, to 0.

- $\int_1^\infty \frac{dx}{x^r}$ converges (and equals $\frac{1}{r-1}$) if and only if $r > 1$.

## INFINITE INTERVAL

**INFINITE RAY TO THE RIGHT**

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

**INFINITE RAY TO THE LEFT**

$$\int_{-\infty}^b f(x)\,dx = \lim_{t \to -\infty} \int_t^b f(x)\,dx$$

**WHOLE REAL LINE**

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

for any $c$. The original integral converges only if both half-interval integrals converge.

- $\int_0^1 \frac{dx}{x^r}$ converges (and is equal to $\frac{1}{1-r}$) if and only if $r < 1$.

# PARAMETRIC EQUATIONS and POLAR CURVES

## PARAMETRIC EQUATIONS

A parametric curve defines both the $x$- and the $y$-coordinates in terms of a third variable, often $t$ for *time*.

- **Ex:** $x = \cos t$ and $y = \sin t$ for $0 \le t < 2\pi$ are **parametric equations** that describe the unit circle.

### CONVERTING

To convert a curve described parametrically to Cartesian coordinates, eliminate $t$ to relate $x$ and $y$ directly. You may or may find that $y$ is a proper function of $x$.

- **Ex:** Use $\cos^2\theta + \sin^2\theta = 1$ for to relate $x$ and $y$ with the familiar $x^2 + y^2 = 1$ equation for the unit circle.

## GEOMETRY OF THE CURVE

### SLOPE OF THE TANGENT LINE

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

If $\frac{dx}{dt} = 0$ but $\frac{dy}{dt} \neq 0$, then the tangent is vertical.

### AREA DEFINED BY CURVE

The area between the $x$-axis and the curve traced out from $t = a$ to $t = b$ is

$$A = \int_a^b y(t)x'(t)dt.$$

This integral is positive if the curve is moving forward (if $\frac{dx}{dt} > 0$ for $a < t < b$) and negative if the curve is moving backwards.

### ARC LENGTH

The length of a parametric curve traced out from $t = a$ to $t = b$ is

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt.$$

## POLAR COORDINATES

**Polar coordinates** describe a point $P = (r, \theta)$ on a plane in terms of its distance $r$ from the **pole** (usually, the origin $O$) and the (counterclockwise) angle $\theta$ that the line $\overline{OP}$ makes with a reference line (usually, the positive $x$-axis).

### CONVERTING FROM CARTESIAN TO POLAR

Use $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$.

### CONVERTING FROM POLAR TO CARTESIAN

Use $x = r\cos\theta$ and $y = r\sin\theta$.

## GEOMETRY OF THE CURVE

### SLOPE OF THE TANGENT LINE

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta}$$

Derived by considering the curve $r = f(\theta)$ as a parametric function with $x = r\cos\theta = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$.

### AREA

The area enclosed by rays at $\theta = \alpha$ and $\theta = \beta$ bounded by the curve $r = f(\theta)$ is

$$A = \int_\alpha^\beta \frac{1}{2}r^2\, d\theta.$$

# SEQUENCES and SERIES

Complicated functions can sometimes be approximated with infinite polynomials called **power series**, which are easy to differentiate and integrate.

## SEQUENCES

A **sequence** is an ordered list of real numbers, called **terms**. An **infinite sequence** has infinitely many terms.

- **Notation:** The infinite sequence $a_1, a_2, a_3, \ldots$ is represented by $\{a_k\}_{k=1}^{\infty}$ or just $\{a_k\}$.
- A sequence is defined **explicitly** with a formula for each term. **Ex:** $\{n^2\}_1^\infty$ explicitly defines the sequence $1, 4, 9, 16, \ldots$.
- A sequence is defined **recursively** if each term is defined using preceding term(s). **Ex:** $\{a_n\}$ with $a_1 = 1$ and $a_n = a_{n-1} + (2n-1)$ is a recursive definition for $1, 4, 9, \ldots$.

### LIMIT OF A SEQUENCE

Intuitively, the **limit** of an infinite sequence, denoted $\lim_{n\to\infty} a_n$, is the value that the sequence approaches.

- If the limit exists (and is finite), the sequence **converges**. Otherwise the sequence **diverges**. **Ex:** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ converges to 1. But both $\{(-1)^n\}$ and $\{n^2\}$ diverge.
- Formally, $\lim_{n\to\infty} a_n$ exists and is equal to $a$ if for all $\varepsilon > 0$ there exists an $N$ so that $|a_n - a| < \varepsilon$ whenever $n > N$.
- If the terms of sequence $\{a_n\}$ tend to infinity, we can say that $\lim_{n\to\infty} a_n = \infty$. But $\{a_n\}$ is still called divergent.

### INCREASING AND DECREASING SEQUENCES

Sequence $\{a_n\}$ is called **increasing** if $a_k \le a_{k+1}$ for all $k$. Similarly, $\{a_n\}$ is **decreasing** if $a_k \ge a_{k+1}$ for all $k$.

- A **monotonic** sequence is either increasing or decreasing.
- A sequence is **bounded above** if every term is smaller than a fixed constant. Similarly, a sequence is **bounded below** if every term is greater than a fixed constant. A **bounded** sequence is bounded both above and below.
- **Monotonic Sequence Theorem:** All bounded, monotonic sequences are convergent.

## SERIES

A **series** is a summed sequence. An **infinite series** has infinitely many terms.

- **Notation:** The infinite series $a_1 + a_2 + a_3 + \cdots$ is represented as $\sum_{k=1}^{\infty} a_k$ or just $\sum a_k$.
- A **partial sum** of a series is a cutoff series sum.
$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$
- A series **converges** if the sequence of partial sums has a finite limit. This limit is the **sum** of the series. If the partial sums do not converge, the series **diverges**.

### GEOMETRIC SERIES

A **geometric series** has the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{k=0}^{\infty} ar^k.$$

- **Convergence:** The series $\sum ar^k$ converges if and only if $|r| < 1$.
- **Sum:** If $|r| < 1$, the series converges to $\frac{a}{1-r}$.

### *p*-SERIES

A ***p*-series** has the form $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^p}$.

- **Convergence:** The series $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.
- **Harmonic series:** Byname for the divergent $p$-series with $p = 1$:
$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots = \sum_{n=1}^{\infty} \tfrac{1}{n}.$$

### ALTERNATING SERIES

An **alternating series** has terms with alternating $\pm$ signs:

$$\sum_{n=0}^{\infty} (-1)^n a_n \text{ with all } a_n \text{ positive.}$$

- **Alternating Series Test:** The alternating series converges if both
  1. The terms are decreasing: $a_{n+1} \le a_n$ for all $n$.
  2. The terms tend to zero: $\lim_{n\to\infty} a_n = 0$.
- **Error bound:** If $\sum(-1)^k a_k$ converges to $a$ and satisfies the conditions of the Alternating Series Test, then the error from truncation less than the next term: if $s_n = \sum_{k=0}^{n}(-1)^k a_k$ then $|s_n - a| < a_{n+1}$.

## CONVERGENCE TESTS

### DIVERGENCE TEST

If $\lim_{n\to\infty} a_n \neq 0$ (or if $\lim_{n\to\infty} a_n$ does not exist), then the series $\sum a_n$ diverges.

### COMPARISON TESTS

If $\sum a_n$ and $\sum b_n$ are series with positive terms . . .

- **Convergence:** If $\sum b_n$ converges, and $a_n \le b_n$ for all $n$, then $\sum a_n$ converges.
- **Divergence:** If $\sum b_n$ diverges and $a_n \ge b_n$ for all $n$, then $\sum a_n$ diverges.
- **Limit comparison test:** If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite and nonzero, then either both series converge or both series diverge.

### INTEGRAL TEST

If $\{a_n\}$ is a monotonically decreasing positive sequence and $f(x)$ is a continuous function with $f(n) = a_n$ for all $n$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^\infty f(x)\,dx$ converges.

### ABSOLUTE CONVERGENCE TEST

The series $\sum a_n$ **converges absolutely** if the series of absolute values $\sum |a_n|$ converges.

- **Absolute Convergence Test:** If a series conveges absolutely, then it is convergent.
- If $\sum |a_n|$ does not converge but the original series $\sum a_n$ does, then we say that $\sum a_n$ **converges conditionally**.

### RATIO TEST

Suppose $\sum a_n$ is a series and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ exists.

- If $L < 1$ then the series converges absolutely.
- If $L > 1$, or if the limit is infinite, then the series diverges.
- If $L = 1$ then the test is inconclusive.

## POWER SERIES

A **power series** is a formal function in the form of an infinite polynomial:

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots.$$

The $a_n$ are **coefficients** and $x$ is the variable. This series is **centered at** $a$. It may converge for some $x$ and diverge for others.

### CONVERGENCE

A power series about $a$ converges in one of three ways.

1. Only at $x = a$. Corresponds to $R = 0$.
2. For all real $x$. Corresponds to $R = \infty$.
3. In an interval of radius $R$ around $a$. That is, the series converges for $|x - a| < R$ and diverges for $|x - a| > a$. It may or may not converge at endpoints $a - R$ and $a + R$.

The number $R$ is the **radius of convergence**. It is related to the **interval of convergence**, but the interval also takes into account convergence at $a - R$ and $a + R$.

### FUNCTIONS AS POWER SERIES

The function $f$ given by $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is defined and differentiable on $(a - R, a + R)$, where $R$ is the radius of convergence of the power series. It may be integrated or differentiated **term by term**:

- **Derivative:** $f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$
- **Integral:** $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$

The radius of convergence of both $f'(x)$ and $\int f(x)\,dx$ is $R$ (but no prediction about convergence at $a - R$ and $a + R$).

## TAYLOR AND MACLAURIN SERIES

If the function $f$ can be represented by a power series around $a$, then the coefficients $a_n$ are given by

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

Here, $f^{(n)}$ is the $n^{\text{th}}$ derivative of $f$, and $n!$ is $n$ **factorial**: $n! = 1 \times 2 \times \cdots \times n$, with $0! \stackrel{\text{def}}{=} 1$.

### TAYLOR SERIES

The **Taylor series for** $f$ **centered at** $a$ has the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

It converges to $f$ at $a$ or in some interval around $a$.

- **Maclaurin series:** Byname for a Taylor series centered at 0:
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$
- **Taylor polynomials:** A Taylor series for $f$ cut off after the $x^n$ term gives an $n^{\text{th}}$-degree polynomial approximation to $f$:
$$f(x) \approx T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k.$$

**Ex:** The third degree Taylor polynomial for $f(x) = \sin x$ is $x + \frac{x^3}{6}$. **Ex:** The tangent line is the first-degree approximation.

### LAGRANGE ERROR BOUND (a.k.a. **Taylor's Inequality**)

Let $c$ be a point within the interval of convergence of the Taylor series for $f$ about $a$. If the $(n+1)^{\text{st}}$ derivative of $f$ is bounded, specifically, if $|f^{(n+1)}(x)| \le M$ for all $|x - a| \le c$, then:

- The error in evaluating $f(c)$ by the Taylor series truncated after the $n^{\text{th}}$-degree term is bounded by $\frac{Mc^{n+1}}{(n+1)!}$. In other words, $|f(c) - T_n(c)| \le \frac{Mc^{n+1}}{(n+1)!}$.
- In general,
$$|f(x) - T_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!} \text{ whenever } |x - a| \le c.$$

**Rule of thumb:** The error of a truncated Taylor series is less than something a lot like the next term after the cutoff.

### IMPORTANT MACLAURIN SERIES

| Function / Interval of convergence | Maclaurin series | |
|---|---|---|
| $e^x$ <br> all real $x$ | $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ | $= \sum_{n=0}^{\infty} \frac{x^n}{n!}$ |
| $\sin x$ <br> all real $x$ | $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ | $= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ |
| $\cos x$ <br> all real $x$ | $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ | $= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ |
| $\frac{1}{1-x}$ <br> $\lvert x\rvert < 1$ | $1 + x + x^2 + x^3 + \cdots$ | $= \sum_{n=0}^{\infty} x^n$ |
| $\ln(1-x)$ <br> $-1 \le x < 1$ | $-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ | $= -\sum_{n=1}^{\infty} \frac{x^n}{n}$ |

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